Asymptotic analysis of a Selberg-type integral via hypergeometrics

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Abstract. We show how to determine the asymptotics of a certain Selberg-type integral by means of tools available in the theory of (generalised) hypergeometric series. This provides an alternative derivation of a result of Carré, Deneufchâtel, Luque and Vivo [arχiv:1003.5996].

In [2], Carré, Deneufchâtel, Luque and Vivo consider the Selberg-type integral

$$S_k(a,b)=\frac{1}{N!}\int_{[0,1]^N} x_1^k\left(\prod_{1\le i<j\le N}(x_i-x_j)^2\right)\left(\prod_{i=1}^N x_i^{a-1}(1-x_i)^{b-1}\,dx_i\right),$$

and they determine its asymptotic behaviour when N,a,b all tend to infinity so that $a\sim a_1N$ and $b\sim b_1N$, where a_1 and b_1 are given non-negative real numbers. The reader is referred to the introduction of [2] for information on motivation from random matrix theory connected to random scattering theory to investigate this question.

It should be noted that $S_0(a,b)$ *is* a Selberg integral, which can be evaluated in a product/quotient of gamma functions (cf. [3]). This being the case, the asymptotics of $S_0(a,b)$ is easily determined by means of known asymptotic formulae for the Barnes G-function (see [7]). Thus, it suffices to consider the quotient

$$J_k=\frac{S_k(a,b)}{S_0(a,b)}$$

(this quotient is denoted by I_k/N in [2]) and determine its asymptotic behaviour. By (now) classical identities in the theory of symmetric functions, it is shown in [2, Cor. II.3] that

$$J_k=\frac{1}{N\cdot k!}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}\frac{(N-i)_k\,(a+N-i-1)_k}{(a+b+2N-i-2)_k}, \tag{1}$$

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where $(\alpha)_m$ is the usual Pochhammer symbol defined by $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$ for $m \ge 1$, and $(\alpha)_0 = 1$. In view of this rather simple expression, which, after substitution of a by a_1N and of b by b_1N, is a rational function in N, the problem of determining the asymptotic behaviour of the resulting expression looks innocent. However, inspection of the expression (1) shows that this is not so innocent at all: the summands are all of the order of magnitude N^k for large N, while (see Theorem and Corollary below), in fact, the expression in (1) has a *finite* limit as $N \to \infty$ in the described situation. Hence, what happens here is a large amount of cancellation of leading asymptotic terms. This is what one has to get under control, and this constitutes the challenge in determining this limit.

To solve the problem, Carré, Deneufchâtel, Luque and Vivo develop a difference operator calculus in [2], which is of interest in its own right.

The purpose of the present note is to demonstrate that the tools for solving this problem are readily available in the literature on (*generalised*) *hypergeometric series*, and it is clear that they may serve as well to solve problems of similar type.

Namely, using the standard hypergeometric notation

$$ {}_pF_q\left[\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}; z\right] = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{m!\,(b_1)_m \cdots (b_q)_m} z^m $$

where $(a_1)_m$, etc. are again Pochhammer symbols, Equation (1) becomes

$$ J_k = \frac{(N+1)_{k-1}\,(a+N-1)_k}{k!\,(2N+a+b-2)_k}\,{}_4F_3\left[\begin{matrix} 1-N, 1-k, 2-a-N, 3-a-b-k-2N \\ 2-a-k-N, 1-k-N, 3-a-b-2N \end{matrix}; 1\right]. \quad (2) $$

The point of this conversion is that one may now look in the literature of hypergeometric series, which provides a large arsenal of utilities, such as transformation formulae for hypergeometric series etc., and see whether one (or a combination of more) of these formulae enable us to convert (2) into a different form from which the asymptotics of J_k can be directly read off. As I show below, this is indeed the case. To the informed reader it will come as no surprise that the computations below have been carried out using the *Mathematica* package HYP [5].

The crucial observation to make is that the ${}_4F_3$-series in (2) is not just an "arbitrary" hypergeometric series, but that it is a (-1)-*balanced* series (cf. [1, p. 140]), that is, that the sum of the lower parameters of the hypergeometric series minus the sum of the upper parameters is -1. Unfortunately, there are no transformation formulae available for (-1)-balanced series, but there are plenty of them available for 1-balanced series (or, as is common to say, for *balanced series*), such as, for example (see [4, Appendix (III.16), $q \uparrow 1$]; this is T4302 in HYP),

$$ {}_4F_3\left[\begin{matrix} A, B, C, -n \\ E, F, 1+A+B+C-E-F-n \end{matrix}; 1\right] \\ = \frac{(A)_n\,(E+F-A-B)_n\,(E+F-A-C)_n}{(E)_n\,(F)_n\,(E+F-A-B-C)_n} \\ \times {}_4F_3\left[\begin{matrix} -n, E-A, F-A, E+F-A-B-C \\ E+F-A-B, E+F-A-C, 1-A-n \end{matrix}; 1\right], \quad (3) $$

where n is a non-negative integer. That is, we are just by "2 off" in applying a transformation formula.

In order to relate series with (small) integral differences between the parameters, the hypergeometric literature offers *contiguous relations*. One of the simplest such relations is (this is C14 in HYP),

$$
{}_4F_3\left[\begin{matrix}A,B,C,D\\ E,F,G\end{matrix};z\right]=z\frac{BCD}{EFG}\,{}_4F_3\left[\begin{matrix}A,B+1,C+1,D+1\\ E+1,F+1,G+1\end{matrix};z\right]+{}_4F_3\left[\begin{matrix}A-1,B,C,D\\ E,F,G\end{matrix};z\right].
$$

If we iterate this contiguous relation, then we arrive at

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix}A,B,C,D\\ E,F,G\end{matrix};z\right]&=z^r\frac{(B)_r\,(C)_r\,(D)_r}{(E)_r\,(F)_r\,(G)_r}\,{}_4F_3\left[\begin{matrix}A,B+r,C+r,D+r\\ E+r,F+r,G+r\end{matrix};z\right]\\
&\quad+\sum_{s=0}^{r-1}z^s\frac{(B)_s\,(C)_s\,(D)_s}{(E)_s\,(F)_s\,(G)_s}\,{}_4F_3\left[\begin{matrix}A-1,B+s,C+s,D+s\\ E+s,F+s,G+s\end{matrix};z\right].
\end{aligned}\tag{4}
$$

We now apply (4) with $r=k-1$ to the ${}_4F_3$-series in (2). The reader should observe that thereby all the ${}_4F_3$-series in the sum over s will be 0-balanced, while the single ${}_4F_3$-series outside of the sum will reduce to 1 because one of the upper parameters is $1-k+r=0$. To be precise, after writing the result in compact form, application of (4) to (2) yields

$$
\begin{aligned}
J_k=\sum_{s=0}^{k-1}&\frac{(N+1)_{k-1}\,(a+N-1)_k\,(1-k)_s\,(2-a-N)_s\,(3-a-b-k-2N)_s}{k!\,(a+b+2N-2)_k\,(2-a-k-N)_s\,(1-k-N)_s\,(3-a-b-2N)_s}\\
&\times{}_4F_3\left[\begin{matrix}-N,1-k+s,2-a-N+s,3-a-b-k-2N+s\\ 2-a-k-N+s,1-k-N+s,3-a-b-2N+s\end{matrix};1\right]
\end{aligned}
$$

Continuing in the same spirit, if we apply the contiguous relation (4) a second time, this time with $r=k-s-1$, then this will produce a double sum in which all appearing ${}_4F_3$-series are balanced. To be precise, we obtain

$$
\begin{aligned}
&J_k\\
&=\sum_{s=0}^{k-1}\sum_{t=0}^{k-s-1}\frac{(N+1)_{k-1}\,(a+N-1)_k\,(1-k)_{s+t}\,(2-a-N)_{s+t}\,(3-a-b-k-2N)_{s+t}}{k!\,(a+b+2N-2)_k\,(2-a-k-N)_{s+t}\,(1-k-N)_{s+t}\,(3-a-b-2N)_{s+t}}\\
&\qquad\times{}_4F_3\left[\begin{matrix}3-a-b-k-2N+s+t,-1-N,2-a-N+s+t,1-k+s+t\\ 3-a-b-2N+s+t,1-k-N+s+t,2-a-k-N+s+t\end{matrix};1\right].
\end{aligned}
$$

Now the transformation formula (3) can be applied, and, after some simplification, the resulting expression collapses to

$$J_k = \sum_{s=0}^{k-1} \sum_{t=0}^{k-s-1} \frac{(a-1)_{k-s-t-1}\,(1-a-N)_{s+t+1}\,(k-s-t)_{s+t}\,(s+t+2)_{k-s-t-1}}{k!\,(2-a-b-2N)_k}$$
$$\cdot\, {}_4F_3\left[\begin{matrix}1-k+s+t, k, a+b+N-2, a+N\\ s+t+2, a-1, a+b+2N-1\end{matrix}; 1\right]$$
$$= \sum_{m=0}^{k-1} \frac{(a-1)_{k-m-1}\,(1-a-N)_{m+1}\,(k-m)_m\,(m+1)_{k-m}}{k!\,(2-a-b-2N)_k}$$
$$\cdot\, {}_4F_3\left[\begin{matrix}-k+m+1, k, a+b+N-2, a+N\\ m+2, a-1, a+b+2N-1\end{matrix}; 1\right].$$

Now the limit $N, a, b \to \infty$, in which $a \sim a_1 N$ and $b \sim b_1 N$, can be safely performed. Hence, we have proven the following result.

Theorem. *The limit of the quantity J_k as $N, a, b \to \infty$ such that $a \sim a_1 N$ and $b \sim b_1 N$ is equal to*

$$\lim_{N\to\infty} J_k = \sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k-1}{m} \left(\frac{a_1}{a_1+b_1+2}\right)^k \left(\frac{a_1+1}{a_1}\right)^{m+1}$$
$$\cdot \sum_{\ell=0}^{k-m-1} (-1)^\ell \binom{k-m-1}{\ell} \frac{(k+\ell-1)!(m+1)!}{(k-1)!(m+\ell+1)!} \left(\frac{(a_1+1)(a_1+b_1+1)}{a_1(a_1+b_1+2)}\right)^\ell. \qquad (5)$$

This theorem does not give the same expression as in [2, Theorem IV.4]. However, once one has found *some* expression, it is easy to pass to an equivalent one by using hypergeometric transformation and summation formulae. (Since, in the above derivation, there is plenty of room for variation — for example, the transformation formula (3) could be applied in different ways, or we could have applied a different transformation formula for balanced ${}_4F_3$-series, see [6, Sec. 4.3.5] —, it is conceivable that there is a direct route leading to the corollary below, using the above ideas. We did not pursue this, however; in particular, we did not do a systematic search of such an alternative route.)

Corollary ([2, Theorem IV.4]). *The limit of the quantity J_k as $N, a, b \to \infty$ such that $a \sim a_1 N$ and $b \sim b_1 N$ is equal to*

$$\lim_{N\to\infty} J_k = \frac{1}{k} \sum_{j=0}^{k-1} (-1)^j \binom{k+j-1}{j} \frac{(a_1+1)^{j+1}}{(a_1+b_1+2)^{k+j}} \sum_{i=0}^{k-j-1} \binom{k}{i} \binom{k}{i+j+1} (a_1+1)^i.$$

Proof. In the expression on the right-hand side of (5) we interchange the summations over m and ℓ, and we write the (now) inner sum over m in hypergeometric notation. This leads to

$$\lim_{N\to\infty} J_k = \sum_{\ell=0}^{k-1} (-1)^{k-\ell-1} \frac{a_1^{k-\ell-1}\,(a_1+1)^{\ell+1}\,(a_1+b_1+1)^\ell}{(a_1+b_1+2)^{k+\ell}} \frac{(k+\ell-1)!}{\ell!\,(\ell+1)!\,(k-\ell-1)!}$$
$$\cdot\, {}_2F_1\left[\begin{matrix}2, -k+\ell+1\\ \ell+2\end{matrix}; \frac{a_1+1}{a_1}\right].$$

To the ${}_2F_1$-series we apply the transformation formula (see [6, (1.7.1.3), sum reversed at the right-hand side]; this is T2106 in HYP)

$$ {}_2F_1\left[\begin{matrix} A, -n \\ C \end{matrix}; z\right] = z^n \frac{(C-A)_n}{(C)_n} {}_2F_1\left[\begin{matrix} -n, 1-C-n \\ 1+A-C-n \end{matrix}; -\frac{1-z}{z}\right], $$

where n is a non-negative integer. Writing the resulting ${}_2F_1$-series as a sum over i, this gives

$$ \lim_{N\to\infty} J_k = \sum_{\ell=0}^{k-1} \sum_{i=0}^{k-\ell-1} (-1)^{k-\ell-i-1} \binom{k-\ell-1}{i} \frac{(k+\ell-1)!}{(\ell-1)!\,\ell!\,(k-\ell-1)!\,(k-i-1)(k-i)} \cdot \frac{(a_1+1)^{k-i}\,(a_1+b_1+1)^{\ell}}{(a_1+b_1+2)^{k+\ell}}. $$

Now we replace the term $(a_1+b_1+1)^\ell$ by $\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j(a_1+b_1+2)^{\ell-j}$, we make the sum over ℓ the inner-most sum, and finally write the sum over ℓ in hypergeometric notation. In this way, the last equation becomes

$$ \lim_{N\to\infty} J_k = \sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} (-1)^{k-i-1} \binom{k-j-1}{i} \frac{(k+j-1)!}{(j-1)!\,j!\,(k-j-1)!\,(k-i-1)(k-i)} \cdot \frac{(a_1+1)^{k-i}}{(a_1+b_1+2)^{k+j}} {}_2F_1\left[\begin{matrix} -k+i+j+1, k+j \\ j \end{matrix}; 1\right]. $$

The ${}_2F_1$-series can be evaluated by the Chu-Vandermonde summation (see [6, (1.7.7); Appendix (III.4)]; this is S2101 in HYP)

$$ {}_2F_1\left[\begin{matrix} A, -n \\ C \end{matrix}; 1\right] = \frac{(C-A)_n}{(C)_n}, $$

where n is a non-negative integer. After finally replacing i by $k-i-j-1$, the claimed result follows immediately upon minor rearrangement of terms. $\square$

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