SUMMATION FORMULAS FOR GJMS-OPERATORS AND Q-CURVATURES ON THE MÖBIUS SPHERE

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Dedicated to Richard Askey

Abstract. For the Möbius spheres $\mathbb{S}^{q,p}$, we give alternative elementary proofs of the recursive formulas for GJMS-operators and Q-curvatures due to the first author [*Geom. Funct. Anal.* **23**, (2013), 1278–1370]. These proofs make essential use of the theory of hypergeometric series.

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1. Introduction and statement of results

In [J10] and [J13], one of the authors proved recursive formulas for GJMS-operators and Branson's Q-curvatures. In the present article, we give alternative proofs of these results for the space $\mathbb{S}^{q,p} = \mathbb{S}^q \times \mathbb{S}^p$ with the signature (q,p)-metric $g_{\mathbb{S}^q} - g_{\mathbb{S}^p}$ given by the round metrics on the factors. The corresponding (explicit) formulas are found in Theorems 1.1 and 1.2. The proofs which we give here provide another example of the extreme usefulness of hypergeometric series identities. For other recent instances of their crucial role in conformal differential geometry we refer to [CKOP11] and [K13].

In the following, we write $n = q + p$.

A GJMS-operator P_{2N}, $N \geq 1$, is a specific rule which associates to any pseudo-Riemannian manifold (M, g) a differential operator of the form

$$P_{2N}(g) = \Delta_g^N + \text{lower-order terms},$$

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where $\Delta_g = -\delta_g d$ is the Laplace–Beltrami operator of g. The operators $P_{2N}(g)$ are given by universal polynomial formulas in terms of the metric g, its inverse, the curvature tensor of g, and its covariant derivatives. Moreover, the operators $P_{2N}(g)$ are conformally covariant in the sense that

$$e^{(\frac{n}{2}+N)\varphi} P_{2N}(e^{2\varphi}g)(u) = P_{2N}(g)\left(e^{(\frac{n}{2}-N)\varphi}u\right) \tag{1.1}$$

for all $\varphi \in C^\infty(M)$ and $u \in C^\infty(M)$. The operators $P_{2N}(g)$ were derived in [GJMS92] from the N^{th} powers of the Laplace–Beltrami operator of the Fefferman–Graham ambient metric associated to g [FG12a]. On manifolds of even dimension n, this construction is obstructed at $2N = n$, i.e., it only yields a finite sequence $P_2(g), \dots, P_n(g)$ of operators for which (1.1) is valid for *arbitrary* metrics. However, the obstructions should not be attributed to the method of construction. In fact, Graham [G92] proved that on manifolds of dimension 4, there is no conformally covariant cube of the Laplacian. More generally, Gover and Hirachi [GH04] proved that on manifolds of even dimension n, there is no conformally covariant power of the Laplacian the order of which exceeds n. Nevertheless, for even n and locally conformally flat metrics, the obstruction vanishes, and the construction of [GJMS92] still yields an infinite sequence of conformally covariant operators. In the present work, we will be concerned with such a case. For odd n, the operators $P_{2N}(g)$ exist for general metrics g and all even orders $2N \geq 2$.

The GJMS-operators are not uniquely determined by the requirement of their conformal covariance. It remains a challenge to characterize them among other conformally covariant operators with leading part a power of the Laplacian.

Explicit formulas for P_2 and P_4 are well-known. These operators coincide with the Yamabe-operator

$$\Delta - \left(\frac{n}{2} - 1\right)\mathsf{J}, \quad \mathsf{J} = \frac{\mathrm{scal}}{2(n-1)}, \tag{1.2}$$

and the Paneitz-operator

$$\Delta^2 + \delta\left((n-2)\mathsf{J}g - 4\mathsf{P}\right)d + \left(\frac{n}{2} - 2\right)\left(\frac{n}{2}\mathsf{J}^2 - 2\mathsf{P}^2 - \Delta\mathsf{J}\right). \tag{1.3}$$

Here

$$\mathsf{P} = \frac{1}{n-2}\left(\mathrm{Ric} - \mathsf{J}g\right)$$

denotes the Schouten tensor; in (1.3), it is regarded as an endomorphism on one-forms.

Explicit formulas for higher-order GJMS-operators are much more complicated. In [J09], one of the authors introduced new ideas for unveiling the structure of high-order GJMS-operators. Among other things, these led to the discovery of a recursive structure among these operators [J13].

Special cases of recursive formulas for GJMS-operators are the relations

$$P_4 - P_2^2 = -4\delta(\mathsf{P}d) + \mu_4 \tag{1.4}$$

for the Paneitz operator P_4 and

$$P_6 - (2P_2P_4 + 2P_4P_2 - 3P_2^3) = -48\delta(\mathsf{P}^2 d) + \mu_6 \tag{1.5}$$

for the GJMS-operator P_6 of locally conformally flat metrics. Here the scalar functions μ_4 and μ_6 are given in terms of the curvature of g; for explicit formulas see [J13].

In order to describe the general recursive formula for GJMS-operators, we need to introduce some notation. A sequence $I = (I_1, \dots, I_r)$ of integers $I_j \geq 1$ will be regarded as a composition of the sum $|I| = I_1 + I_2 + \cdots + I_r$. As usual, a composition of a positive integer m is a representation of m as a sum $m = m_1 + m_2 + \cdots + m_r$ of positive integers $m_1, m_2, \dots, m_r$, where two representations which contain the same summands but differ in the order of the summands are regarded as different. $|I|$ will be called the size of I. For any composition I, we set

$$P_{2I} = P_{2I_1} \circ \cdots \circ P_{2I_r}. \tag{1.6}$$

For $N \geq 1$, let

$$\mathcal{M}_{2N} \stackrel{\text{def}}{=} \sum_{|I|=N} m_I P_{2I} \tag{1.7}$$

with the multiplicities m_I defined by

$$m_I = -(-1)^r |I|! \, (|I|-1)! \prod_{j=1}^{r} \frac{1}{I_j! \, (I_j - 1)!} \prod_{j=1}^{r-1} \frac{1}{I_j + I_{j+1}}. \tag{1.8}$$

Here, an empty product has to be interpreted as 1. In particular, $m_{(N)} = 1$ for all $N \geq 1$, and $m_I = m_{I^{-1}}$, where I^{-1} is the reverse composition of $I = (I_1, I_2, \dots, I_r)$, that is, $I^{-1} = (I_r, I_{r-1}, \dots, I_1)$.

Theorem 1.1 of [J13] implies that, for locally conformally flat Riemannian manifolds of dimension $n \geq 3$, we have

$$\mathcal{M}_{2N} = -c_N \delta(\mathsf{P}^{N-1} d) + \text{zeroth-order part}, \tag{1.9}$$

where

$$c_N = 2^{N-1} N! \, (N-1)!. \tag{1.10}$$

Eq. (1.9) may be regarded as a summation formula for the operator $\mathcal{M}_{2N}$. We note that the sum (1.7) contains 2^{N-1} terms. Since one of these terms is P_{2N}, the operator $\mathcal{M}_{2N}$ is of order $2N$. But (1.9) shows that the operator $\mathcal{M}_{2N}$ is actually only second-order. In other words, in the sum (1.7), huge cancellations take place. Since $m_{(N)} = 1$, Eq. (1.9) can be regarded as a formula for $P_{2N}^0 \stackrel{\text{def}}{=} P_{2N} - P_{2N}(1)$ in terms of a linear combination of the second-order operator $\delta(\mathsf{P}^{N-1} d)$ and compositions of lower-order GJMS-operators. In that sense, it is a recursive formula for P_{2N}^0. Eqs. (1.4) and (1.5) are the first two special cases.

The proof of (1.9) in [J13] rests on the theory of residue families as developed in [J09]. Residue families are generalizations of certain differential intertwining operators in representation theory ([J09], [KOSS13]). It is the recursive structure of these families which is responsible for the identities (1.9). An alternative proof of (1.9) was given in [FG12b]. It rests on the original definition of the GJMS-operators in terms of the Fefferman–Graham ambient metric as given in [GJMS92]. Note that the formulations in [J10] and [J13] restrict to Riemannian metrics while in [FG12b] the signature of the metric is arbitrary.

Branson [B95] used representation theory to derive explicit expressions for the GJMS-operators of the spaces $\mathbb{S}^{q,p}$. In the present paper, we use these formulas to give a direct proof of the relation (1.9) for $\mathbb{S}^{q,p}$. More precisely, we prove the following summation formula.

Theorem 1.1. *On $\mathbb{S}^{q,p}$, we have*

$$\mathcal{M}_{4N} = (2N)!\,(2N-1)!\left(\frac{1}{2}-B^2-C^2\right),\ N \geq 1 \tag{1.11}$$

and

$$\mathcal{M}_{4N+2} = (2N+1)!\,(2N)!\,(-B^2+C^2),\ N \geq 0. \tag{1.12}$$

Here

$$B^2 = -\Delta_{\mathbb{S}^q} + \left(\frac{q-1}{2}\right)^2 \quad \textit{and} \quad C^2 = -\Delta_{\mathbb{S}^p} + \left(\frac{p-1}{2}\right)^2. \tag{1.13}$$

Theorem 1.1 implies

$$\begin{aligned}\mathcal{M}^0_{4N} &= (2N)!\,(2N-1)!\,(\Delta_{\mathbb{S}^q} + \Delta_{\mathbb{S}^p}),\ N \geq 1,\\ \mathcal{M}^0_{4N+2} &= (2N+1)!\,(2N)!\,(\Delta_{\mathbb{S}^q} - \Delta_{\mathbb{S}^p}),\ N \geq 0\end{aligned}$$

for the non-constant parts of the operators $\mathcal{M}_{2N}$. But using

$$\mathsf{P} = \frac{1}{2}\begin{pmatrix} 1_{\mathbb{S}^q} & 0 \\ 0 & -1_{\mathbb{S}^p}\end{pmatrix}, \tag{1.14}$$

these two identities can be written in the form

$$\begin{aligned}\mathcal{M}^0_{4N} &= -c_{2N}\delta(\mathsf{P}^{2N-1}d),\\ \mathcal{M}^0_{4N+2} &= -c_{2N+1}\delta(\mathsf{P}^{2N}d),\end{aligned}$$

where c_{2N} and c_{2N+1} are defined in (1.10). Thus, Theorem 1.1 actually implies (1.9). Note that Theorem 1.1 also determines the zeroth-order parts of all $\mathcal{M}_{2N}$. It is easy to verify that the result fits with the description of that part as given in [J13, Ex. 8.2].

We continue with the discussion of Q-curvatures. The main result of [J10] concerns the zeroth-order parts of the GJMS-operators. These define Branson's Q-curvatures as follows. For even n, general metrics and $2N < n$, the equation

$$P_{2N}(1) = (-1)^N\left(\frac{n}{2}-N\right)Q_{2N}$$

defines a scalar curvature quantity Q_{2N}. The quantities $Q_2, \dots, Q_{n-2}$ are called the subcritical Q-curvatures. The critical Q-curvature Q_n can be defined through the subcritical ones by continuation in dimension. In odd dimensions as well as for locally conformally flat metrics in even dimensions, the Q-curvatures Q_{2N} are well defined for all $N \geq 1$. For details we refer to [B95] and [J09].

Explicit formulas for Q_2 and Q_4 follow from (1.2) and (1.3):

$$Q_2 = \mathsf{J} \quad \text{and} \quad Q_4 = \frac{n}{2}\mathsf{J}^2 - 2|\mathsf{P}|^2 - \Delta\mathsf{J}.$$

For $N \geq 3$, explicit formulas for Q_{2N} are substantially more complicated. Therefore, it is of some interest to establish recursive formulas for Q-curvatures. Such a formula was found in [J10].

In order to motivate its formulation, we start with the description of two special cases. First of all, we rewrite Q_4 in the form

$$Q_4 = -P_2(Q_2) - Q_2^2 + 2!\,2^3\,v_4, \tag{1.15}$$

where

$$v_4 = \frac{1}{4}\operatorname{tr}(\wedge^2 \mathsf{P}).$$

Next, for locally conformally flat metrics, we have (see [J09])

$$Q_6 = \left[-2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2)\right] - 6\left[Q_4 + P_2(Q_2)\right]Q_2 - 2!\,3!\,2^5\,v_6, \tag{1.16}$$

where

$$v_6 = -\frac{1}{8}\operatorname{tr}(\wedge^3 \mathsf{P}).$$

More conceptually, the coefficients v_4 and v_6 are Taylor coefficients of the function

$$v(r) = \det\left(1 - \frac{r^2}{4}\mathsf{P}\right) = 1 + v_2 r^2 + v_4 r^4 + v_6 r^6 + \cdots, \tag{1.17}$$

which describes the volume form of a Poincaré–Einstein metric associated to the locally conformally flat metric g (see [G00], [FG12a]). Eq. (1.17) implies

$$v_{2k} = \left(-\frac{1}{2}\right)^k \operatorname{tr}(\wedge^k \mathsf{P}).$$

Now, one can prove that the formulas (1.15) and (1.16) are equivalent to the respective identities

$$Q_4 + P_2(Q_2) = 2!\,2^4\,w_4 \tag{1.18}$$

and

$$-Q_6 - 2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2) = 2!\,3!\,2^6\,w_6, \tag{1.19}$$

where the quantities w_4 and w_6 are Taylor coefficients of

$$w(r) = \sqrt{v(r)},$$

i.e.,

$$\left(1 + r^2 w_2 + r^4 w_4 + r^6 w_6 + \cdots\right)^2 = v(r).$$

More explicitly, we have

$$\begin{aligned} 2w_2 &= v_2, \\ 2w_4 &= \frac{1}{4}(4v_4 - v_2^2), \\ 2w_6 &= \frac{1}{8}(8v_6 - 4v_4 v_2 + v_2^3). \end{aligned}$$

Now, in these terms, the main result of [J10] implies that

$$\sum_{|I|=N} m_I \frac{P_{2I}(1)}{\frac{n}{2} - I_{\mathrm{last}}} = N!\,(N-1)!\,w_{2N} \tag{1.20}$$

for locally conformally flat metrics and $N \geq 1$. Here, I_{last} denotes the last entry of the composition I. For $I = (J, \frac{n}{2})$, the quotient in (1.20) is to be interpreted as

$$(-1)^{\frac{n}{2}} P_{2J}(Q_n).$$

Again, the sum in (1.20) contains 2^{N-1} terms. By separating the contribution for $I = (N)$, Eq. (1.20) can be regarded as a formula for Q_{2N}. It expresses Q_{2N} in terms of lower-order Q-curvatures, lower-order GJMS-operators and w_{2N}. Eqs. (1.18) and (1.19) are the first two special cases.

The proof of (1.20) in [J10] rests on the recursive structure of residue families. This result played an important role in the discussion of the recursive structure of GJMS-operators in [J13]. Again, an alternative proof of (1.20) was given in [FG12b].

The following result proves (1.20) for $\mathbb{S}^{q,p}$ by proving the following summation formula.

Theorem 1.2. *On $\mathbb{S}^{q,p}$, we have*

$$\sum_{|I|=N} m_I \frac{P_{2I}(1)}{\frac{n}{2} - I_{last}} = N!\,(N-1)!\,\sum_{M=0}^{N} (-1)^M \binom{\frac{q}{2}}{M} \binom{\frac{p}{2}}{N-M} \tag{1.21}$$

for all $N \geq 1$.

In order to see that Theorem 1.2 implies (1.20), we note that (1.14) and (1.17) imply

$$v(r) = (1 - r^2/4)^q (1 + r^2/4)^p.$$

Hence

$$w(r) = (1 - r^2/4)^{q/2} (1 + r^2/4)^{p/2}.$$

It follows that the right-hand side of (1.21) coincides with

$$N!\,(N-1)!\,2^{2N}\,w_{2N}.$$

The remaining part of the paper is organized as follows. In Section 2, we prove Theorem 1.1. Section 3 contains the proof of a summation formula which contains Theorem 1.2 as a special case. The proofs of some technical results which are used in the course of these proofs are collected in an appendix.

2. Proof of Theorem 1.1

We start with a description of the GJMS-operators on $\mathbb{S}^{q,p}$.

Theorem 2.1 ([B95], Theorem 6.2)**.** *On $\mathbb{S}^{q,p}$, the GJMS-operators are given by the product formulas*

$$P_{4N} = \prod_{j=1}^{N} (B+C+(2j-1))(B-C-(2j-1))(B+C-(2j-1))(B-C+(2j-1)) \tag{2.1}$$

and

$$P_{4N+2} = (-B^2+C^2)\prod_{j=1}^{N}(B+C+2j)(B-C-2j)(B+C-2j)(B-C+2j). \tag{2.2}$$

Here, the operators B and C are defined as the positive square roots of the non-negative operators B^2 and C^2.

The reader should note that the formulas (2.1) and (2.2) are equivalent to the product representations

$$P_{4N} = \prod_{j=1}^{N}\left((B^2-C^2)^2 - 2(2j-1)^2(B^2+C^2) + (2j-1)^4\right) \tag{2.3}$$

and

$$P_{4N+2} = (-B^2+C^2)\prod_{j=1}^{N}\left((B^2-C^2)^2 - 2(2j)^2(B^2+C^2) + (2j)^4\right). \tag{2.4}$$

An advantage of the latter formulas is that they do not require to leave the framework of differential operators.

The formulas (2.1) and (2.2) can be restated uniformly as

$$P_{2N} = 2^{2N}\left((C+B+1-N)/2\right)_N\left((C-B+1-N)/2\right)_N, \tag{2.5}$$

where $(\alpha)_m$ is the usual Pochhammer symbol defined by $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$ for $m \ge 1$, and $(\alpha)_0 = 1$. In terms of the variables

$$X = C+B \quad \text{and} \quad Y = C-B, \tag{2.6}$$

the product formula (2.5) is equivalent to

$$P_{2N} = 2^{2N}\left((X+1-N)/2\right)_N\left((Y+1-N)/2\right)_N. \tag{2.7}$$

In the sequel, we shall regard P_{2N} as this polynomial in the variables X and Y.

In the proofs of Theorems 1.1 and 1.2, the basis $\left(((X+1-A)/2)_A\right)_{A=0,1,\dots}$ of the linear space of all polynomials in X will play an essential role. (A glance at (2.7) may suggest why this could be the case.) In the following lemma, we compute the structure coefficients with respect to multiplication of this basis of polynomials. They will enter the inductive proofs of Lemma 2.2 and of Theorems 2.2 and 3.1.

Lemma 2.1. *For all non-negative integers A and B, we have*

$$\begin{aligned}&((X+1-A)/2)_A\,((X+1-B)/2)_B\\ &= \sum_{j=0}^{\lfloor (A+B)/2\rfloor}(-1)^j\frac{(-A/2)_j\,(-B/2)_j\,(-(A+B)/2)_j}{j!}\,((X+1-A-B+2j)/2)_{A+B-2j}.\end{aligned} \tag{2.8}$$

Remark 2.1. *The sum on the right-hand side of* (2.8) *terminates at the latest at $j = \lfloor (A+B)/2\rfloor$. If A or B should be even, it terminates already at $j = A/2$, respectively at $j = B/2$.*

Proof. The reader should recall the standard hypergeometric notation

$$ {}_pF_q\left[\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z\right]=\sum_{m=0}^{\infty}\frac{(a_1)_m\cdots(a_p)_m}{m!\,(b_1)_m\cdots(b_q)_m}z^m $$

where $(a_1)_m$, etc. are again Pochhammer symbols. In terms of this notation, the right-hand side of (2.8) reads

$$ \big((X+1-A-B)/2\big)_{A+B}\,{}_3F_2\left[\begin{matrix}-(A+B)/2,-A/2,-B/2\\ (1+X-A-B)/2,(1-X-A-B)/2\end{matrix};1\right]. $$

The ${}_3F_2$-series can be evaluated by means of the Pfaff–Saalschütz summation (cf. [S66, (2.3.1.3); Appendix (III.2)])

$$ {}_3F_2\left[\begin{matrix}a,b,-n\\ c,1+a+b-c-n\end{matrix};1\right]=\frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n}, $$

where n is a non-negative integer. If we apply the formula, then we obtain the left-hand side of (2.8) after little simplification. □

Lemma 2.2. *For all positive integers $a<N$, the partial sum*

$$ S(N,a)=\sum_{J:|J|+a=N}m_{(J,a)}P_{2J} \tag{2.9} $$

satisfies

$$ \begin{aligned} S(N,a)=\binom{N-1}{a-1}&\sum_{k=0}^{\lfloor (N-a)/2\rfloor}\sum_{l=0}^{\lfloor (N-a)/2\rfloor}(-1)^{N+k+l+a}2^{2N-2k-2l-2a}\\ &\cdot\big((X+1-N+a+2k)/2\big)_{N-a-2k}\big((Y+1-N+a+2l)/2\big)_{N-a-2l}\\ \cdot\frac{(-N+a)_{2k}\,(-N+a)_{2l}\,(-N/2)_k\,(-N/2)_l}{k!\,l!}&\,{}_4F_3\left[\begin{matrix}-\frac12,-k,-l,\frac12-\frac N2\\ -\frac N2,\frac a2-\frac N2,\frac12+\frac a2-\frac N2\end{matrix};1\right]. \end{aligned} \tag{2.10} $$

Proof. We prove the claim by induction on N, the case $N=2$ being straightforward to check. Suppose that we have already established (2.10) up to $N-1$. If $a+|J|=N$, we write $m_{(a,J)}$ as

$$ m_{(a,J)}=\begin{cases}-\dfrac{N!\,(N-1)!}{(N-a)!\,(N-a-1)!}\dfrac{1}{N\cdot a!\,(a-1)!}m_{(N-a)}, & \text{if } J=(N-a),\\[2mm] -\dfrac{N!\,(N-1)!}{(N-a)!\,(N-a-1)!}\dfrac{1}{(a+b)\,a!\,(a-1)!}m_{(b,K)}, & \text{if } J=(b,K),\end{cases} $$

and, if $J=(b,K)$, we write $P_{2J}=P_{2b}P_{2K}$. Then the left-hand side of (2.10) becomes

$$ \begin{aligned} &-\frac{N!\,(N-1)!}{N\cdot a!\,(a-1)!\,(N-a)!\,(N-a-1)!}P_{2(N-a)}\\ &-\sum_{b=1}^{N-a-1}\sum_{K:b+|K|=N-a}\frac{N!\,(N-1)!}{(a+b)\,a!\,(a-1)!\,(N-a)!\,(N-a-1)!}m_{(b,K)}P_{2b}P_{2K}. \end{aligned} \tag{2.11} $$

If we now use the induction hypothesis, then this sum simplifies to

$$
\begin{aligned}
-\sum_{b=1}^{N-a} & \frac{N!\,(N-1)!}{(a+b)\,a!\,(a-1)!\,(N-a)!\,(N-a-1)!} 2^{2b}\big((X+1-b)/2\big)_b\,\big((Y+1-b)/2\big)_b \\
&\cdot \binom{N-a-1}{b-1} \sum_{k=0}^{\lfloor (N-a-b)/2\rfloor} \sum_{l=0}^{\lfloor (N-a-b)/2\rfloor} (-1)^{N+k+l+a+b} 2^{2N-2k-2l-2a-2b} \\
&\cdot \big((X+1-N+a+b+2k)/2\big)_{N-a-b-2k}\,\big((Y+1-N+a+b+2l)/2\big)_{N-a-b-2l} \\
&\cdot \frac{(-N+a+b)_{2k}\,(-N+a+b)_{2l}\,((-N+a)/2)_k\,((-N+a)/2)_l}{k!\,l!} \\
&\cdot {}_4F_3\left[\begin{matrix} -\frac{1}{2}, -k, -l, \frac{1}{2}+\frac{a}{2}-\frac{N}{2} \\ \frac{a}{2}-\frac{N}{2}, \frac{a}{2}+\frac{b}{2}-\frac{N}{2}, \frac{1}{2}+\frac{a}{2}+\frac{b}{2}-\frac{N}{2}\end{matrix}; 1\right]. \qquad (2.12)
\end{aligned}
$$

Here, the term containing $P_{2(N-a)}$, appearing in (2.11), arises as the summand for $b=N-a$, since this forces k and l to be zero.

Now, Lemma 2.1 implies the expansions

$$
\begin{aligned}
&\big((X+1-b)/2\big)_b\,\big((X+1-N+a+b+2k)/2\big)_{N-a-b-2k} \\
&\quad = \sum_{j_1=0}^{\lfloor (N-a-2k)/2\rfloor} (-1)^{j_1} \frac{(-b/2)_{j_1}\,(-(N-a-b-2k)/2)_{j_1}\,(-(N-a-2k)/2)_{j_1}}{j_1!} \\
&\qquad\qquad \cdot \big((X+1-N+a+2k+2j_1)/2\big)_{N+a-2k-2j_1}.
\end{aligned}
$$

and

$$
\begin{aligned}
&\big((Y+1-b)/2\big)_b\,\big((Y+1-N+a+b+2l)/2\big)_{N-a-b-2l} \\
&\quad = \sum_{j_2=0}^{\lfloor (N-a-2l)/2\rfloor} (-1)^{j_2} \frac{(-b/2)_{j_2}\,(-(N-a-b-2l)/2)_{j_2}\,(-(N-a-2l)/2)_{j_2}}{j_2!} \\
&\qquad\qquad \cdot \big((Y+1-N+a+2l+2j_2)/2\big)_{N+a-2l-2j_2}.
\end{aligned}
$$

We use these in (2.12) and, in addition, perform the index transformation $s_1=k+j_1$ and $s_2=l+j_2$. Thus, we arrive at the expression

$$
\begin{aligned}
-&\sum_{s_1=0}^{\lfloor (N-a)/2\rfloor} \sum_{s_2=0}^{\lfloor (N-a)/2\rfloor} \big((X+1-N+a+2s_1)/2\big)_{N-a-2s_1}\,\big((Y+1-N+a+2s_2)/2\big)_{N-a-2s_2} \\
&\cdot \sum_{b=1}^{N-a} \frac{N!\,(N-1)!}{(a+b)\,a!\,(a-1)!\,(N-a)!\,(N-a-1)!} \binom{N-a-1}{b-1} (-1)^{N+s_1+s_2+a+b} 2^{2N-2a} \\
&\cdot \sum_{k=0}^{\lfloor (N-a-b)/2\rfloor} \sum_{l=0}^{\lfloor (N-a-b)/2\rfloor} (-b/2)_{s_1-k}\,(-b/2)_{s_2-l}\,((-N+a)/2)_{s_1}\,((-N+a)/2)_{s_2}
\end{aligned}
$$

$$\cdot\,((-N+a+b)/2)_{s_1}\,((-N+a+b)/2)_{s_2}\frac{((-N+a+b+1)/2)_k\,((-N+a+b+1)/2)_l}{k!\,l!\,(s_1-k)!\,(s_2-l)!}$$
$$\cdot\,{}_4F_3\left[\begin{matrix}-\frac{1}{2},-k,-l,\frac{1}{2}+\frac{a}{2}-\frac{N}{2}\\ \frac{a}{2}-\frac{N}{2},\frac{a}{2}+\frac{b}{2}-\frac{N}{2},\frac{1}{2}+\frac{a}{2}+\frac{b}{2}-\frac{N}{2}\end{matrix};1\right]. \tag{2.13}$$

In this expression, we now concentrate on the terms involving the summation index k only:

$$\sum_{k=0}^{s_1}\frac{(-b/2)_{s_1-k}\,(-(N-a-b-1)/2)_k\,(-k)_s}{k!\,(s_1-k)!}, \tag{2.14}$$

where s stands for the summation index of the ${}_4F_3$-series in (2.19). Because of the term $(-k)_s$ in the numerator of the summand, we may start the summation at $k=s$ (instead of at $k=0$). Hence, if we write this sum in hypergeometric notation, we obtain

$$\frac{(-b/2)_{s_1-s}\,(-(N-a-b-1)/2)_s\,(-s)_s}{s!}\,{}_2F_1\left[\begin{matrix}-s_1+s,\frac{1}{2}+\frac{a}{2}-\frac{N}{2}+\frac{b}{2}+s\\ 1+\frac{b}{2}-s_1+s\end{matrix};1\right].$$

This ${}_2F_1$-series can be evaluated by means of the Chu–Vandermonde summation formula (see [S66, (1.7.7), Appendix (III.4)]), so that the sum in (2.14) equals

$$\frac{(-s_1)_s\,(-(N-a-b-1)/2)_s\,((1+a-N)/2)_{s_1}}{s_1!\,((1+a-N)/2)_s}.$$

Clearly, an analogous computation can be done for the sum over l in (2.13). Altogether, we see that (2.13) simplifies to

$$-\sum_{s_1=0}^{\lfloor (N-a)/2\rfloor}\sum_{s_2=0}^{\lfloor (N-a)/2\rfloor}\big((X+1-N+a+2s_1)/2\big)_{N-a-2s_1}\big((Y+1-N+a+2s_2)/2\big)_{N-a-2s_2}$$
$$\cdot\sum_{b=1}^{N-a}(-1)^{N+s_1+s_2+a+b}2^{2N-2s_1-2s_2-2a}\frac{N!\,(N-1)!\,\binom{N-a-1}{b-1}}{(a+b)\,a!\,(a-1)!\,(N-a)!\,(N-a-1)!}$$
$$\cdot\,\frac{(-N+a)_{2s_1}\,(-N+a)_{2s_2}\,((-N+a+b)/2)_{s_1}\,((-N+a+b)/2)_{s_2}}{s_1!\,s_2!}$$
$$\cdot\,{}_4F_3\left[\begin{matrix}-\frac{1}{2},-s_1,-s_2,\frac{1}{2}+\frac{a}{2}+\frac{b}{2}-\frac{N}{2}\\ \frac{a}{2}-\frac{N}{2},\frac{1}{2}+\frac{a}{2}-\frac{N}{2},\frac{a}{2}+\frac{b}{2}-\frac{N}{2}\end{matrix};1\right]$$
$$=-\sum_{s_1=0}^{\lfloor (N-a)/2\rfloor}\sum_{s_2=0}^{\lfloor (N-a)/2\rfloor}\big((X+1-N+a+2s_1)/2\big)_{N-a-2s_1}\big((Y+1-N+a+2s_2)/2\big)_{N-a-2s_2}$$
$$\cdot\sum_{b=1}^{N-a}(-1)^{N+s_1+s_2+a+b}2^{2N-2s_1-2s_2-2a}\frac{N!\,(N-1)!}{(a+b)\,a!\,(a-1)!\,(N-a)!\,(b-1)!}$$
$$\cdot\sum_{s=0}^{s_1}\frac{(-N+a)_{2s_1}\,(-N+a)_{2s_2}\,((-N+a+b+2s)/2)_{s_1-s}\,((-N+a+b+2s)/2)_{s_2-s}}{s_1!\,s_2!}$$

$$\cdot \frac{(-1/2)_s\,(-s_1)_s\,(-s_2)_s\,(N-a-2s)!}{s!\,(N-a)!\,(N-a-b-2s)!}. \quad (2.15)$$

Next we concentrate on the terms involving the summation index b only:

$$\frac{1}{(N-a-2s-1)!}\sum_{b=1}^{N-a}\frac{(-1)^b}{a+b}\binom{N-a-2s-1}{b-1} \cdot ((-N+a+b+2s)/2)_{s_1-s}\,((-N+a+b+2s)/2)_{s_2-s}.$$

By Lemma A.2 with $X=-a-1$ and $M=N-a-2s-1$, this is equal to

$$\frac{(-1)^{N-a-2s-1}}{(-N+2s)_{N-a-2s}}((-N+2s)/2)_{s_1-s}((-N+2s)/2)_{s_2-s} + \chi(s_1=s_2=(N-a)/2)\cdot 2^{-N+a+2s}$$
$$= -\frac{a!}{(N-2s)!}\frac{(-N/2)_{s_1}\,(-N/2)_{s_2}}{(-N/2)_s^2} + \chi(s_1=s_2=(N-a)/2)\cdot 2^{-N+a+2s},$$

where $\chi(\mathcal{S})=1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S})=0$ otherwise. If we substitute this in (2.15), then we obtain

$$\sum_{s_1=0}^{\lfloor (N-a)/2\rfloor}\sum_{s_2=0}^{\lfloor (N-a)/2\rfloor}\big((X+1-N+a+2s_1)/2\big)_{N-a-2s_1}\big((Y+1-N+a+2s_2)/2\big)_{N-a-2s_2}$$
$$\cdot (-1)^{N+s_1+s_2+a}2^{2N-2s_1-2s_2-2a}\frac{(N-1)!}{(a-1)!\,(N-a)!}$$
$$\cdot \frac{(-N+a)_{2s_1}\,(-N+a)_{2s_2}\,(-N/2)_{s_1}\,(-N/2)_{s_2}}{s_1!\,s_2!}$$
$$\cdot \sum_{s=0}^{s_1}\frac{(-1/2)_s\,(-s_1)_s\,(-s_2)_s\,N!\,(N-a-2s)!}{s!\,(N-2s)!\,(-N/2)_s^2\,(N-a)!}$$
$$+\chi(N-a \text{ is even})\cdot\frac{N!\,(N-1)!}{a!\,(a-1)!\,((N-a)/2)!^2}$$
$$\times \sum_{s=0}^{(N-a)/2}\frac{2^{-N+a+2s}(-1/2)_s\,(-(N-a)/2)_s^2\,(N-a-2s)!}{s!}.$$

Here, the first sum is, upon rewriting, exactly equal to the right-hand side of (2.10) (except that s_1 and s_2 took over the role of k and l). On the other hand, if we write the second sum in hypergeometric notation, we obtain

$$\chi(N-a \text{ is even})\cdot\frac{N!\,(N-1)!}{a!\,(a-1)!\,((N-a)/2)!^2}$$
$$\times \sum_{s=0}^{(N-a)/2}\frac{2^{-N+a+2s}(-1/2)_s\,(-(N-a)/2)_s^2\,(N-a-2s)!}{s!}$$

$$= \chi(N-a \text{ is even}) \cdot \frac{N!\,(N-1)!}{a!\,(a-1)!\,((N-a)/2)!^2} 2^{-N+a}\,(N-a)!\,{}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{a}{2}-\frac{N}{2} \\ \frac{a}{2}-\frac{N}{2}+\frac{1}{2} \end{matrix}; 1\right].$$

The ${}_2F_1$-series can be evaluated by the Chu–Vandermonde summation formula, so that the above expression becomes

$$\chi(N-a \text{ is even}) \cdot \frac{N!\,(N-1)!}{a!\,(a-1)!\,((N-a)/2)!^2} 2^{-N+a}\,(N-a)!\frac{((a-N+2)/2)_{(N-a)/2}}{((a-N+1)/2)_{(N-a)/2}},$$

which vanishes because of the term $((a-N+2)/2)_{(N-a)/2}$. This concludes the induction step. ☐

Theorem 2.2. *For all positive integers N, we have*

$$\sum_{|I|=N} m_I P_{2I} = \begin{cases} N!\,(N-1)!\,XY, & \text{if } N \text{ is odd,} \\ \frac{1}{2}N!\,(N-1)!\,\left(1-X^2-Y^2\right), & \text{if } N \text{ is even,} \end{cases} \tag{2.16}$$

where P_{2N} is defined in (2.7).

Proof. By the definition of $S(N,a)$ in (2.9), we have

$$\sum_{|I|=N} m_I P_{2I} = P_{2N} + \sum_{a=1}^{N-1} S(N,a) P_{2a}. \tag{2.17}$$

If, in this equation, we substitute the expressions for P_{2a} and $S(N,a)$ given by (2.7) and (2.10), then we see that the left-hand side of (2.16) is equal to

$$\begin{aligned} &\sum_{a=1}^{N} \big((X+1-a)/2\big)_a \big((Y+1-a)/2\big)_a \binom{N-1}{a-1} \\ &\cdot \sum_{k=0}^{\lfloor (N-a)/2 \rfloor} \sum_{l=0}^{\lfloor (N-a)/2 \rfloor} (-1)^{N+k+l+a} 2^{2N-2k-2l} \\ &\qquad \cdot \big((X+1-N+a+2k)/2\big)_{N-a-2k} \big((Y+1-N+a+2l)/2\big)_{N-a-2l} \\ &\cdot \frac{(-N+a)_{2k}\,(-N+a)_{2l}\,(-N/2)_k\,(-N/2)_l}{k!\,l!}\,{}_4F_3\left[\begin{matrix} -\frac{1}{2}, -k, -l, \frac{1}{2}-\frac{N}{2} \\ -\frac{N}{2}, \frac{a}{2}-\frac{N}{2}, \frac{1}{2}+\frac{a}{2}-\frac{N}{2} \end{matrix}; 1\right]. \end{aligned} \tag{2.18}$$

Here, the term P_{2N}, appearing on the left-hand side of (2.16), arises as the summand for $a=N$, since this forces k and l to be zero.

Lemma 2.1 implies the expansions

$$\begin{aligned} &\big((X+1-a)/2\big)_a \big((X+1-N+a+2k)/2\big)_{N-a-2k} \\ &\qquad = \sum_{j_1=0}^{\lfloor (N-2k)/2 \rfloor} (-1)^{j_1} \frac{(-a/2)_{j_1}\,(-(N-a-2k)/2)_{j_1}\,(-(N-2k)/2)_{j_1}}{j_1!} \\ &\qquad\qquad \cdot \big((X+1-N+2k+2j_1)/2\big)_{N-2k-2j_1}. \end{aligned}$$

and

$$\big((Y+1-a)/2\big)_a\,\big((Y+1-N+a+2l)/2\big)_{N-a-2l}$$
$$=\sum_{j_2=0}^{\lfloor (N-2l)/2\rfloor}(-1)^{j_2}\frac{(-a/2)_{j_2}\,(-(N-a-2l)/2)_{j_2}\,(-(N-2l)/2)_{j_2}}{j_2!}$$
$$\cdot\big((Y+1-N+2l+2j_2)/2\big)_{N-2l-2j_2}.$$

We use these in (2.18) and, in addition, perform the index transformation $s_1=k+j_1$ and $s_2=l+j_2$. Thus, the left-hand side in (2.16) can be written in the form

$$\sum_{s_1=0}^{\lfloor N/2\rfloor}\sum_{s_2=0}^{\lfloor N/2\rfloor}\sum_{a=1}^{N}(-1)^{N+s_1+s_2+a}2^{2N-2k-2l}\binom{N-1}{a-1}$$
$$\cdot\big((X+1-N+2s_1)/2\big)_{N-2s_1}\big((Y+1-N+2s_2)/2\big)_{N-2s_2}$$
$$\cdot\sum_{k=0}^{\lfloor (N-a)/2\rfloor}\sum_{l=0}^{\lfloor (N-a)/2\rfloor}\frac{(-a/2)_{s_1-k}\,(-a/2)_{s_2-l}\,(-(N-a)/2)_{s_1}\,(-(N-a)/2)_{s_2}}{(s_1-k)!\,(s_2-l)!\,(-(N-a)/2)_k\,(-(N-a)/2)_l}$$
$$\cdot\frac{(-N+a)_{2k}\,(-N+a)_{2l}\,(-N/2)_{s_1}\,(-N/2)_{s_2}}{k!\,l!}\,{}_4F_3\!\left[\begin{matrix}-\frac12,-k,-l,\frac12-\frac N2\\ -\frac N2,\frac a2-\frac N2,\frac12+\frac a2-\frac N2\end{matrix};1\right].\qquad(2.19)$$

In this expression, we now concentrate on the terms involving the summation index k only:

$$\sum_{k=0}^{\lfloor (N-a)/2\rfloor}2^{-2k}\frac{(-a/2)_{s_1-k}\,(-N+a)_{2k}\,(-k)_s}{k!\,(s_1-k)!\,(-(N-a)/2)_k}$$
$$=\sum_{k=0}^{s_1}\frac{(-a/2)_{s_1-k}\,(-(N-a-1)/2)_k\,(-k)_s}{k!\,(s_1-k)!},\qquad(2.20)$$

where s stands for the summation index of the ${}_4F_3$-series in (2.19). Because of the term $(-k)_s$ in the numerator of the summand, we may start the summation at $k=s$ (instead of at $k=0$). Hence, if we write this sum in hypergeometric notation, we obtain

$$\frac{(-a/2)_{s_1-s}\,(-(N-a-1)/2)_s\,(-s)_s}{s!}{}_2F_1\!\left[\begin{matrix}-s_1+s,\frac12-\frac N2+\frac a2+s\\ 1+\frac a2-s_1+s\end{matrix};1\right].$$

This ${}_2F_1$-series can be evaluated by means of the Chu–Vandermonde summation formula, so that the sum in (2.20) equals

$$\frac{(-s_1)_s\,(-(N-a-1)/2)_s\,((1-N)/2)_{s_1}}{s_1!\,((1-N)/2)_s}.$$

Clearly, an analogous computation can be done for the sum over l in (2.19). Altogether, we see that (2.19) simplifies to

$$\sum_{s_1=0}^{\lfloor N/2\rfloor}\sum_{s_2=0}^{\lfloor N/2\rfloor}\sum_{a=1}^{N}(-1)^{N+s_1+s_2+a}2^{2N}\binom{N-1}{a-1}$$

$$\cdot\,(-N/2)_{s_1}\,((1-N)/2)_{s_1}\,(-N/2)_{s_2}\,((1-N)/2)_{s_2}$$
$$\cdot\,\big((X+1-N+2s_1)/2\big)_{N-2s_1}\big((Y+1-N+2s_2)/2\big)_{N-2s_2}$$
$$\cdot\,\frac{(-(N-a)/2)_{s_1}\,(-(N-a)/2)_{s_2}}{s_1!\,s_2!}\,{}_4F_3\left[\begin{matrix}-\frac{1}{2},-s_1,-s_2,\frac{1}{2}+\frac{a}{2}-\frac{N}{2}\\ -\frac{N}{2},\frac{1}{2}-\frac{N}{2},\frac{a}{2}-\frac{N}{2}\end{matrix};1\right]. \tag{2.21}$$

Next we concentrate on the terms involving the summation index a only:

$$\begin{aligned}\sum_{a=1}^{N}&(-1)^a\binom{N-1}{a-1}\frac{(-(N-a)/2)_{s_1}\,(-(N-a)/2)_{s_2}\,(-(N-a-1)/2)_s}{(-(N-a)/2)_s}\\
&=\frac{(N-1)!}{2^{2s}\,(N-2s-1)!}\sum_{a=1}^{N-2s}(-1)^a\binom{N-2s-1}{a-1}\\
&\qquad\qquad\cdot\,(-(N-a-2s)/2)_{s_1-s}\,(-(N-a-2s)/2)_{s_2-s}\\
&=-\frac{(N-1)!}{2^{2s}\,(N-2s-1)!}\sum_{a=0}^{N-2s-1}(-1)^a\binom{N-2s-1}{a}\\
&\qquad\qquad\cdot\,((a-N+2s+1)/2)_{s_1-s}\,((a-N+2s+1)/2)_{s_2-s},\end{aligned} \tag{2.22}$$

where, in abuse of notation, we wrote again s for the summation index of the ${}_4F_3$-series in (2.21). By Lemma A.1, this sum vanishes if the degree of

$$((a-N+2s+1)/2)_{s_1-s}\,((a-N+2s+1)/2)_{s_2-s}$$

as a polynomial in a should be less than $N-2s-1$. In fact, this is almost always the case since $s_1-s+s_2-s\le 2\lfloor N/2\rfloor-2s\le N-2s$. More precisely, the sum in (2.22) is non-zero only if:

(1) N is odd and $s_1=s_2=(N-1)/2$;
(2) N is even and $s_1=s_2=N/2$ or $\{s_1,s_2\}=\{N/2,(N-2)/2\}$.

We now discuss these cases separately.

(1) Let N be odd. Then, as we observed above, in the sum (2.21) only the terms corresponding to $s_1=s_2=(N-1)/2$ survive. Using Lemma A.1, we see that (2.21) reduces to

$$\begin{aligned}&-4\,(-N)_{N-1}^2\,(X/2)\,(Y/2)\\
&\quad\times\sum_{s=0}^{(N-1)/2}(-1)^{N-2s}2^{-N+2s+1}\frac{(N-1)!}{2^{2s}\,((N-1)/2)!^2}\,\frac{(-1/2)_s\,((1-N)/2)_s}{s!\,(-N/2)_s}\\
&\qquad\qquad=2^{1-N}XY\frac{N!^2\,(N-1)!}{((N-1)/2)!^2}\,{}_2F_1\left[\begin{matrix}-\frac{1}{2},\frac{1}{2}-\frac{N}{2}\\ -\frac{N}{2}\end{matrix};1\right].\end{aligned}$$

The ${}_2F_1$-series can be evaluated by means of the Chu–Vandermonde summation formula. After some simplification, we obtain $N!\,(N-1)!\,XY$, in accordance with the claim in (2.16).

(2) Let N be even. As we observed above, there are now three terms in (2.21) which survive: the terms for $s_1 = s_2 = N/2$, for $(s_1, s_2) = (N/2, (N-2)/2)$, and for $(s_1, s_2) = ((N-2)/2, N/2)$.

We begin with the term for $s_1 = N/2$ and $s_2 = (N-2)/2$. By Lemma A.1, this term equals

$$\begin{aligned}
&-4\,(-N)_N\,(-N)_{N-2}\,((Y-1)/2)_2 \\
&\quad\times \sum_{s=0}^{(N-2)/2} (-1)^{N-2s} 2^{-N+2s+1} \frac{(N-1)!}{2^{2s}\,(N/2)!\,((N-2)/2)!} \frac{(-1/2)_s\,((2-N)/2)_s}{s!\,((1-N)/2)_s} \\
&\qquad\qquad = -2^{-N}(Y^2-1)\frac{N!^2\,(N-1)!}{(N/2)!\,((N-2)/2)!}\,{}_2F_1\!\left[\begin{matrix}-\frac{1}{2}, 1-\frac{N}{2}\\ \frac{1}{2}-\frac{N}{2}\end{matrix}; 1\right].
\end{aligned}$$

Again, the ${}_2F_1$-series can be evaluated by means of the Chu–Vandermonde summation formula. After some simplification, we obtain $-\frac{1}{2}N!\,(N-1)!\,(Y^2-1)$. Clearly, an analogous computation yields that the term for $s_1 = (N-2)/2$ and $s_2 = N/2$ equals $-\frac{1}{2}N!\,(N-1)!\,(X^2-1)$. Finally, we consider the term in (2.21) corresponding to $s_1 = s_2 = N/2$. Here, in order to apply Lemma A.1, we need to compute the coefficient c_{N-2s-1} in the expansion

$$((a-N+2s+1)/2)_{(N-2s)/2}\,((a-N+2s+1)/2)_{(N-2s)/2} = \sum_{k=0}^{N-2s} c_k \binom{a}{k}. \tag{2.23}$$

We have $c_{N-2s} = 2^{-N+2s}(N-2s)!$. Comparison of coefficients of a^{N-2s-1} on both sides of (2.23) then yields that

$$\begin{aligned}
c_{N-2s-1} &= 2^{-N+2s}\left(-2\sum_{\ell=1}^{(N-2s)/2}(2\ell-1) + \sum_{\ell=1}^{N-2s-1}\ell\right) \\
&= 2^{-N+2s}\left(-2\left(\frac{N-2s}{2}\right)^2 + \frac{(N-2s)(N-2s-1)}{2}\right) \\
&= -2^{-N+2s-1}(N-2s).
\end{aligned}$$

Hence, if we use this, together with Lemma A.1, we obtain that the term corresponding to $s_1 = s_2 = N/2$ in (2.21) equals

$$\begin{aligned}
&(-N)_N^2 \sum_{s=0}^{N/2}(-1)^{N-2s+1}2^{-N+2s-1}(N-2s)\frac{(N-1)!}{2^{2s}\,(N/2)!^2}\frac{(-1/2)_s\,(-N/2)_s}{s!\,((1-N)/2)_s} \\
&\quad = -2^{-N-1}\frac{N!^2\,(N-1)!}{(N/2)!^2}\left(N\,{}_2F_1\!\left[\begin{matrix}-\frac{1}{2}, -\frac{N}{2}\\ \frac{1}{2}-\frac{N}{2}\end{matrix}; 1\right] + \frac{N}{N-1}\,{}_2F_1\!\left[\begin{matrix}\frac{1}{2}, 1-\frac{N}{2}\\ \frac{3}{2}-\frac{N}{2}\end{matrix}; 1\right]\right).
\end{aligned}$$

Both ${}_2F_1$-series can be evaluated by means of the Chu–Vandermonde summation formula. As it turns out, the first one sums to zero. If the result is simplified, one obtains $-\frac{1}{2}N!\,(N-1)!$ eventually.

Putting our results together, we obtain

$$\frac{1}{2}N!\,(N-1)!(-X^2+1-Y^2+1-1)=\frac{1}{2}N!\,(N-1)!(1-X^2-Y^2)$$

for the left-hand side of (2.16), which is in accordance with the right-hand side. $\square$

Now, Theorem 1.1 follows from Theorem 2.2 by recalling (2.6).

3. Proof of Theorem 1.2

As in Section 2 we regard all GJMS-operators as polynomials in the variables X and Y (see (2.7)). In particular, P_{2N} is divisible by the monomial $(X+1-N)$.

Theorem 3.1. *For all positive integers N, we have*

$$\sum_{a=1}^{N}\sum_{|J|=N-a} m_{(J,a)}\frac{1}{X+1-a}P_{2J}P_{2a}\\
=N!(N-1)!\sum_{k=0}^{N}(-1)^k\binom{(X-Y+1)/2}{k}\binom{(X+Y+1)/2}{N-k}. \tag{3.1}$$

Proof. By the definition of $S(N,a)$ in (2.9), we have

$$\sum_{a=1}^{N}\sum_{|J|=N-a} m_{(J,a)}\frac{1}{X+1-a}P_{2J}P_{2a}=\frac{1}{X+1-N}P_{2N}+\sum_{a=1}^{N-1}\frac{1}{X+1-a}S(N,a)P_{2a}.$$

The reader should note that the only difference to (2.17) are the factors $1/(X+1-N)$ in front of P_{2N} and the factor $1/(X+1-a)$ in front of the summand of the sum over a. We may therefore proceed as in the proof of Theorem 2.2, as long as the summation over a does not come into play. More precisely, by comparing with (2.21), we see that the left-hand side of (3.1) simplifies to

$$\sum_{s_1=0}^{\lfloor N/2\rfloor}\sum_{s_2=0}^{\lfloor N/2\rfloor}\sum_{a=1}^{N}(-1)^{N+s_1+s_2+a}2^{2N}\binom{N-1}{a-1}\\
\cdot(-N/2)_{s_1}\,((1-N)/2)_{s_1}\,(-N/2)_{s_2}\,((1-N)/2)_{s_2}\\
\cdot\frac{1}{X+1-a}\big((X+1-N+2s_1)/2\big)_{N-2s_1}\big((Y+1-N+2s_2)/2\big)_{N-2s_2}\\
\cdot\frac{(-(N-a)/2)_{s_1}\,(-(N-a)/2)_{s_2}}{s_1!\,s_2!}\,{}_4F_3\!\left[\begin{matrix}-\frac{1}{2},-s_1,-s_2,\frac{1}{2}+\frac{a}{2}-\frac{N}{2}\\ -\frac{N}{2},\frac{1}{2}-\frac{N}{2},\frac{a}{2}-\frac{N}{2}\end{matrix};1\right]. \tag{3.2}$$

As in the proof of Theorem 2.2, we now concentrate on the terms involving the summation index a only:

$$\sum_{a=1}^{N}(-1)^a\frac{1}{X+1-a}\binom{N-1}{a-1}\frac{(-(N-a)/2)_{s_1}\,(-(N-a)/2)_{s_2}\,(-(N-a-1)/2)_s}{(-(N-a)/2)_s}$$

$$= \frac{(N-1)!}{2^{2s}\,(N-2s-1)!} \sum_{a=1}^{N-2s} (-1)^a \frac{1}{X+1-a} \binom{N-2s-1}{a-1} \\ \cdot (-(N-a-2s)/2)_{s_1-s}\,(-(N-a-2s)/2)_{s_2-s}$$

$$= -\frac{(N-1)!}{2^{2s}\,(N-2s-1)!} \sum_{a=0}^{N-2s-1} (-1)^a \frac{1}{X-a} \binom{N-2s-1}{a} \\ \cdot ((a-N+2s+1)/2)_{s_1-s}\,((a-N+2s+1)/2)_{s_2-s},$$

where, in abuse of notation, we wrote again s for the summation index of the ${}_4F_3$-series in (3.2). By Lemma A.2 with $M = N-2s-1$, this is equal to

$$\frac{(-1)^{N-2s}\,(N-1)!}{2^{2s}(X-N+2s+1)_{N-2s}}((X+1-N+2s)/2)_{s_1-s}((X+1-N+2s)/2)_{s_2-s} \\ -\chi(s_1=s_2=N/2)\cdot 2^{-N}(N-1)!. \quad (3.3)$$

We substitute this in (3.2). As before in the proof of Lemma 2.2, the second term in (3.3) does not contribute anything since the (remaining) sum over s vanishes. Therefore, substitution of (3.3) in (3.2) leads to the expression

$$\sum_{s_1=0}^{\lfloor N/2\rfloor} \sum_{s_2=0}^{\lfloor N/2\rfloor} \sum_{s=0}^{s_1} (-1)^{s_1+s_2+s} 2^{2N-2s} (-N/2)_{s_1}\,((1-N)/2)_{s_1}\,(-N/2)_{s_2}\,((1-N)/2)_{s_2} \\ \cdot \frac{\big((X+1-N+2s)/2\big)_{N-s-s_1}\big((X+1-N+2s)/2\big)_{s_2-s}}{(s_1-s)!\,(X-N+2s+1)_{N-2s}} \\ \cdot \big((Y+1-N+2s_2)/2\big)_{N-2s_2} \frac{(N-1)!\,(-1/2)_s\,(-s_2)_s}{s_2!\,s!\,(-N/2)_s\,((1-N)/2)_s}.$$

We write the sum over s_1 in hypergeometric notation to obtain

$$\sum_{s_2=0}^{\lfloor N/2\rfloor} \sum_{s=0}^{s_1} (-1)^{s_2} 2^{2N-2s} (-N/2)_{s_2}\,((1-N)/2)_{s_2} \\ \cdot \frac{\big((X+1-N+2s)/2\big)_{N-2s}\big((X+1-N+2s)/2\big)_{s_2-s}}{(X-N+2s+1)_{N-2s}} \\ \cdot \big((Y+1-N+2s_2)/2\big)_{N-2s_2} \frac{(N-1)!\,(-1/2)_s\,(-s_2)_s}{s_2!\,s!} {}_2F_1\left[\begin{matrix} -\frac{N}{2}+s, \frac{1}{2}-\frac{N}{2}+s \\ -\frac{X}{2}+\frac{1}{2}-\frac{N}{2}+s \end{matrix}; 1\right].$$

The ${}_2F_1$-series can be evaluated by means of the Chu–Vandermonde summation formula. After some simplification, we arrive at

$$\sum_{s_2=0}^{\lfloor N/2\rfloor} \sum_{s=0}^{s_1} (-1)^{s_2} 2^{N} (-N/2)_{s_2}\,((1-N)/2)_{s_2} \\ \cdot \big((X+1-N+2s)/2\big)_{s_2-s}\big((Y+1-N+2s_2)/2\big)_{N-2s_2} \frac{(N-1)!\,(-1/2)_s\,(-s_2)_s}{s_2!\,s!}$$

$$= \sum_{s_2=0}^{\lfloor N/2\rfloor} (-1)^{s_2} 2^N \frac{(N-1)!}{s_2!} (-N/2)_{s_2} \,((1-N)/2)_{s_2} \cdot \big((X+1-N)/2\big)_{s_2} \big((Y+1-N+2s_2)/2\big)_{N-2s_2}\, {}_2F_1\left[\begin{matrix} -\frac{1}{2}, -s_2 \\ \frac{X}{2}+\frac{1}{2}-\frac{N}{2}\end{matrix}; 1\right].$$

After having evaluated the ${}_2F_1$-series by means of the Chu–Vandermonde summation formula, we are left with

$$\sum_{s_2=0}^{\lfloor N/2\rfloor} (-1)^{s_2} 2^N \frac{(N-1)!}{s_2!} (-N/2)_{s_2} \,((1-N)/2)_{s_2} \cdot \big((X+2-N)/2\big)_{s_2} \big((Y+1-N+2s_2)/2\big)_{N-2s_2}$$
$$= 2^N (N-1)! \big((Y+1-N)/2\big)_N \, {}_3F_2\left[\begin{matrix} \frac{X}{2}-\frac{N}{2}+1, -\frac{N}{2}, \frac{1}{2}-\frac{N}{2} \\ -\frac{Y}{2}-\frac{N}{2}+\frac{1}{2}, \frac{Y}{2}-\frac{N}{2}+\frac{1}{2}\end{matrix}; 1\right].$$

By the transformation formula in Lemma A.3, this is equal to

$$(N-1)!\,((X+Y-2N+3)/2)_N \, {}_2F_1\left[\begin{matrix} -\frac{X}{2}+\frac{Y}{2}-\frac{1}{2}, -N \\ \frac{X}{2}+\frac{Y}{2}-N+\frac{3}{2}\end{matrix}; -1\right].$$

which agrees with the right-hand side of (3.1). This finishes the proof. $\square$

Now we are finally in the position to establish Theorem 1.2.

Proof of Theorem 1.2. Using $n = q + p$, we find

$$X(1) = (C+B)(1) = \frac{n}{2} - 1 \quad \text{and} \quad Y(1) = (C-B)(1) = \frac{p-q}{2}.$$

Thus, we can write the constant term of P_{2N} in the form

$$P_{2N}\left(\frac{n}{2}-1, \frac{p-q}{2}\right).$$

Now we calculate

$$\sum_{|I|=N} m_I \frac{1}{\frac{n}{2} - I_{\text{last}}} P_{2I}(1)$$
$$= \sum_{a=1}^{N} \sum_{|J|=N-a} m_{(J,a)} \frac{1}{\frac{n}{2}-a} P_{2J}\left(\frac{n}{2}-1, \frac{p-q}{2}\right) P_{2a}\left(\frac{n}{2}-1, \frac{p-q}{2}\right)$$
$$= N!(N-1)! \sum_{k=0}^{N} (-1)^k \binom{\frac{q}{2}}{k} \binom{\frac{p}{2}}{N-k}.$$

In the last step we have applied Theorem 3.1. $\square$

Appendix. Proofs of some auxiliary results

Here we prove three technical lemmas which were used in Sections 2 and 3.

Lemma A.1. *Let $p(x)$ be a polynomial in x, and suppose that*

$$p(x) = \sum_{k\geq 0} c_k \binom{x}{k},$$

for (uniquely determined) coefficients c_k. Furthermore, let M be an integer. Then

$$\sum_{a=0}^{M} (-1)^a \binom{M}{a} p(a) = (-1)^M c_M.$$

Proof. This is a classical fact from finite difference calculus. For the convenience of the reader, we give the simple proof. We calculate

$$\begin{aligned}\sum_{a=0}^{M} (-1)^a \binom{M}{a} p(a) &= \sum_{a=0}^{M} (-1)^a \binom{M}{a} \sum_{k\geq 0} c_k \binom{a}{k} \\ &= \sum_{k\geq 0} c_k \binom{M}{k} \sum_{a=0}^{M} (-1)^a \binom{M-k}{a-k} \\ &= \sum_{k\geq 0} c_k \binom{M}{k} (-1)^k \delta_{M,k} = (-1)^M c_M.\end{aligned}$$

The proof is complete. □

Lemma A.2. *Let $p(x)$ be a polynomial in x, and suppose that*

$$p(x) = \sum_{k\geq 0} c_k \binom{x}{k}$$

for (uniquely determined) c_k. Furthermore, let M be an integer. If the degree of $p(x)$ is at most M, then

$$\sum_{b=0}^{M} \frac{(-1)^b}{X-b} \binom{M}{b} p(b) = (-1)^M \frac{M!}{(X-M)_{M+1}} p(X).$$

If the degree of $p(x)$ equals $M+1$, then

$$\sum_{b=0}^{M} \frac{(-1)^b}{X-b} \binom{M}{b} p(b) = (-1)^M \frac{M!}{(X-M)_{M+1}} p(X) + \frac{c_{M+1}}{M+1}.$$

Proof. It suffices to verify the claim for a basis of the vector space of polynomials in x of degree at most $M+1$. We choose $\left\{\binom{x}{k}\right\}_{0\leq k\leq M+1}$ as such a basis. If $k \leq M$, we have

$$\sum_{b=0}^{M} \frac{(-1)^b}{X-b} \binom{M}{b} \binom{b}{k} = \frac{(-1)^k}{X-k} \binom{M}{k} {}_2F_1\left[\begin{matrix}-M+k, k-X \\ k-X+1\end{matrix}; 1\right]$$

$$= \frac{(-1)^k}{X-k}\binom{M}{k}\frac{(M-k)!}{(-X+k+1)_{M-k}}$$
$$= (-1)^M\frac{M!}{k!}\frac{(X-k+1)_k}{(X-M)_{M+1}}$$
$$= (-1)^M\frac{M!}{(X-M)_{M+1}}\binom{X}{k},$$

while if $k = M+1$ we obtain 0. Here, we used the Chu–Vandermonde summation formula to evaluate the ${}_2F_1$-series. All this is in agreement with our claims. ☐

Lemma A.3. *For all non-negative integers N, we have*

$$ {}_3F_2\left[\begin{matrix}a, \frac{1}{2}-\frac{N}{2}, -\frac{N}{2}\\ e, 1-N-e\end{matrix}; 1\right] = 2^{-N}\frac{(e-a)_N}{(e)_N}\,{}_2F_1\left[\begin{matrix}1-a-e-N, -N\\ 1+a-e-N\end{matrix}; -1\right]. \tag{A.1}$$

Proof. In [KR03, (3.14)],

$$ {}_4F_3\left[\begin{matrix}a, b, \frac{1}{2}-\frac{N}{2}, -\frac{N}{2}\\ \frac{1}{2}+a+b, e, 1-N-e\end{matrix}; 1\right] = \frac{(a+e)_N}{(e)_N}\,{}_3F_2\left[\begin{matrix}2\,a, a+b, -N\\ 2\,a+2\,b, a+e\end{matrix}; 1\right],$$

we let b tend to infinity. As a result, we obtain the transformation formula

$$ {}_3F_2\left[\begin{matrix}a, \frac{1}{2}-\frac{N}{2}, -\frac{N}{2}\\ e, 1-N-e\end{matrix}; 1\right] = \frac{(a+e)_N}{(e)_N}\,{}_2F_1\left[\begin{matrix}2\,a, -N\\ a+e\end{matrix}; \frac{1}{2}\right].$$

If we now apply the transformation formula (see [S66, (1.7.1.3)])

$$ {}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; z\right] = (1-z)^{-B}\,{}_2F_1\left[\begin{matrix}C-A, B\\ C\end{matrix}; -\frac{z}{1-z}\right]$$

to the ${}_2F_1$-series on the right-hand side, then we obtain

$$2^{-N}\frac{(a+e)_N}{(e)_N}\,{}_2F_1\left[\begin{matrix}e-a, -N\\ a+e\end{matrix}; -1\right].$$

Finally, if we reverse the order of summands in this ${}_2F_1$-series (that is, if we denote the summation index in the ${}_2F_1$-series by s, then we replace s by $N-s$ and rewrite the result again in hypergeometric notation), then we arrive at the right-hand side of (A.1). ☐

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