RECENT PROGRESS IN THE STUDY OF REPRESENTATIONS OF INTEGERS AS SUMS OF SQUARES

HENG HUAT CHAN AND CHRISTIAN KRATTENTHALER

Abstract. In this article, we collect the recent results concerning the representations of integers as sums of an even number of squares that are inspired by conjectures of Kac and Wakimoto. We start with a sketch of Milne's proof of two of these conjectures. We also show an alternative route to deduce these two conjectures from Milne's determinant formulas for sums of $4s^2$, respectively $4s(s+1)$, triangular numbers. This approach is inspired by Zagier's proof of the Kac–Wakimoto formulas via modular forms. We end the survey with recent conjectures of the first author and Chua.

1. Introduction

The problem of finding explicit formulas for the number of representations of an integer n as a sum of s squares is an old one. The first formula of this kind is due to Legendre and Gauß. If $r_s(n)$ denotes the number of representations of n as a sum of s squares, Legendre and Gauß proved that

$$r_2(n) = 4(d_1(n) - d_3(n)), \tag{1.1}$$

where $d_j(n)$ denotes the number of divisors of n of the form $4k+j$. For example, if n is a prime p of the form $4k+1$, then $r_2(p) = 8$ since $d_1(p) = 2$ and $d_3(p) = 0$. On the other hand, if n is a prime p of the form $4k+3$, then $r_2(p) = 0$ since $d_1(p) = d_3(p) = 1$. This, of course, leads to the well known result of Fermat, which states that a prime p is of the form $x^2 + y^2$ if and only if p is of the form $4k+1$. Fermat's result led mathematicians to explore and characterize primes of the form $x^2 + ny^2$, $n \geq 1$. For more information on such characterizations, the reader is encouraged to consult the excellent book by D. A. Cox [5].

Let

$$\varphi(q) = \sum_{k=-\infty}^{\infty} q^{k^2}.$$

2000 *Mathematics Subject Classification.* 11E25, 11F11.

It is clear that

$$\varphi^s(q) = \sum_{k\ge 0} r_s(k) q^k.$$

As a result, to obtain expressions for $r_s(n)$, it suffices to obtain expressions for $\varphi^s(q)$. The first identity of this kind is due to Jacobi, namely,

$$\varphi^2(q) = 1 - 4\sum_{k=1}^{\infty}(-1)^k \frac{q^{2k-1}}{1-q^{2k-1}}. \tag{1.2}$$

Note that (1.1) is a direct consequence of (1.2). Using the theory of elliptic functions, Jacobi also found formulas for $r_4(n)$, $r_6(n)$ and $r_8(n)$, namely,

$$\varphi^4(q) = 1 + 8\sum_{k=1}^{\infty} \frac{kq^k}{1+(-q)^k}, \tag{1.3}$$

$$\varphi^6(q) = 1 + 16\sum_{k=1}^{\infty} \frac{k^2q^k}{1+q^{2k}} + 4\sum_{k=1}^{\infty}(-1)^k \frac{(2k-1)^2 q^{2k-1}}{1-q^{2k-1}}, \tag{1.4}$$

and

$$\varphi^8(q) = 1 + 16\sum_{k=1}^{\infty} \frac{k^3q^k}{1-(-q)^k}. \tag{1.5}$$

From (1.3), we find that

$$r_4(n) = 8 \sum_{\substack{d\mid n\\ d\not\equiv 0 \pmod 4}} d,$$

and this immediately implies that every positive integer is a sum of four squares, a famous result of Lagrange.

We call series of the type

$$A + B\sum_{k\ge 1} a_k \frac{q^k}{1-q^k}$$

generalized Lambert series. Note that for even $s \le 8$, we are able to express $\varphi^s(q)$ in terms of generalized Lambert series. This does not seem possible when $s = 10$. In fact, Liouville showed that

$$\begin{aligned}\varphi^{10}(q) = 1 &+ \frac{4}{5}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2k-1)^4 q^{2k-1}}{1-q^{2k-1}} + \frac{64}{5}\sum_{k=1}^{\infty}\frac{k^4q^k}{1+q^{2k}}\\ &+ \frac{32}{5} q\varphi^2(q)\varphi^4(-q)\psi^4(q^2),\end{aligned}$$

where

$$\psi(q) = \sum_{k=0}^{\infty} q^{k(k+1)/2}.$$

Indeed for any even $s \geq 10$, $\varphi^s(q)$ is a sum of generalized Lambert series and a "cusp form".

Recently, new formulas for $r_s(n)$ were discovered. One common feature of these formulas is the absence of "cusp forms".[1] The new formulas involve only generalized Lambert series. The purpose of this article is to describe these recent discoveries.

Before we proceed with our discussion, we make the following observation: It is known that [2, p. 43, Entry 27 (ii)]

$$4e^{\pi i/(2\tau)}\psi^2(e^{-2\pi i/\tau}) = \frac{\tau}{i}\varphi^2(-e^{\pi i\tau}). \tag{1.6}$$

Suppose we have a relation

$$4^s q^{s/2}\psi^{2s}(q^2) = F(L_1(q^2), L_2(q^2), \ldots, L_m(q^2)), \quad \text{with } q = e^{\pi i\tau},$$

where each L_j is a generalized Lambert series or a product of generalized Lambert series satisfying

$$L_j(e^{-2\pi i/\tau}) = \left(\frac{\tau}{i}\right)^s L_j^*(-e^{\pi i\tau})$$

for some $L_j^*(-q)$ (which is also a generalized Lambert series or a product of generalized Lambert series), then we would have

$$\varphi^{2s}(-q) = F(L_1^*(-q), \ldots, L_m^*(-q)).$$

Conversely if we have a formula for sums of squares, we will have a formula for sums of triangular numbers. We illustrate the above observation by the following identities:

Suppose for $q = e^{\pi i\tau}$, we have

$$\begin{aligned} 4^4 e^{2\pi i\tau}\psi^8(e^{2\pi i\tau}) &= 256\sum_{k\geq 0}\frac{k^3q^{2k}}{1-q^{4k}} \\ &= \frac{16}{15}\left(E_4(\tau) - E_4(2\tau)\right), \end{aligned} \tag{1.7}$$

where

$$E_4(\tau) = 1 + 240\sum_{k\geq 1}\frac{k^3e^{2\pi ik\tau}}{1-e^{2\pi ik\tau}}. \tag{1.8}$$

[1]Note that the coefficients of q^n in Lambert series can be calculated once we know the factorization of n. In general, this is impossible for cusp forms. Hence, these new formulas are more "effective" if one wants to determine $r_s(n)$.

The Eisenstein series $E_4(\tau)$ satisfies the transformation formula [1, p. 24, Ex. 12]

$$E_4\left(-\frac{1}{\tau}\right) = \tau^4 E_4(\tau).$$

Replacing τ by $-1/\tau$ in (1.7), the left hand side of (1.7) is $\tau^4\varphi^8(-q)$ by (1.6), with $q = e^{\pi i\tau}$. Now, by (1.8), we have

$$\begin{aligned}\frac{16}{15}\left(E_4(-1/\tau) - E_4(-2/\tau)\right) &= \tau^4\frac{16}{15}\left(E_4(\tau) - \frac{1}{2^4}E_4(\tau/2)\right)\\ &= \tau^4\left(1 + 128\sum_{k\ge1}\frac{k^3q^k}{1-q^k} - 16\sum_{k\ge1}\frac{(2k+1)^3q^{2k+1}}{1-q^{2k+1}}\right).\end{aligned}$$

Hence, we conclude that

$$\varphi^8(-q) = 1 + 16\sum_{k=1}^{\infty}\frac{k^3(-q)^k}{1-q^k}.$$

Replacing q by $-q$, we obtain the formula for sums of 8 squares. For more details of the relations between formulas associated with squares and triangular numbers see [8] and [10].

Note that the identity for $\psi^8(q)$ is much simpler than that for $\varphi^8(q)$. This is in fact a general phenomenon: the identity for $\psi^{2s}(q)$ will be much simpler than that for $\varphi^{2s}(q)$ for any $s \in \mathbb{N}$. For the rest of this article, we will therefore only present identities associated with $\psi(q)$.

2. The formulas of Kac and Wakimoto

Let $t_k(n)$ be the number of representations of n as the sum of k triangular numbers. In 1994, V. G. Kac and M. Wakimoto [7] conjectured that

$$t_{4s^2}(n) = \frac{1}{s!}\frac{4^{-s(s-1)}}{\prod_{j=1}^{2s-1} j!}\sum_{\substack{a_1,\cdots,a_s\in\mathbb{N},\ a_i\text{ odd}\\ r_1,\cdots,r_s\in\mathbb{N},\ r_i\text{ odd}\\ a_1r_1+\cdots a_sr_s=2n+s^2}} a_1\cdots a_s\prod_{i<j}(a_i^2-a_j^2)^2 \tag{2.1}$$

and

$$t_{4s(s+1)}(n) = \frac{1}{s!}\frac{2^s}{\prod_{j=1}^{2s} j!}\sum_{\substack{a_1,\cdots,a_s\in\mathbb{N}\\ r_1,\cdots,r_s\in\mathbb{N},\ r_i\text{ odd}\\ a_1r_1+\cdots a_sr_s=n+\frac12 s(s+1)}} (a_1\cdots a_s)^3\prod_{i<j}(a_i^2-a_j^2)^2. \tag{2.2}$$

These formulas follow from a conjectural affine denominator formula for simple Lie superalgebras of type $Q(m)$. (For the definition of $Q(m)$, see [6].)

Identities (2.1) and (2.2) were first proved by S. C. Milne [9], using results on continued fractions and elliptic functions. For example, Milne showed using Schur functions, that (2.1) is a consequence of his determinant formula [9, (5.107)]

$$\left(q\psi^4(q^2)\right)^{s^2} = \frac{4^{-s(s-1)}}{\prod_{j=1}^{2s-1} j!}\det(C_{2(u+v-1)-1})_{1\le u,v\le s}\ , \tag{2.3}$$

where

$$C_{2j-1} = \sum_{r=1}^{\infty} \frac{(2r-1)^{2j-1}q^{2r-1}}{1-q^{2(2r-1)}}, \quad j \geq 1.$$

We now briefly describe Milne's proof of (2.3).

Milne first showed that if $\mathrm{sn}(u) := \mathrm{sn}(u,\mathbf{k})$, $\mathrm{dn}(u) := \mathrm{dn}(u,\mathbf{k})$, and $\mathrm{cn}(u) := \mathrm{cn}(u,\mathbf{k})$ are the classical Jacobi elliptic functions, then [9, (2.44), (2.68)]

$$\begin{aligned} \frac{\mathrm{sn}(u)\mathrm{cn}(u)}{\mathrm{dn}(u)} &= \frac{1}{\mathbf{k}^2}\sum_{m\geq 1} \frac{2^{2m+2}(-1)^{m-1}}{z^{2m}} C_{2m-1}\frac{u^{2m-1}}{(2m-1)!} \\ &=: \sum_{m\geq 1} c_m \frac{u^{2m-1}}{(2m-1)!}, \end{aligned} \tag{2.4}$$

where

$$z = \varphi^2(q) \quad \text{and} \quad \mathbf{k}^2 = 16q\frac{\psi^4(q^2)}{\varphi^4(q)}. \tag{2.5}$$

Milne then showed that

$$\begin{aligned} &\int_0^{\infty} \frac{\mathrm{sn}(u)\mathrm{cn}(u)}{\mathrm{dn}(u)} e^{-u/t}\, du \\ &= \frac{t^2}{1+(4-2\mathbf{k}^2)t^2 + \mathbf{K}_{n=2}^{\infty} \dfrac{-(2n-1)(2n-2)^2(2n-3)\mathbf{k}^4t^4}{1+(2n-1)^2(4-2\mathbf{k}^2)t^2}}. \end{aligned} \tag{2.6}$$

Here, $\mathbf{K}_{n=2}^{\infty}$ is the notation for continued fractions,

$$\mathbf{K}_{n=2}^{\infty} \frac{a_n}{b_n} := \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}.$$

Using [9, Theorem 3.4] and (2.6), Milne deduced the Hankel determinant evaluation [9, (4.9)]

$$H_n^{(1)}(\{c_m\}) := \det \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{pmatrix} = (\mathbf{k}^2)^{n(n-1)} \prod_{r=1}^{2n-1} r!. \tag{2.7}$$

Simplifying the left hand side of (2.7) using the definition (2.4) of the c_i's and making use of the relations [9, (3.66), (5.11)]

$$H_n^{(1)}(\{t^m a_m\}) = t^{n^2} H_n^{(1)}(\{a_m\}),$$

and (2.5), we deduce (2.3).

We now describe a simplification of Milne's Schur function argument that allowed him to deduce (2.1) from (2.3). As a side result, we also obtain a new expression for $t_{4s^2}(n)$ (see (2.8) below).

In a recent paper [11], D. Zagier gave a direct proof of the above formulas of Kac and Wakimoto using the theory of modular forms. In that paper, he constructed a certain map sending the monomials $X_1^{k_1-1}\cdots X_s^{k_s-1}$ (here, the X_i's are indeterminates) to the product of Eisenstein series $g_{k_1}^+\cdots g_{k_s}^+$ (the quantities g_{2j}^+ being, up to scaling, the quantities C_{2j-1} in Milne's formula). It turns out that if we apply a variant of that map, Φ_s say, defined by sending the product $X_1^{2k_1-1}\cdots X_s^{2k_s-1}$ to the product $C_{2k_1-1}\cdots C_{2k_s-1}$, in Milne's formula, then we get the new formula

$$\begin{aligned} t_{4s^2}(n) = {} & \frac{(-1)^{s(s-1)/2}}{4^{s(s-1)}\prod_{j=1}^{2s-1} j!} \\ & \times \sum_{\substack{a_i,\, r_i \in \mathbb{N} \text{ odd} \\ a_1r_1+\cdots+a_sr_s=2n+s^2}} a_1a_2^3\cdots a_s^{2s-1} \prod_{1\le i<j\le s} (a_i^2-a_j^2). \end{aligned} \tag{2.8}$$

This is seen as follows: By series expansion, we have

$$\begin{aligned} C_{2j-1} &= \sum_{r=1}^{\infty} (2r-1)^{2j-1} q^{2r-1} \sum_{k=0}^{\infty} q^{2k(2r-1)} \\ &= \sum_{r,k\ge 1} (2r-1)^{2j-1} q^{(2k-1)(2r-1)} \\ &= \sum_{\substack{m \text{ odd} \\ a|m}} a^{2j-1} q^m. \end{aligned}$$

Thus, we obtain

$$\Phi_s(X_1^{2k_1-1}\cdots X_s^{2k_s-1}) = C_{2k_1-1}\cdots C_{2k_s-1} = \sum_{\substack{m_1,\dots,m_s \text{ odd} \\ a_1|m_1,\ \dots,\ a_s|m_s}} q^{m_1+\cdots+m_s} a_1^{2k_1-1}\cdots a_s^{2k_s-1}.$$

Returning to Milne's formula (2.3), this implies that

$$\begin{aligned}
\det\left(C_{2(u+v-1)-1}\right)_{1\le u,v\le s} &= \Phi_s\left(\det\left(X_u^{2(u+v-1)-1}\right)_{1\le u,v\le s}\right) \\
&= \Phi_s\left(\prod_{i=1}^s X_i^{2i-1}\det\left(X_u^{2(v-1)}\right)_{1\le u,v\le s}\right) \\
&= \Phi_s\left((-1)^{s(s-1)/2}\prod_{i=1}^s X_i^{2i-1}\prod_{1\le i<j\le s}(X_i^2-X_j^2)\right) \\
&= (-1)^{s(s-1)/2}\sum_{\substack{m_1,\dots,m_s \text{ odd} \\ a_1|m_1,\ \dots,\ a_s|m_s}} q^{m_1+\cdots+m_s}\prod_{i=1}^s a_i^{2i-1}\prod_{1\le i<j\le s}(a_i^2-a_j^2),
\end{aligned}$$

where we have used the Vandermonde determinant evaluation to evaluate the determinant in going from the second to the third line. Now a comparison of coefficients of q^{2n+s^2} leads us to (2.8).

We now show that (2.1) follows from (2.8) using an elementary combinatorial argument.

For each positive integer s, let

$$P_s(X_1,\dots,X_s) = \prod_{i=1}^s X_i\prod_{i<j}(X_i^2-X_j^2)^2$$

and

$$P'_s(X_1,\dots,X_s) = \prod_{i=1}^s X_i^{2i-1}\prod_{i<j}(X_i^2-X_j^2).$$

For positive integers s and m, let

$$R_s(m, P(X_1,\dots,X_s)) = \sum_{\substack{a_i, r_i\in\mathbb{N}\text{ odd} \\ a_1r_1+\cdots+a_sr_s=m}} P(a_1,\dots,a_s).$$

We want to show that

$$R_s(m,P_s) = (-1)^{s(s-1)/2}s!\,R_s(m,P'_s).$$

Now, if S_s denotes the symmetric group of s elements and $\sigma\in S_s$, then

$$R_s(m, P'_s(X_{\sigma(1)},\dots,X_{\sigma(s)})) = R_s(m, P'_s(X_1,\dots,X_s)),$$

since, whenever $(a_1,\dots,a_s)$ is an s-tuple of odd nonnegative integers for which there are $r_1,\dots r_s$ such that $a_1r_1+\dots+a_sr_s=m$, then $(a_{\sigma(1)},\dots,a_{\sigma(s)})$ is an s-tuple with the same property. Hence,

$$R_s\left(m,\sum_{\sigma\in S_s}P_s'(X_{\sigma(1)},\dots,X_{\sigma(s)})\right)=s!\,R_s(m,P_s'(X_1,\dots,X_s)).$$

On the other hand,

$$\begin{aligned}R_s\left(m,\sum_{\sigma\in S_s}P_s'(X_{\sigma(1)},\dots,X_{\sigma(s)})\right)\\&=R_s\left(m,\det(X_j^{2i-2})_{1\le i,j\le s}\prod_{i=1}^{s}X_i\prod_{1\le i<j\le s}(X_i^2-X_j^2)\right)\\&=(-1)^{s(s-1)/2}R_s(m,P_s(X_1,\dots,X_s)),\end{aligned}$$

by the Vandermonde determinant evaluation. Thus, we have shown how (2.1) follows from Milne's determinant formula (2.3) by passing via (2.8), thereby providing an alternative to Milne's (somewhat more involved) Schur function argument.

3. A conjecture for the sum of $8s$ triangular numbers

We now state Milne's determinant formula for $4s(s+1)$ triangles:

$$\left(16q\psi^4(q^2)\right)^{s(s+1)}=\left(2^{s(4s+5)}\right)\prod_{j=1}^{2s}(j!)^{-1}\det\left(D_{2(u+v-1)+1}\right)_{1\le u,v\le s},\tag{3.1}$$

where

$$D_{2j+1}=\sum_{r=1}^{\infty}\frac{r^{2j+1}q^{2r}}{1-q^{4r}},j\ge 1.$$

This formula led to the first proof of (2.2). Using arguments analogous to the ones given in the last section, one can deduce (2.2) from (3.1).

When $s=2$, this leads to the following beautiful formula:

$$q^6\psi^{24}(q^2)=\frac{1}{72}\left(T_8T_4-T_6^2\right),$$

where

$$T_{2k}(q):=\sum_{n=1}^{\infty}\frac{n^{2k-1}q^{2n}}{1-q^{4n}},\ \ k>1.$$

Note the resemblance of this formula with the well-known formula

$$q\prod_{n=1}^{\infty}(1-q^n)^{24}=\frac{1}{1728}\left(E_4E_8-E_6^2\right),$$

where the E_i's are the classical Eisenstein series.[2] Note that, as indicated in the introduction, one obtains Milne's new formula for 24 squares, namely, if

$$S_4(q) = 1 + 16\sum_{k=1}^{\infty} \frac{k^3 q^k}{1-(-q)^k},$$
$$S_6(q) = 1 - 8\sum_{k=1}^{\infty} \frac{k^5 q^k}{1-(-q)^k},$$

and

$$S_8(q) = 17 + 32\sum_{k=1}^{\infty} \frac{k^7 q^k}{1-(-q)^k},$$

then [9, Theorem 1.6, (1.25)]

$$\varphi^{24}(q) = \frac{1}{9}\left\{S_4(q)S_8(q) - 8S_6^2(q)\right\}.$$

Comparing this with the "old" formula

$$\varphi^{24}(q) = 1 + \frac{16}{691}E_{11}(q) + \frac{33152}{691}qf^{24}(q) - \frac{65536}{691}q^2 f^{24}(-q^2),$$

where

$$f(-q) = \prod_{k=1}^{\infty}(1-q^k),$$

we find that Milne's formula requires less terms. Moreover, if we know the factorization of n, then we can calculate $r_{24}(n)$ explicitly from the new formula, since the terms are all Eisenstein series. The new formula for 24 squares and a recent paper of Z.-G. Liu [8] led the first author and Chua [3] to formulate the following conjecture for $8s$ triangular numbers:

Conjecture. For any positive integer $s > 1$, we have

$$q^{2s}\psi^{8s}(q^2) = \sum_{\substack{m+n=2s\\ m\geq n\geq 2}} a_{m,n}T_{2m}T_{2n},$$

for some rational numbers $a_{m,n}$.

The above conjecture is equivalent to showing that the forms $T_{2m}T_{2n}$ are linearly independent over the complex numbers $\mathbb{C}$.

[2]This was probably first observed by F. G. Garvan.

For a fixed s one can verify the corresponding identity. For example, when $s = 4$ we are led to the following new identity:

$$q^8\psi^{32}(q^2) = \frac{1}{75600}\left(T_4(q)T_{12}(q) - \frac{25}{4}T_{10}(q)T_6(q) + \frac{21}{4}T_8^2(q)\right). \tag{3.2}$$

Note that the above identity does not follow from any of the formulas of Kac and Wakimoto or Milne, since 32 is not of the form $4s^2$ or $4s(s+1)$.

The first proof of this result proceeds by expressing the T_{2m}'s in terms of $\mathbf{k}^2$ and z (see (2.5) for their definitions). The corresponding expressions are found by using the theory of modular forms, as well as the following recurrence satisfied by T_{2m}'s:

$$T_{2n+8}(q) = T_2(q)T_{2n+6}(q) + 12\sum_{j=0}^{n}\binom{2n+4}{2j+2}T_{2j+4}(q)T_{2n-2j+4}(q),$$

where

$$T_2(q) = 1 + 24\sum_{j=0}^{\infty}\frac{jq^{2j}}{1+q^{2j}}.$$

The above recurrence follows from the differential equation satisfied by the Jacobi elliptic function $M := M(u) = \operatorname{sn}^2(u)$, namely,

$$\left(\frac{dM}{du}\right)^2 = 4M(1-M)(1-\mathbf{k}^2M).$$

We end this article with a sketch of a new proof of (3.2). It is known that the constant term of $M(u)$ is T_4 and that

$$T_4 = q^2\psi^8(q^2).$$

One can verify that

$$\begin{aligned} 3840M^4 =& (7M^{(2)} + 68T_4)(M^{(2)} - 4T_4) \\ &- 9M^{(3)}M^{(1)} + M(2M^{(4)} + 128T_6), \end{aligned} \tag{3.3}$$

where $M^{(i)}$ is the ith derivative of M with respect to u.

By comparing the constant term on both sides we are immediately led to (3.2). The above identity is motivated by recent work of the first author and Liu [4], where a proof of an identity similar to (3.3) is illustrated.

Acknowledgements. It is our pleasure to thank Professors Bruce C. Berndt and Stephen C. Milne for pointing out several misprints in an earlier version of this article and for their fruitful suggestions. We are also grateful to Professor Robin J. Chapman for pointing out some misprints in the article. The preliminary version of this article was written during the first author's stay at the East China Normal University. He would like to thank Professor Zhi-Guo Liu and the University for their hospitality and support.

References

[1] T.M. Apostol, *Modular functions and Dirichlet series in Number Theory*, Springer-Verlag, 2nd ed., New York, 1990.

[2] B.C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[3] H. H. Chan and K.S. Chua, *Representations of integers as sums of 32 squares*, The Ramanujan J. **7** (2003), 79–89.

[4] H. H. Chan and Z.-G. Liu, *Elliptic functions to the quintic base,* preprint.

[5] D. A. Cox, *Primes of the form* $x^2 + ny^2$, John Wiley & Sons, New York, 1989.

[6] V. G. Kac, Lie superalgebras, Adv. Math. **26** (1997), pp. 8–96.

[7] V. G. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory,* in: *Lie Theory and Geometry,* in honor of Bertram Kostant (J. L. Brylinski, R. Brylinski, V. Guillemin and V. Kac, eds.) **123**, Prog. in Math., Birkhäuser Boston, Inc., Boston, MA., 1994, pp. 415–456.

[8] Z.-G. Liu, *On the representations of integers as sums of squares,* in *q-series with applications to Combinatorics, Number Theory, and Physics* (B. C. Berndt and K. Ono, eds.), **291**, Contemp. Math., AMS, Providence, R.I., 2001, pp. 163–176.

[9] S. Milne, *Infinite families of exact sums of squares formulas, Jacobi elliptic functions, continued fractions, and Schur functions*, The Ramanujan J. **6** (2002), 7–149.

[10] K. Ono, *Representations of integers as sums of squares,* J. Number Theory **95** (2002), 253–258.

[11] D. Zagier, *A proof of the Kac-Wakimoto affine denominator formula for the strange series,* Math. Res. Lett. **7** (2000), 597–604.

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543

E-mail address: chanhh@math.nus.edu.sg

Institut Girard Desargues, Université Claude Bernard Lyon-I, 21, avenue Claude Bernard, F-69622 Villeurbanne Cedex, France

E-mail address: kratt@euler.univ-lyon1.fr