SCHUR LOG-CONCAVITY AND THE QUANTUM PASCAL TRIANGLE

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ABSTRACT. We say a sequence $f_0, f_1, f_2, \dots$ of symmetric functions is *Schur log-concave* if $f_n^2 - f_{n-1}f_{n+1}$ is Schur positive for all $n \geqslant 1$. We conjecture that a very general class of sequences of Schur functions satisfies this property, and show it for sequences of Schur functions indexed by partitions with growing first part and column. Our findings are related to work of Lam, Postnikov and Pylyavskyy on Schur positivity, and of Butler, Sagan, and the second author on q-log-concavity.

1. INTRODUCTION

We say a sequence $f_0, f_1, f_2, \dots$ of symmetric functions is *Schur log-concave* if $f_n^2 - f_{n-1}f_{n+1}$ is Schur positive for all $n \geqslant 1$. It is *strongly Schur log-concave* if $f_n f_{n+i} - f_{n-1}f_{n+i+1}$ is Schur positive for all $i \geqslant 0$ and $n \geqslant 1$. We put forward the following conjecture. (We refer the reader to §2 for all terminology and notation.)

Conjecture 1. *Let λ, β be partitions, α be an integral vector. Suppose $\ell(\lambda) \gg \ell(\alpha)$ and $\lambda_{\ell(\lambda)} \gg \beta_1$. Then, the sequence*

$$s_\lambda, \quad s_{\lambda\cup\beta+\alpha}, \quad s_{\lambda\cup^2\beta+2\alpha}, \quad s_{\lambda\cup^3\beta+3\alpha}, \quad \cdots$$

is strongly Schur log-concave.

We adopt the convention that $s_\gamma = 0$ if γ is not a partition. Hence, the sequence in the conjecture might be finite or infinite. For $|\lambda| \leqslant 6$ and $\ell(\lambda) \leqslant 3$, the conjecture holds for all tested α and β such that $\ell(\lambda) > \ell(\alpha)$ and $\lambda_{\ell(\lambda)} \geqslant \beta_1$. If $\ell(\lambda) = \ell(\alpha)$, however, the claim is no longer true: consider e.g. $\lambda = (3,3),\ \alpha = (1,1),\ \beta = (3)$.

Our goal is to provide evidence supporting the conjecture. When $\lambda = \beta = \varnothing$ and $\ell(\alpha) = 1$, the conjecture is true by the Jacobi–Trudi formula (see Corollary 4.2 below). When λ is a hook, $\beta = \varnothing$, and $\ell(\alpha) = 1$, the conjecture is essentially known —it follows from either [BBR06, Thm. 3.1(1)], [BM04, Prop. 3.3], or [LPP07, Thm. 5].

More generally, we show Conjecture 1 for the case $\beta_1 \leqslant 1$, $\ell(\alpha) \leqslant 1$ of growing first part and first column.

Theorem A. *The following sequence is strongly Schur log-concave:*

$$s_\lambda, \quad s_{\lambda\cup(1^j)+(k)}, \quad s_{\lambda\cup(1^{2j})+(2k)}, \quad s_{\lambda\cup(1^{3j})+(3k)}, \quad \cdots$$

for any $k = 0$ or $k \geqslant \lambda_2$, and any $j = 0$ or $j \geqslant \lambda_2'$.

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To our knowledge, Schur log-concavity of sequences has not been studied explicitly. However, the Schur positivity of expressions of the form $s_\mu s_\nu - s_\rho s_\delta$ has an extensive literature: see for instance [BO14, LPP07] and the references therein. Although sequences are not usually discussed, some of these results can be reframed in our language. The aforementioned case when λ is a hook, $\beta = \varnothing$ and $\ell(\alpha) = 1$ appears in this context as a particular case of a conjecture of [FFLP05], which was ultimately established in [LPP07].

Symmetric polynomials are characters of GL_n representations. A related problem is to study sequences of SL_n characters, which are the images of symmetric polynomials in $\mathbb{Z}[x_1, \dots x_n]/(x_1 \cdots x_n - 1)$. For SL_2, the variable q is usually preferred; $\mathbb{Z}[q, q^{-1}] \cong \mathbb{Z}[x_1, x_2]/(x_1 x_2 - 1)$. A *quantum integer* is defined as $[n] = \frac{q^{-n} - q^n}{q^{-1} - q} = q^{-n+1} + q^{-n+3} + \cdots + q^{n-3} + q^{n-1}$, for $n \geqslant 1$, with $[0] = 0$, and a *quantum binomial* as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n][n-1]\cdots[n-k+1]}{[k][k-1]\cdots[1]}$$

for $n \geqslant k \geqslant 0$. They are Laurent polynomials in $\mathbb{N}_0[q, q^{-1}]$. Moreover, they are centred —invariant under $q \mapsto q^{-1}$— and unimodal —that is, the coefficients are non-negative, (weakly) monotone increasing up to a certain point, and from there on (weakly) monotone decreasing. They can be realised via the following specialisation of Schur polynomials:

$$\begin{bmatrix} n \\ k \end{bmatrix} = s_{(1^k)}(q^{-n+1}, \dots, q^{n-3}, q^{n-1}). \tag{1}$$

A sequence $a_0(q), a_1(q), a_2(q), \dots$ of (Laurent) polynomials in $\mathbb{N}_0[q, q^{-1}]$ is *Schur log-concave* if for $n \geqslant 1$ the difference $a_n(q)^2 - a_{n+1}(q)a_{n-1}(q)$ is a unimodal centred polynomial with non-negative coefficients. It is *strongly Schur log-concave* if $a_{n+i}(q)a_n(q) - a_{n+i+1}(q)a_{n-1}(q)$ is a unimodal centred polynomial with non-negative coefficients for $n \geqslant 1$ and $i \geqslant 0$.

Our next conjecture, based on data gathered with SageMath [Sag25], concerns diagonals in the quantum Pascal triangle (see Figure 1).

Conjecture 2. *For all choices of $n \geqslant k \geqslant 0$, $\alpha \geqslant -1$, and $\beta \geqslant 0$, the diagonal of the quantum Pascal triangle starting at $\begin{bmatrix} n \\ k \end{bmatrix}$ and of slope α/β, namely*

$$\begin{bmatrix} n \\ k \end{bmatrix}, \begin{bmatrix} n-\alpha \\ k+\beta \end{bmatrix}, \begin{bmatrix} n-2\alpha \\ k+2\beta \end{bmatrix}, \begin{bmatrix} n-3\alpha \\ k+3\beta \end{bmatrix}, \begin{bmatrix} n-4\alpha \\ k+4\beta \end{bmatrix}, \dots,$$

is strongly Schur log-concave.

The case $(n, k, \alpha, \beta) = (n, 0, 0, 1)$ of the conjecture refers to the rows. The case $(n, k, \alpha, \beta) = (0, k, -1, 0)$ of the conjecture refers to the columns. Both follow from Theorem A but are of independent interest.

Corollary A.1. *For all n, the nth row $\begin{bmatrix} n \\ 0 \end{bmatrix}, \begin{bmatrix} n \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} n \\ n \end{bmatrix}$ of the quantum Pascal triangle is strongly Schur log-concave.*

Corollary A.2. *For all k, the kth column $\begin{bmatrix} k+0 \\ k \end{bmatrix}, \begin{bmatrix} k+1 \\ k \end{bmatrix}, \begin{bmatrix} k+2 \\ k \end{bmatrix}, \dots$ of the quantum Pascal triangle is strongly Schur log-concave.*

Conjecture 2 refers to many other studied sequences, such as the nth Fibonacci diagonal $(n, k, \alpha, \beta) = (n, 0, 1, 1)$. See Figure 1. The general form of Conjecture 2 does however not follow from Conjecture 1.

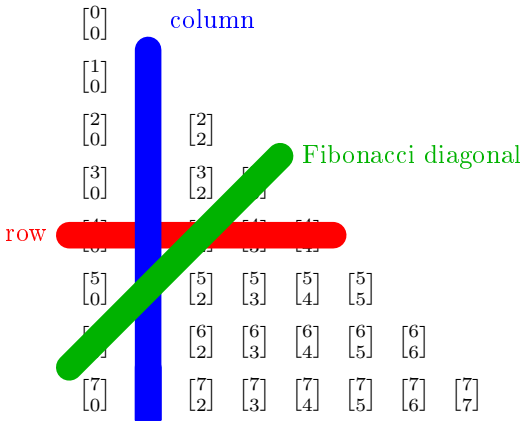


FIGURE 1. The first eight rows of the quantum Pascal triangle. A row, a column, and a Fibonacci diagonal are highlighted.

Corollaries A.1 and A.2 are related to a series of papers by Butler [But90], Sagan [Sag92a, Sag92b], and the second author [Kra89] in the 90s. However, these authors studied q-analogues (as opposed to quantum analogues). A *q-integer* (without "quantum") is defined as $[\![n]\!] = 1 + q + q^2 + \cdots + q^{n-1}$, and *$q$-binomial coefficients* (again without "quantum") are realised via the principal specialisation of Schur polynomials, $q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} = s_{(1^k)}(1, q, \ldots, q^{n-1})$.

A sequence $a_0(q), a_1(q), \ldots$ of $\mathbb{N}_0[q]$ is q-log-concave if $a_n^2(q) - a_{n-1}(q)a_{n+1}(q) \in \mathbb{N}_0[q]$, and strong q-log-concavity is defined in the obvious way. Butler showed (weak) q-log-concavity of the columns of the q-Pascal triangle and the second author the strong version; for us it follows from the main theorem (see Note 6.1). The strong q-log-concavity of rows and columns of the q-Pascal triangle were shown combinatorially by Sagan [Sag92b]. Our methods cannot establish the q-log-concavity of the rows in a straightforward way.

Structure of the paper. In §2 we give some background on symmetric functions, symmetric polynomials, and representation theory of SL_n. In §3, we define log-concavity of sequences in any algebra over an ordered ring. Lemma 3.2 provides a powerful tool for translating log-concavity from one algebra to another. We conclude the section by giving (non-)examples of properties of Schur log-concavity. We establish Theorem A in §4, and deduce some corollaries. In §5, we show that Conjecture 1 satisfies necessary conditions for Schur log-concavity derived from work of McNamara [McN08] and of the first author and Rosas [GR23]. In §6 we show the results concerning quantum binomials.

2. Background

2.1. **Symmetric functions.** We follow [Sta99]. Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ be a set of variables. The algebra of symmetric polynomials in $\mathbf{x}$ is the invariant space $\Lambda[\mathbf{x}] = \mathbb{Q}[\mathbf{x}]^{S_n}$, where the symmetric group acts by permuting variables. The

(a) (b)

FIGURE 2

elementary and *complete homogeneous* polynomials are defined by

$$e_k(\mathbf{x}) = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} x_{i_1}x_{i_2}\cdots x_{i_k}, \quad h_k(\mathbf{x}) = \sum_{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_k\leqslant n} x_{i_1}x_{i_2}\cdots x_{i_k}.$$

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a weakly decreasing finite sequence of non-negative integers. The components λ_i are called *parts* of the partition λ. The *length* $\ell(\lambda)$ is the number of non-zero parts. We write $\mathrm{Par}_{\leqslant n}$ for the set of partitions of length at most n. We usually omit 0s and use exponents to denote repeated parts, so that e.g. $(4, 4, 2, 1, 1, 1, 0)$ is abbreviated by $(4^2, 2, 1^3)$. The set of cells of λ, also called *Young diagram of* λ, is $Y(\lambda) = \{(i, j) \ : \ 1 \leqslant j \leqslant \lambda_i\}$. We draw Young diagrams according to the English convention; the coordinates of the cells are matrix-like. For instance, the Young diagram of $(6^2, 3, 2, 1)$ is shown in Figure 2a.

The *transpose* of λ is the partition λ' such that $Y(\lambda') = \{(i, j) \ : \ 1 \leqslant i \leqslant \lambda_j\}$. The addition of partitions is entry-wise. This generalises to addition with an integral vector: given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of length $\ell(\lambda) \leqslant k$ and a vector $v \in \mathbb{Z}^k$, we let $\lambda + v = (\lambda_1 + v_1, \lambda_2 + v_2, \ldots, \lambda_n + v_k) \in \mathbb{Z}^k$. For a scalar $n \in \mathbb{N}$, we let $n\lambda = (n\lambda_1, \ldots, n\lambda_k)$. The union $\lambda \cup \mu$ of two partitions λ and μ is the partition formed by sorting the parts of λ and μ. We let $\lambda \cup^n \mu = \lambda \cup \mu \cup \cdots \cup \mu$ with n copies of μ.

The antisymmetric polynomial $a_\mu(\mathbf{x})$ indexed by the partition μ is $\det(x_j^{\mu_i})_{i,j}$. Note that letting $\rho = (n-1, \ldots, 2, 1)$ the determinant $a_\rho(\mathbf{x})$ is a Vandermonde determinant. The *Schur polynomial* s_λ indexed by λ is the quotient

$$s_\lambda(\mathbf{x}) = \frac{a_{\lambda+\rho}(\mathbf{x})}{a_\rho(\mathbf{x})}.$$

Note that (1) is a direct consequence of the definitions.

The Schur polynomial has a combinatorial interpretation in terms of tableaux, which is further generalised to a *skew Schur polynomial* below. A *skew partition* λ/μ is a pair of partitions such that $\lambda_i \geqslant \mu_i$ for all i. A partition λ is identified with the skew partition $\lambda/\varnothing$. A *Young tableau* of shape λ/μ in the alphabet $[n]$ is a function $T : Y(\lambda) \setminus Y(\mu) \to [n]$. It is *semistandard* if $T(i, j) \leqslant T(i, j+1)$ and $T(i, j) < T(i+1, j)$ whenever the expressions are defined. The set of semistandard Young tableaux of shape λ/μ in the alphabet $[n]$ is denoted $\mathrm{SSYT}_n(\lambda/\mu)$. The *content* $\mathrm{ct}(T)$ is the multiset of its entries; the *weight* of T is $\mathbf{x}^T = \prod_{i\in\mathrm{ct}(T)} x_i$. See Figure 2b for a semistandard Young tableau of shape $(5, 5, 3, 2, 2)/(4, 3, 1, 1)$ and weight $x_1^5x_2^2x_3$. The following result can be found in [Sta99, Thm. 7.15.1].

Theorem 2.1. *Let* $|\mathbf{x}| = n$. *Then* $s_\lambda(\mathbf{x}) = \sum_{T\in\mathrm{SSYT}_n(\lambda/\varnothing)} \mathbf{x}^T$.

One can more generally define $s_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}_n(\lambda/\mu)} \mathbf{x}^T$. We direct the reader to [GV85, FK97] for combinatorial proofs of the Jacobi–Trudi identities, given in the next two theorems.

Theorem 2.2 (Jacobi–Trudi identity). $s_\lambda(\mathbf{x}) = \det(h_{\lambda_i - i+j}(\mathbf{x}))_{i,j}$.

Theorem 2.3 (Dual Jacobi–Trudi identity). $s_{\lambda'}(\mathbf{x}) = \det(e_{\lambda_i - i+j}(\mathbf{x}))_{i,j}$.

It follows that $s_{(k)} = h_k$ and $s_{(1^k)} = e_k$. Moreover, we obtain the following lemma by collecting the above remarks.

Lemma 2.4. *We have* $\begin{bmatrix} k+\ell \\ k \end{bmatrix} = h_\ell(q^{-k}, \ldots, q^{k-2}, q^k)$.

The *algebra of symmetric functions* is the colimit $\Lambda = \lim_{n\to\infty} \Lambda[x_1, \ldots, x_n] = \mathbb{Q}[x_1, x_2, \ldots]^{S_\infty}$. The elementary symmetric function e_k, complete homogeneous symmetric function h_k, and skew Schur function $s_{\lambda/\mu}$ are the limits of the respective sequences of polynomials $\big(e_k(x_1, \ldots, x_n)\big)_n, \big(h_k(x_1, \ldots, x_n)\big)_n, \big(s_{\lambda/\mu}(x_1, \ldots, x_n)\big)_n$. We similarly consider $\mathrm{SSYT}(\lambda/\mu) = \bigcup_n \mathrm{SSYT}_n(\lambda/\mu)$.

The *Littlewood–Richardson coefficients* $c^\lambda_{\mu,\nu}$ are the coefficients in the expansion $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$. The *reverse reading word* $\mathrm{rrw}(T)$ of a tableau T is obtained by reading the entries of T from right to left in each row, and arranging the (ordered) rows from top to bottom. For instance, the reverse reading word of the tableau in Figure 2b is 12111231. A (reverse reading) word is *Yamanouchi* if every prefix contains at least as many 1s as 2s, at least as many 2s as 3s, etc. For a proof of the next theorem, see e.g. [Sta99, §A1.3].

Theorem 2.5 (Littlewood–Richardson rule). *The coefficient $c^\lambda_{\mu,\nu}$ is the cardinality of the set*

$$\mathrm{LR}^\lambda_{\mu,\nu} = \{T \in \mathrm{SSYT}(\lambda/\mu) \ : \ \mathrm{ct}(T) = \nu \text{ and } \mathrm{rrw}(T) \text{ is Yamanouchi}\}.$$

These objects are called *Littlewood–Richardson tableaux.*

2.2. **Representation theory.** We follow [FH91, PW21]. Let $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{C})$ be the group of 2×2 matrices with determinant 1. A *representation* of SL_2 is a vector space V together with a homomorphism $\rho : \mathrm{SL}_2 \to \mathrm{GL}(V)$. The *Weyl character* $\mathrm{ch}(V)$ of a representation V is the unique polynomial in $\mathbb{N}_0[x_1, x_2]^{S_2}/(x_1x_2 - 1) \cong \Lambda[q, q^{-1}]$ such that

$$\mathrm{tr}\Big(\rho\Big(\begin{smallmatrix} q & 0 \\ 0 & q^{-1} \end{smallmatrix}\Big)\Big) = \mathrm{ch}(V)(q, q^{-1})$$

for all q. Weyl characters form the Grothendieck group of the category of finite dimensional representations of SL_2.

Example 2.6. Let $L(k)$ be the irreducible representation of SL_2 of dimension k. Its Weyl character is $q^{-k+1} + q^{-k+3} + \cdots + q^{k-1}$.

Theorem 2.7 ([Sta85]). *A Laurent polynomial $a(q)$ is unimodal, centred, and has non-negative coefficients if and only if it is the Weyl character of an SL_2 representation.*

The *Schur functor* S^λ is a functor from the category of GL_n representations to itself. It is uniquely characterised by the property

$$\mathrm{ch}(S^\lambda V) = s_\lambda \circ \mathrm{ch}(V),$$

where $\circ$ denotes *plethysm*. For a definition of plethysm in generality, see [Sta99, p. 447]. For us, it suffices to provide a rule to compute some cases of it.

Lemma 2.8. *Let $g(x_1,\dots,x_n)$ and $f(y_1,\dots,y_m)$ be symmetric polynomials.*

If $g(x_1,\dots,x_n) = g_1 + g_2 + \cdots + g_m$ is a sum of m monic monomials, then $(f\circ g)(x_1,\dots,x_n) = f(g_1,\dots,g_m)$.

3. Observations on Schur log-concavity

The notion of Schur log-concavity makes sense in a few different contexts (symmetric polynomials, symmetric functions, characters of SL_n). Hence we take a more general approach.

Definition 3.1. Let $\mathcal{A}_R$ be an algebra over an ordered ring R. Given a basis B of $\mathcal{A}_R$, we say an element a is *B-positive* (and write $a \geqslant_B 0$) if the coefficients in the expansion $a = \sum_{b\in B} c_b b$ are all non-negative. The notation $a_1 \geqslant_B a_2$ stands for $a_1 - a_2 \geqslant_B 0$. A sequence $(a_0, a_1, a_2, \dots)$ of elements of $\mathcal{A}_R$ is *B-log-concave* if

$$a_n^2 \underset{B}{\geqslant} a_{n+1}a_{n-1},$$

and *strongly B-log-concave* if

$$a_{n+i}a_n \underset{B}{\geqslant} a_{n+i+1}a_{n-1},$$

for all $n \geqslant 1$ and all $i \geqslant 0$.

For the present paper, we mainly study the following pairs $(\mathcal{A}_{\mathbb{Q}}, B)$:

(1) $\big(\Lambda,\ \{s_\lambda\ :\ \lambda \text{ a partition}\}\big)$,
(2) $\big(\Lambda[x_1,\dots,x_n],\ \{s_\lambda\ :\ \ell(\lambda)\leqslant n\}\big)$,
(3) $\big(\Lambda[q,q^{-1}],\ \{[n]\ :\ n\geqslant 1\}\big)$.

In each of the cases, we refer to the basis as the *Schur basis*, and write $\geqslant_s 0$ for "Schur positive".

Lemma 3.2. *Let $\mathcal{A}_R$ and $\mathcal{A}'_R$ be algebras over R, with bases B and B', respectively. Suppose $b_1b_2 \geqslant_B 0$ for all $b_1, b_2 \in B$, and $b'_1b'_2 \geqslant_{B'} 0$ for all $b'_1, b'_2 \in B'$. Let $\pi : \mathcal{A}_R \to \mathcal{A}'_R$ be an algebra homomorphism such that $\pi(b) \geqslant_{B'} 0$ for all $b \in B$. Then π sends (strongly) B-log-concave sequences to (strongly) B'-log-concave sequences.*

Proof. Consider a B-positive expansion $a_{n+i}a_n - a_{n+i+1}a_{n-1} = \sum_{b\in B} c_b b$. Taking the image under π, we get $\pi(a_{n+i})\pi(a_n) - \pi(a_{n+i+1})\pi(a_{n-1}) = \sum_{b\in B} c_b\pi(b)$. Since each $\pi(b)$ is B'-positive, so is the right-hand side. ∎

Examples of such homomorphisms include

(1) $\omega : \Lambda \to \Lambda, \quad s_\lambda \mapsto s_{\lambda'}$,
(2) $\Lambda \to \Lambda[\mathbf{x}], \quad s_\lambda \mapsto s_\lambda(\mathbf{x})$,
(3) $\Lambda \to \Lambda, \quad s_\lambda \mapsto s_\lambda \circ g$ for any fixed Schur positive $g \in \Lambda$,
(4) $\Lambda[x_1,x_2] \to \Lambda[q,q^{-1}], \quad s_\lambda(x_1,x_2) \mapsto s_\lambda(q,q^{-1})$.

as well as their compositions. See [Ber18, Thm. 2.5.1] for more examples.

3.1. Properties and non-properties of Schur log-concavity.

A finite sequence $f_0, f_1, f_2, \dots, f_n$ is said to be B-unimodal if

$$f_0 \leqslant_B f_1 \leqslant_B \cdots \leqslant_B f_k \geqslant_B f_{k+1} \geqslant_B \cdots \geqslant_B f_n$$

for some k. Note that, unlike in the case of sequences of natural numbers, Schur log-concavity does not imply Schur unimodality.

Example 3.3 (Log-concave but not unimodal)**.** The sequence $1, s_1, s_2$ is Schur log-concave. But $1 - s_1$ is not Schur positive nor Schur negative.

Another example, in which all elements are of the same degree, is $s_4, s_{3,1}, s_{2,2}$.

For sequences of integers, the notion of "strong log-concavity" is equivalent to that of "log-concavity". We now show by example that for our three algebras $\Lambda, \Lambda[\mathbf{x}]$ and $\Lambda[q, q^{-1}]$, the notion of strong Schur log-concavity is indeed stronger than the notion of Schur log-concavity.

Example 3.4 (Weak but not strong log-concavity in Λ)**.** Consider the sequence

$$3s_{2,1}, \quad 2(s_3 + s_{2,1} + s_{1,1,1}), \quad 2(s_3 + s_{2,1} + s_{1,1,1}), \quad 3s_{2,1}.$$

Computer algebra reveals that it is Schur log-concave, but not strongly so. To give an example: $s_{3,3}$ appears with multiplicity -1 in $4(s_3 + s_{2,1} + s_{1,1,1})^2 - 9s_{2,1}^2$.

Example 3.5 (Weak but not strong log-concavity in $\Lambda[\mathbf{x}]$)**.** Let $\mathbf{x} = (x_1, x_2)$ and consider the sequence

$$s_{3,3}(\mathbf{x}), \quad s_{4,2}(\mathbf{x}), \quad s_{5,1}(\mathbf{x}), \quad s_{3,3}(\mathbf{x}).$$

It is Schur log-concave:

$$\begin{aligned} s_{4,2}(\mathbf{x})^2 - s_{3,3}(\mathbf{x})s_{5,1}(\mathbf{x}) &= s_{7,5}(\mathbf{x}) + s_{6,6}(\mathbf{x}) && \underset{s}{\geqslant} 0, \\ s_{5,1}(\mathbf{x})^2 - s_{4,2}(\mathbf{x})s_{3,3}(\mathbf{x}) &= s_{10,2}(\mathbf{x}) + s_{9,3}(\mathbf{x}) + s_{8,4}(\mathbf{x}) + s_{6,6}(\mathbf{x}) && \underset{s}{\geqslant} 0. \end{aligned}$$

However, it is not strongly Schur log-concave:

$$s_{5,1}(\mathbf{x})\ s_{4,2}(\mathbf{x}) = s_{9,3}(\mathbf{x}) + s_{8,4}(\mathbf{x}) + s_{7,5}(\mathbf{x}) \underset{s}{\not\geqslant} s_{6,6}(\mathbf{x}) = s_{3,3}(\mathbf{x})^2.$$

Example 3.6 (Weak but not strong log-concavity in $\Lambda[q, q^{-1}]$)**.** We take the sequence from Example 3.5 and specialise $\mathbf{x} \mapsto (q, q^{-1})$ to obtain the sequence $[1], [3], [5], [1]$. By the Clebsch–Gordan rule,

$$\begin{aligned} [3]^2 - [5][1] &= ([5] + [3] + [1]) - [5] \quad \text{and} \\ [5]^2 - [3][1] &= ([9] + [7] + [5] + [3] + [1]) - [3] \end{aligned}$$

are Schur positive, and therefore the sequence is Schur log-concave. But

$$[3][5] - [1]^2 = ([7] + [5] + [3]) - [1]$$

which is not Schur positive: the sequence is not strongly Schur log-concave.

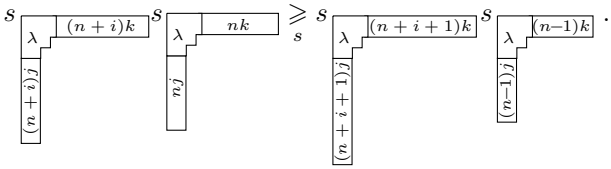
4. Proofs of Theorem A and its corollaries

Theorem A. *The following sequence is strongly Schur log-concave:*

$$s_\lambda, \quad s_{\lambda \cup (1^j) + (k)}, \quad s_{\lambda \cup (1^{2j}) + (2k)}, \quad s_{\lambda \cup (1^{3j}) + (3k)}, \quad \cdots$$

for any $k = 0$ or $k \geqslant \lambda_2$, and any $j = 0$ or $j \geqslant \lambda'_2$.

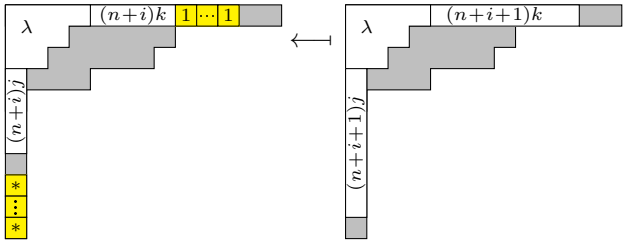
Proof. Fix $k \geqslant \lambda_2$, $j \geqslant \lambda'_2$, $n \geqslant 0$ and $i \geqslant 0$. The cases in which $k = 0$ or $j = 0$ will follow immediately from the proof of the more general case. We aim to show



Let μ, ν, ρ, δ denote the four partitions in the above equation, in order. An equivalent restatement is that for all θ, we have to show the inequality $c^{\theta}_{\mu\nu} \geqslant c^{\theta}_{\rho\delta}$. of Littlewood–Richardson coefficients. It suffices to see that there is an injective map of Littlewood–Richardson tableaux

$$\mathrm{LR}^{\theta}_{\mu\nu} \longleftarrow \mathrm{LR}^{\theta}_{\rho\delta}\,.$$

The map is defined as follows:



Let T be a Littlewood–Richardson tableau of shape $\theta/(\lambda \cup (1^{(n+i+1)j}) + ((n+i+1)k))$ and content $\lambda \cup (1^{(n-1)j}) + ((n-1)k)$. The map has two steps: (i) insert k copies of 1 in the first row of the tableau, and (ii) insert

$$\lambda'_1 + (n-1)j + 1, \quad \lambda'_1 + (n-1)j + 2, \quad \ldots, \quad \lambda'_1 + nj$$

in the first column. (In the diagram above, the relevant cells are filled with stars.)

The map is clearly injective; we conclude by showing that the map is well-defined. Note that if the image of the map is semistandard then it is of the correct shape and content, and that the reverse reading word is trivially Yamanouchi. So it suffices to check that it is semistandard.

The image of T under (i) is semistandard unless T has a $\boxed{1}$ in row 2 and column $\lambda_1 + (n+i)k + 1$. The greatest column in which a $\boxed{1}$ can exist in row 2 is

$$\lambda_2 + \underbrace{\lambda_1 + (n-1)k}_{\#\boxed{1}\text{s}}\,.$$

But if $\lambda_2 + (\lambda_1 + (n-1)k) \geqslant \lambda_1 + (n+i)k + 1$ then $\lambda_2 \geqslant (i+1)k + 1 > k$. This contradicts the hypothesis of the theorem. Hence the tableau P obtained by applying (i) to T is semistandard.

The image of P under (ii) is clearly semistandard if T has no entries in column 2 and row $\lambda'_1 + (n+i)j + 1$. The total number of entries in column 2 is at most

$$\lambda'_2 + \underbrace{\lambda'_1 + (n-1)j}_{\#\text{ of distinct entries}}\,.$$

We reach a contradiction $\lambda'_2 > j$ as before. ∎

Corollary 4.1. *The sequence $e_0, e_1, e_2, \ldots$ is strongly Schur log-concave.*

Proof. This is the case $\lambda = (1)$, $k = 0$, $j = 1$ of the theorem. ∎

Another proof of the corollary is by observing that

$$e_{n+i}e_n - e_{n+i+1}e_{n-1} = \det \left(\begin{smallmatrix} e_{n+i} & e_{n+i+1} \\ e_{n-1} & e_n \end{smallmatrix} \right) = s_{(2^n,1^i)},$$

by the dual Jacobi–Trudi identity. Note that this gives a second combinatorial proof of the corollary by path enumeration arguments (cf. [GV85, FK97]). The next corollary can be shown similarly using the usual Jacobi–Trudi identity.

Corollary 4.2. *The sequence $h_0, h_1, h_2, \ldots$ is strongly Schur log-concave.*

Proof. This follows from Lemma 3.2 after applying ω to the previous corollary. ∎

An interesting family of corollaries of Theorem A concerns the case in which λ is a hook. For instance:

Corollary 4.3. *For all $n \geqslant 0$ and $m \geqslant 0$, the sequence $s_{(n,1^m)}, s_{(n+1,1^{m+1})}, s_{(n+2,1^{m+2})}, \ldots$ is strongly Schur log-concave.*

Proof. This is the case $\lambda = (n, 1^m)$, $k = 1$, $j = 1$ of the theorem. ∎

Corollary 4.4 ([BBR06, LPP07])**.** *For all $n \geqslant 0$, the sequence $s_{(n,1^0)}, s_{(n,1^1)}, s_{(n,1^2)}, \ldots$ is strongly Schur log-concave.*

Proof. This is the case $\lambda = (n)$, $k = 1$, $j = 0$ of the theorem. ∎

5. Further evidence for Conjecture 1

In [McN08], McNamara gives necessary conditions for partitions μ, ν, ρ, δ to satisfy $s_\mu s_\nu - s_\rho s_\delta \geqslant_s 0$. For fixed λ, β, and α, let

$$s_{\lambda^{(0)}}, \quad s_{\lambda^{(1)}}, \quad s_{\lambda^{(2)}}, \quad s_{\lambda^{(3)}}, \quad \ldots$$

be a shorthand for the sequence of Conjecture 1. As further supporting evidence for the conjecture, we show in Proposition 5.5 that the expressions

$$s_{\lambda^{(n)}} s_{\lambda^{(n+i)}} - s_{\lambda^{(n-1)}} s_{\lambda^{(n+i+1)}} \tag{2}$$

satisfy one of these necessary conditions. In Proposition 5.7, we derive another necessary condition for Schur positivity of $s_\mu s_\nu - s_\rho s_\delta \geqslant_s 0$. We show in Proposition 5.8 that it also holds for the expression (2) in the case $\beta = \varnothing$.

5.1. The condition of McNamara.

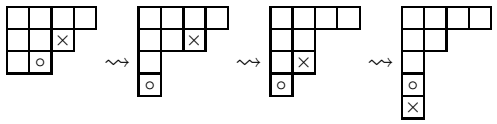
Definition 5.1 (Dominance order)**.** Given partitions μ and ν, we write $\mu \preceq \nu$ and say ν *dominates* μ if $\mu_1 + \cdots + \mu_j \leqslant \nu_1 + \cdots + \nu_j$ for all j.

Proposition 5.2 ([McN08, Prop. 3.1])**.** *If $s_\mu s_\nu \geqslant_s s_\rho s_\delta$ then $\mu \cup \nu \preceq \rho \cup \delta$.*

We require a combinatorial description of the dominance order. An *inner corner* of λ is a cell of $Y(\lambda)$ such that $Y(\lambda) \setminus \{c\}$ is the set of cells of a partition. An *outer corner* of λ is a tuple $c \in \mathbb{N}^2$ such that $Y(\lambda) \cup \{c\}$ is the set of cells of a partition.

Proposition 5.3 ([Bry73, Prop. 2.3])**.** *Given partitions μ and ν of the same size, $\mu \preceq \nu$ if and only if one can reach μ by starting with ν and successively moving an inner corner to the next outer corner immediately below.*

For instance, $(4,2,1^3) \prec (4,3,2)$ since we have the following sequence of moves:



A *composition* is a tuple of non-negative integers. Given compositions ρ and δ, we let $\mathrm{sort}(\rho,\beta)$ be the partition obtained by sorting the parts of ρ and δ in weakly decreasing order. Note that, if α and δ are partitions, then $\mathrm{sort}(\rho,\beta) = \rho \cup \delta$.

Lemma 5.4. *Let μ,ν,ρ,δ be compositions of length n satisfying*

$$\mu_i + \nu_i = \rho_i + \delta_i \quad \text{and} \quad \rho_i \leqslant \mu_i \leqslant \nu_i \leqslant \delta_i \quad \text{for all } i.$$

Then $\mathrm{sort}(\mu,\nu) \preceq \mathrm{sort}(\rho,\delta)$.

Proof. For $1 \leqslant i \leqslant n$, we have

$$\begin{aligned}\mathrm{sort}\Big((\mu_1,\dots,\mu_{i-1},\mu_i,\rho_{i+1},\dots,\rho_n),(\nu_1,\dots,\nu_{i-1},\nu_i,\delta_{i+1},\dots,\delta_n)\Big)\\ \preceq \mathrm{sort}\Big((\mu_1,\dots,\mu_{i-1},\rho_i,\rho_{i+1},\dots,\rho_n),(\nu_1,\dots,\nu_{i-1},\delta_i,\delta_{i+1},\dots,\delta_n)\Big)\end{aligned}$$

by Lemma 5.3, since one can reach the former by moving $\delta_i - \nu_i$ $(= \mu_i - \rho_i)$ corners of the latter down. By transitivity of the dominance order, we obtain the claim. ∎

The next proposition shows that the differences of products of Schur functions involved in Conjecture 1 satisfy the necessary condition of McNamara. Recall that $\lambda^{(n)}$ is shorthand for $\lambda \cup^n \beta + n\alpha$ when α and β are clear from context.

Proposition 5.5. *Fix partitions λ,β and an integral vector α such that $\beta_1 \leqslant \lambda_{\ell(\lambda)}$ and $\ell(\alpha) < \ell(\lambda)$. Then*

$$\lambda^{(n)} \cup \lambda^{(n+i)} \preceq \lambda^{(n-1)} \cup \lambda^{(n+i+1)}$$

for all n and i such that the $\lambda^{(*)}$ are partitions.

Proof. Consider the four compositions μ,ν,ρ,δ of length $\ell(\lambda)$ with parts

$$\begin{aligned}\mu_j &= \min\Big\{\lambda_j^{(n)},\ \lambda_j^{(n+i)}\Big\}, & \rho_j &= \min\Big\{\lambda_j^{(n-1)},\ \lambda_j^{(n+i+1)}\Big\},\\ \nu_j &= \max\Big\{\lambda_j^{(n)},\ \lambda_j^{(n+i)}\Big\}, & \delta_j &= \max\Big\{\lambda_j^{(n-1)},\ \lambda_j^{(n+i+1)}\Big\}.\end{aligned}$$

It is routine to check that they satisfy the hypotheses of Lemma 5.4. Hence $\mathrm{sort}(\mu,\nu) \preceq \mathrm{sort}(\rho,\delta)$. The claim is then that $\mathrm{sort}(\mu,\nu) \cup^{2n+i} \beta \preceq \mathrm{sort}(\rho,\delta) \cup^{2n+i} \beta$, which is clear since $\beta_1 \leqslant \mathrm{sort}(\mu,\nu)_{2\ell(\lambda)}$ and $\beta_1 \leqslant \mathrm{sort}(\rho,\delta)_{2\ell(\lambda)}$ by hypothesis. ∎

5.2. **The condition of Rosas and the first author.** Let $\mathrm{Out}(\mu)$ be the set of outer corners of μ. Consider the sum in $\mathbb{N}^2$ defined by

$$(i,j) + (\ell,k) := (i+\ell-1, j+k-1).$$

Given a partition λ, its complement $Y(\lambda)^c$ is an ideal of $\mathbb{N}^2$ with respect to this sum. Given two sets A and $B \subseteq \mathbb{N}^2$, let $A + B = \{a + b \ :\ a \in A,\ b \in B\}$.

Theorem 5.6 ([GR23, Thm. 3.5])**.** *If $c^{\theta}_{\mu\nu} > 0$ then $Y(\theta)^c \supseteq \mathrm{Out}(\mu) + \mathrm{Out}(\nu) + \mathbb{N}^2$.*

Proposition 5.7. *Suppose $s_\mu s_\nu \geqslant_s s_\rho s_\delta$. Then, for each θ such that $c^{\theta}_{\rho,\delta} > 0$, we have $Y(\theta)^c \supseteq \mathrm{Out}(\mu) + \mathrm{Out}(\nu) + \mathbb{N}^2$.*

Proof. Since $s_\mu s_\nu \geqslant_s s_\rho s_\delta$ we deduce $c^\theta_{\mu\nu} \geqslant c^\theta_{\rho\delta}$ for all θ. In particular, if $c^\theta_{\rho\delta} > 0$ then $c^\theta_{\mu\nu} > 0$, which by Theorem 5.7 implies the claim. ∎

The following result supports Conjecture 1 in the case $\beta = \varnothing$. After applying Lemma 3.2, it also applies to the case $\alpha = \varnothing$.

Proposition 5.8. *Fix a partition λ and an integral vector α. For each θ,*

if $c^\theta_{\lambda+(n-1)\alpha,\ \lambda+(n+i+1)\alpha} > 0$ *then* $Y(\theta)^c \supseteq \mathrm{Out}(\lambda+n\alpha)+\mathrm{Out}(\lambda+(n+i)\alpha)+\mathbb{N}^2$ *for all n and i such that all of the relevant sums are partitions.*

Proof. Note that for any partition μ we have $\mathrm{Out}(\mu)+\mathbb{N}^2 = \{(j, \mu_j+1)\}_{j\leqslant \ell(\mu)}+\mathbb{N}^2$. We define four partitions

$$\rho = \lambda + (n-1)\alpha, \quad \mu = \lambda + n\alpha, \quad \nu = \lambda + (n+i)\alpha, \quad \delta = \lambda + (n+i+1)\alpha.$$

Assume $c^\theta_{\rho\delta} > 0$. Then,

$$\begin{aligned} Y(\theta)^c &\supseteq \{(j, \rho_j + 1)\}_j + \{(\ell, \delta_\ell + 1)\}_{\ell\leqslant n} + \mathbb{N}^2 \\ &= \{(j+\ell-1, \rho_j + \delta_\ell + 1)\}_{j,\ell} + \mathbb{N}^2. \end{aligned}$$

We aim to show

$$\{(j+\ell-1, \rho_j + \delta_\ell + 1)\}_{j,\ell} + \mathbb{N}^2 \supseteq \{(j+\ell-1, \mu_j + \nu_\ell + 1)\}_{j,\ell},$$

which implies the claim.

The containment follows from the following observation: for all j and ℓ,

(1) if $\alpha_j \geqslant \alpha_\ell$ then $\mu_j + \nu_\ell \geqslant \rho_j + \delta_\ell$,
(2) if $\alpha_j < \alpha_\ell$ then $\mu_j + \nu_\ell \geqslant \rho_\ell + \delta_j$. ∎

6. The quantum Pascal triangle

In this section we show the results concerning quantum binomials.

Corollary A.1. *For all n, the nth row $\begin{bmatrix} n \\ 0 \end{bmatrix}, \begin{bmatrix} n \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} n \\ n \end{bmatrix}$ of the quantum Pascal triangle is strongly Schur log-concave.*

Proof. Since $\begin{bmatrix} n \\ k \end{bmatrix}$ is the image of $e_k \circ s_{(n-1)}$ in the Grothendieck group of the category of SL_2 representations, the lemma follows from Corollary 4.1 and Lemma 3.2 by the composition $f \mapsto (f \circ s_{(n-1)})(q, q^{-1})$. ∎

Note that a direct proof using the dual Jacobi–Trudi identity is also possible, and gives

$$\left(s_{(\lambda_1,\lambda_2)'} \circ s_{(n-1)}\right)(q, q^{-1}) = \begin{bmatrix} n \\ \lambda_1 \end{bmatrix} \begin{bmatrix} n \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} n \\ \lambda_1 + 1 \end{bmatrix} \begin{bmatrix} n \\ \lambda_2 - 1 \end{bmatrix}. \tag{3}$$

Corollary A.2. *For all k, the kth column $\begin{bmatrix} k+0 \\ k \end{bmatrix}, \begin{bmatrix} k+1 \\ k \end{bmatrix}, \begin{bmatrix} k+2 \\ k \end{bmatrix}, \ldots$ of the quantum Pascal triangle is strongly Schur log-concave.*

Proof. Since $\begin{bmatrix} k+\ell \\ k \end{bmatrix}$ is the image of $h_\ell \circ s_{(k)}$ in the Grothendieck group of the category of SL_2 representations, the lemma follows from Corollary 4.2 and Lemma 3.2 by the composition $f \mapsto (f \circ s_{(k)})(q, q^{-1})$. ∎

A direct proof using the Jacobi–Trudi identity gives

$$\left(s_{(\lambda_1,\lambda_2)} \circ s_{(k)}\right)(q, q^{-1}) = \begin{bmatrix} k+\lambda_1 \\ k \end{bmatrix} \begin{bmatrix} k+\lambda_2 \\ k \end{bmatrix} - \begin{bmatrix} k+\lambda_1+1 \\ k \end{bmatrix} \begin{bmatrix} k+\lambda_2-1 \\ k \end{bmatrix}. \tag{4}$$

Note 6.1 (*q*-ANALOGUES). The q-analogue of Corollary A.2 follows similarly:

> *For all k, the kth column $\begin{bmatrix}k+0\\k\end{bmatrix}$, $\begin{bmatrix}k+1\\k\end{bmatrix}$, $\begin{bmatrix}k+2\\k\end{bmatrix}$, ... of the q-Pascal triangle is strongly q-log-concave.*

Indeed, the ℓth term is $(h_\ell \circ s_{(k)})(1,q)$, and $f \mapsto f(1,q)$ sends Schur functions to polynomials with positive coefficients. Hence the claim follows from the Schur log-concavity of $h_0, h_1, h_2, \ldots$ after applying $f \mapsto (f \circ s_{(k)})(1,q)$.

However, note that the q-analogue of Corollary A.1 does not follow in the same manner: application of $f \mapsto (f \circ s_{(n-1)})(1,q)$ to the sequence $e_0, e_1, e_2, \ldots$ gives

$$q^{\binom{0}{2}}\begin{bmatrix}n\\0\end{bmatrix}, \quad q^{\binom{1}{2}}\begin{bmatrix}n\\1\end{bmatrix}, \quad q^{\binom{2}{2}}\begin{bmatrix}n\\2\end{bmatrix}, \quad q^{\binom{3}{2}}\begin{bmatrix}n\\3\end{bmatrix}, \quad \ldots$$

which is not the nth row of the q-Pascal triangle. A combinatorial proof with Jacobi–Trudi determinants is however possible; see [Sag92b] for details.

Although we have mainly been interested in sequences of irreducible characters, there is no reason to not explore other sequences.

Corollary 6.2. *For any $n, m, d \geqslant 0$, the sequence of $\mathbb{Z}[q,q^{-1}]$ whose kth term is $\begin{bmatrix}n+m+2k-1\\m+k\end{bmatrix}\begin{bmatrix}n+d+k\\n+m+2k\end{bmatrix}$ is strongly Schur log-concave.*

Proof. A quantum version of Stanley's hook-content formula [Sta99, Cor. 7.21.4] for hook shapes says

$$\big(s_{(n,1^m)} \circ s_{(d)}\big)(q,q^{-1}) = \begin{bmatrix}n+m-1\\m\end{bmatrix}\begin{bmatrix}n+d\\n+m\end{bmatrix}.$$

See for instance [GMSW25]. The claim then follows from Corollary 4.3 and Lemma 3.2 after applying $f \mapsto (f \circ s_{(d)})(q,q^{-1})$. ∎

7. Concluding remarks

An unpublished conjecture of Lam, Postnikov and Pylyavskyy is the following. We point to [DP07] for the definition of an alcoved polytope. Denote by χ_λ the SL_n character indexed by λ, which is the image of s_λ in $\mathbb{Z}[x_1,\ldots,x_n]/(x_1\cdots x_n - 1)$.

Conjecture 3. *Let ρ, μ, ν, ρ be partitions. Suppose $\mu + \nu = \rho + \delta$, and that μ and ν belong to the minimal alcoved polytope containing ρ and δ. Then $\chi_\mu\chi_\nu \geqslant_s \chi_\rho\chi_\delta$.*

The corresponding conjecture for symmetric functions implies the following:

Conjecture 4. *For all partitions λ and all integral vectors α, the sequence*

$$s_\lambda, \quad s_{\lambda+\alpha}, \quad s_{\lambda+2\alpha}, \quad s_{\lambda+3\alpha}, \quad \ldots$$

is strongly Schur log-concave.

This is the case $\beta = \varnothing$ and $\alpha \in \mathbb{Z}^n$ (without restriction on the length of α) of Conjecture 1. We ask, more generally:

Problem. *Under which conditions on λ, β, α is the sequence*

$$s_\lambda, \quad s_{\lambda\cup\beta+\alpha}, \quad s_{\lambda\cup^2\beta+2\alpha}, \quad s_{\lambda\cup^3\beta+3\alpha}, \quad \ldots$$

strongly Schur log-concave?

To illustrate the subtleties in the problem, we provide a counterexample to a relaxed version of Conjecture 1.

Example 7.1. Let $\lambda = (3,3)$, $\beta = (3)$, $\alpha = (1,1)$. Note that $\ell(\alpha) = \ell(\lambda)$. The sequence

$$s_{(3^2)}, \quad s_{(4^2,3)}, \quad s_{(5^2,3^2)}, \quad s_{(6^2,3^3)}, \quad \dots$$

is not Schur log-concave, since $s^2_{(4^2,3)} \not\geq_s s_{(3^2)} s_{(5^2,3^2)}$.

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References

[BBR06] François Bergeron, Riccardo Biagioli, and Mercedes H. Rosas. Inequalities between Littlewood–Richardson coefficients. *J. Combin. Theory Ser. A*, 113(4):567–590, 2006.

[Ber18] François Bergeron. Symmetric functions and rectangular Catalan combinatorics, 2018. Notes, available as Open Math Notes of the AMS, OMN:202206.111331.

[BM04] François Bergeron and Peter McNamara. Some positive differences of products of Schur functions, 2004.

[BO14] Cristina Ballantine and Rosa Orellana. Schur-positivity in a square. *Electron. J. Combin.*, 21, 2014.

[Bry73] Thomas Brylawski. The lattice of integer partitions. *Discrete Math.*, 6:201–219, 1973.

[But90] Lynne M. Butler. The q-log-concavity of q-binomial coefficients. *J. Combin. Theory, Ser. A*, 54(1):54–63, 1990.

[DP07] Galyna Dobrovolska and Pavlo Pylyavskyy. On products of $\mathfrak{sl}_n$ characters and support containment. *J. Algebra*, 316(2):706–714, 2007.

[FFLP05] Sergey Fomin, William Fulton, Chi-Kwong Li, and Yiu-Tung Poon. Eigenvalues, singular values, and Littlewood–Richardson coefficients. *Amer. J. Math.*, 127(1):101–127, 2005.

[FH91] William Fulton and Joe Harris. *Representation Theory: A First Course*. Graduate Texts in Mathematics. Springer New York, 1991.

[FK97] Markus Fulmek and Christian Krattenthaler. Lattice path proofs for determinantal formulas for symplectic and orthogonal characters. *J. Combin. Theory, Ser. A*, 77(1):3–50, 1997.

[GMSW25] Álvaro Gutiérrez, Alvaro Martinez, Michał Szwej, and Mark Wildon. Modular isomorphisms of $\mathrm{SL}_2(\mathbb{F})$-plethysms for Weyl modules labelled by hook partitions. 2025. Preprint, arXiv:2509.01490.

[GR23] Álvaro Gutiérrez and Mercedes H. Rosas. Necessary conditions for the positivity of Littlewood–Richardson and plethystic coefficients. *C. R. Math. Acad. Sci. Paris*, 361:1163–1173, 2023.

[GV85] Ira Gessel and Gérard Viennot. Binomial determinants, paths, and hook length formulae. *Adv. in Math.*, 58(3):300–321, 1985.

[Kra89] Christian Krattenthaler. On the q-log-concavity of Gaussian binomial coefficients. *Monatsh. Math.*, 107(4):333–339, Dec 1989.

[LPP07] Thomas Lam, Alexander Postnikov, and Pavlo Pylyavskyy. Schur positivity and Schur log-concavity. *Amer. J. Math.*, 129(6):1611–1622, 2007.

[McN08] Peter McNamara. Necessary conditions for Schur-positivity. *J. Algebr. Combin.*, 28:495–507, 2008.

[PW21] Rowena Paget and Mark Wildon. Plethysms of symmetric functions and representations of $\mathrm{SL}_2(\mathbf{C})$. *Algebr. Comb.*, 4(1):27–68, 2021.

[Sag92a] Bruce E. Sagan. Inductive proofs of q-log concavity. *Discrete Math.*, 99(1-3):289–306, 1992.

[Sag92b] Bruce E. Sagan. Log concave sequences of symmetric functions and analogs of the Jacobi–Trudi determinants. *Trans. Amer. Math. Soc.*, 329(2):795–811, 1992.

[Sag25] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.5)*, 2025. `https://www.sagemath.org`.

[Sta85] Richard Stanley. Unimodality and Lie superalgebras. *Stud. Appl. Math.*, 72(3):263–281, 1985.

[Sta99] Richard Stanley. *Enumerative Combinatorics Vol. 2*. Cambridge Studies in Advanced Mathematics, 1999.

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