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One-Variable q-Analogues for Abhyankar's Inversion Formula

CHRISTIAN KRATTENTHALER

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Vienna, Austria

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Two types of q-extensions of Abhyankar's inversion formula for formal power series in a single variable are obtained. One type represents a new contribution to the Garsia–Gessel q-Lagrange inversion theory, the second to that of Hofbauer and the author. © 1989 Academic Press, Inc.

1. INTRODUCTION

Let $f(z)$ be a formal power series (fps) over a field K_0 with characteristic zero subject to $f(0)=0$ and $f'(0)=1$. Then there exists the compositional inverse fps $F(z)$ meaning $f(F(z))=\sum_{l=1}^{\infty} f_l F^l(z)=F(f(z))=z$. Given a formal Laurent series (fLs) $g(z)$ over K_0 of the form

$$g(z)=\sum_{k\geqslant d} g_k z^k, \tag{1.1}$$

for a $d\in\mathbb{Z}$ (integers), and given $f(z)$ as above, the formula of Abhyankar [1], rediscovered independently by Garsia and Joni [5, 6, 12] and Viskov [15], in the one variable case gives an expression in terms of $f(z)$ and $g(z)$ for the substitution of $F(z)$ in $g(z)$, namely

$$g(F(z))=\sum_{k=0}^{\infty}\frac{D^k}{k!}\,(g(z)\,f'(z)\,G^k(z)), \tag{1.2}$$

where $G(z)=z-f(z)$ and D denotes the differential operator acting on z. (Note that, since the order of $G(z)$ is at least two, the right-hand side of (1.2) is summable in the formal sense. The order of an fLs is the smallest integer d for which the coefficient of z^d is different from zero.) It turns out that Abhyankar's formula is equivalent with the Lagrange–Good formula

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[9; 12; 7, Part II], which in the one-variable case can be rewritten as [4, Identity I.5(a)]

$$\langle z^n\rangle\, g(F(z)) = \langle z^0\rangle\, g(z)\frac{zf'(z)}{f^{n+1}(z)}, \qquad n\in\mathbb{Z}, \tag{1.3}$$

where $\langle z^n\rangle\, a(z)$ means the coefficient of z^n in $a(z)$.

The crucial point for finding q-analogues of (1.2) is the existence of q-analogues of (1.3). Up to now three different types of q-analogues of the Lagrange formula have been discovered (see [11, 14] and the references cited there), two of which have general character. The first is due to Garsia [4]. The q-analogue of (1.2) coming out of his formula will be derived in Section 3. In Section 4 the q-Lagrange formula of Hofbauer [10] further developed by the author [13] will be applied to deduce a q-analogue of a formula similar to (1.2):

$$g(F(z)) = z\sum_{k=0}^{\infty}\frac{D^k}{k!}\left(g(z)\frac{f'(z)}{f(z)}\,G^k(z)\right). \tag{1.4}$$

Obviously (1.4) comes out of (1.2) by substituting $g(z)/f(z)$ for $g(z)$ in (1.2). Moreover, (1.4) and (1.2) are equivalent. In [5] Garsia and Joni give an alternative form of (1.2),

$$g(F(z)) = g(z) + \sum_{k=1}^{\infty}\frac{D^{k-1}}{k!}\,(g'(z)\,G^k(z)), \tag{1.5}$$

which corresponds to a second form of the Lagrange formula in one variable [4, Identity I.5(b)]

$$\langle z^n\rangle\, g(F(z)) = \frac{1}{n}\langle z^{-1}\rangle\frac{g'(z)}{f^n(z)}, \qquad n\neq 0. \tag{1.6}$$

In Garsia's q-Lagrange theory an analogue of (1.6) could not be found, but there is one for Hofbauer's. From this we are able to derive a q-analogue of (1.5), which also will be given in Section 4.

2. Notation and Preliminaries

We use the familiar standard q-notation $[\alpha]_q = (q^\alpha - 1)/(q-1)$, $[n]_q! = [n]_q\,[n-1]_q\cdots[1]_q$, $[0]_q! = 1$, $(x;q)_\infty = \prod_{i=0}^{\infty}(1-q^ix)$ and

$$(x;q)_\alpha = (x;q)_\infty/(xq^\alpha;q)_\infty = \sum_{k=0}^{\infty}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}\alpha\\ k\end{bmatrix}_q x^k, \tag{2.1}$$

where

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = [\alpha]_q [\alpha-1]_q \cdots [\alpha-k+1]_q/[k]_q!.$$

The q-exponential function is $e_q(z)=\sum_{k=0}^{\infty} z^k/[k]_q!$. Alternative expressions are $e_q(z)=\prod_{k=0}^{\infty}(1+(q-1)q^kz)^{-1}$ and $e_{1/q}(z)=\prod_{k=0}^{\infty}(1+(1-q)q^kz)$. Finally we introduce the q-difference operator by

$$D_q f(z)=(f(qz)-f(z))/(q-1)z. \tag{2.2}$$

In the q-analogues of the Lagrange formula the powers $f^k(z)$ are replaced by q-powers, say $f_k(z)$, having the form $f_k(z)=\sum_{n\geqslant k} f_{nk}z^n$, where $k\in\mathbb{Z}$. In the limiting case $q\to 1$ the fLs $f_k(z)$ become powers of a single fps. Substitution of the sequence $\not{f}=(f_k(z))_{k\in\mathbb{Z}}$ of fLs into an fLs of the form (1.1) is defined by

$$g(\not{f})(z)=\sum_{k\geqslant d} g_k f_k(z).$$

The inverse sequence $\mathscr{F}=(F_l(z))_{l\in\mathbb{Z}}$ of $\not{f}$ is the unique solution of the equations

$$F_l(\not{f})(z)=z^l \qquad \text{for} \quad l\in\mathbb{Z}.$$

It is easy to show [14, Section 3] that $\not{f}$ is the inverse of $\mathscr{F}$, too, thus establishing

$$f_k(\mathscr{F})=z^k \qquad \text{for} \quad k\in\mathbb{Z}.$$

Now, following Henrici [9], let us recall the proof of (1.2) starting from (1.3). Let $f(z)=z-G(z)$; then by (1.3)

$$\begin{aligned}
\langle z^n\rangle\, g(F(z)) &= \langle z^0\rangle\, g(z)\frac{zf'(z)}{f^{n+1}(z)}\\
&= \langle z^0\rangle\, g(z)\, f'(z)\frac{1}{z^n(1-G(z)/z)^{n+1}}\\
&= \langle z^0\rangle\, g(z)\, f'(z) z^{-n}\sum_{m=0}^{\infty}\binom{n+m}{m}(G(z)/z)^m\\
&= \sum_{m=0}^{\infty}\langle z^{n+m}\rangle\binom{n+m}{m} g(z)\, f'(z)\, G(z)^m\\
&= \langle z^n\rangle\sum_{m=0}^{\infty}\frac{D^m}{m!}(g(z)\, f'(z)\, G(z)^m).
\end{aligned}$$

As this is valid for all $n \in \mathbb{Z}$, (1.2) follows. A proof of (1.5) starting from (1.6) proceeds quite analogously.

Considering this calculation we recognize that for transferring this proof to the q-case it is necessary to find a q-analogue of

$$z^{n+1} f^{-n-1}(z) = \sum_{m=0}^{\infty} \binom{n+m}{m} (G(z)/z)^m$$

for all $n \in \mathbb{Z}$, or, what is the same, for

$$z^{-n} f^n(z) = \sum_{m=0}^{\infty} (-1)^m \binom{n}{m} (G(z)/z)^m \tag{2.3}$$

for all $n \in \mathbb{Z}$. Indeed after having found the "right" q-analogues of $(G(z)/z)^m$ this can easily be done.

3. The q-Analogue Using Garsia's q-Powers

Here the powers $h^k(z)$ are replaced by the q-powers

$$h^{[k,q]}(z) = \begin{cases} h(z)\, h(qz) \cdots h(q^{k-1} z), & k > 0 \\ 1, & k = 0 \\ 1/h(z/q)\, h(z/q^2) \cdots h(z/q^{-k}), & k < 0, \end{cases}$$

where $h(z)$ is an arbitrary fps. (For properties of these q-powers see [14, Section 6].) With the help of Garsia's [4] starring operator this could be written in a closed expression,

$$h^{[k,q]}(z) = h^*(z)/h^*(q^k z) \qquad \text{for} \quad k \in \mathbb{Z}.$$

Let $f(z)$ be an fps with $f(0) = 0$ and $f'(0) = 1$. It is the surprising result of Garsia [4, Theorem 1.1] that the inverse sequence of $\not{f} = (f^{[k,q]}(z))_{k \in \mathbb{Z}}$ also can be written in terms of q-powers, namely $\mathscr{F} = (F^{[l,1/q]}(z))_{l \in \mathbb{Z}}$, where the so called "right inverse" of $f(z)$, $F(z)$, satisfies $F(\not{f})(z) = z$ (vice versa, $f(z)$ is called the "left inverse" of $F(z)$). The q-Lagrange formula [4, Theorem 1.2] reads

$$\langle z^n \rangle\, g(\mathscr{F})(z) = \langle z^0 \rangle\, g(z) \frac{q^n z f^0(q^n z)}{f^{[n+1,q]}(z)}, \tag{3.1}$$

where the fps $f^0(z)$ is the q-analogue for $f'(z)$ and is uniquely determined by

$$\langle z^{-1}\rangle \frac{f^0(q^n z)}{f^{[n+1,q]}(z)} = \delta_{n0}. \tag{3.2}$$

(δ_{kl} is the Kronecker delta.)

The next lemma essentially contains the wanted q-analogue of (2.3).

LEMMA 1. *If $h(z)$ is an fps with $h(0)=1$ then for $k \in \mathbb{N}_0$ (non-negative integers) the fps*

$$H^{(k,q)}(z) = \sum_{j=0}^{k} (-1)^j q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q h^{[j,q]}(z), \tag{3.3}$$

a q-analogue of $(1-h(z))^k$, is of order k. Moreover, there holds

$$h^{[n,q]}(z) = \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q H^{(k,q)}(z) \tag{3.4}$$

for all $n \in \mathbb{Z}$.

Proof. First observe that, by using

$$\begin{bmatrix} k \\ j \end{bmatrix}_q = q^j \begin{bmatrix} k-1 \\ j \end{bmatrix}_q + \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q \tag{3.5}$$

[2, p. 89], we get the recurrence relation

$$H^{(k,q)}(z) = q^{k-1} H^{(k-1,q)}(z) - H^{(k-1,q)}(qz)\, h(z), \tag{3.6}$$

for $k \geqslant 1$. From this identity, by an inductive argument, it can be derived that the order of $H^{(k,q)}(z)$ is k. Therefore the infinite formal sum

$$E^{[n,q]}(z) = \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q H^{(k,q)}(z)$$

is well defined. Rewriting (3.6), with k replaced by $k+1$, as

$$H^{(k,q)}(qz)\, h(z) = q^k H^{(k,q)}(z) - H^{(k+1,q)}(z),$$

multiplying both sides of this identity by $(-1)^k \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$, and summing up over all $k \in \mathbb{N}_0$, leads to

$$E^{[n,q]}(qz)\,h(z)=\sum_{k=0}^{\infty}(-1)^k q^k \begin{bmatrix} n \\ k \end{bmatrix}_q H^{(k,q)}(z)$$
$$-\sum_{k=0}^{\infty}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} H^{(k+1,q)}(z)$$
$$=\sum_{k=0}^{\infty}(-1)^k \left(q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q\right) H^{(k,q)}(z)$$
$$=E^{[n+1,q]}(z),$$

by (3.5). As from the definition $E^{[0,q]}=h^{[0,q]}(z)$, by induction, $E^{[n,q]}(z)=h^{[n,q]}(z)$ for all $n\in\mathbb{Z}$ is proved, which is (3.4). ▌

Before formulating the q-analogue of Abhyankar's formula it is convenient to adopt two further notations of Garsia's paper [4]. The unroofing operator $^\vee$, acting on fLs of the form (1.1), is defined by

$$(g(z))^\vee=\sum_{k\geqslant d} g_k q^{\binom{k}{2}} z^k. \tag{3.7}$$

The q-substitution of $F(z)$ into $g(z)$ is denoted by

$$g(\bar{F})(z)=\sum_{k\geqslant d} g_k F^{[k,q]}(z). \tag{3.8}$$

THEOREM 2. *For a given fps $f(z)=zh(z)$ with $h(0)=1$ let $F(z)$ be the right inverse of $f(z)$, $f^0(z)$ the q-analogue of $f'(z)$ defined by* (3.2). *If*

$$G^{(k,q)}(z)=q^{-2k}z^k H^{(k,q)}(z)$$
$$=q^{-2k}\sum_{j=0}^{\infty}(-1)^j q^{\binom{k}{2}-j(k-1)} \begin{bmatrix} k \\ j \end{bmatrix}_q z^{k-j} f^{[j,q]}(z), \tag{3.9}$$

($H^{(k,q)}(z)$ is given by (3.3)), *then for $g(z)$, an fLs of the form* (1.1), *holds*

$$g(\bar{F})(z)=\left(\sum_{k=0}^{\infty}\frac{D_{1/q}^k}{[k]_q!}\left(g(z)\,f^0(z)\,G^{(k,q)}(qz)\right)\right)^\vee. \tag{3.10}$$

Proof. Starting point is (3.1) with q replaced by $1/q$. If ${}_1f(z)$ denotes the right inverse of $F(z)$ then (3.1) reads

$$\langle z^n\rangle\, g(\bar{F})(z)=\langle z^0\rangle\, g(z)\,\frac{q^{-n}z\;{}_1f^0(q^{-n}z)}{{}_1f^{[n+1,1/q]}(z)}. \tag{3.11}$$

The connection between the right and left inverse of $F(z)$ was discovered in [14, identity (6.19)]:

$$ {}_1f(z)=q\frac{\bar{f}_0(z/q)}{\bar{f}_0(z)}f(z/q), \tag{3.12}$$

where

$$\bar{f}_0(z)=\frac{zf^0(z)}{f(z)} \tag{3.13}$$

[14, identity (6.8)]. Moreover, it is proved in [14, Remark to identity (6.15)] that $\bar{f}_0(z)$ is the same for the left and right inverse of $F(z)$, i.e., $\bar{f}_0(z)={}_1\bar{f}_0(z)$, therefore

$$\bar{f}_0(z)=\frac{z\,{}_1f^0(z)}{{}_1f(z)}. \tag{3.14}$$

Use of (3.12) turns (3.11) into

$$\langle z^n\rangle\; g(\bar{F})(z)=\langle z^0\rangle\; g(z)\,\frac{q^{-n}z\,{}_1f^0(q^{-n}z)\,\bar{f}_0(z)}{q^n\bar{f}_0(q^{-n}z)\,f^{[n,1/q]}(z/q)\,{}_1f(q^{-n}z)}.$$

By (3.13) and (3.14) this is

$$\langle z^n\rangle\; g(\bar{F})(z)=\langle z^0\rangle\; g(z)q^{-n}\,\frac{zf^0(z)}{f^{[n+1,1/q]}(z)},$$

and, after having replaced $f(z)$ by $zh(z)$,

$$\langle z^n\rangle\; g(\bar{F})(z)=q^{\binom{n}{2}}\langle z^0\rangle\; g(z)\,f^0(z)z^{-n}h^{[-n-1,q]}(qz).$$

Application of (3.4) gives

$$\begin{aligned}\langle z^n\rangle\; g(\bar{F}(z))&=q^{\binom{n}{2}}\langle z^0\rangle\; g(z)\,f^0(z)z^{-n}\sum_{k=0}^{\infty}(-1)^k\begin{bmatrix}-n-1\\ k\end{bmatrix}_q H^{(k,q)}(qz)\\
&=q^{\binom{n}{2}}\sum_{k=0}^{\infty}\langle z^{n+k}\rangle q^{-k}\,\frac{[n+k]_{1/q}\cdots[n+1]_{1/q}}{[k]_q!}\\
&\quad\times g(z)\,f^0(z)z^kH^{(k,q)}(qz)\\
&=q^{\binom{n}{2}}\langle z^n\rangle\sum_{k=0}^{\infty}\frac{D_{1/q}^k}{[k]_q!}\,(g(z)\,f^0(z)\,G^{(k,q)}(qz)).\end{aligned}$$

To establish (3.10), both sides of the last equation have to be multiplied by z^n and then summed up over all $n\in\mathbb{Z}$. ∎

EXAMPLE 3. The standard example for Garsia's q-theory is the case $f(z)=z/(1-z)$, thus $h(z)=1/(1-z)$. From (3.6), by induction, we gain

$$H^{(k,q)}(z)=(-1)^k q^{k^2-k}\frac{z^k}{(z;q)_k}, \tag{3.15}$$

so

$$G^{(k,q)}(qz)=(-1)^k q^{k^2-k}\frac{z^{2k}}{(qz;q)_k}. \tag{3.16}$$

Since $f^0(z)=1/(1-z)(1-z/q)$, which can easily be checked in (3.2), by (3.10) we obtain the expansion

$$g(\bar{F})(z)=\left(\sum_{k=0}^{\infty}(-1)^k q^{\binom{k}{2}}\frac{D_{1/q}^k}{[k]_{1/q}!}\left(g(z)\frac{z^{2k}}{(z/q;q)_{k+2}}\right)\right)^{\vee}. \tag{3.17}$$

By the Lagrange formula (3.1) for $g(z)=z$, it turns out that $F(z)=z/(1+z/q)$, therefore (3.17) for $g(z)=z^l$ leads to

$$\frac{q^{\binom{l}{2}}z^l}{(-z/q;q)_l}=\left(\sum_{k=0}^{\infty}(-1)^k q^{\binom{k}{2}}\frac{D_{1/q}^k}{[k]_{1/q}!}\left(\frac{z^{2k+l}}{(z/q;q)_{k+2}}\right)\right)^{\vee}. \tag{3.18}$$

Equating coefficients of z^n and some manipulation furnish the q-binomial identity

$$\begin{bmatrix}-l\\ n-l\end{bmatrix}_q=\sum_{n=0}^{n-l}q^{k(k+1)}\begin{bmatrix}-n-1\\ k\end{bmatrix}_q\begin{bmatrix}n-l+1\\ n-k-l\end{bmatrix}_q, \tag{3.19}$$

which is a special case of q-Vandermonde convolution [2, identity (18)].

Another choice is

$$g(z)=(z/q;q)_{-l}=1/(1-z/q^2)(1-z/q^3)\cdots(1-z/q^{l+1}).$$

Again for $g(\bar{F})(z)$ a closed expression can be obtained. The equation $f_l(\underline{F})(z)=z^l$ in our example is

$$q^{\binom{l}{2}}\sum_{i=0}^{\infty}\begin{bmatrix}l+i-1\\ i\end{bmatrix}_q q^{-\binom{i+l}{2}}z^{i+l}(-z;q)_{-i-l}=z^l. \tag{3.20}$$

After division of z^l, substitution of $q^l z$, and multiplication of $(-z;q)_l$ on both sides of this identity, we obtain

$$\sum_{i=0}^{\infty}\begin{bmatrix}l+i-1\\ i\end{bmatrix}_q q^{-\binom{i}{2}}z^i(-z;q)_{-i}=(-z;q)_l. \tag{3.21}$$

Changing q into $1/q$ and replacing z by z/q^2 turns (3.21) into

$$\sum_{i=0}^{\infty} q^{-i(l+1)} \begin{bmatrix} l+i-1 \\ i \end{bmatrix}_q q^{\binom{i}{2}} z^i/(-z/q;q)_i$$
$$= (-z/q^{l+1};q)_l,$$

which is equivalent with

$$g(\bar{F})(z) = (-z/q^{l+1};q)_l. \tag{3.22}$$

Therefore, for $g(z) = (z/q;q)_{-l}$, (3.17) yields

$$(-z/q^{l+1};q)_l = \left(\sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} \frac{D_{1/q}^k}{[k]_{1/q}!} \left(\frac{z^{2k}}{(z/q^{l+1};q)_{k+l+2}} \right) \right)^{\vee}. \tag{3.23}$$

Next we consider the uniform example for $g(z)$ containing the preceding two choices of $g(z)$ as special cases. Set $g(z) = z^l(z/q;q)_m$; then the generalization of (3.22) is

$$g(\bar{F})(z) = q^{\binom{l}{2}} z^l/(-z/q;q)_{l+m}, \tag{3.24}$$

valid for all $l, m \in \mathbb{Z}$. Indeed, to establish (3.24), quite similar considerations like that which led from (3.20) to (3.22), have to be done. Hence, by combining (3.17) and (3.24), we get the expansion

$$\frac{q^{\binom{l}{2}} z^l}{(-z/q;q)_{l+m}} = \left(\sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} \frac{D_{1/q}^k}{[k]_{1/q}!} \left(\frac{z^{2k+l}}{(q^{m-1}z;q)_{k-m+2}} \right) \right)^{\vee}. \tag{3.25}$$

This time, equating coefficients of z^n on both sides of (3.25) leads to

$$\begin{bmatrix} -l-m \\ n-l \end{bmatrix}_q = \sum_{k=0}^{l} q^{k(k-m+1)} \begin{bmatrix} -n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n-l-m+1 \\ n-k-l \end{bmatrix}_q, \tag{3.26}$$

which by change of variables is seen to be equivalent with the q-Vandermonde convolution formula [2, identity (18)].

4. The q-Analogue Involving Hofbauer's q-Powers

The essential definition is

Definition 4. The fps $\varphi_\alpha(z), \alpha \in \mathbb{R}$ (real numbers), are called q-powers for a fixed fps $\varphi(z)$, if $\varphi_\alpha(0) = 1$ for all α and

$$D_q \varphi_\alpha(z) = [\alpha]_q\, \varphi(z)\, \varphi_\alpha(z). \tag{4.1}$$

By (4.1) $\varphi_\alpha(z)$ is uniquely determined for all $\alpha \in \mathbb{R}$. Obviously in the case $q=1$, where D_q becomes the ordinary derivative, the fps $\varphi_\alpha(z)$ are powers of an fps $\bar{\varphi}(z)$ with $\varphi(z)=\bar{\varphi}'(z)/\bar{\varphi}(z)$.

This time the q-analogue of (2.3) reads

LEMMA 5. *If $\varphi_\alpha(z)$ are q-powers for $\varphi(z)$ then for all $k \in \mathbb{N}_0$ the fps*

$$H_k^{(q)}(z)=\sum_{j=0}^{k} (-1)^j q^{-\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{1/q} \varphi_{-j}(z) \tag{4.2}$$

(the q-analogue of $(1-h(z))^k$, where $h(z)$ corresponds to $1/\bar{\varphi}(z)$) has order k. Moreover,

$$\varphi_{-n}(z)=\sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} H_k^{(q)}(z) \tag{4.3}$$

for all $n \in \mathbb{Z}$ (even for all $n \in \mathbb{R}$).

Proof. By the defining relation (4.1), we have

$$\begin{aligned} D_q H_k^{(q)}(z) &= \sum_{j=0}^{k} (-1)^j q^{-\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{1/q} [-j]_q\, \varphi(z)\, \varphi_{-j}(z) \\ &= [-k]_q\, \varphi(z) \sum_{j=1}^{k} (-1)^j q^{-\binom{k-j}{2}} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_{1/q} \varphi_{-j}(z), \end{aligned}$$

and after remembering (3.5)

$$\begin{aligned} D_q H_k^{(q)}(z) &= [-k]_q\, \varphi(z) \sum_{j=0}^{k} (-1)^j q^{-\binom{k-j}{2}} \\ &\quad \times \left(\begin{bmatrix} k \\ j \end{bmatrix}_{1/q} - q^{-j} \begin{bmatrix} k-1 \\ j \end{bmatrix}_{1/q} \right) \varphi_{-j}(z) \\ &= [-k]_q\, \varphi(z) (H_k^{(q)}(z) - q^{-k+1} H_{k-1}^{(q)}(z)). \end{aligned} \tag{4.4}$$

First the constant term of $H_k^{(q)}(z)$ has to be evaluated:

$$H_k^{(q)}(0)=\sum_{j=0}^{k} (-1)^j q^{-\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{1/q} = (-1)^k (1; 1/q)_k = \delta_{k0}.$$

Again, reasoning inductively, $H_k^{(q)}(z)$ is seen to be of order k. Therefore the infinite formal sum

$$F_n^{(q)}(z)=\sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} H_k^{(q)}(z)$$

is well defined. After having multiplied both sides of (4.4) by $(-1)^k \left[{n \atop k}\right]_{1/q}$ and after summation with respect to $k \in \mathbb{N}_0$, we get by renewed use of (3.5)

$$\begin{aligned} D_q F_n^{(q)}(z) &= [-n]_q\, \varphi(z) \sum_{k=0}^{\infty} (-1)^k \left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{1/q} \right. \\ &\qquad \left. + q^{-k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{1/q} \right) H_k^{(q)}(z) \\ &= [-n]_q\, \varphi(z)\, F_n^{(q)}(z). \end{aligned}$$

Comparison with (4.1) completes the proof of (4.3) because of the uniqueness of the q-powers $\varphi_\alpha(z)$. ▌

Perhaps it is interesting to note that there is a result which in a way is dual to Lemma 5. We will state it without proof, because it runs in the same manner as that of Lemma 5.

PROPOSITION 6. *With the assumptions of Lemma 5 the fps*

$$\bar{H}_k^{(q)}(z) = \sum_{j=0}^{k} (-1)^j q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \varphi_j(z) \tag{4.5}$$

are of order k. Moreover, there holds

$$\varphi_n(z) = \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \bar{H}_k^{(q)}(z). \quad ▌ \tag{4.6}$$

The reason why by Lemma 5 we do get a q-analogue of Abhyankar's formula, but we do not by trying with Proposition 6 is that there exists a q-Lagrange formula for the sequence $(z^k/\varphi_k(z))_{k\in\mathbb{Z}}$, but there is none for the sequence $(z^k/\varphi_{-k}(z))_{k\in\mathbb{Z}}$.

Let $\mathscr{f} = (f_k(z))_{k\in\mathbb{Z}}$ be the sequence of fLs defined by

$$f_k(z) = z^k/\varphi_k(z). \tag{4.7}$$

$\mathscr{F} = (F_l(z))_{l\in\mathbb{Z}}$ denotes the inverse sequence of $\mathscr{f}$. Then the Lagrange formula [13, Theorem 1 (A) for $\lambda = \mu = 0$, $\Phi(z) = 0$] can be rewritten as

$$\langle z^n \rangle\, g(\mathscr{F})(z) = \langle z^0 \rangle\, g(z)(1 - (z/q)\, \varphi(z/q)) z^{-n} \varphi_n(z/q). \tag{4.8}$$

The q-analogue of (1.4) is the contents of

THEOREM 7. *For a given fps $\varphi(z)$ let the sequence $\mathscr{f} = (f_k(z))_{k\in\mathbb{Z}}$ be defined by* (4.7), *where the q-powers $\varphi_\alpha(z)$ are given by* (4.1). *If*

$$\begin{aligned} G_k^{(q)}(z) &= q^{2k} z^k H_k^{(q)}(z) \\ &= q^{2k} z^k \sum_{j=0}^{k} (-1)^j q^{-\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{1/q} \varphi_{-j}(z), \end{aligned} \tag{4.9}$$

then for $g(z)$, *an fLs of the form* (1.1), *holds*

$$g(\mathscr{F})(z)=z\sum_{k=0}^{\infty}\frac{D_q^k}{[k]_{1/q}!}\times(g(z)\,z^{-1}(1-(z/q)\,\varphi(z/q))\,G_k^{(q)}(z/q)). \tag{4.10}$$

Remark. In the case $q=1$ as mentioned above, $\varphi_k(z)$ are powers of an fps $\bar{\varphi}(z)$ and $\varphi(z)$ is the analogue for $\bar{\varphi}'(z)/\bar{\varphi}(z)$. Therefore $f_k(z)$ are powers of $f(z)=z/\bar{\varphi}(z)$ and $z^{-1}(1-(z/q)\,\varphi(z/q))\to f'(z)/f(z)$ for $q\to 1$.

Proof. According to (4.8) and (4.3),

$$\begin{aligned}\langle z^n\rangle\, g(\mathscr{F})(z)&=\langle z^0\rangle\, g(z)(1-(z/q)\,\varphi(z/q))z^{-n}\varphi_n(z/q)\\ &=\langle z^0\rangle\, g(z)(1-(z/q)\,\varphi(z/q))z^{-n}\sum_{k=0}^{\infty}(-1)^k\\ &\quad\times\begin{bmatrix}-n\\ k\end{bmatrix}_{1/q}H_k^{(q)}(z/q)\\ &=\sum_{k=0}^{\infty}\langle z^{n+k-1}\rangle q^k\,\frac{[n+k-1]_q\,[n+k-2]_q\cdots[n]_q}{[k]_{1/q}!}\\ &\quad\times g(z)\,z^{-1}(1-(z/q)\,\varphi(z/q))z^kH_k^{(q)}(z/q)\\ &=\langle z^{n-1}\rangle\sum_{k=0}^{\infty}\frac{D_q^k}{[k]_{1/q}!}\\ &\quad\times(g(z)\,z^{-1}(1-(z/q)\,\varphi(z/q))\,G_k^{(q)}(z/q)).\end{aligned}$$

Multiplication of both sides of this equation by z^n and summation over all $n\in\mathbb{Z}$ furnish (4.10). ▮

Next we derive the q-analogue of (1.5).

THEOREM 8. *With the assumptions of Theorem* 7,

$$g(\mathscr{F})(z)=g(z)+\sum_{k=1}^{\infty}q^{-k}\frac{D_q^{k-1}}{[k]_{1/q}!}(D_q(g(z))\,G_k^{(q)}(z)). \tag{4.11}$$

Proof. The q-Lagrange formula corresponding to (1.6) is

$$\langle z^n\rangle\, g(\mathscr{F})(z)=\frac{1}{[n]_q}\langle z^{-1}\rangle\, D_q(g(z))z^{-n}\varphi_n(z),\qquad n\neq 0 \tag{4.12}$$

[13, Theorem 1 (B) for $\Phi(z)=0$). Therefore for $n\neq 0$, by (4.3), we get

$$\langle z^n\rangle\, g(\mathscr{F})(z) = \frac{1}{[n]_q} \langle z^{-1}\rangle\, D_q(g(z))z^{-n} \sum_{k=0}^{\infty} (-1)^k \times \begin{bmatrix} -n \\ k \end{bmatrix}_{1/q} H_k^{(q)}(z)$$

$$= \frac{1}{[n]_q} \langle z^{n-1}\rangle\, D_q(g(z)) + \sum_{k=1}^{\infty} \langle z^{n+k-1}\rangle q^k \times \frac{[n+k-1]_q\,[n+k-2]_q \cdots [n+1]_q}{[k]_{1/q}!} D_q(g(z))z^k H_k^{(q)}(z) \tag{4.13}$$

$$= \langle z^n\rangle\, g(z) + \langle z^n\rangle \sum_{k=1}^{\infty} q^{-k} \frac{D_q^{k-1}}{[k]_{1/q}!} (D_q(g(z))\, G_k^{(q)}(z)). \tag{4.14}$$

This leaves us to prove that (4.14) is true for $n=0$, too.

By (4.13) and (4.4) the right-hand side of (4.14) for $n=0$ can be transformed as follows:

$$\langle z^0\rangle\, g(z) + \langle z^0\rangle \sum_{k=1}^{\infty} q^{-k} \frac{D_q^{k-1}}{[k]_{1/q}!} (D_q(g(z))\, G_k^{(q)}(z))$$

$$= \langle z^0\rangle\, g(z) + \langle z^{-1}\rangle\, D_q(g(z)) \sum_{k=1}^{\infty} \frac{q^{\binom{k}{2}+1}}{[k]_{1/q}} H_k^{(q)}(z)$$

$$= \langle z^0\rangle\, g(z) - \langle z^{-1}\rangle\, g(z) q^{-1} D_{1/q} \left(\sum_{k=1}^{\infty} \frac{q^{\binom{k}{2}+1}}{[k]_{1/q}} H_k^{(q)}(z) \right)$$

$$= \langle z^0\rangle\, g(z) - \langle z^{-1}\rangle\, g(z)\, D_q \left(\sum_{k=1}^{\infty} \frac{q^{\binom{k}{2}+1}}{[k]_{1/q}} H_k^{(q)}(z/q) \right)$$

$$= \langle z^0\rangle\, g(z) - \langle z^{-1}\rangle\, g(z) \sum_{k=1}^{\infty} \frac{q^{\binom{k}{2}}[-k]_q}{[k]_{1/q}} \times (H_k^{(q)}(z/q) - q^{-k+1} H_{k-1}^{(q)}(z/q))\, \varphi(z/q)$$

$$= \langle z^0\rangle\, g(z) - \langle z^{-1}\rangle\, g(z) q^{-1} \varphi(z/q)$$

$$= \langle z^0\rangle\, g(z)(1-(z/q)\,\varphi(z/q)) = \langle z^0\rangle\, g(\mathscr{F})(z).$$

The last step was performed by remembering (4.8). The second step used the fact that the adjoint of D relative to the bilinear form $\langle a(z), b(z)\rangle = \langle z^{-1}\rangle\, a(z)\cdot b(z)$ is $-q^{-1}D_{1/q}$. (For a more detailed discussion of this concept concerning Lagrange inversion we refer the reader to [11, 14].) ∎

Concluding we give some examples for these two theorems.

EXAMPLE 9. For $\varphi(z) = -1/(1-z)$ the corresponding q-powers are given by $\varphi_\alpha(z) = (z;q)_\alpha$ [13, Example 3]. From (4.4) we get by induction

$$H_k^{(q)}(z) = (-1)^k q^{-k^2} z^k (z;q)_{-k} = \frac{(-1)^k q^{-k^2} z^k}{(1-z/q)(1-z/q^2)\cdots(1-z/q^k)}; \tag{4.15}$$

thus

$$G_k^{(q)}(z/q) = (-1)^k q^{-k^2} z^{2k} (z/q;q)_{-k} = (-1)^k q^{-k^2} \frac{z^{2k}}{(1-z/q^2)\cdots(1-z/q^{k+1})}. \tag{4.16}$$

Since $1 - z\varphi(z) = 1/(1-z)$, by (4.10),

$$g(\mathscr{F})(z) = z \sum_{k=0}^{\infty} (-1)^k q^{-k^2} \frac{D_q^k}{[k]_{1/q}!} \times \left(g(z) \frac{z^{2k-1}}{(1-z/q)\cdots(1-z/q^{k+1})} \right). \tag{4.17}$$

The Lagrange formula (4.8) for $g(z) = z^l$ leads to

$$F_l(z) = \sum_{n=l}^{\infty} (-1)^{n-l} q^{\binom{n-l}{2}} \begin{bmatrix} n-1 \\ n-l \end{bmatrix}_q z^n$$

which, of course, in view of Example 3 is the same as

$$F_l(z) = \left(\left(\frac{z}{1+z/q} \right)^{[l,1/q]} \right)^{\vee}.$$

Therefore the q-Abhyankar formula (4.17) for $g(z) = z^l$ gives

$$F_l(z) = z \sum_{k=0}^{\infty} (-1)^k q^{-\binom{k+1}{2}} \frac{D_q^k}{[k]_q!} \times \left(\frac{z^{2k+l-1}}{(1-z/q)\cdots(1-z/q^{k+1})} \right). \tag{4.18}$$

Equating coefficients of z^n on both sides of (4.18) leads after some simplification to the q-Vandermonde convolution

$$\begin{bmatrix} -l \\ n-l \end{bmatrix}_q = \sum_{k=0}^{n-l} q^{-(n+k)(n-k-l)} \begin{bmatrix} -n \\ k \end{bmatrix}_q \begin{bmatrix} n-l \\ n-k-l \end{bmatrix}_q, \tag{4.19}$$

which again is a special case of [2, identity (18)]. (4.11), the second form of the q-Abhyankar formula, reads

$$g(\mathscr{F})(z) = g(z) + \sum_{k=1}^{\infty} (-1)^k q^{-\binom{k}{2}} \frac{D_q^{k-1}}{[k]_q!} \times (D_q(g(z))) \frac{z^{2k}}{(1-z/q)\cdots(1-z/q^k)}. \tag{4.20}$$

Next we take $g(z) = z^l/(z;q)_m$. For this choice of $g(z)$ we obtain just in the same manner as (3.24),

$$g(\mathscr{F})(z) = q^{-\binom{l}{2}} (z^l(-z;q)_{m-l})^{\vee}. \tag{4.21}$$

From (4.17) and (4.20) therefore we get the expansions

$$q^{-\binom{l}{2}} (z^l(-z;q)_{m-l})^{\vee} = z \sum_{k=0}^{\infty} (-1)^k q^{-\binom{k+1}{2}} \frac{D_q^k}{[k]_q!} \left(\frac{z^{2k+l-1}}{(z/q^{k+1};q)_{k+m+1}} \right) \tag{4.22}$$

and

$$q^{-\binom{l}{2}} (z^l(-z;q)_{m-l})^{\vee} = \frac{z^l}{(z;q)_m} + \sum_{k=1}^{\infty} (-1)^k q^{-\binom{k}{2}} \times \frac{D_q^{k-1}}{[k]_q!} \left(\frac{([l] + q^l[m-l]z) z^{2k+l-1}}{(z/q^k;q)_{k+m+1}} \right). \tag{4.23}$$

For the latter expression we used

$$D_q \frac{z^l}{(z;q)_m} = \frac{([l] + q^l[m-l]z) z^{l-1}}{(z;q)_{m+1}}.$$

EXAMPLE 10. Take $\varphi(z) = 1$, then [13, Example 2] $\varphi_\alpha(z) = e_q([\alpha]_q z)$; hence in this case,

$$H_k^{(q)}(z) = \sum_{j=0}^{k} (-1)^j q^{-\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{1/q} e_q([-j]_q z). \tag{4.24}$$

Comparison with [3, Example at the bottom of p. 536] unveils the connection with the q-Stirling numbers of second kind introduced by Gould [8]. In [3] Cigler obtains an expression for the generating function of Gould's q-Stirling numbers of the second kind which is quite similar to (4.24):

$$[k]_q! \sum_{n=k}^{\infty} \frac{S_q(n,k)}{[n]_q!} z^n = \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{j}{2}-j(k-1)} \begin{bmatrix} k \\ j \end{bmatrix}_q e_q([j]_q z). \tag{4.25}$$

Comparing the coefficients of z^m in (4.24) and (4.25) implies

$$H_k^{(q)}(-qz) = (-1)^k q^{-\binom{k}{2}} [k]_{1/q}! \sum_{n=k}^{\infty} \frac{S_{1/q}(n,k)}{[n]_q!} z^n; \tag{4.26}$$

therefore

$$G_k^{(q)}(z/q) = q^{-\binom{k}{2}} [k]_{1/q}! \sum_{n=2k}^{\infty} (-1)^n q^{3k-2n} \frac{S_{1/q}(n-k,k)}{[n-k]_q!} z^n. \tag{4.27}$$

Now we are ready to apply Theorems 7 and 8. (4.10) reads

$$g(\mathscr{F})(z) = z \sum_{k=0}^{\infty} q^{-\binom{k}{2}} D_q^k \Bigg(g(z) z^{-1}(1-z/q) \times \sum_{n=2k}^{\infty} (-1)^n q^{3k-2n} \frac{S_{1/q}(n-k,k)}{[n-k]_q!} z^n \Bigg); \tag{4.28}$$

by (4.11) we get

$$g(\mathscr{F})(z) = g(z) + \sum_{k=1}^{\infty} q^{-\binom{k}{2}} D_q^{k-1} \Bigg(D_q(g(z)) \times \sum_{n=2k}^{\infty} (-1)^n q^{2k-n} \frac{S_{1/q}(n-k,k)}{[n-k]_q!} z^n \Bigg). \tag{4.29}$$

By the q-Lagrange formula (4.12) it is possible to evaluate $F_l(z)$:

$$F_l(z) = \sum_{n=l}^{\infty} \frac{[l]_q [n]_q^{n-l-1}}{[n-l]_q!} z^n. \tag{4.30}$$

Setting $g(z) = z^l$ in (4.29) and equating coefficients of z^n of both sides leads by a short calculation finally to the well-known identity [8, identity (3.7)]

$$[-n]_q^j = \sum_{k=0}^{j} q^{\binom{k}{2}} \begin{bmatrix} -n \\ k \end{bmatrix}_q [k]_q! \, S_q(j,k), \tag{4.31}$$

where we set $n-l=j$.

By (4.28), similarly, a q-identity involving q-Stirling numbers could be derived. We will omit it here because it is more complicated than (4.31) (which is due to the term $(1-z/q)$ on the right-hand side of (4.28)) and therefore of less interest.

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