ON THE PROBLEM OF ELECTROMAGNETIC-FIELD QUANTIZATION

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Abstract. We consider the radiation field operators in a cavity with varying dielectric medium in terms of solutions of Heisenberg's equations of motion for the most general one-dimensional quadratic Hamiltonian. Explicit solutions of these equations are obtained and applications to the radiation field quantization, including randomly varying media, are briefly discussed.

1. Canonical Quantization

Radiation field quantization in the vacuum was introduced in original works of Born, Heisenberg and Jordan [10], [11] on matrix mechanics (see also books on quantum electrodynamics [1], [3], [6] and quantum optics [49], [66], [85], [88], [97]). A modern mathematical approach to quantization of mechanical systems is discussed in detail, for example, in [9], [26], [27], [93], [94], and/or [98] (see also [48] and the references therein). For a classical Hamiltonian system one replaces canonically conjugate coordinates and momenta by time-dependent operators $q_\lambda(t)$ and $p_\lambda(t)$ that satisfy the commutation rules

$$[q_\lambda(t),\, q_\mu(t)] = [p_\lambda(t),\, p_\mu(t)] = 0, \qquad [q_\lambda(t),\, p_\mu(t)] = i\hbar\delta_{\lambda\mu}. \tag{1.1}$$

The time-evolution is determined by the Heisenberg equations of motion [38]:

$$\frac{d}{dt}p_\lambda(t) = \frac{i}{\hbar}\left[p_\lambda(t),\, \mathcal{H}\right], \qquad \frac{d}{dt}q_\lambda(t) = \frac{i}{\hbar}\left[q_\lambda(t),\, \mathcal{H}\right], \tag{1.2}$$

with appropriate initial conditions.[1]

Traditionally, the electromagnetic-field quantization is considered under the assumption that the field occupies an empty box [1], [3], [20], [28]. The quantization of the field in a uniform dielectric medium was discussed in [32], [40], [42], [43], [44], [50] (see also [20], [24], [92] and the references therein). Yet the problem of electromagnetic-field quantization in time-dependent nonuniform linear nondispersive media remains an active research topic up to now [2], [7], [12], [13], [20], [33], [39], [41], [58], [60], [67], [77], [78], [79], [80], [92], [96].

In the present article, we study the radiation field operators in a varying dispersive medium, which is mathematically described by the most general phenomenological quadratic Hamiltonian $\mathcal{H}$ in an abstract Hilbert space. (We concentrate on a single photon cavity mode, say υ, with frequency $\omega_\upsilon = 1$ and use the units $c = \hbar = 1$. From now on, we shall usually omit the indices when dealing with the single mode under consideration.) In particular, our approach gives a natural

Date: July 23, 2013.

2010 *Mathematics Subject Classification.* Primary 81Q05, 35C05. Secondary 42A38.

Key words and phrases. Generalized harmonic oscillators, time-dependent Schrödinger equation, Heisenberg equations of motion, dynamic invariants, radiation field operators, Bogoliubov transformation, quantization in randomly varying media, Berry's phase, uncertainty relation, minimum-uncertainty squeezed states.

[1] The standard form of Heisenberg's equations can be obtained by the time reversal $t \to -t$.

description of squeezed photons that can be created as a result of parametric amplification of quantum fluctuations in the dynamic Casimir effect [17], [18], [54], [70], [72], [87], [99] and/or by similar dynamical amplification mechanisms including the Unruh effect [95] and Hawking radiation [8], [36], [37]. It is also useful for quantum fields propagating in nonstationary external potentials [7], [62], [96], [87], and for photon quantization in randomly varying media.

2. Solution of Heisenberg Equations for Nonautonomous Quadratic Systems

Our main result is the following.

Theorem 1. *The solution of the Heisenberg equations of motion* (1.2) *for the nonautonomous quadratic Hamiltonian*

$$H = a(t)p^2 + b(t)x^2 + c(t)xp - id(t) - f(t)x - g(t)p \tag{2.1}$$

(a, b, c, d, f, and g are suitable real-valued functions of time only) has the form

$$p_\lambda(t) = \frac{\widehat{b}(t) - \widehat{b}^\dagger(t)}{i\sqrt{2}}, \qquad q_\lambda(t) = \frac{\widehat{b}(t) + \widehat{b}^\dagger(t)}{\sqrt{2}}. \tag{2.2}$$

Here, the time-dependent annihilation $\widehat{b}(t)$ and creation $\widehat{b}^\dagger(t)$ operators are given by the Ansatz

$$\begin{aligned} \widehat{b}(t) &= \frac{e^{-2i\gamma(t)}}{\sqrt{2}} \left(\beta(t)\, x + \varepsilon(t) + i\frac{p - 2\alpha(t)\, x - \delta(t)}{\beta(t)} \right), \\ \widehat{b}^\dagger(t) &= \frac{e^{2i\gamma(t)}}{\sqrt{2}} \left(\beta(t)\, x + \varepsilon(t) - i\frac{p - 2\alpha(t)\, x - \delta(t)}{\beta(t)} \right) \end{aligned} \tag{2.3}$$

in terms of solutions of the Ermakov-type system

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = a\beta^4, \tag{2.4}$$

$$\frac{d\beta}{dt} + (c + 4a\alpha)\, \beta = 0, \tag{2.5}$$

$$\frac{d\gamma}{dt} + a\beta^2 = 0 \tag{2.6}$$

and

$$\frac{d\delta}{dt} + (c + 4a\alpha)\, \delta - f - 2g\alpha = 2a\beta^3\varepsilon, \tag{2.7}$$

$$\frac{d\varepsilon}{dt} - (g - 2a\delta)\, \beta = 0, \tag{2.8}$$

$$\frac{d\kappa}{dt} - g\delta + a\delta^2 = a\beta^2\varepsilon^2. \tag{2.9}$$

The time-independent (self-adjoint) operators x and p obey the canonical commutation rule $[x,p] = i$ in an abstract (complex) Hilbert space which implies that the relation

$$\widehat{b}(t)\widehat{b}^\dagger(t) - \widehat{b}^\dagger(t)\widehat{b}(t) = 1 \tag{2.10}$$

holds at all times.

Proof. These results can be verified by a direct, but somewhat tedious, calculation when one expands the solution in generators of the Heisenberg–Weyl algebra, namely $\{1, x, p\}$, with undetermined time-dependent complex coefficients and simplifies the commutators. The substitutions (2.2)–(2.3) allow us to derive equations (2.4)–(2.8), say from the first Heisenberg equation (2.11) below. (A *Mathematica* based proof is available on the article's website; see notebook *HeisenbergOscillators.nb* and [46] for an important program ingredient.[2]) Equation (2.9), which determines the global phase of the corresponding Fock states in the Schrödinger representation, does not show up in this proof, but will appear later (see Lemma 2). ∎

By back substitution, we see that the operators $\widehat{b}(t)$ and $\widehat{b}^{\dagger}(t)$ are solutions of the Heisenberg equations

$$\frac{d}{dt}\widehat{b}(t) = i\left[\,\widehat{b}(t),\, H\right], \qquad \frac{d}{dt}\widehat{b}^{\dagger}(t) = i\left[\,\widehat{b}^{\dagger}(t),\, H\right], \tag{2.11}$$

subject to the initial conditions

$$\begin{aligned}\widehat{b}(0) &= \frac{e^{-2i\gamma(0)}}{\sqrt{2}}\left(\beta(0)x + \varepsilon(0) + i\frac{p - 2\alpha(0)x - \delta(0)}{\beta(0)}\right),\\ \widehat{b}^{\dagger}(0) &= \frac{e^{2i\gamma(0)}}{\sqrt{2}}\left(\beta(0)x + \varepsilon(0) - i\frac{p - 2\alpha(0)x - \delta(0)}{\beta(0)}\right).\end{aligned} \tag{2.12}$$

To a certain extent, the creation and annihilation operators (2.3) allow us to incorporate the Schrödinger symmetry group of the harmonic oscillator, originally found in coordinate representation [73], [74], into a more abstract Heisenberg picture. (For the sake of simplicity, we have restricted ourselves to a single photon mode υ with frequency $\omega_{\upsilon} = 1$; see [52] for a detailed investigation of the special case of uniform media.)

A concept of dynamical invariants for generalized harmonic oscillators, which is crucial for constructing the corresponding Fock states from our creation and annihilation operators, has been recently revisited in [15], [84], and [90] (see [21], [23], [29], [59], [68], [69], [86] and the references therein for classical accounts).

3. Solving The Ermakov-Type System

A general solution of (2.4)–(2.9) is given by Lemma 3 of [57] in a real form (see also [51] and [64]). In order to proceed to a more compact form, one needs to recall some notation. The substitution

$$\alpha = \frac{1}{4a}\frac{\mu'}{\mu} - \frac{d}{2a} \tag{3.1}$$

reduces the inhomogeneous equation (2.4) to the second order ordinary differential equation

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = c_0(2a)^2\beta^4\mu, \tag{3.2}$$

which has the familiar time-varying coefficients

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \qquad \sigma(t) = ab - cd + d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right). \tag{3.3}$$

(In (3.2) and in the rest of the paper, we use a formal 'binary' parameter $c_0 = 0, 1$ for the sake of convenience.)

[2]See also *Koutschan.nb* [51].

The time-dependent coefficients α_0, β_0, γ_0, δ_0, ε_0, κ_0, which satisfy the homogeneous (Riccati-type) system (2.4)–(2.9), are given by (cf. [14], [57], [90])

$$\alpha_0(t) = \frac{1}{4a(t)} \frac{\mu_0'(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \tag{3.4}$$

$$\beta_0(t) = -\frac{\lambda(t)}{\mu_0(t)}, \qquad \lambda(t) = \exp\left(-\int_0^t (c(s) - 2d(s))\, ds\right), \tag{3.5}$$

$$\gamma_0(t) = \frac{1}{2\mu_1(0)} \frac{\mu_1(t)}{\mu_0(t)} + \frac{d(0)}{2a(0)}, \tag{3.6}$$

and

$$\delta_0(t) = \frac{\lambda(t)}{\mu_0(t)} \int_0^t \left[\left(f(s) - \frac{d(s)}{a(s)} g(s) \right) \mu_0(s) + \frac{g(s)}{2a(s)} \mu_0'(s) \right] \frac{ds}{\lambda(s)}, \tag{3.7}$$

$$\begin{aligned} \varepsilon_0(t) = {} & -\frac{2a(t)\lambda(t)}{\mu_0'(t)} \delta_0(t) + 8 \int_0^t \frac{a(s)\sigma(s)\lambda(s)}{(\mu_0'(s))^2} (\mu_0(s)\delta_0(s))\, ds \\ & + 2 \int_0^t \frac{a(s)\lambda(s)}{\mu_0'(s)} \left(f(s) - \frac{d(s)}{a(s)} g(s) \right) ds, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \kappa_0(t) = {} & \frac{a(t)\mu_0(t)}{\mu_0'(t)} \delta_0^2(t) - 4 \int_0^t \frac{a(s)\sigma(s)}{(\mu_0'(s))^2} (\mu_0(s)\delta_0(s))^2\, ds \\ & - 2 \int_0^t \frac{a(s)}{\mu_0'(s)} (\mu_0(s)\delta_0(s)) \left(f(s) - \frac{d(s)}{a(s)} g(s) \right) ds \end{aligned} \tag{3.9}$$

($\delta_0(0) = -\varepsilon_0(0) = g(0)/(2a(0))$ and $\kappa_0(0) = 0$), provided that μ_0 and μ_1 are the standard (real-valued) solutions of equation (3.2) when $c_0 = 0$ corresponding to the initial conditions $\mu_0(0) = 0$, $\mu_0'(0) = 2a(0) \neq 0$ and $\mu_1(0) \neq 0$, $\mu_1'(0) = 0$. (Proofs of these facts are outlined in [14] and [16]. The integrals are treated in the most general way, which may include stochastic calculus; see, for example, [76].)

Here, we would like to present a new compact form of these solutions. Let us introduce the complex-valued function

$$z(t) = \left(2\alpha(0) + \frac{d(0)}{a(0)}\right) \mu_0(t) + \frac{\mu_1(t)}{\mu_1(0)} + i\beta^2(0)\mu_0(t) \tag{3.10}$$

(a complex parametrization of Green's function, linear invariants, and Wigner functions of generalized harmonic oscillators are also discussed in [22], [23], [34], and [86]). Then

$$z(t) = c_1 E(t) + c_2 E^*(t), \tag{3.11}$$

where the complex-valued solutions are given by

$$E(t) = \frac{\mu_1(t)}{\mu_1(0)} + i\mu_0(t), \qquad E^*(t) = \frac{\mu_1(t)}{\mu_1(0)} - i\mu_0(t), \tag{3.12}$$

and the corresponding complex-valued parameters are defined by

$$c_1 = \frac{1 + \beta^2(0)}{2} - i\left(\alpha(0) + \frac{d(0)}{2a(0)}\right), \qquad c_2 = \frac{1 - \beta^2(0)}{2} + i\left(\alpha(0) + \frac{d(0)}{2a(0)}\right), \tag{3.13}$$

with

$$c_1 + c_2 = 1, \qquad |c_1|^2 - |c_2|^2 = c_1 - c_2^* = \beta^2(0). \tag{3.14}$$

In addition,

$$z(0) = c_1 + c_2 = 1, \qquad z'(0) = 2ia(0)\,(c_1 - c_2)\,. \tag{3.15}$$

When written in terms of the complex function z in (3.10), the complex conjugate functions E and E^* defined in (3.12) become

$$E = \frac{c_1^* z - c_2 z^*}{|c_1|^2 - |c_2|^2}, \qquad E^* = \frac{c_1 z^* - c_2^* z}{|c_1|^2 - |c_2|^2} \tag{3.16}$$

and

$$\mu_0 = \frac{z - z^*}{2i\,(c_1 - c_2^*)}, \qquad \frac{\mu_1}{\mu_1(0)} = \frac{(c_1^* - c_2^*)\,z + (c_1 - c_2)\,z^*}{2\,(c_1 - c_2^*)}. \tag{3.17}$$

One can readily verify that

$$\alpha_0 = \frac{1}{4a}\frac{(z - z^*)'}{z - z^*} + \frac{d}{2a}, \qquad \beta_0 = -2i\lambda\frac{c_1 - c_2^*}{z - z^*},$$
$$\gamma_0 = \frac{(c_1^* - c_2^*)\,z + (c_1 - c_2)\,z^*}{2i\,(z - z^*)} + \frac{d(0)}{2a(0)}, \tag{3.18}$$

and equations (3.7)–(3.9) can also be rewritten in terms of the function z in view of (3.17). Finally, we introduce a second complex function,

$$\zeta(t) = \varepsilon(0)\beta(0) + i\,(\delta(0) + \varepsilon_0\,(t)) = c_3 + i\varepsilon_0, \qquad c_3 = \varepsilon(0)\beta(0) + i\delta(0), \tag{3.19}$$

and indicate the inverse relations between the essential, real and complex, parameters:

$$\alpha(0) = \frac{c_1^* - c_1}{2i} - \frac{d(0)}{2a(0)}, \qquad \beta^2(0) = c_1 - c_2^* = |c_1|^2 - |c_2|^2\,, \tag{3.20}$$

and

$$\delta(0) = \frac{c_3 - c_3^*}{2i}, \qquad \varepsilon(0) = \pm\frac{c_3 + c_3^*}{2\sqrt{|c_1|^2 - |c_2|^2}}. \tag{3.21}$$

Then the solution of the initial value problem for the Ermakov-type system can be expressed in terms of the complex function z in (3.10) as given in the lemma below.

Lemma 1. *The system* (2.4)–(2.9) *is solved by*

$$\alpha = \alpha_0 + \lambda^2\frac{c_1 - c_2^*}{2i\,|z|^2}\,\frac{z + z^*}{z - z^*}, \qquad \beta = \pm\lambda\frac{\sqrt{|c_1|^2 - |c_2|^2}}{|z|}, \qquad \gamma = \gamma(0) - \frac{1}{2}\arg z, \tag{3.22}$$

$$\delta = \delta_0 + \lambda\frac{\zeta z - \zeta^* z^*}{2i\,|z|^2}, \qquad \varepsilon = \pm\frac{\zeta z + \zeta^* z^*}{2\,|z|\,\sqrt{|c_1|^2 - |c_2|^2}}, \tag{3.23}$$

$$\kappa = \kappa(0) + \kappa_0 + \frac{\left(\zeta^2 z + \zeta^{*\,2} z^*\right)(z - z^*)}{8i\,(c_1 - c_2^*)\,|z|^2}, \tag{3.24}$$

with z and ζ as in (3.10) *and* (3.19), *respectively. (The solution of the Ermakov-type equation* (3.2) *is given by $\mu = \mu(0)\,|z|$.)*

Proof. This amounts to a straightforward calculation using Lemma 3 in [57]. ∎

As a consequence, one gets

$$\frac{2i\,(\alpha-\alpha_0)}{\beta^2}=\frac{z+z^*}{z-z^*}, \tag{3.25}$$

$$i\,(\alpha-\alpha_0)+\frac{\beta^2}{2}=\beta^2\frac{z}{z-z^*}, \qquad i\,(\alpha-\alpha_0)-\frac{\beta^2}{2}=\beta^2\frac{z^*}{z-z^*}, \tag{3.26}$$

$$\varepsilon+i\frac{\delta-\delta_0}{\beta}=\frac{\zeta z}{\beta(0)\,|z|}, \qquad \varepsilon-i\frac{\delta-\delta_0}{\beta}=\frac{\zeta^* z^*}{\beta(0)\,|z|}, \tag{3.27}$$

$$\varepsilon^2+\left(\frac{\delta-\delta_0}{\beta}\right)^2=\varepsilon^2(0)+\left(\frac{\delta(0)+\varepsilon_0}{\beta(0)}\right)^2, \tag{3.28}$$

and

$$\kappa=\kappa(0)+\kappa_0+\frac{\delta-\delta_0}{2\beta}\varepsilon-\frac{\varepsilon_0+\delta(0)}{2\beta(0)}\varepsilon(0). \tag{3.29}$$

These "quasi-invariants" can be useful, for example, when making a comparison of calculations done by different approximation methods.

Examples of explicitly integrable quadratic systems are discussed in [15], [16], [22], [23], [59], [63], [64], [65], [68], [86], and [100] (see also the references therein).

4. Single Mode Fock States for Nonautonomous Quadratic Hamiltonians

The time-dependent quadratic operator (see [84])

$$\widehat{E}(t)=\frac{1}{2}\left[\frac{(p-2\alpha x-\delta)^2}{\beta^2}+(\beta x+\varepsilon)^2\right]=\frac{1}{2}\left[\widehat{a}(t)\widehat{a}^\dagger(t)+\widehat{a}^\dagger(t)\widehat{a}(t)\right], \tag{4.1}$$

with the defining property

$$i\frac{d\widehat{E}}{dt}+\widehat{E}H-H\widehat{E}=0, \tag{4.2}$$

extends the standard Hamiltonian (and/or number operator) for any given solution of the Ermakov-type system. (The initial data play a role of integrals/constants of motion and/or quantum numbers for the creation and annihilation operators in the Heisenberg representation.) We use the substitution $\widehat{b}(t)=e^{-2i\gamma}\widehat{a}(t)$ and $\widehat{b}^\dagger(t)=\widehat{a}^\dagger(t)e^{2i\gamma}$ with $\left[\widehat{a}(t),\widehat{a}^\dagger(t)\right]=1$. The oscillator-type spectrum,

$$\widehat{E}(t)\,|\Psi_n(t)\rangle=\left(n+\frac{1}{2}\right)|\Psi_n(t)\rangle\,, \tag{4.3}$$

can be obtained in a standard fashion (with the aid of modified variable creation and annihilation operators; cf. [1], [9], [30], [31]):

$$\widehat{a}(t)\,|\Psi_n(t)\rangle=\sqrt{n}\ |\Psi_{n-1}(t)\rangle\,, \quad \widehat{a}^\dagger(t)\,|\Psi_n(t)\rangle=\sqrt{n+1}\ |\Psi_{n+1}(t)\rangle\,. \tag{4.4}$$

Here and in what follows, it is convenient to use the orthogonality relation $\langle\Psi_m(t),\Psi_n(t)\rangle=\delta_{mn}\,\lambda^{-1}$ with $\beta(0)\mu(0)=1$.

Now we can analyze abstract Fock states in the Schrödinger representation (see also [29]).

Lemma 2. *Let* $\widehat{a}(t)\,|\Psi_0(t)\rangle = 0$. *The dynamic Fock states given by*

$$|\psi_n(t)\rangle = e^{i(2n+1)\gamma+i\kappa}\,|\Psi_n(t)\rangle = \frac{e^{i(2n+1)\gamma+i\kappa}}{\sqrt{n!}}\left(\widehat{a}^{\dagger}(t)\right)^n|\Psi_0(t)\rangle \tag{4.5}$$

satisfy the time-dependent Schrödinger equation

$$i\frac{d}{dt}\,|\psi_n(t)\rangle = H\,|\psi_n(t)\rangle \tag{4.6}$$

with the general quadratic Hamiltonian (2.1) *provided that equation* (2.9) *for the global phase holds and* $\langle\Phi_n, d\Phi_n/dt\rangle = 0$ *for* $\Phi_n = \lambda^{1/2}e^{-i\left(\alpha x^2+\delta x\right)}\Psi_n$.

Proof. From (4.2)–(4.3), one gets, formally,

$$\widehat{E}\left(i\frac{d}{dt}\,|\psi_n(t)\rangle - H\,|\psi_n(t)\rangle\right) = \left(n+\frac{1}{2}\right)\left(i\frac{d}{dt}\,|\psi_n(t)\rangle - H\,|\psi_n(t)\rangle\right). \tag{4.7}$$

Therefore

$$i\frac{d}{dt}\,|\psi_n(t)\rangle - H\,|\psi_n(t)\rangle = c_n\,|\psi_n(t)\rangle\,, \tag{4.8}$$

where $\operatorname{Im} c_n = 0$ in view of the normalization condition; see [15], [29], [59]. (We assume also that the vacuum state is nondegenerate.)

Here,

$$H = ap^2 + bx^2 + \frac{c}{2}\,(px+xp) + \frac{i}{2}\,(c-2d) - fx - gp, \tag{4.9}$$

and the position and linear momentum operators are given by

$$x = \frac{1}{\beta}\left[\frac{1}{\sqrt{2}}\left(\widehat{a}+\widehat{a}^{\dagger}\right) - \varepsilon\right], \tag{4.10}$$

$$p = \frac{\beta}{i\sqrt{2}}\left(\widehat{a}-\widehat{a}^{\dagger}\right) + \frac{\sqrt{2}\alpha}{\beta}\left(\widehat{a}+\widehat{a}^{\dagger}\right) + \delta - \frac{2\alpha\varepsilon}{\beta}. \tag{4.11}$$

In terms of creation and annihilation operators, the Hamiltonian takes the form

$$\begin{aligned}
H = {} & \left[\frac{a}{2}\left(\frac{4\alpha^2}{\beta^2}-\beta^2\right) + \frac{b+2c\alpha}{2\beta^2} - \frac{i}{2}\,(c+4a\alpha)\right](\widehat{a})^2 \\
& + \left[\frac{a}{2}\left(\frac{4\alpha^2}{\beta^2}-\beta^2\right) + \frac{b+2c\alpha}{2\beta^2} + \frac{i}{2}\,(c+4a\alpha)\right]\left(\widehat{a}^{\dagger}\right)^2 \\
& + \frac{1}{2}\left[a\left(\beta^2+\frac{4\alpha^2}{\beta^2}\right) + \frac{b+2c\alpha}{\beta^2}\right]\left(\widehat{a}\widehat{a}^{\dagger}+\widehat{a}^{\dagger}\widehat{a}\right) + \frac{i}{2}\,(c-2d) \\
& + \sqrt{2}\left[\frac{4a\alpha+c}{2\beta}\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right) - \frac{\varepsilon}{\beta^2}\,(b+c\alpha) - \frac{f+2g\alpha}{2\beta}\right. \\
& \qquad \left. + \frac{i}{2}\left(\beta\,(g-2a\delta) + \varepsilon\,(c+4a\alpha)\right)\right]\widehat{a} \\
& + \sqrt{2}\left[\frac{4a\alpha+c}{2\beta}\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right) - \frac{\varepsilon}{\beta^2}\,(b+c\alpha) - \frac{f+2g\alpha}{2\beta}\right. \\
& \qquad \left. - \frac{i}{2}\left(\beta\,(g-2a\delta) + \varepsilon\,(c+4a\alpha)\right)\right]\widehat{a}^{\dagger}
\end{aligned}$$

$$+ a\left(\delta - \frac{2\alpha\varepsilon}{\beta}\right)^2 + \frac{\varepsilon}{\beta}\left(f + \frac{b\varepsilon}{\beta}\right) - \left(\delta - \frac{2\alpha\varepsilon}{\beta}\right)\left(g + \frac{c\varepsilon}{\beta}\right), \tag{4.12}$$

where we have corrected a typo in [84]. As a result, by (4.4)–(4.3) we have

$$\begin{aligned} \lambda \operatorname{Re}\langle \Psi_n, H\Psi_n\rangle = \left(n + \frac{1}{2}\right)\left[a\left(\beta^2 + \frac{4\alpha^2}{\beta^2}\right) + \frac{b + 2c\alpha}{\beta^2}\right] \\ + a\left(\delta - \frac{2\alpha\varepsilon}{\beta}\right)^2 + \frac{\varepsilon}{\beta}\left(f + \frac{b\varepsilon}{\beta}\right) - \left(\delta - \frac{2\alpha\varepsilon}{\beta}\right)\left(g + \frac{c\varepsilon}{\beta}\right) \end{aligned} \tag{4.13}$$

in terms of solutions of the Ermakov-type system.

In order to complete the proof, one can repeat the evaluation of Berry's phase [4] for generalized harmonic oscillators, given in [84] in coordinate representation, in a more abstract form. Indeed,

$$\langle \psi_n, H\psi_n\rangle = \left\langle \psi_n, i\frac{d}{dt}\psi_n\right\rangle = -\lambda^{-1}\left[(2n+1)\frac{d\gamma}{dt} + \frac{d\kappa}{dt}\right] + \left\langle \Psi_n, i\frac{d}{dt}\Psi_n\right\rangle = \langle \Psi_n, H\Psi_n\rangle. \tag{4.14}$$

In general,

$$\Psi_n = e^{(1/2)\int(c-2d)\ dt} e^{i\left(\alpha x^2 + \delta x\right)}\Phi_n, \qquad \langle \Phi_m, \Phi_n\rangle = \delta_{mn}, \tag{4.15}$$

and

$$\begin{aligned} \lambda\left\langle \Psi_n, \frac{d\Psi_n}{dt}\right\rangle &= i\left\langle \Phi_n, \left(\frac{d\alpha}{dt}x^2 + \frac{d\delta}{dt}x\right)\Phi_n\right\rangle + \frac{1}{2}(c-2d) + \left\langle \Phi_n, \frac{d\Phi_n}{dt}\right\rangle \\ &= i\frac{d\alpha}{dt}\lambda\left\langle \Psi_n, x^2\Psi_n\right\rangle + i\frac{d\delta}{dt}\lambda\langle \Psi_n, x\Psi_n\rangle + \frac{1}{2}(c-2d), \end{aligned} \tag{4.16}$$

where $\langle \Phi_n, d\Phi_n/dt\rangle = 0$ by our hypothesis. Here,

$$\lambda\langle \Psi_n, x\Psi_n\rangle = -\varepsilon\beta^{-1}, \qquad \lambda\left\langle \Psi_n, x^2\Psi_n\right\rangle = \beta^{-2}\left(\varepsilon^2 + n + \frac{1}{2}\right). \tag{4.17}$$

As a result,

$$\lambda \operatorname{Re}\left(i\left\langle \Psi_n, \frac{d\Psi_n}{dt}\right\rangle\right) = -\beta^{-2}\left(\varepsilon^2 + n + \frac{1}{2}\right)\frac{d\alpha}{dt} + \varepsilon\beta^{-1}\frac{d\delta}{dt}. \tag{4.18}$$

Finally, in view of (4.8), (4.13) and (4.18), we obtain (after some simplification) that

$$c_n = \langle \Phi_n, id\Phi_n/dt\rangle = 0$$

for any given solution of the Ermakov-type system (2.4)–(2.9). This completes the proof. ∎

Remark. In coordinate representation, when Φ_n are, essentially, the real-valued stationary orthonormal wave functions for the simple harmonic oscillator with respect to the new variable $\xi = \beta x + \varepsilon$ (see [55], [57], [75], and [84] for more details), the equation $\langle \Phi_n, d\Phi_n/dt\rangle = 0$ is valid due to the normalization condition $\|\Phi_n\|^2 = 1$. In general, one gets $c_n = \langle \Phi_n, id\Phi_n/dt\rangle$, and the previous connection is associated with a transport law for line bundles in the Hilbert space, namely, the change $d\Phi_n$ is orthogonal to Φ_n [89].

The last lemma can be reformulated in terms of an analog of Berry's phase [4],

$$\frac{d\theta_n}{dt} = \lambda \operatorname{Re}\left\langle \psi_n, i\frac{d}{dt}\psi_n\right\rangle - \frac{d\varphi_n}{dt}, \tag{4.19}$$

where $\varphi_n(t) = -(2n+1)\gamma(t)$ is the dynamical phase [84] (see also [91] for an example).

Lemma 3. *The Fock states, given by* (4.5) *in terms of solutions of the Ermakov-type system, satisfy the Schrödinger equation* (4.6) *if and only if the two expressions* (40) *and* (48) *in* [84] *for the derivative of the Berry phase become equivalent.*

Remark. In [84] it is shown that the two expressions (40) and (48) are equivalent in coordinate representation, when a particular form of coordinate and momentum are taken together with the L^2 inner product. The point of the above lemma is that this happens in an abstract Hilbert space if and only if the Fock states satisfy the Schrödinger equation (4.6).

5. Expectation Values and Variances

By (4.10)–(4.11) and (4.4), one gets

$$\langle \psi_n(t), x\psi_n(t)\rangle = -\frac{\lambda\varepsilon}{\beta}, \qquad \overline{x} = \frac{\langle x\rangle}{\langle 1\rangle} = \frac{\langle \psi_n, x\psi_n\rangle}{\langle \psi_n, \psi_n\rangle} = -\frac{\varepsilon(t)}{\beta(t)}, \tag{5.1}$$

$$\langle \psi_n(t), p\psi_n(t)\rangle = \lambda\left(\delta - \frac{2\alpha\varepsilon}{\beta}\right), \quad \overline{p} = \frac{\langle p\rangle}{\langle 1\rangle} = \frac{\langle \psi_n, p\psi_n\rangle}{\langle \psi_n, \psi_n\rangle} = \delta(t) - \frac{2\alpha(t)\varepsilon(t)}{\beta(t)}. \tag{5.2}$$

(The relation to the Ehrenfest theorem is discussed in [57].)

The standard deviations,

$$(\delta p)^2 = \frac{\left\langle (\Delta p)^2\right\rangle}{\langle 1\rangle} = \overline{(p^2)} - (\overline{p})^2, \qquad (\delta x)^2 = \frac{\left\langle (\Delta x)^2\right\rangle}{\langle 1\rangle} = \overline{(x^2)} - (\overline{x})^2, \tag{5.3}$$

are given by

$$(\delta p)^2 = \left(n+\frac{1}{2}\right)\left(\beta^2 + \frac{4\alpha^2}{\beta^2}\right), \qquad (\delta x)^2 = \left(n+\frac{1}{2}\right)\beta^{-2} \tag{5.4}$$

in terms of solutions of the Ermakov-type system. In particular,

$$(\delta p)^2 (\delta x)^2 = \left(n+\frac{1}{2}\right)^2 \left(1 + \frac{4\alpha^2}{\beta^4}\right) \geq \frac{1}{4} \tag{5.5}$$

as required by the fundamental Heisenberg uncertainty relation [7], [15], [38]. The minimum-uncertainty squeezed states occur for $n = 0$ if $\alpha(t_{\min}) = 0$.

6. Electromagnetic-Field Quantization in Varying Media

In the macroscopic approach, one can present the (noncommuting) vector field operators of the electric displacement $\boldsymbol{D}(\boldsymbol{r}, t)$ and the magnetic induction $\boldsymbol{B}(\boldsymbol{r}, t)$, which fully describe the properties of the quantized electromagnetic radiation inside a cavity filled with linear nonstationary dielectric material (with factorized electric permittivity and magnetic permeability tensors [20]), by the eigenfunction expansions (cf. [7], [25], [38], [45], [78], [85])

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{r}, t) &= \sum_v \varpi_v(t) p_v(t) \boldsymbol{D}_v(\boldsymbol{r}), \\ \boldsymbol{B}(\boldsymbol{r}, t) &= \sum_v \omega_v(t) q_v(t) \boldsymbol{B}_v(\boldsymbol{r}), \end{aligned} \tag{6.1}$$

in the Heisenberg picture when the time evolution is introduced through the equations (2.2)–(2.3), which provides a more direct analogy between quantum and classical physics [7], [35]. For a discussion of properties of the stationary orthonormal eigenfunctions $\boldsymbol{D}_\upsilon(\boldsymbol{r})$ and $\boldsymbol{B}_\upsilon(\boldsymbol{r})$ defined by the geometry of the cavity and given boundary conditions, see [2], [7], [12], [13], [19], [20], [25], [32], [50], [60], [67], [77], [78], and [81].

In view of the phenomenological Maxwell equations, the single electromagnetic radiation mode υ in a cavity resonator is analogous to a parametric (driven) harmonic oscillator [13], [20], [28], [67], [77], [78]. This analogy between classical mechanics and electrodynamics allows one to determine functions $\omega_\upsilon(t)$ and $\varpi_\upsilon(t)$ from the electric permittivity, magnetic permeability, and conductivity of the (slowly-)varying medium in connection with the quadratic Hamiltonian (2.1) (see Appendix A for more details). After quantization of the field Hamiltonian, the time-dependent operators $p_\upsilon(t)$ and $q_\upsilon(t)$ are determined by Theorem 1, and the corresponding Fock states are constructed in Lemma 2. (The average fields obey the classical Maxwell equations [7].) Methods of stochastic calculus [76] can be used in the case of randomly varying media.

In Schrödinger's picture, for the diagonal matrix elements of the field oscillators, we get

$$\begin{aligned} \langle \boldsymbol{D}(\boldsymbol{r},t)\rangle &= \boldsymbol{D}_\upsilon(\boldsymbol{r})\varpi_\upsilon(t)\,\langle \psi_n(t), p\psi_n(t)\rangle\,, \\ \langle \boldsymbol{B}(\boldsymbol{r},t)\rangle &= \boldsymbol{B}_\upsilon(\boldsymbol{r})\omega_\upsilon(t)\,\langle \psi_n(t), x\psi_n(t)\rangle\,, \end{aligned} \tag{6.2}$$

with (5.1)–(5.2) for a single mode υ, and the corresponding variances can be obtained with the help of (5.4). In the autonomous case (see [52]), the variances are given (up to a normalization) by equations (A.4)–(A.5) of [65].

7. Summary and Applications

In this article, the radiation field operators in a cavity with varying dielectric medium are constructed in terms of explicit solutions of Heisenberg's equations of motion. The phenomenological quadratic Hamiltonian under consideration corresponds to the most general (single mode) one-dimensional mathematical model of quantization in an abstract Hilbert space. Nonstandard solutions of these equations are obtained (in terms of quadratures), and applications to the radiation field quantization, including randomly varying media, are briefly discussed.

For most applications in (nonlinear) optics, the electromagnetic field can be treated classically. But, when quantum limits are approached and one is interested in the photon statistics of the field, a quantum description is required (see [24], [32], [39], [40] and the references therein). The explicit form of the Bogoliubov transformation (2.3) (for nonautonomous quadratic systems) is one of the starting points in this approach [7], [20], [77], [100]. The interaction of multiple modes, on the one hand, and microscopic (lossy) medium models, on the other hand, are also under consideration [42], [47], [50], [58], [60], [61], [80], [87], [92].

The problem of quantization of electromagnetic field in material media remains important in view of recent trends in the flourishing cavity QED [17], [19], [47], [72], [80] and for experiments in quantum optics in media [7], [32], [33], [58], [81], [96], which may result in a better understanding of the interaction of light with matter. Among other possible applications of the electromagnetic wave propagation in time-dependent media are the modulation of microwave power [71], wave propagation in ionized plasma [53], and magnetoelastic delay lines [82], [83] (see also [13] and the references therein).

Acknowledgments. We thank Michael Berry for bringing Ref. [5] to our attention and Victor Dodonov and Luc Vinet for valuable discussions. This research was partially supported by an AFOSR grant FA9550-11-1-0220 and by the Austrian Science Foundation FWF, grants Z130-N13 and S50-N15, the latter in the framework of the Special Research Program "Algorithmic and Enumerative Combinatorics".

Appendix: Factorized Media

For the phenomenological Maxwell equations in linear, dispersive, time-varying media, namely

$$\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \operatorname{div} \boldsymbol{D} = 4\pi\rho, \tag{A.1}$$

$$\operatorname{curl} \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}, \qquad \operatorname{div} \boldsymbol{B} = 0, \tag{A.2}$$

$$\boldsymbol{D} = \widetilde{\varepsilon}(\boldsymbol{r},t)\boldsymbol{E}, \qquad \boldsymbol{B} = \widetilde{\mu}(\boldsymbol{r},t)\boldsymbol{H}, \qquad \boldsymbol{j} = \widetilde{\sigma}(\boldsymbol{r},t)\boldsymbol{E}, \tag{A.3}$$

the continuity equation,

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{j} = \frac{\partial \rho}{\partial t} + \frac{4\pi\widetilde{\sigma}}{\widetilde{\varepsilon}}\rho + \boldsymbol{D}\cdot \operatorname{grad}\left(\frac{\widetilde{\sigma}}{\widetilde{\varepsilon}}\right) = 0, \tag{A.4}$$

has the stationary solution $\rho \equiv 0$ under the condition $\operatorname{grad}(\widetilde{\sigma}/\widetilde{\varepsilon}) = 0$.

With the help of the vector $\boldsymbol{A}$ and scalar φ potentials,

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}, \qquad \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \operatorname{grad}\varphi, \tag{A.5}$$

the Maxwell equations can be reduced to the gauge condition

$$\frac{1}{c}\operatorname{div}\left(\widetilde{\varepsilon}\frac{\partial \boldsymbol{A}}{\partial t}\right) = \operatorname{div}\left(\widetilde{\varepsilon}\operatorname{grad}\varphi\right) \tag{A.6}$$

and the single second-order generalized wave equation

$$\operatorname{curl}\left(\widetilde{\mu}^{-1}\operatorname{curl}\boldsymbol{A}\right) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\widetilde{\varepsilon}\frac{\partial \boldsymbol{A}}{\partial t}\right) + \frac{4\pi\widetilde{\sigma}}{c^2}\frac{\partial \boldsymbol{A}}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\left(\widetilde{\varepsilon}\operatorname{grad}\varphi\right) + \frac{4\pi\widetilde{\sigma}}{c}\operatorname{grad}\varphi. \tag{A.7}$$

Here, we consider the factorized (real-valued) dielectric permittivity, the magnetic permeability, and the conductivity (tensors)

$$\widetilde{\varepsilon}(\boldsymbol{r},t) = \xi(t)\overline{\varepsilon}(\boldsymbol{r}), \qquad \widetilde{\mu}(\boldsymbol{r},t) = \eta(t)\overline{\mu}(\boldsymbol{r}), \qquad \widetilde{\sigma}(\boldsymbol{r},t) = \chi(t)\overline{\sigma}(\boldsymbol{r}) \tag{A.8}$$

(the case $\widetilde{\sigma} \equiv 0$ was discussed in [20]). Under the imposed condition of $\operatorname{grad}(\widetilde{\sigma}/\widetilde{\varepsilon}) = 0$, one can choose $4\pi\overline{\sigma} = \overline{\varepsilon}$ without loss of generality.

The solution of the classical problem for a given single mode, υ say, has the form

$$\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{u}(\boldsymbol{r})q(t), \qquad \varphi(\boldsymbol{r},t) = (k/c)\frac{dq}{dt}\phi(\boldsymbol{r})$$

(k is a constant), and

$$\boldsymbol{B} = q\operatorname{curl}\boldsymbol{U}, \qquad \boldsymbol{D} = -\frac{\xi}{c}\frac{dq}{dt}\,\overline{\varepsilon}\boldsymbol{U}, \tag{A.9}$$

provided that

$$\operatorname{curl}\left(\frac{1}{\overline{\mu}}\operatorname{curl}\boldsymbol{U}\right) = \upsilon^2\,\overline{\varepsilon}\boldsymbol{U}, \qquad \boldsymbol{U} = \boldsymbol{u} - k\operatorname{grad}\phi,$$

$$\frac{d^2q}{dt^2} + \frac{\xi' + \chi}{\xi}\frac{dq}{dt} + \frac{c^2\upsilon^2}{\xi\eta}q = 0, \qquad \upsilon = \text{constant}, \tag{A.10}$$

and certain required boundary conditions are satisfied on the boundary of the cavity (see [20], [25], [32], [50], [85] for more details).

As a result, we can choose $c = d = f = g = 0$ and

$$a = \frac{1}{2\xi}e^{-\int(\chi/\xi)\,dt}, \qquad b = \frac{c^2\upsilon^2}{2\eta}e^{\int(\chi/\xi)\,dt} \tag{A.11}$$

in Theorem 1 for the quantization of the mode of the electromagnetic field under consideration. (See also [2], [13], [67], [77], and [78].)

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