The F-triangle of the generalised cluster complex

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Abstract. The F-triangle is a refined face count for the generalised cluster complex of Fomin and Reading. We compute the F-triangle explicitly for all irreducible finite root systems. Furthermore, we use these results to partially prove the "$F = M$ Conjecture" of Armstrong which predicts a surprising relation between the F-triangle and the Möbius function of his m-divisible partition poset associated to a finite root system.

1. Introduction. Fomin and Zelevinsky created a new exciting research field when they invented *cluster algebras* in [11]. The classification of cluster algebras *of finite type* from [12] says that there is a one-to-one correspondence between finite-type cluster algebras and finite root systems. Furthermore, for each finite root system Φ, Fomin and Zelevinsky [13] defined a simplicial complex corresponding to the associated cluster algebra, the *cluster complex* $\Delta(\Phi)$. This is a simplicial complex on a subset of the set of roots Φ. As they showed, this complex has many remarkable properties. In particular, the number of facets is given by the *Catalan number for the root system* Φ, and, moreover, all the face numbers are given by elegant product formulae. Further remarkable (originally, conjectural) properties have been discovered by Chapoton in [8]. In this paper, he refines the face enumeration to, what he calls, the "F-triangle." He computed the F-triangle for all types (and revealed his findings partially in [8]) and observed a surprising relationship (see [8, Conjecture 1]) between the F-triangle and the Möbius function of the non-crossing partition lattice $NC(\Phi)$ associated to Φ, the latter being due to Bessis [4] and Brady and Watt [5]. This relationship, to which we shall refer in the sequel as the "$F = M$ Conjecture," has been recently proved by Athanasiadis [2]. For further fascinating properties of the F-triangle see [8].

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The subject of this paper is "m-generalisations" of the cluster complex $\Delta(\Phi)$ and of the non-crossing partition lattice $NC(\Phi)$. More precisely, in [10], Fomin and Reading introduce the generalised cluster complex $\Delta^m(\Phi)$, where m is some non-negative integer. This is, again, a simplicial complex, now on *coloured* roots, and for $m = 1$ it reduces to the (ordinary) cluster complex $\Delta(\Phi)$. As they show, this generalised complex has again remarkable properties. In particular, the number of facets is given by the *Fuss–Catalan number for the root system* Φ, and, moreover, again, all the face numbers are given by elegant product formulae.

Going one step further, Sergey Fomin suggested to the author to investigate the "Chapoton-like" refinement of the face numbers of the generalised cluster complex $\Delta^m(\Phi)$, that is to say, to study the "F-triangle" for $\Delta^m(\Phi)$ (see Section 2 for the definition). This is what we do in this paper. We compute the F-triangle of $\Delta^m(\Phi)$ for all types of irreducible root systems Φ, see Sections 4–7. We do this case-by-case. While a case-independent formula would certainly be desirable, certain features of our results (in particular, the appearance of the Kronecker delta in the formula in Theorem FD in Section 6 for type D_n) make it highly unlikely that such a case-independent formula exists. As an aside, we draw the reader's attention to the unexpected outcome of our results that the refined face numbers are all polynomials in m *with non-negative coefficients*, a phenomenon for which we have no intrinsic explanation.

One may then ask if there is also an "$F = M$ Conjecture" in this generalised context. This would, first of all, require an "m-extension" of the non-crossing partition lattice. Indeed, Armstrong [1] has recently introduced the "m-divisible non-crossing partition poset" $NC^m(\Phi)$, generalising an earlier construction of Edelman [9] in type A_n. He shows that this poset has also remarkable properties, resembling those of the non-crossing partition lattices. Moreover, he observed that there is also a rather straight-forward extension of the $F = M$ Conjecture relating the F-triangle of the generalised cluster complex $\Delta^m(\Phi)$ to the Möbius function of the corresponding m-divisible non-crossing partitions poset $NC^m(\Phi)$. We reproduce this conjecture in Section 8. (We refer the reader to [1, Sec. 4] and [21] for further fascinating properties of the F-triangle of $\Delta^m(\Phi)$.)

With the explicit formulae for the F-triangle in hand, we are able to prove this "m-version" of the $F = M$ Conjecture in types A_n and B_n, for the dihedral root systems $I_2(a)$, for the hyperbolic root systems H_3 and H_4, and for F_4 and E_6, see Sections 9, 10, 13–17. In types A_n and B_n, the proofs depend crucially on results about rank selected chain enumeration in $NC^m(A_n)$ and $NC^m(B_n)$ due to Edelman [9] and Armstrong [1], respectively. Moreover, in Section 11, we provide a calculation in type D_n which will prove the conjecture also for this type once the corresponding rank selected chain enumeration result analogous to the ones by Edelman and Armstrong is available for $NC^m(D_n)$. In view of the results of Athanasiadis and Reiner [3] on rank selected chain enumeration in $NC^1(D_n) = NC(D_n)$, this argument does accomplish the proof for $m = 1$. As we explain in Section 12, the verification of the (generalised) $F = M$ Conjecture in the exceptional types is a routine task which can, in principle, be carried out on a computer. To do this in practice for the root systems E_7 or E_8, say, may however require additional simplifications of the proposed procedure.[1]

[1]Using some additional ideas, this has been carried through in *"The M-triangle of generalised non-*

In the final Section 18, we prove Armstrong's conjecture [1, Sec. 4] on the form of, what he calls, the *dual F-triangle* in a case-by-case fashion. For the exceptional root systems this is just a routine calculation, while for A_n, B_n and D_n this requires only the Chu–Vandermonde summation formula.

We conclude the introduction by saying a few words how the F-triangles for $\Delta^m(\Phi)$ are found and the corresponding results are proved in this paper. The main tool in [8] for finding F-triangles for the (ordinary) cluster complex $\Delta(\Phi)$ consists in two recurrence formulae (see [8, Prop. 3]). These recurrences carry over verbatim to the generalised cluster complex $\Delta^m(\Phi)$, see Proposition F in Section 2. Indeed, in the exceptional types, the formula for the F-triangle for $\Delta^m(\Phi)$ can be found in a routine fashion by using these recurrences, see Section 7. In types A_n, B_n, and D_n, however, the recurrences can only be used to compute the F-triangle for the corresponding generalised cluster complex for *specific* n. By doing this for sufficiently many n, we first worked out *guesses* for the F-triangle for *generic* n. (In types A_n and B_n, this has also been done independently by Tzanaki [21].) Subsequently, one tries to verify these guesses by checking the recurrences. As it turns out, this requires multivariate summation formulae due to Carlitz [7], which we restate here in Section 3 for the convenience of the reader. An interesting detail is the fact that Chapoton's proofs in [8] for the F-triangle for $\Delta(A_n)$ and $\Delta(B_n)$, respectively, which also use Carlitz's summation formulae, do *not* extend to $\Delta^m(A_n)$ and $\Delta^m(B_n)$, for the following reason. In order to do the above described verification using the recurrences, he has to evaluate a triple sum. He does this by first simplifying one sum by means of the Chu–Vandermonde summation, and by using subsequently one of Carlitz's summation formulae to evaluate the remaining double sum. However, if $m \neq 1$, the Chu–Vandermonde summation is not applicable to the triple sum that we encounter at the start. Remarkably, it is possible to apply Carlitz's summation formula *directly*, in a *different* way than in [8]. The use of the Chu–Vandermonde summation is then not necessary anymore.

2. Preliminaries. Let Φ be a finite root system of rank n. (We refer the reader to [15] for all root system terminology.) For a non-negative integer m, the generalised cluster complex $\Delta^m(\Phi)$ is a certain simplicial complex on a certain set of "coloured" roots, the roots being from Φ. The precise definition will not be important here, we refer the reader to [10, Sec. 2]. The only fact which is important here is that some of the coloured roots can be positive, others negative. Let $f_{k,l}(\Phi, m)$ denote the number of faces of $\Delta^m(\Phi)$ which contain exactly k positive and l negative coloured roots. Define the F-triangle of $\Delta^m(\Phi)$, denoted by $F_\Phi^m(x, y)$, as the two-variable polynomial

$$F_\Phi^m(x,y) = \sum_{k,l\geq 0} f_{k,l}(\Phi,m)\, x^k y^l. \tag{1}$$

It is called "triangle" because all faces have cardinality at most n and, thus, in the summation in (1) we can restrict the summation indices to the triangle $k + l \leq n$, $k, l \geq 0$.

Then, in this generalised context, the arguments from [8, Prop. 3] carry over verbatim to prove the following properties of the F-triangle of $\Delta^m(\Phi)$.

crossing partitions for the types E_7 and E_8" (preprint; `arXiv:math.CO/0601676`).

Proposition F. *The F-triangle $F_\Phi^m(x,y)$ satisfies the following three properties:*

(1) *If Φ and Φ' are two root systems, then*

$$F^m_{\Phi\times\Phi'}(x,y) = F^m_{\Phi}(x,y)F^m_{\Phi'}(x,y), \tag{2}$$

where $\Phi\times\Phi'$ denotes the orthogonal product of the two root systems.

(2) *If $\Phi=\Phi(S)$ is an irreducible root system with simple roots S, then*

$$\frac{\partial}{\partial y}F^m_{\Phi(S)}(x,y) = \sum_{\alpha\in S} F^m_{\Phi(S\backslash\{\alpha\})}(x,y), \tag{3}$$

where $\Phi(S\backslash\{\alpha\})$ denotes the root system generated by the simple roots $S\backslash\{\alpha\}$.

(3) *The specialisation $x=y$ is given by*

$$F^m_\Phi(x,x) = \sum_{k\ge 0} f_k(\Phi,m)\,x^k, \tag{4}$$

where the coefficients $f_k(\Phi,m)$ are the face numbers of the cluster complex $\Delta^m(\Phi)$, summarised in [10, Theorem 7.5] *for the irreducible root systems.*

We remark that an equivalent statement of (3) is

$$l\cdot f_{k,l}(\Phi(S),m) = \sum_{\alpha\in S} f_{k,l-1}(\Phi(S\backslash\{\alpha\}),m), \quad k,l\ge 0. \tag{5}$$

Moreover, in view of the multiplicativity property (2), it suffices to compute the F-triangle for the irreducible root systems, which we do in Sections 4–7.

3. Carlitz's summation formulae. Crucial in the proofs of our claims for the F-triangle in types A_n, B_n and D_n are the following two double sum evaluations due to Carlitz [7]. (He has in fact extensions for any number of summations, see [7, Sec. 6].) Let

$$A_{k,n}(\alpha,\beta) = \frac{bk\alpha+cn\beta+\alpha\beta}{(ak+cn+\alpha)(bk+dn+\beta)}\binom{ak+cn+\alpha}{k}\binom{bk+dn+\beta}{n}.$$

Here, and in the sequel, for integers N and K the binomial coefficient $\binom{N}{K}$ is understood according to the definition

$$\binom{N}{K} = \begin{cases} \frac{N(N-1)\cdots(N-K+1)}{K!} & \text{if } K\ge 0,\\ 0 & \text{if } K<0.\end{cases} \tag{6}$$

Then (see [7, (5.14)]),

$$\sum_{k_1,n_1\ge 0} A_{k_1,n_1}(\alpha,\beta)A_{k-k_1,n-n_1}(\alpha',\beta') = A_{k,n}(\alpha+\alpha',\beta+\beta'). \tag{7}$$

Furthermore (see [7, (5.15); the minus sign in front of cn must be replaced by a plus sign there]),

$$\sum_{k_1,n_1\ge 0}\binom{ak_1+cn_1+\alpha-1}{k_1}\binom{bk_1+dn_1+\beta-1}{n_1}A_{k-k_1,n-n_1}(\alpha',\beta')\\ = \binom{ak+cn+\alpha+\alpha'-1}{k}\binom{bk+dn+\beta+\beta'-1}{n}. \tag{8}$$

4. The F-triangle for A_n. The theorem below gives an explicit expression for the refined face numbers $f_{k,l}(A_n,m)$, and, thus, of the F-triangle in type A_n.

Theorem FA. *For $n\ge 1$, the face numbers $f_{k,l}(A_n,m)$ are given by*

$$f_{k,l}(A_n,m)=\frac{l+1}{k+l+1}\binom{n}{k+l}\binom{m(n+1)+k-1}{k}.$$

Proof. In view of Proposition F and (5) in Section 2, in order to prove this claim we have to show

$$l\cdot f_{k,l}(A_n,m)=\sum_{\substack{n_1+n_2=n-1\\ k_1+k_2=k\\ l_1+l_2=l-1}} f_{k_1,l_1}(A_{n_1},m)\cdot f_{k_2,l_2}(A_{n_2},m) \tag{9}$$

and

$$\sum_{k_1+l_1=k} f_{k_1,l_1}(A_n,m)=\frac{1}{k+1}\binom{n}{k}\binom{m(n+1)+k+1}{k}. \tag{10}$$

The triple sum on the right-hand side of (9) is

$$\sum_{k_1,l_1,n_1\ge 0}\frac{l_1+1}{k_1+l_1+1}\binom{n_1}{k_1+l_1}\binom{m(n_1+1)+k_1-1}{k_1}\\ \cdot\frac{l-l_1}{k-k_1+l-l_1}\binom{n-n_1-1}{k-k_1+l-l_1-1}\binom{m(n-n_1)+k-k_1-1}{k-k_1}.$$

We replace n_1 by $n_1+k_1+l_1$ and rewrite the resulting expression in the form

$$\sum_{k_1,l_1,n_1\ge 0}\frac{m(l_1+1)}{m(k_1+l_1+n_1+1)+k_1}\binom{n_1+k_1+l_1+1}{n_1}\binom{m(n_1+k_1+l_1+1)+k_1}{k_1}\\ \cdot\frac{m(l-l_1)}{m(n-n_1-k_1-l_1)+k-k_1}\binom{n-n_1-k_1-l_1}{n-k-l-n_1}\\ \cdot\binom{m(n-n_1-k_1-l_1)+k-k_1}{k-k_1}. \tag{11}$$

Forgetting the sum over l_1, this is now exactly in the form of the left-hand side of (7) with n replaced by $n-k-l$, $a=m+1$, $c=m$, $\alpha=m(l_1+1)$, $\alpha'=m(l-l_1)$, $b=d=1$, $\beta=l_1+1$, and $\beta'=l-l_1$. Substituting the right-hand side, we obtain

$$\sum_{l_1=0}^{l-1} A_{k,n-k-l}(m(l+1),l+1)$$

for the sum in (11), or, equivalently,

$$l\cdot\frac{m(l+1)}{m(n+1)+k}\binom{m(n+1)+k}{k}\binom{n+1}{n-k-l},$$

which is indeed equal to $l\cdot f_{k,l}(A_n,m)$. This proves (9).

Next we compute the sum on the left-hand side of (10),

$$\begin{aligned}
&\sum_{k_1=0}^{k}\frac{k-k_1+1}{k+1}\binom{n}{k}\binom{m(n+1)+k_1-1}{k_1}\\
&\qquad=\sum_{k_1=0}^{k}\binom{n}{k}\binom{m(n+1)+k_1-1}{k_1}-\sum_{k_1=0}^{k}\frac{m(n+1)}{k+1}\binom{n}{k}\binom{m(n+1)+k_1-1}{k_1-1}\\
&\qquad=\binom{n}{k}\binom{m(n+1)+k}{m(n+1)}-\frac{m(n+1)}{k+1}\binom{n}{k}\binom{m(n+1)+k}{m(n+1)+1}\\
&\qquad=\frac{1}{k+1}\binom{n}{k}\binom{m(n+1)+k+1}{k},
\end{aligned}$$

the simplification of summations being due to the Chu–Vandermonde summation (see [14, Sec. 5.1, (5.27)] or [18, (1.7.7), Appendix (III.4)]). This completes the proof. $\square$

5. The F-triangle for B_n. The theorem below gives an explicit expression for the refined face numbers $f_{k,l}(B_n,m)$, and, thus, of the F-triangle in type B_n.

Theorem FB. *For $n\ge 1$, the face numbers $f_{k,l}(B_n,m)$ are given by*

$$f_{k,l}(B_n,m)=\binom{n}{k+l}\binom{mn+k-1}{k}.$$

Here we identify B_1 with A_1.

PROOF. By inspection, the formula for $f_{k,l}(B_1,m)$ given in the theorem agrees with the formula for $f_{k,l}(A_1,m)$ in Theorem FA. Hence, in view of Proposition F and (5) in Section 2, in order to prove this claim we have to show

$$l\cdot f_{k,l}(B_n,m)=\sum_{\substack{n_1+n_2=n-1\\k_1+k_2=k\\l_1+l_2=l-1}}f_{k_1,l_1}(B_{n_1},m)\cdot f_{k_2,l_2}(A_{n_2},m)\tag{12}$$

and

$$\sum_{k_1+l_1=k}f_{k_1,l_1}(B_n,m)=\binom{n}{k}\binom{mn+k}{k}.\tag{13}$$

The triple sum on the right-hand side of (12) is

$$\sum_{k_1,l_1,n_1\ge 0}\binom{n_1}{k_1+l_1}\binom{mn_1+k_1-1}{k_1}\\ \cdot\frac{l-l_1}{k-k_1+l-l_1}\binom{n-n_1-1}{k-k_1+l-l_1-1}\binom{m(n-n_1)+k-k_1-1}{k-k_1}.$$

We replace n_1 by $n_1+k_1+l_1$ and rewrite the resulting expression in the form

$$\sum_{k_1,l_1,n_1\ge 0}\binom{n_1+k_1+l_1}{n_1}\binom{m(n_1+k_1+l_1)+k_1-1}{k_1}\\ \cdot\frac{m(l-l_1)}{m(n-n_1-k_1-l_1)+k-k_1}\binom{n-n_1-k_1-l_1}{n-k-l-n_1}\\ \cdot\binom{m(n-n_1-k_1-l_1)+k-k_1}{k-k_1}. \qquad (14)$$

Forgetting the sum over l_1, this is now in the form of the left-hand side of (8) with n replaced by $n-k-l$, $a=m+1$, $c=m$, $\alpha=ml_1$, $\alpha'=m(l-l_1)$, $b=d=1$, $\beta=l_1+1$, $\beta'=l-l_1$. Substituting the right-hand side, we obtain

$$\sum_{l_1=0}^{l-1}\binom{mn+k-1}{k}\binom{n}{n-k-l}$$

for the sum in (14), or, equivalently,

$$l\cdot\binom{n}{k+l}\binom{mn+k-1}{k},$$

which is indeed equal to $l\cdot f_{k,l}(B_n,m)$. This proves (12).

Next we compute the sum on the left-hand side of (13),

$$\sum_{k_1=0}^{k}\binom{n}{k}\binom{mn+k_1-1}{k_1}=\binom{n}{k}\binom{mn+k}{k},$$

the simplification of summation being due to the Chu–Vandermonde summation. This completes the proof. $\square$

6. The F-triangle for D_n. The theorem below gives an explicit expression for the refined face numbers $f_{k,l}(D_n,m)$, and, thus, of the F-triangle in type D_n.

Theorem FD. *For $n \ge 2$, the face numbers $f_{k,l}(D_n, m)$ are given by*

$$f_{k,l}(D_n, m) = \binom{n}{k+l}\binom{m(n-1)+k-1}{k} + m\binom{n-1}{k+l-1}\binom{m(n-1)+k-2}{k-1}$$
$$- \delta_{l,0}\frac{1}{n-1}\binom{n-1}{k-1}\binom{m(n-1)+k-1}{k},$$

where $\delta_{l,0}$ is the Kronecker delta, that is, it is equal to 1 if $l = 0$, and it is equal to 0 otherwise. Here we identify D_2 with A_1^2, and we identify D_3 with A_3.

PROOF. By inspection, for $n = 2$ the formula for $f_{k,l}(D_2, m)$ given in the theorem yields for the F-triangle

$$F^m_{D_2}(x, y) = \sum_{k,l \ge 0} f_{k,l}(D_2, m)\, x^k y^l = m^2x^2 + 2mxy + y^2 + 2mx + 2y + 1 = (mx + y + 1)^2,$$

which, according to Theorem FA, is indeed the F-triangle of A_1^2. Furthermore, again by inspection, the formula for $f_{k,l}(B_3, m)$ given in the theorem agrees with the formula for $f_{k,l}(A_3, m)$ in Theorem FA. Hence, in view of Proposition F and (5) in Section 2, in order to prove this claim we have to show

$$l \cdot f_{k,l}(D_n, m) = \sum_{\substack{n_1+n_2=n-1,\ n_1\ge 2\\ k_1+k_2=k\\ l_1+l_2=l-1}} f_{k_1,l_1}(D_{n_1}, m) \cdot f_{k_2,l_2}(A_{n_2}, m) + 2 \cdot f_{k,l-1}(A_{n-1}, m) \tag{15}$$

and

$$\sum_{k_1+l_1=k} f_{k_1,l_1}(D_n, m) = \binom{n}{k}\binom{m(n-1)+k}{k} + \binom{n-2}{k-2}\binom{m(n-1)+k-1}{k}. \tag{16}$$

We start with the proof of (15). Clearly, it suffices to consider the case $l \ge 1$ because (15) is trivially true for $l = 0$. We shall therefore assume $l \ge 1$ from now on.

Using the rewriting

$$\frac{l+1}{k+l+1}\binom{n}{k+l}\binom{m(n+1)+k-1}{k} = \frac{m(l+1)}{m(n+1)+k}\binom{n+1}{n-k-l}\binom{m(n+1)+k}{k}$$

of the defining expression for $f_{k,l}(A_n, m)$ in Theorem FA, the expression on the right-hand side of (15) is

$$\sum_{l_1=0}^{l-1}\sum_{k_1=0}^{k}\sum_{n_1=2}^{n-1}\binom{n_1}{k_1+l_1}\binom{m(n_1-1)+k_1-1}{k_1}$$

$$\cdot \frac{m(l-l_1)}{m(n-n_1)+k-k_1}\binom{n-n_1}{n-n_1-k+k_1-l+l_1}\binom{m(n-n_1)+k-k_1}{k-k_1} \tag{17a}$$

$$+m\sum_{l_1=0}^{l-1}\sum_{k_1=0}^{k}\sum_{n_1=2}^{n-1}\binom{n_1-1}{k_1+l_1-1}\binom{m(n_1-1)+k_1-2}{k_1-1}$$
$$\cdot \frac{m(l-l_1)}{m(n-n_1)+k-k_1}\binom{n-n_1}{n-n_1-k+k_1-l+l_1}\binom{m(n-n_1)+k-k_1}{k-k_1} \tag{17b}$$

$$-\sum_{k_1=0}^{k}\sum_{n_1=2}^{n-1}\frac{1}{n_1-1}\binom{n_1-1}{k_1-1}\binom{m(n_1-1)+k_1-1}{k_1}$$
$$\cdot \frac{ml}{m(n-n_1)+k-k_1}\binom{n-n_1}{n-n_1-k+k_1-l}\binom{m(n-n_1)+k-k_1}{k-k_1} \tag{17c}$$

$$+\frac{2l}{k+l}\binom{n-1}{k+l-1}\binom{mn+k-1}{k}. \tag{17d}$$

We treat the three sums in (17) separately. We begin with the sum (17a). We extend the sum to all $n_1 \ge 0$. In order to do so, we must subtract the terms with $n_1 = 0$ and $n_1 = 1$. If $n_1 = 0$, the summand is only non-zero for $k_1 = l_1 = 0$ because of the presence of the binomial $\binom{n_1}{k_1+l_1}$. If $n_1 = 1$, then, because of the presence of the binomial coefficient $\binom{m(n_1-1)+k_1-1}{k_1} = \binom{k_1-1}{k_1}$, the summand can be non-zero only if $k_1 = 0$. On the other hand, in that case, the summand can be only non-zero for $l_1 = 0$ and $l_1 = 1$, again because of the presence of the binomial $\binom{n_1}{k_1+l_1} = \binom{1}{l_1}$. In summary, the sum (17a) is equal to

$$\sum_{l_1=0}^{l-1}\sum_{k_1=0}^{k}\sum_{n_1=0}^{n-1}\left(\binom{n_1}{k_1+l_1}\binom{m(n_1-1)+k_1-1}{k_1}\right.$$
$$\left.\cdot \frac{m(l-l_1)}{m(n-n_1)+k-k_1}\binom{n-n_1}{n-n_1-k+k_1-l+l_1}\binom{m(n-n_1)+k-k_1}{k-k_1}\right)$$
$$-\frac{l}{k+l}\binom{n-1}{k+l-1}\binom{mn+k-1}{k}-\frac{l}{k+l}\binom{n-2}{k+l-1}\binom{m(n-1)+k-1}{k}$$
$$-\frac{l-1}{k+l-1}\binom{n-2}{k+l-2}\binom{m(n-1)+k-1}{k},$$

where the second-to-last term corresponds to the summand for $n_1 = k_1 = l_1 = 0$, the next-to-last term corresponds to the summand for $n_1 = 1$, $k_1 = 0$, $l_1 = 0$, and the last term corresponds to the summand for $n_1 = 1$, $k_1 = 0$, $l_1 = 1$. In the sum over n_1, k_1, l_1, we replace n_1 by $n_1+k_1+l_1$. Forgetting the sum over l_1, we see that it is then in the form of the left-hand side of (8) with n replaced by $n-k-l$, $a = m+1$, $c = m$, $\alpha = m(l_1-1)$,

$\alpha' = m(l-l_1)$, $b=d=1$, $\beta = l_1+1$, $\beta' = l-l_1$. Hence, if we substitute the right-hand side, the expression simplifies to

$$\sum_{l_1=0}^{l-1} \binom{n}{k+l}\binom{m(n-1)+k-1}{k} - \frac{l}{k+l}\binom{n-1}{k+l-1}\binom{mn+k-1}{k} \\ - \frac{l}{k+l}\binom{n-2}{k+l-1}\binom{m(n-1)+k-1}{k} - \frac{l-1}{k+l-1}\binom{n-2}{k+l-2}\binom{m(n-1)+k-1}{k}. \tag{18}$$

Clearly, the sum over l_1 sums the same summand for each l_1, so that the result is that summand multiplied by l.

We next turn our attention to the sum (17b). The first observation is that for $k_1 = 0$ the summand vanishes because of the presence of the binomial coefficient $\binom{m(n_1-1)+k_1-2}{k_1-1}$. We therefore replace k_1 by k_1+1 to obtain

$$m\sum_{l_1=0}^{l-1}\sum_{k_1=0}^{k-1}\sum_{n_1=2}^{n-1} \binom{n_1-1}{k_1+l_1}\binom{m(n_1-1)+k_1-1}{k_1} \\ \cdot \frac{m(l-l_1)}{m(n-n_1)+k-k_1-1}\binom{n-n_1}{n-n_1-k+k_1-l+l_1+1}\binom{m(n-n_1)+k-k_1-1}{k-k_1-1}.$$

This time we extend the sum to $n_1 \ge 1$. In order to do so, we must subtract the terms with $n_1 = 1$. In the latter case, because of the presence of the binomial coefficient $\binom{n_1-1}{k_1+l_1}$, the summand will vanish except if $k_1 = l_1 = 0$. Thus, we obtain

$$m\sum_{l_1=0}^{l-1}\sum_{k_1=0}^{k-1}\sum_{n_1=1}^{n-1} \Bigg(\binom{n_1-1}{k_1+l_1}\binom{m(n_1-1)+k_1-1}{k_1} \\ \cdot \frac{m(l-l_1)}{m(n-n_1)+k-k_1-1}\binom{n-n_1}{n-n_1-k+k_1-l+l_1+1}\binom{m(n-n_1)+k-k_1-1}{k-k_1-1}\Bigg) \\ - \frac{ml}{k+l-1}\binom{n-2}{k+l-2}\binom{m(n-1)+k-2}{k-1}$$

for (17b). In the triple sum, we replace n_1 by $n_1+k_1+l_1+1$. Forgetting the sum over l_1, we see that it is then in the form of the left-hand side of (8) with n replaced by $n-k-l$, $a = m+1$, $c = m$, $\alpha = ml_1$, $\alpha' = m(l-l_1)$, $b = d = 1$, $\beta = l_1+1$, $\beta' = l-l_1$, and k replaced by $k-1$. Hence, if we substitute the right-hand side, the expression simplifies to

$$m\sum_{l_1=0}^{l-1} \binom{n-1}{k+l-1}\binom{m(n-1)+k-2}{k-1} - \frac{ml}{k+l-1}\binom{n-2}{k+l-2}\binom{m(n-1)+k-2}{k-1}. \tag{19}$$

Also here, the sum over l_1 sums the same summand for each l_1, so that the result is that summand multiplied by l.

Finally we treat the sum (17c). Again, we want to extend the sum over n_1 to $n_1 \geq 0$. In order to do so, we would have to subtract the terms for $n_1 = 1$ and $n_1 = 0$. However, it is somewhat unclear which values we should give the summand for these choices of n_1. To obtain a partial answer, we rewrite

$$\frac{1}{n_1-1}\binom{n_1-1}{k_1-1}\binom{m(n_1-1)+k_1-1}{k_1} = \frac{m}{m(n_1-1)+k_1}\binom{n_1-1}{n_1-k_1}\binom{m(n_1-1)+k_1}{k_1}. \tag{20}$$

(This rewriting is already in the spirit of the forth-coming application of Carlitz's identity (7). We alert the reader that, according to our convention (6), the rewriting $\binom{n_1-1}{k_1-1} = \binom{n_1-1}{n_1-k_1}$ is without problem as long as $n_1 \geq 1$, which is the case in (17c). However, it becomes wrong if $1 > n_1 \geq k_1$, in which case $\binom{n_1-1}{k_1-1} = 0$ while $\binom{n_1-1}{n_1-k_1} \neq 0$. In the following considerations, whenever we talk about cases where $1 > n_1 \geq k_1$, we shall talk about the *right-hand side* in (20).) If $n_1 = 0$, then, because of the presence of the binomial coefficient $\binom{n_1-1}{n_1-k_1}$, the summand is only non-zero if $k_1 = 0$, in which case it equals

$$-\frac{ml}{mn+k}\binom{n}{n-k-l}\binom{mn+k}{k} = -\frac{l}{k+l}\binom{n-1}{k+l-1}\binom{mn+k-1}{k}.$$

For $n_1 = 1$, the above expression vanishes certainly if $k_1 > 1$. If $k_1 = 1$, it is equal to m. But if $k_1 = 0$, it is still not clear which value to assign to it. Leaving this question open for the moment, the arguments so far show that the expression (17c) is equal to

$$\begin{aligned}
-\sum_{k_1=0}^{k}\sum_{n_1=0}^{n-1}\Bigg(&\frac{m}{m(n_1-1)+k_1}\binom{n_1-1}{n_1-k_1}\binom{m(n_1-1)+k_1}{k_1}\\
&\cdot\frac{ml}{m(n-n_1)+k-k_1}\binom{n-n_1}{n-n_1-k+k_1-l}\binom{m(n-n_1)+k-k_1}{k-k_1}\Bigg)\\
&-\frac{l}{k+l}\binom{n-1}{k+l-1}\binom{mn+k-1}{k}+\frac{ml}{k+l-1}\binom{n-2}{k+l-2}\binom{m(n-1)+k-2}{k-1}\\
&\qquad+(\text{summand for } n_1=1,\ k_1=0).
\end{aligned}$$

We now replace n_1 by $n_1 + k_1$ in the double sum. This leads to the expression

$$\begin{aligned}
-\sum_{n_1,k_1\geq 0}\Bigg(&\frac{m}{m(n_1+k_1-1)+k_1}\binom{n_1+k_1-1}{n_1}\binom{m(n_1+k_1-1)+k_1}{k_1}\\
&\cdot\frac{ml}{m(n-n_1-k_1)+k-k_1}\binom{n-n_1-k_1}{n-n_1-k-l}\binom{m(n-n_1-k_1)+k-k_1}{k-k_1}\Bigg)\\
&-\frac{l}{k+l}\binom{n-1}{k+l-1}\binom{mn+k-1}{k}+\frac{ml}{k+l-1}\binom{n-2}{k+l-2}\binom{m(n-1)+k-2}{k-1}\\
&\qquad+(\text{summand for } n_1=1,\ k_1=0).
\end{aligned}$$

The double sum is now exactly equal to the negative of the left-hand side of (7) with n replaced by $n-k-l$, $a=m+1$, $c=m$, $\alpha=-m$, $\alpha'=ml$, $b=d=1$, $\beta=-1$, $\beta'=l$. From there, we can also determine the missing value of the summand for $n_1=1$ and $k_1=0$. Namely, we have

$$A_{0,1}(\alpha,\beta)=\beta=-1.$$

Thus, if we substitute the right-hand side of (7), we obtain

$$\begin{aligned}\frac{l-1}{k+l-1}\binom{n-2}{k+l-2}\binom{m(n-1)+k-1}{k}-\frac{l}{k+l}\binom{n-1}{k+l-1}\binom{mn+k-1}{k}\\+\frac{ml}{k+l-1}\binom{n-2}{k+l-2}\binom{m(n-1)+k-2}{k-1}+\frac{l}{k+l}\binom{n-2}{k+l-1}\binom{m(n-1)+k-1}{k}\end{aligned}\tag{21}$$

for (17c).

Adding the expressions (18), (19), (21) and (17d), we obtain that the sum in (17) is equal to

$$l\binom{n}{k+l}\binom{m(n-1)+k-1}{k}+ml\binom{n-1}{k+l-1}\binom{m(n-1)+k-2}{k-1},$$

which is indeed equal to $l\cdot f_{k,l}(D_n,m)$ if $l\ge 1$. This proves (15).

Next we compute the sum on the left-hand side of (16),

$$\begin{aligned}\sum_{k_1=0}^{k}\binom{n}{k}\binom{m(n-1)+k_1-1}{k_1}+m\sum_{k_1=0}^{k}\binom{n-1}{k-1}\binom{m(n-1)+k_1-2}{k_1-1}\\-\frac{1}{n-1}\binom{n-1}{k-1}\binom{m(n-1)+k-1}{k}\\=\binom{n}{k}\binom{m(n-1)+k}{k}+m\binom{n-1}{k-1}\binom{m(n-1)+k-1}{k-1}\\-\frac{1}{n-1}\binom{n-1}{k-1}\binom{m(n-1)+k-1}{k}\\=\binom{n}{k}\binom{m(n-1)+k}{k}+\binom{n-2}{k-2}\binom{m(n-1)+k-1}{k},\end{aligned}$$

the simplification of summations being due to the Chu–Vandermonde summation. This completes the proof. $\square$

7. The F-triangle in the exceptional cases. It is a routine matter to use Proposition F in Section 2 (and a computer algebra package) to find the F-triangles for the exceptional root systems. We list our findings below.

The F-triangle for $I_2(a)$:

$$F^m_{I_2(a)}(x,y) = \frac{m(ma+a-2)}{2}x^2 + 2mxy + amx + y^2 + 2y + 1. \tag{22}$$

The F-triangle for H_3:

$$\begin{aligned} F^m_{H_3}(x,y) = {} & \frac{m(5m+2)(5m+4)}{3}x^3 + m(5m+2)x^2y \\ & + 5m(5m+2)x^2 + 3mxy^2 + 10mxy + 15mx + y^3 + 3y^2 + 3y + 1. \end{aligned} \tag{23}$$

The F-triangle for H_4:

$$\begin{aligned} F^m_{H_4}(x,y) = {} & \frac{m(3m+1)(5m+3)(15m+14)}{4}x^4 + m(3m+1)(5m+3)x^3y \\ & + 15m(3m+1)(5m+3)x^3 + \frac{1}{2}m(17m+5)x^2y^2 + m(45m+14)x^2y \\ & + \frac{1}{2}m(465m+149)x^2 + 4mxy^3 + 17mxy^2 + 31mxy + 60mx + y^4 + 4y^3 + 6y^2 + 4y + 1. \end{aligned} \tag{24}$$

The F-triangle for F_4:

$$\begin{aligned} F^m_{F_4}(x,y) = {} & \frac{m(2m+1)(3m+1)(6m+5)}{2}x^4 + 2m(2m+1)(3m+1)x^3y \\ & + 12m(2m+1)(3m+1)x^3 + 2m(4m+1)x^2y^2 + 2m(18m+5)x^2y + m(78m+23)x^2 \\ & + 4mxy^3 + 16mxy^2 + 26mxy + 24mx + y^4 + 4y^3 + 6y^2 + 4y + 1. \end{aligned} \tag{25}$$

The F-triangle for E_6:

$$\begin{aligned} F^m_{E_6}(x,y) = {} & \frac{1}{30}m(2m+1)(3m+1)(4m+1)(6m+5)(12m+7)x^6 \\ & + \frac{1}{5}m(2m+1)(3m+1)(4m+1)(12m+7)x^5y + \frac{6}{5}m(2m+1)(3m+1)(4m+1)(12m+7)x^5 \\ & + \frac{1}{2}m(3m+1)(4m+1)(8m+3)x^4y^2 + 2m(3m+1)(4m+1)(12m+5)x^4y \\ & + 2m(3m+1)(4m+1)(30m+13)x^4 + \frac{5}{3}m(4m+1)(5m+1)x^3y^3 \\ & + m(4m+1)(48m+11)x^3y^2 + m(4m+1)(120m+31)x^3y + 9m(4m+1)(18m+5)x^3 \\ & + \frac{5}{2}m(7m+1)x^2y^4 + 5m(20m+3)x^2y^3 + m(242m+39)x^2y^2 + 3m(108m+19)x^2y \\ & + 12m(21m+4)x^2 + 6mxy^5 + 35mxy^4 + 85mxy^3 + 111mxy^2 + 84mxy + 36mx \\ & + y^6 + 6y^5 + 15y^4 + 20y^3 + 15y^2 + 6y + 1. \end{aligned} \tag{26}$$

The F-triangle for E_7:

$$\begin{aligned}
F^m_{E_7}(x,y) &= \frac{1}{280}m(3m+1)(3m+2)(9m+2)(9m+4)(9m+5)(9m+8)x^7\\
&+\frac{9}{40}m(3m+1)(3m+2)(9m+2)(9m+4)(9m+5)x^6+\frac{1}{40}m(3m+1)(3m+2)(9m+2)(9m+4)(9m+5)x^6y\\
&+\frac{3}{40}m(3m+1)(7m+3)(9m+2)(9m+4)x^5y^2+\frac{3}{20}m(3m+1)(9m+2)(9m+4)(27m+13)x^5y\\
&+\frac{3}{40}m(3m+1)(9m+2)(9m+4)(207m+103)x^5+\frac{1}{8}m(3m+1)(9m+2)(27m+7)x^4y^3\\
&+\frac{3}{8}m(3m+1)(9m+2)(63m+19)x^4y^2+\frac{3}{8}m(3m+1)(9m+2)(207m+71)x^4y\\
&+\frac{21}{8}m(3m+1)(9m+2)(63m+23)x^4+m(6m+1)(9m+2)x^3y^4+\frac{3}{2}m(9m+2)(27m+5)x^3y^3\\
&+\frac{3}{2}m(9m+2)(81m+17)x^3y^2+\frac{21}{2}m(9m+2)(21m+5)x^3y+\frac{21}{2}m(9m+2)(27m+7)x^3+3m(8m+1)x^2y^5\\
&+3m(54m+7)x^2y^4+\frac{3}{2}m(315m+43)x^2y^3+\frac{21}{2}m(75m+11)x^2y^2+\frac{21}{2}m(81m+13)x^2y+\frac{21}{2}m(63m+11)x^2\\
&+7mxy^6+48mxy^5+141mxy^4+231mxy^3+231mxy^2+147mxy+63mx\\
&+y^7+7y^6+21y^5+35y^4+35y^3+21y^2+7y+1. \qquad (27)
\end{aligned}$$

The F-triangle for E_8:

$$\begin{aligned}
F^m_{E_8}(x,y) &= \frac{m(3m+1)(5m+1)(5m+2)(5m+3)(15m+8)(15m+11)(15m+14)x^8}{1344}\\
&+\frac{1}{168}m(3m+1)(5m+1)(5m+2)(5m+3)(15m+8)(15m+11)x^7y\\
&+\frac{5}{56}m(3m+1)(5m+1)(5m+2)(5m+3)(15m+8)(15m+11)x^7\\
&+\frac{1}{48}m(3m+1)(5m+1)(5m+2)(15m+7)(15m+8)x^6y^2\\
&+\frac{5}{24}m(3m+1)(5m+1)(5m+2)(15m+8)^2x^6y+\frac{5}{48}m(3m+1)(5m+1)(5m+2)(15m+8)(195m+107)x^6\\
&+\frac{1}{3}m(3m+1)(5m+1)(5m+2)(10m+3)x^5y^3+\frac{5}{8}m(3m+1)(5m+1)(5m+2)(45m+16)x^5y^2\\
&+\frac{25}{8}m(3m+1)(5m+1)(5m+2)(39m+16)x^5y+15m(3m+1)(5m+1)(5m+2)(30m+13)x^5\\
&+\frac{1}{6}m(5m+1)(10m+3)(19m+4)x^4y^4+m(5m+1)(10m+3)(25m+6)x^4y^3\\
&+\frac{1}{4}m(5m+1)\left(3675m^2+2125m+308\right)x^4y^2+m(5m+1)\left(2250m^2+1395m+218\right)x^4y\\
&+\frac{1}{2}m(5m+1)\left(10350m^2+6675m+1084\right)x^4+\frac{7}{3}m(5m+1)(7m+1)x^3y^5\\
&+\frac{1}{3}m(5m+1)(380m+59)x^3y^4+\frac{1}{3}m(5m+1)(1315m+226)x^3y^3+m(5m+1)(915m+178)x^3y^2\\
&+m(5m+1)(1380m+307)x^3y+45m(5m+1)(45m+11)x^3+\frac{7}{2}m(9m+1)x^2y^6+7m(35m+4)x^2y^5\\
&+\frac{1}{2}m(1675m+199)x^2y^4+4m(415m+52)x^2y^3+\frac{1}{2}m(4295m+579)x^2y^2+75m(27m+4)x^2y\\
&+\frac{35}{2}m(105m+17)x^2+8mxy^7+63mxy^6+217mxy^5+428mxy^4+532mxy^3+435mxy^2+245mxy+120mx\\
&+y^8+8y^7+28y^6+56y^5+70y^4+56y^3+28y^2+8y+1. \qquad (28)
\end{aligned}$$

8. The $F = M$ Conjecture. In order to state the "$F = M$ Conjecture" for generalised cluster complexes, we need to first introduce Armstrong's [1] m-divisible non-crossing partition posets.

Given a root system Φ and an element $\alpha \in \Phi$, let t_α denote the corresponding reflection in the central hyperplane perpendicular to α. Let $W = W(\Phi)$ be the group generated by these reflections. By definition, any element w of W can be represented as a product $w = t_1 t_2 \cdots t_\ell$, where the t_i's are reflections. We call the minimal number of reflections which is needed for such a product representation the *absolute length* of w, and we denote it by $\ell_T(w)$. We then define the *absolute order* on W, denoted by $\le_T$, by

$$u \le_T w \quad \text{if and only if} \quad \ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w).$$

It can be shown that this is equivalent to the statement that any shortest product representation of u by reflections occurs as an initial segment in some shortest product representation of w by reflections.

We can now define the *non-crossing partition lattice* $NC(\Phi)$. Let c be a *Coxeter element* in W, that is, the product of all reflections corresponding to the simple roots. Then $NC(\Phi)$ is defined to be the restriction of the partial order $\le_T$ to the set of all elements which are less than or equal to c in absolute order. This definition makes sense because, regardless of the chosen Coxeter element c, the resulting poset is always the same up to isomorphism. It can be shown that $NC(\Phi)$ is indeed a lattice. (See [6] for a uniform proof.) The term "non-crossing partition lattice" is used because $NC(A_n)$ is isomorphic to the lattice of non-crossing partitions originally introduced by Kreweras [16], and because also $NC(B_n)$ and $NC(D_n)$ can be realized as lattices of non-crossing partitions (see [3, 17]).

The poset of *m-divisible non-crossing partitions* has as a groundset the following subset of $(NC(\Phi))^{m+1}$,

$$NC^m(\Phi) = \{(w_0; w_1, \dots, w_m) : w_0 w_1 \cdots w_m = c \text{ and} \\ \ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c)\}. \quad (29)$$

The order relation is defined by

$$(u_0; u_1, \dots, u_m) \le (w_0; w_1, \dots, w_m) \quad \text{if and only if} \quad u_i \ge_T w_i,\ 1 \le i \le m.$$

We emphasize that, according to this definition, u_0 and w_0 need not be related in any way. The poset $NC^m(\Phi)$ is graded by the rank function

$$\operatorname{rk}\big((w_0; w_1, \dots, w_m)\big) = \ell_T(w_0).$$

Thus, there is a unique maximal element, namely $(c; \varepsilon, \dots, \varepsilon)$, where ε stands for the identity element in W, but, if $m > 1$, there are many different minimal elements. In particular, there is no global minimum in $NC^m(\Phi)$ if $m > 1$ and, hence, $NC^m(\Phi)$ is not a lattice for $m > 1$. (It is, however, a graded join-semilattice, see [1, Theorem 2.2.7].) We remark that for $NC^m(A_n)$ and $NC^m(B_n)$ combinatorial realisations are available as subposets of non-crossing partitions in which each block has a size which is divisible by m. The corresponding translations are due to Armstrong [1, Sec. 3]. In type A_n, the resulting poset had been earlier studied by Edelman [9]. The analogous combinatorial realisation of $NC^m(D_n)$, generalising the one of Athanasiadis and Reiner [3] for $m = 1$, has not yet been worked out.

Next, we define the "M-triangle" of $NC^m(\Phi)$ as

$$M_\Phi^m(x,y) = \sum_{u,w\in NC^m(\Phi)} \mu(u,w)\, x^{\operatorname{rk} u} y^{\operatorname{rk} w},$$

where $\mu(u,w)$ is the Möbius function in $NC^m(\Phi)$.

The generalised version of Chapoton's (ex-)conjecture [8, Conjecture 1], due to Armstrong [1, Sec. 4], is the following.

Conjecture FM. *For any finite root system Φ of rank n, we have*

$$F_\Phi^m(x,y) = y^n\, M_\Phi^m\left(\frac{1+y}{y-x}, \frac{y-x}{y}\right).$$

Equivalently,

$$(1-xy)^n F_\Phi^m\left(\frac{x(1+y)}{1-xy}, \frac{xy}{1-xy}\right) = \sum_{u,w\in(NC^m(\Phi))^*} \mu^*(u,w)\,(-x)^{\operatorname{rk}^* w}(-y)^{\operatorname{rk}^* u}, \tag{30}$$

where $(NC^m(\Phi))^*$ denotes the poset dual *to $NC^m(\Phi)$ (i.e., the poset which arises from $NC^m(\Phi)$ by reversing all order relations), where μ^* denotes the Möbius function in $(NC^m(\Phi))^*$, and where* rk^* *denotes the rank function in $(NC^m(\Phi))^*$.*

Since the Möbius function is multiplicative (see e.g. [19, Prop. 3.8.2]), the multiplicativity property (2) for the F-triangle holds also for the M-triangle. Therefore, it is enough to prove the conjecture for the irreducible root systems. In Sections 9–17, we provide proofs for the root systems of type A_n, B_n, $I_2(a)$, H_3, H_4, F_4, and E_6, and a partial proof for the root system of type D_n.

9. Proof of the $F = M$ Conjecture for A_n. In this section we prove Conjecture FM for the type A_n. In the spirit of this paper, we follow a computational approach. We first simplify the left-hand side of (30) by a double application of the Chu–Vandermonde summation. Subsequently, we compute the right-hand side of (30) by relying on a result on rank selected chain enumeration in the m-divisible non-crossing partition lattice in type A_n due to Edelman [9].

The link between chain enumeration and the Möbius function is the following. (The reader should consult [19, Sec. 3.11] for more information on this topic.) Given a poset P and two elements u and w, $u \le w$, in the poset, the *zeta polynomial* of the interval $[u,w]$, denoted by $Z(u,w;z)$, is the number of (multi)chains from u to w of length z. (It can be shown that this is indeed a polynomial in z.) Then the Möbius function of u and w is equal to $\mu(u,w) = Z(u,w;-1)$.

Proposition A. *In type A_n, the left-hand side of* (30) *is equal to*

$$\sum_{r,s\ge 0} x^s y^r \frac{1}{s+1}\binom{n}{s}\binom{m(n+1)}{r}\binom{m(n+1)+s-r-1}{s-r}. \tag{31}$$

PROOF. By definition of $F^m_{A_n}(x,y)$, and by Theorem FA in Section 4, the left-hand side of (30) in type A_n is equal to

$$\sum_{k,l,r,s\ge 0} \frac{l+1}{k+l+1}\binom{n}{k+l}\binom{m(n+1)+k-1}{k}\binom{k}{r}\binom{n-k-l}{s}(-1)^s x^{k+l+s} y^{l+r+s}.$$

Fixing $L=s+l$, we rewrite this as

$$\sum_{k,L,r\ge 0} \frac{n!\,(m(n+1)+k-1)!}{(m(n+1)-1)!\,r!\,(k-r)!\,(n-k-L)!\,(k+L+1)!} x^{k+L} y^{r+L} \\ \cdot \sum_{s=0}^{L}(L-s+1)(-1)^s\binom{k+L+1}{s}.$$

We compute the sum over s by the Chu–Vandermonde summation. Thus, we arrive at

$$\sum_{k,L,r\ge 0} \frac{n!\,(m(n+1)+k-1)!}{(m(n+1)-1)!\,r!\,(k-r)!\,(n-k-L)!\,(k+L+1)!} x^{k+L} y^{r+L}(-1)^L\binom{k+L-1}{L}.$$

We now write $K=k+L$ and $R=r+L$. Subsequently, the sum over L can be computed using the Chu–Vandermonde summation. The result is

$$\sum_{K,R\ge 0} \frac{n!\,(m(n+1)+K-R-1)!\,(m(n+1))!}{(m(n+1)-1)!\,(m(n+1)-R)!\,R!\,(K-R)!\,(n-K)!\,(K+1)!} x^K y^R.$$

Aside from a parameter replacement, this is exactly the expression (31). □

For the computation of the right-hand side of (30) we require the following theorem due to Edelman [9].

Theorem NA. *The number of chains in $(NC^m(A_n))^*$ with successive rank jumps $s_1, s_2, \dots, s_\ell$, $s_1+s_2+\dots+s_\ell=n$, is*

$$\binom{m(n+1)}{s_1}\cdots\binom{m(n+1)}{s_{\ell-1}}\frac{1}{n+1}\binom{n+1}{s_\ell}. \tag{32}$$

PROOF OF CONJECTURE FM IN TYPE A_n. We now compute the right-hand side of (30), that is,

$$\sum_{u,w\in(NC^m(A_n))^*} \mu^*(u,w)(-x)^{\operatorname{rk}^* w}(-y)^{\operatorname{rk}^* u}.$$

In order to compute the coefficient of $x^s y^r$ in this expression,

$$(-1)^{r+s} \sum_{\substack{u,w\in(NC^m(A_n))^* \\ \text{with } \mathrm{rk}^* u=r \text{ and } \mathrm{rk}^* w=s}} \mu^*(u,w),$$

we compute the sum of all corresponding zeta polynomials (in the variable z), multiplied by $(-1)^{r+s}$,

$$(-1)^{r+s} \sum_{\substack{u,w\in(NC^m(A_n))^* \\ \text{with } \mathrm{rk}^* u=r \text{ and } \mathrm{rk}^* w=s}} Z(u,w;z),$$

and then put $z=-1$.

For computing this sum of zeta polynomials, we must set $\ell = z+2$, $s_1 = r$, $n - s_\ell = s$, $s_2 + s_3 + \cdots + s_{\ell-1} = s - r$ in (32), and then sum the resulting expression over all possible $s_2, s_3, \dots, s_{\ell-1}$. By using the Chu–Vandermonde summation, one obtains

$$\frac{1}{n+1}\binom{m(n+1)}{r}\binom{zm(n+1)}{s-r}\binom{n+1}{n-s}.$$

If we put $z=-1$ in this expression and multiply it by $(-1)^{r+s}$, then we obtain exactly the coefficient of $x^s y^r$ in (31). $\square$

10. Proof of the $F=M$ Conjecture for B_n. In this section we prove Conjecture FM for the type B_n, by following the same approach as the one for type A_n in the previous section.

Proposition B. *In type B_n, the left-hand side of* (30) *is equal to*

$$\sum_{r,s\ge 0} x^s y^r \binom{n}{s}\binom{mn}{r}\binom{mn+s-r-1}{s-r}. \tag{33}$$

Proof. By definition of $F^m_{B_n}(x,y)$, and by Theorem FB in Section 5, the left-hand side of (30) in type B_n is equal to

$$\sum_{k,l,r,s\ge 0} \binom{n}{k+l}\binom{mn+k-1}{k}\binom{k}{r}\binom{n-k-l}{s}(-1)^s x^{k+l+s} y^{l+r+s}.$$

Fixing $L = s+l$, we rewrite this as

$$\sum_{k,L,r\ge 0} \frac{n!\,(mn+k-1)!}{(mn-1)!\,r!\,(k-r)!\,(n-k-L)!\,(k+L)!} x^{k+L} y^{L+r} \sum_{s=0}^{L} (-1)^s \binom{k+L}{s}.$$

We compute the sum over s by the Chu–Vandermonde summation. Thus, we arrive at

$$\sum_{k,L,r\ge 0} \frac{n!\,(mn+k-1)!}{(mn-1)!\,r!\,(k-r)!\,(n-k-L)!\,(k+L)!} x^{k+L} y^{L+r} (-1)^L \binom{k+L-1}{L}.$$

We now write $K = k + L$ and $R = r + L$. Subsequently, the sum over L can be computed using the Chu–Vandermonde summation. The result is

$$\sum_{K,R\ge 0} \frac{n!\,(mn+K-R-1)!\,(mn)!}{(mn-1)!\,(mn-R)!\,R!\,(K-R)!\,(n-K)!\,K!} x^K y^R.$$

Aside from a parameter replacement, this is exactly the expression (33). $\square$

For the computation of the right-hand side of (30) we require the following theorem due to Armstrong [1, Theorem 3.5.7].

Theorem NB. *The number of chains in $(NC^m(B_n))^*$ with successive rank jumps $s_1, s_2, \dots, s_\ell$, $s_1 + s_2 + \cdots + s_\ell = n$, is*

$$\binom{mn}{s_1} \cdots \binom{mn}{s_{\ell-1}} \binom{n}{s_\ell}. \tag{34}$$

Proof of Conjecture FM in type B_n. We now compute the right-hand side of (30), that is,

$$\sum_{u,w\in(NC^m(B_n))^*} \mu^*(u,w)(-x)^{\operatorname{rk}^* w}(-y)^{\operatorname{rk}^* u}.$$

In order to compute the coefficient of $x^s y^r$ in this expression,

$$(-1)^{r+s} \sum_{\substack{u,w\in(NC^m(B_n))^* \\ \text{with } \operatorname{rk}^* u=r \text{ and } \operatorname{rk}^* w=s}} \mu^*(u,w),$$

we compute the sum of all corresponding zeta polynomials (in the variable z), multiplied by $(-1)^{r+s}$,

$$(-1)^{r+s} \sum_{\substack{u,w\in(NC^m(B_n))^* \\ \text{with } \operatorname{rk}^* u=r \text{ and } \operatorname{rk}^* w=s}} Z(u,w;z),$$

and then put $z = -1$.

For computing this sum of zeta polynomials, we must set $\ell = z + 2$, $s_1 = r$, $n - s_\ell = s$, $s_2 + s_3 + \cdots + s_{\ell-1} = s - r$ in (34), and then sum the resulting expression over all possible $s_2, s_3, \dots, s_{\ell-1}$. By using the Chu–Vandermonde summation, one obtains

$$\binom{mn}{r}\binom{zmn}{s-r}\binom{n}{n-s}.$$

If we put $z = -1$ in this expression and multiply it by $(-1)^{r+s}$, then we obtain exactly the coefficient of $x^s y^r$ in (33). $\square$

11. Towards a proof of the $F = M$ Conjecture for D_n. This section exhibits how far the approach of the previous two sections of proving Conjecture FM in types A_n and B_n can take us in type D_n. The simplification of the left-hand side of (30) along the lines of the proofs of Propositions A and B goes through smoothly. The problem which we face in type D_n is that, up to this date, the rank selected chain enumeration result for $NC^m(D_n)$ has not been found yet. Thus, we do not have the means to compute the M-triangle for $NC^m(D_n)$. The only exception is for $m = 1$. Namely, for the (ordinary) non-crossing partition lattice $NC(D_n) = NC^1(D_n)$, Athanasiadis and Reiner [3] have done the rank selected chain enumeration as we need it in our application. Hence, we are able to prove Conjecture FM in type D_n if $m = 1$.

Proposition D. *In type D_n, the left-hand side of* (30) *is equal to*

$$\sum_{r,s\ge 0} x^s y^r \left(2\binom{n-1}{s-1}\binom{m(n-1)}{r}\binom{m(n-1)+s-r-1}{s-r} \right.$$
$$+\binom{n-2}{s}\binom{m(n-1)}{r}\binom{m(n-1)+s-r-1}{s-r}$$
$$+m\binom{n-1}{s-1}\binom{m(n-1)-1}{r-2}\binom{m(n-1)+s-r-1}{s-r}$$
$$\left. -m\binom{n-1}{s-1}\binom{m(n-1)}{r}\binom{m(n-1)+s-r-2}{s-r-2}\right). \tag{35}$$

Proof. By definition of $F^m_{D_n}(x,y)$, and by Theorem FD in Section 6, the left-hand side of (30) in type D_n is equal to

$$\sum_{k,l,r,s\ge 0}\binom{n}{k+l}\binom{m(n-1)+k-1}{k}\binom{k}{r}\binom{n-k-l}{s}(-1)^s x^{k+l+s}y^{l+r+s} \tag{36a}$$
$$+m\sum_{k,l,r,s\ge 0}\binom{n-1}{k+l-1}\binom{m(n-1)+k-2}{k-1}\binom{k}{r}\binom{n-k-l}{s}(-1)^s x^{k+l+s}y^{l+r+s} \tag{36b}$$
$$-\frac{1}{n-1}\sum_{k,r,s\ge 0}\binom{n-1}{k-1}\binom{m(n-1)+k-1}{k}\binom{k}{r}\binom{n-k}{s}(-1)^s x^{k+s}y^{r+s}. \tag{36c}$$

We treat the three sums in (36) separately. We begin with the sum (36a). Fixing $L = s+l$, we rewrite it as

$$\sum_{k,L,r\ge 0}\frac{n!\,(m(n-1)+k-1)!}{(m(n-1)-1)!\,r!\,(k-r)!\,(n-k-L)!\,(k+L)!}x^{k+L}y^{L+r}\sum_{s=0}^{L}(-1)^s\binom{k+L}{s}.$$

We compute the sum over s by the Chu–Vandermonde summation. Thus, we arrive at

$$\sum_{k,L,r\ge 0}\frac{n!\,(m(n-1)+k-1)!}{(m(n-1)-1)!\,r!\,(k-r)!\,(n-k-L)!\,(k+L)!}x^{k+L}y^{L+r}(-1)^L\binom{k+L-1}{L}.$$

We now write $K=k+L$ and $R=r+L$. Subsequently, the sum over L can be computed using the Chu–Vandermonde summation. The result is

$$\sum_{K,R\ge0}\frac{n!\,(m(n-1)+K-R-1)!\,(m(n-1))!}{(m(n-1)-1)!\,(m(n-1)-R)!\,R!\,(K-R)!\,(n-K)!\,K!}x^Ky^R. \tag{37}$$

Next we consider the sum (36b). Again fixing $L=s+l$, we rewrite it as

$$\sum_{k,L,r\ge0}\frac{mk\,(n-1)!\,(m(n-1)+k-2)!}{(m(n-1)-1)!\,r!\,(k-r)!\,(n-k-L)!\,(k+L-1)!}x^{k+L}y^{L+r}\\
\cdot\sum_{s=0}^{L}(-1)^s\binom{k+L-1}{s}.$$

We compute the sum over s by the Chu–Vandermonde summation. Thus, we arrive at

$$\sum_{k,L,r\ge0}\frac{mk\,(n-1)!\,(m(n-1)+k-2)!}{(m(n-1)-1)!\,r!\,(k-r)!\,(n-k-L)!\,(k+L-1)!}x^{k+L}y^{L+r}(-1)^L\binom{k+L-2}{L}.$$

We now write $K=k+L$ and $R=r+L$. Because of the presence of the factor k in the numerator, this makes a factor of $K-L$ appear. We split the sum into two parts accordingly, and then, in both parts, the sum over L can be computed using the Chu–Vandermonde summation. The result is

$$\sum_{K,R\ge0}\frac{mK\,(n-1)!\,(m(n-1)+K-R-2)!\,(m(n-1))!}{(m(n-1)-1)!\,(m(n-1)-R)!\,R!\,(K-R)!\,(n-K)!\,(K-1)!}x^Ky^R\\
-\sum_{K,R\ge0}\frac{m(K-2)\,(n-1)!\,(m(n-1)+K-R-2)!\,(m(n-1))!}{(m(n-1)-1)!\,(m(n-1)-R+1)!\,(R-1)!\,(K-R)!\,(n-K)!\,(K-1)!}x^Ky^R. \tag{38}$$

Finally, we turn to the sum (36c). We write $K=k+s$ and $R=r+s$. Subsequently, the sum over s can be computed using the Chu–Vandermonde summation. The result is

$$-\sum_{K,R\ge0}\frac{1}{n-1}\frac{(n-1)!\,(m(n-1)+K-R-1)!\,(m(n-1))!}{(m(n-1)-1)!\,(m(n-1)-R)!\,R!\,(K-R)!\,(n-K)!\,(K-1)!}x^Ky^R. \tag{39}$$

To complete the proof, we add the expressions (37), (38) and (39). Doing the parameter replacements $K\to s$, $R\to r$ and minor rewriting, this leads to the expression

$$\sum_{r,s\ge0}x^sy^r\Bigg(\binom{n}{s}\binom{m(n-1)}{r}\binom{m(n-1)+s-r-1}{s-r}\\
+\frac{ms}{s-r}\binom{n-1}{s-1}\binom{m(n-1)}{r}\binom{m(n-1)+s-r-2}{s-r-1}\\
+\frac{m(s-2)}{s-r}\binom{n-1}{s-1}\binom{m(n-1)}{r-1}\binom{m(n-1)+s-r-2}{s-r-1}\\
-\frac{1}{n-1}\binom{n-1}{s-1}\binom{m(n-1)}{r}\binom{m(n-1)+s-r-1}{s-r}\Bigg).$$

It is now a routine verification to show that an alternative way to write this is (35). $\square$

For the computation of the right-hand side of (30) in the case that $m = 1$, we require the following result due to Athanasiadis and Reiner [3, Theorem 1.2(ii)].

Theorem ND. *The number of chains in $NC(D_n)$ with successive rank jumps $s_1, s_2, \dots, s_\ell$, $s_1 + s_2 + \cdots + s_\ell = n$, is given by*

$$2\binom{n-1}{s_1}\cdots\binom{n-1}{s_\ell} + \sum_{i=1}^{\ell}\binom{n-1}{s_1}\cdots\binom{n-2}{s_i-2}\cdots\binom{n-1}{s_\ell}. \tag{40}$$

Proof of Conjecture FM for $m = 1$ in type D_n. If $m = 1$, then $NC^m(D_n)$ reduces to the ordinary non-crossing partition lattice $NC(D_n)$, which is self-dual, that is, $(NC(D_n))^* = NC(D_n)$. Hence, the right-hand side of (30) with $m = 1$ in type D_n is equal to

$$\sum_{u,w\in NC(D_n)} \mu(u,w)(-x)^{\operatorname{rk} w}(-y)^{\operatorname{rk} u}.$$

In order to compute the coefficient of $x^s y^r$ in this expression,

$$(-1)^{r+s} \sum_{\substack{u,w\in NC(D_n)\\ \text{with } \operatorname{rk} u=r \text{ and } \operatorname{rk} w=s}} \mu(u,w),$$

we compute the sum of all corresponding zeta polynomials (in the variable z), multiplied by $(-1)^{r+s}$,

$$(-1)^{r+s} \sum_{\substack{u,w\in NC(D_n)\\ \text{with } \operatorname{rk} u=r \text{ and } \operatorname{rk} w=s}} Z(u,w;z),$$

and then put $z = -1$.

For computing this sum of zeta polynomials, we must set $\ell = z + 2$, $s_1 = r$, $n - s_\ell = s$, $s_2 + s_3 + \cdots + s_{\ell-1} = s - r$ in (40), and then sum the resulting expression over all possible $s_2, s_3, \dots, s_{\ell-1}$. By using the Chu–Vandermonde summation again, one obtains

$$2\binom{n-1}{r}\binom{z(n-1)}{s-r}\binom{n-1}{n-s} + \binom{n-2}{r-2}\binom{z(n-1)}{s-r}\binom{n-1}{n-s}\\ + \binom{n-1}{r}\binom{z(n-1)}{s-r}\binom{n-2}{n-s-2} + z\binom{n-1}{r}\binom{z(n-1)-1}{s-r-2}\binom{n-1}{n-s}.$$

If we put $z = -1$ in this expression and multiply it by $(-1)^{r+s}$, then we obtain exactly the coefficient of $x^s y^r$ in (35) with $m = 1$. $\square$

12. How to prove the $F = M$ Conjecture in the exceptional cases. While, at first sight, for a given exceptional root system Φ, it seems that computing the M-triangle (respectively the right-hand side of (30)) for *arbitrary* m is an infinite problem because we have to compute Möbius functions for $NC^m(\Phi)$ (respectively for $(NC^m(\Phi))^*$) for $m = 1, 2, \dots$, this is not really true. We should recall from (29) that an element of $NC^m(\Phi)$ has the form

$$(w_0; w_1, \dots, w_m), \text{ with } w_0 w_1 \cdots w_m = c \text{ and } \ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c) = n, \tag{41}$$

where n is the rank of the root system Φ. Now, n can be at most 8 for an exceptional root system (with equality only for $\Phi = E_8$). This implies that only at most 8 of the w_i's can be different from the identity element ε in $W = W(\Phi)$. Hence, a typical interval in $(NC^m(\Phi))^*$ looks like $[u, w]$, where

$$u = (u_0; u_1, \dots, u_m), \quad w = (w_0; w_1, \dots, w_m),$$

$u_i = w_i = \varepsilon$ for all but at most 8 indices $i \geq 1$, and $u_i \leq w_i$ for these remaining indices. Let these latter indices be $i_1, i_2, \dots, i_d$, with $d \leq 8$. Then, such an interval $[u, w]$ is isomorphic to the "compressed" interval $[u', w']$, where

$$u' = (u_0; u_{i_1}, \dots, u_{i_d}), \quad w' = (w_0; w_{i_1}, \dots, w_{i_d}).$$

Note that "compressed" means that all of $w_{i_1}, w_{i_2}, \dots, w_{i_d}$ are different from ε.

So, what we have to do is to determine all different *compressed* intervals $[u', w']$. The contribution of a compressed interval $[u', w']$ to the right-hand side of (30) is then

$$\binom{m}{d} \cdot \mu^*(u', w')\, x^{\operatorname{rk}^* w'} y^{\operatorname{rk}^* u'}, \tag{42}$$

because there are $\binom{m}{d}$ different ways to choose $\{i_1, i_2, \dots, i_d\}$ out of $\{1, 2, \dots, m\}$. To obtain the M-triangle we "just" have to collect all these contributions and sum them over all possible compressed intervals. Note that this is now a *finite* problem because the number of *compressed* intervals is finite.

Rather than running through all compressed intervals, a more efficient way to implement this is as follows. We rewrite the right-hand side of (30) as

$$\sum_{u,w \in (NC^m(\Phi))^*} \mu^*(u, w)\, (-x)^{\operatorname{rk}^* w} (-y)^{\operatorname{rk}^* u} = \sum_{w \in (NC^m(\Phi))^*} (-x)^{\operatorname{rk}^* w} \cdot \chi^*_{[\hat{0}, w]}(-y), \tag{43}$$

where $\hat{0} = (c; \varepsilon, \dots, \varepsilon)$ is the minimum in $(NC^m(\Phi))^*$, and where

$$\chi^*_{[\hat{0}, w]}(y) = \sum_{u \in (NC^m(\Phi))^*} \mu^*(u, w)\, y^{\operatorname{rk}^* u}$$

is, essentially, the characteristic polynomial of the interval $[\hat{0}, w]$. (To be precise, it is the characteristic polynomial of the interval $[w, \hat{0}]$ in $NC^m(\Phi)$, see [19, Sec. 3.10].) If $w = (w_0; w_1, \dots, w_m)$ with $w_{i_1}, w_{i_2}, \dots, w_{i_d}$ those among $w_1, w_2, \dots, w_m$ which are different from the identity element ε, then

$$[\hat{0}, w] \cong [\varepsilon, w_{i_1}] \times [\varepsilon, w_{i_2}] \times \cdots \times [\varepsilon, w_{i_d}],$$

where each interval $[\varepsilon, w_{i_j}]$ is an interval in $NC(\Phi)$. Since the characteristic polynomial is multiplicative, this implies

$$\chi^*_{[\hat{0},w]}(y) = \chi^*_{[\varepsilon,w_{i_1}]}(y)\chi^*_{[\varepsilon,w_{i_2}]}(y) \cdots \chi^*_{[\varepsilon,w_{i_d}]}(y),$$

where $\chi^*_{[\varepsilon,w_{i_j}]}(y) = \sum_{v \in NC(\Phi)} \mu(v, w_{i_j}) y^{\operatorname{rk} v}$, with μ the Möbius function and rk the rank function in $NC(\Phi)$.

According to a result by Bessis [4, Lemma 1.4.3, Cor. 1.6.2], each element w_{i_j} is some parabolic Coxeter element (that is, a Coxeter element in some parabolic subgroup), and the interval $[\varepsilon, w_{i_j}]$ is isomorphic to some $NC(\Psi)$, where Ψ is the root system of this parabolic subgroup.

If we put all this together, then (43) becomes

$$\sum_{d=0}^{n} \sum_{(T_1,\dots,T_d)} (-x)^{\operatorname{rk} T_1 + \cdots + \operatorname{rk} T_d} \cdot N_\Phi(T_1, T_2, \dots, T_d) \\ \cdot \chi^*_{NC(T_1)}(-y)\chi^*_{NC(T_2)}(-y) \cdots \chi^*_{NC(T_d)}(-y) \binom{m}{d}, \quad (44)$$

where the inner sum is over all possible d-tuples $(T_1, T_2, \dots, T_d)$ of types (not necessarily irreducible types), and where $N_\Phi(T_1, T_2, \dots, T_d)$ is the number of "minimal" products $c_1 c_2 \cdots c_d$ less than or equal to the Coxeter element c in absolute order, "minimal" meaning that all the c_i's are different from ε and that $\ell_T(c_1) + \ell_T(c_2) + \cdots + \ell_T(c_d) = \ell_T(c_1 c_2 \cdots c_d)$, such that the type of c_i as a parabolic Coxeter element is T_i, $i = 1, 2, \dots, d$. The notation $NC(T)$ in (44) means $NC(\Psi)$, where Ψ is a root system of type T, and $\operatorname{rk} T$ denotes the rank of Ψ. We point out that the appearance of the binomial coefficient $\binom{m}{d}$ is explained by (42).

So, what we have to do to apply formula (44) to compute the right-hand side of (30) is, first, to determine all the "decomposition numbers" $N_\Phi(T_1, T_2, \dots, T_d)$. Since we shall refer to it later, we point out that these decomposition numbers have many relations between themselves. For example, the number $N_\Phi(T_1, T_2, \dots, T_d)$ is independent of the order of the types $T_1, T_2, \dots, T_d$, that is, we have

$$N_\Phi(T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(d)}) = N_\Phi(T_1, T_2, \dots, T_d) \quad (45)$$

for any permutation σ of $\{1, 2, \dots, d\}$. This follows from the (proof of the) Shifting Lemma (see [1, Lemma 1.3.1]). Furthermore, by the definition of these numbers, those of "lower rank" can be computed from those of "full rank." To be precise, we have

$$N_\Phi(T_1, T_2, \dots, T_d) = \sum_T N_\Phi(T_1, T_2, \dots, T_d, T), \quad (46)$$

where the sum is over all types T of rank $n-\operatorname{rk}T_1-\operatorname{rk}T_2-\dots-\operatorname{rk}T_d$ (with n still denoting the rank of the fixed root system Φ).

Second, one needs a list of the characteristic polynomials $\chi^*_{NC(\Psi)}(y)$ for all *irreducible* root systems Ψ. (By the multiplicativity of the characteristic polynomial, this then gives also formulae for the characteristic polynomials of all the reducible types.) In fact, the numbers $N_\Psi(T_1,T_2,\dots,T_d)$ carry all the information which is necessary to do this recursively. Namely, by the definition of $NC(\Psi)$ and of the decomposition numbers $N_\Psi(T_1,T_2,\dots,T_d)$, we have

$$\chi^*_{NC(\Psi)}(y)=\sum_{T_1,T_2} N_\Psi(T_1,T_2)\,\mu_{NC(T_2)}\big(\hat0_{NC(T_2)},\hat1_{NC(T_2)}\big)\,y^{\operatorname{rk}T_1},\tag{47}$$

where $\mu_{NC(T_2)}(.,.)$ denotes the Möbius function in $NC(T_2)$, and where $\hat0_{NC(T_2)}$ and $\hat1_{NC(T_2)}$ are, respectively, the minimal and the maximal element in $NC(T_2)$. Indeed, inductively, the Möbius functions $\mu_{NC(T_2)}\big(\hat0_{NC(T_2)},\hat1_{NC(T_2)}\big)$ are already known for all T_2 of lower rank than the rank of Ψ. Hence, the only unknown in (47) is $\mu_{NC(\Psi)}\big(\hat0_{NC(\Psi)},\hat1_{NC(\Psi)}\big)$. However, the latter can be computed by setting $y=1$ in (47) and using the fact that $\chi^*_{NC(\Psi)}(1)=0$ for all root systems Ψ of rank at least 1. (This fact is equivalent to the statement that $\sum_{u\in NC(\Psi)}\mu_{NC(\Psi)}\big(u,\hat1_{NC(\Psi)}\big)=0$, which is nothing but a part of the definition of the Möbius function. Alternatively, one may use the uniform formula for the zeta polynomial of the non-crossing partition lattices, in which one specializes the variable to -1. See [8, Prop. 9]; the reader may be warned that a slightly different convention for the zeta polynomial is used there.)

We show in Sections 13–17 how to implement this procedure for the dihedral root system $I_2(a)$, for the hyperbolic root systems H_3 and H_4, and for F_4 and E_6. We list the values of the characteristic polynomials of the irreducible root systems that we need below.

$$\begin{aligned}
\chi^*_{A_1}(y)&=y-1,\\
\chi^*_{A_2}(y)&=y^2-3y+2,\\
\chi^*_{I_2(a)}(y)&=y^2-ay+a-1,\\
\chi^*_{A_3}(y)&=y^3-6y^2+10y-5,\\
\chi^*_{A_4}(y)&=y^4-10y^3+30y^2-35y+14,\\
\chi^*_{A_5}(y)&=y^5-15y^4+70y^3-140y^2+126y-42,\\
\chi^*_{B_3}(y)&=y^3-9y^2+18y-10,\\
\chi^*_{D_4}(y)&=y^4-12y^3+39y^2-48y+20,\\
\chi^*_{D_5}(y)&=y^5-20y^4+106y^3-230y^2+220y-77,\\
\chi^*_{H_3}(y)&=y^3-15y^2+35y-21,\\
\chi^*_{F_4}(y)&=y^4-24y^3+101y^2-144y+66,\\
\chi^*_{H_4}(y)&=y^4-60y^3+307y^2-480y+232,\\
\chi^*_{E_6}(y)&=y^6-36y^5+300y^4-1035y^3+1720y^2-1368y+418.
\end{aligned}\tag{48}$$

13. Proof of the $F = M$ Conjecture for $I_2(a)$. By (22), we have

$$(1-xy)^2 F^m_{I_2(a)}\left(\frac{x(1+y)}{1-xy}, \frac{xy}{1-xy}\right) = \frac{m(am-a+2)}{2}x^2y^2 + am^2x^2y \\ + \frac{m(am+a-2)}{2}x^2 + amxy + amx + 1 \quad (49)$$

for the left-hand side of (30).

We now compute the right-hand side of (30) following the proposed procedure in the previous section. We have $N_{I_2(a)}(I_2(a)) = 1$, $N_{I_2(a)}(A_1, A_1) = a$, $N_{I_2(a)}(A_1) = a$, $N_{I_2(a)}(\emptyset) = 1$, all other numbers $N_{I_2(a)}(T_1, \dots, T_d)$ being zero. Thus, according to (44) and (48), the right-hand side of (30) is equal to

$$(-x)^2(y^2+ay+a-1)m + (-x)^2 a(-y-1)^2\binom{m}{2} + (-x)a(-y-1)m + 1,$$

which agrees with (49). $\square$

14. Proof of the $F = M$ Conjecture for H_3. By (23), we have

$$(1-xy)^3 F^m_{H_3}\left(\frac{x(1+y)}{1-xy}, \frac{xy}{1-xy}\right) = \frac{m(5m-4)(5m-2)}{3}x^3y^3 + 5m^2(5m-2)x^3y^2 \\ + \frac{m(5m+2)(5m+4)}{3}x^3 + 5m^2(5m+2)x^3y + 5m(5m-2)x^2y^2 \\ + 50m^2x^2y + 5m(5m+2)x^2 + 15mxy + 15mx + 1 \quad (50)$$

for the left-hand side of (30).

We now compute the right-hand side of (30) following the proposed procedure in Section 12. The conclusions which we report here are based on *Maple* computations which we performed using Stembridge's `coxeter` package [20].

We have $N_{H_3}(H_3) = 1$, $N_{H_3}(A_1^2, A_1) = 5$, $N_{H_3}(A_2, A_1) = 5$, $N_{H_3}(I_2(5), A_1) = 5$, $N_{H_3}(A_1, A_1, A_1) = 50$, plus the assignments implied by (45) and (46), all other numbers $N_{H_3}(T_1, \dots, T_d)$ being zero. Thus, according to (44) and (48), the right-hand side of (30) is equal to

$$(-x)^3(-y^3-15y^2-35y-21)m + 2\cdot(-x)^3 5(-y-1)^3\binom{m}{2} \\ + 2\cdot(-x)^3 5(y^2+3y+2)(-y-1)\binom{m}{2} + 2\cdot(-x)^3 5(y^2+5y+4)(-y-1)\binom{m}{2} \\ + (-x)^3 50(-y-1)^3\binom{m}{3} + (-x)^2 5(-y-1)^2 m + (-x)^2 5(y^2+3y+2)m \\ + (-x)^2 5(y^2+5y+4)m + (-x)^2 50(-y-1)^2\binom{m}{2} + (-x)15(-y-1)m + 1,$$

which agrees with (50). $\square$

15. Proof of the $F = M$ Conjecture for H_4. By (24), we have

$$(1-xy)^4 F^m_{H_4}\left(\frac{x(1+y)}{1-xy}, \frac{xy}{1-xy}\right) = \frac{1}{4}m(3m-1)(5m-3)(15m-14)x^4y^4$$
$$+ 15m^2(3m-1)(5m-3)x^4y^3 + \frac{1}{2}m^2\left(675m^2-61\right)x^4y^2$$
$$+ 15m^2(3m+1)(5m+3)x^4y + \frac{1}{4}m(3m+1)(5m+3)(15m+14)x^4$$
$$+ 15m(3m-1)(5m-3)x^3y^3 + 15m^2(45m-14)x^3y^2$$
$$+ 15m^2(45m+14)x^3y + 15m(3m+1)(5m+3)x^3$$
$$+ \frac{1}{2}m(465m-149)x^2y^2 + 465m^2x^2y + \frac{1}{2}m(465m+149)x^2 + 60mxy + 60mx + 1 \quad (51)$$

for the left-hand side of (30).

We now compute the right-hand side of (30) following the proposed procedure in Section 12. The conclusions which we report here are based on *Maple* computations which we performed using Stembridge's `coxeter` package [20].

We have $N_{H_4}(H_4) = 1$, $N_{H_4}(A_1 * A_2, A_1) = 15$, $N_{H_4}(A_3, A_1) = 15$, $N_{H_4}(H_3, A_1) = 15$, $N_{H_4}(A_1 * I_2(5), A_1) = 15$, $N_{H_4}(A_1^2, A_1^2) = 30$, $N_{H_4}(A_1^2, A_2) = 30$, $N_{H_4}(A_1^2, I_2(5)) = 15$, $N_{H_4}(A_2, A_2) = 5$, $N_{H_4}(A_2, I_2(5)) = 15$, $N_{H_4}(I_2(5), I_2(5)) = 3$, $N_{H_4}(A_1^2, A_1, A_1) = 225$, $N_{H_4}(A_2, A_1, A_1) = 150$, $N_{H_4}(I_2(5), A_1, A_1) = 90$, $N_{H_4}(A_1, A_1, A_1, A_1) = 1350$, plus the assignments implied by (45) and (46), all other numbers $N_{H_4}(T_1, \dots, T_d)$ being zero. If one substitutes accordingly in (44), using the information from (48), then one obtains an expression which agrees with (51) after simplification. $\square$

16. Proof of the $F = M$ Conjecture for F_4. By (25), we have

$$(1-xy)^4 F^m_{F_4}\left(\frac{x(1+y)}{1-xy}, \frac{xy}{1-xy}\right) = \frac{1}{2}m(2m-1)(3m-1)(6m-5)x^4y^4$$
$$+ 12m^2(2m-1)(3m-1)x^4y^3 + m^2\left(108m^2-7\right)x^4y^2$$
$$+ 12m^2(2m+1)(3m+1)x^4y + \frac{1}{2}m(2m+1)(3m+1)(6m+5)x^4$$
$$+ 12m(2m-1)(3m-1)x^3y^3 + 12m^2(18m-5)x^3y^2$$
$$+ 12m^2(18m+5)x^3y + 12m(2m+1)(3m+1)x^3 + m(78m-23)x^2y^2$$
$$+ 156m^2x^2y + m(78m+23)x^2 + 24mxy + 24mx + 1 \quad (52)$$

for the left-hand side of (30).

We now compute the right-hand side of (30) following the proposed procedure in Section 12. The conclusions which we report here are based on *Maple* computations which we performed using Stembridge's `coxeter` package [20].

We have $N_{F_4}(F_4) = 1$, $N_{F_4}(A_1 * A_2, A_1) = 12$, $N_{F_4}(B_3, A_1) = 12$, $N_{F_4}(A_1^2, A_1^2) = 12$, $N_{F_4}(A_1^2, B_2) = 12$, $N_{F_4}(A_2, A_2) = 16$, $N_{F_4}(B_2, B_2) = 3$, $N_{F_4}(A_1^2, A_1, A_1) = 72$, $N_{F_4}(A_2, A_1, A_1) = 48$, $N_{F_4}(B_2, A_1, A_1) = 36$, $N_{F_4}(A_1, A_1, A_1, A_1) = 432$, plus the assignments implied by (45) and (46), all other numbers $N_{F_4}(T_1, \dots, T_d)$ being zero. If one substitutes accordingly in (44), using the information from (48), then one obtains an expression which agrees with (52) after simplification. $\square$

17. Proof of the $F = M$ Conjecture for E_6. By (26), we have

$$\begin{multline}
(1-xy)^6 F^m_{E_6}\left(\frac{x(1+y)}{1-xy}, \frac{xy}{1-xy}\right) = \frac{1}{30} m(2m-1)(3m-1)(4m-1)(6m-5)(12m-7)x^6y^6\\
+ \frac{6}{5} m^2(2m-1)(3m-1)(4m-1)(12m-7)x^6y^5 + 2m^2(3m-1)(4m-1)\left(36m^2-9m-2\right)x^6y^4\\
+ 3m^2(4m-1)(4m+1)\left(24m^2-1\right)x^6y^3 + 2m^2(3m+1)(4m+1)\left(36m^2+9m-2\right)x^6y^2\\
+ \frac{6}{5} m^2(2m+1)(3m+1)(4m+1)(12m+7)x^6y + \frac{1}{30} m(2m+1)(3m+1)(4m+1)(6m+5)(12m+7)x^6\\
+ \frac{6}{5} m(2m-1)(3m-1)(4m-1)(12m-7)x^5y^5 + 12m^2(3m-1)(4m-1)(12m-5)x^5y^4\\
+ 6m^2(4m-1)\left(144m^2-12m-5\right)x^5y^3 + 6m^2(4m+1)\left(144m^2+12m-5\right)x^5y^2\\
+ 12m^2(3m+1)(4m+1)(12m+5)x^5y + \frac{6}{5} m(2m+1)(3m+1)(4m+1)(12m+7)x^5\\
+ 2m(3m-1)(4m-1)(30m-13)x^4y^4 + 6m^2(4m-1)(120m-31)x^4y^3 + 16m^2\left(270m^2-7\right)x^4y^2\\
+ 6m^2(4m+1)(120m+31)x^4y + 2m(3m+1)(4m+1)(30m+13)x^4 + 9m(4m-1)(18m-5)x^3y^3\\
+ 18m^2(108m-19)x^3y^2 + 18m^2(108m+19)x^3y + 9m(4m+1)(18m+5)x^3 + 12m(21m-4)x^2y^2\\
+ 504m^2x^2y + 12m(21m+4)x^2 + 36mxy + 36mx + 1
\end{multline} \tag{53}$$

for the left-hand side of (30).

We now compute the right-hand side of (30) following the proposed procedure in Section 12. The conclusions which we report here are based on *Maple* computations which we performed using Stembridge's `coxeter` package [20].

We have $N_{E_6}(E_6) = 1$, $N_{E_6}(A_1 * A_2^2, A_1) = 6$, $N_{E_6}(A_1 * A_4, A_1) = 12$, $N_{E_6}(A_5, A_1) = 6$, $N_{E_6}(D_5, A_1) = 12$, $N_{E_6}(A_1^2 * A_2, A_2) = 36$, $N_{E_6}(A_2^2, A_2) = 8$, $N_{E_6}(A_1 * A_3, A_2) = 24$, $N_{E_6}(A_4, A_2) = 24$, $N_{E_6}(D_4, A_2) = 4$, $N_{E_6}(A_1^2 * A_2, A_1^2) = 18$, $N_{E_6}(A_1 * A_3, A_1^2) = 36$, $N_{E_6}(A_4, A_1^2) = 36$, $N_{E_6}(D_4, A_1^2) = 18$, $N_{E_6}(A_1^3, A_1^3) = 12$, $N_{E_6}(A_1 * A_2, A_1^3) = 24$, $N_{E_6}(A_1 * A_2, A_1 * A_2) = 48$, $N_{E_6}(A_3, A_1^3) = 36$, $N_{E_6}(A_3, A_1 * A_2) = 72$, $N_{E_6}(A_3, A_3) = 27$, $N_{E_6}(A_1^2 * A_2, A_1, A_1) = 144$, $N_{E_6}(A_2^2, A_1, A_1) = 24$, $N_{E_6}(A_1 * A_3, A_1, A_1) = 144$, $N_{E_6}(A_4, A_1, A_1) = 144$, $N_{E_6}(D_4, A_1, A_1) = 48$, $N_{E_6}(A_1^3, A_1^2, A_1) = 180$, $N_{E_6}(A_1^3, A_2, A_1) = 168$, $N_{E_6}(A_1 * A_2, A_1^2, A_1) = 360$, $N_{E_6}(A_1 * A_2, A_2, A_1) = 336$, $N_{E_6}(A_3, A_1^2, A_1) = 378$, $N_{E_6}(A_3, A_2, A_1) = 180$, $N_{E_6}(A_1^2, A_1^2, A_1^2) = 432$, $N_{E_6}(A_2, A_1^2, A_1^2) = 504$, $N_{E_6}(A_2, A_2, A_1^2) = 288$, $N_{E_6}(A_2, A_2, A_2) = 160$, $N_{E_6}(A_1^2, A_1^2, A_1, A_1) = 2376$, $N_{E_6}(A_2, A_1^2, A_1, A_1) = 1872$, $N_{E_6}(A_2, A_2, A_1, A_1) = 1056$, $N_{E_6}(A_1^3, A_1, A_1, A_1) = 864$, $N_{E_6}(A_1 * A_2, A_1, A_1, A_1) = 1728$, $N_{E_6}(A_3, A_1, A_1, A_1) = 1296$, $N_{E_6}(A_1^2, A_1, A_1, A_1, A_1) = 10368$, $N_{E_6}(A_2, A_1, A_1, A_1, A_1) = 6912$, $N_{E_6}(A_1, A_1, A_1, A_1, A_1, A_1) = 41472$, plus the assignments implied by (45) and (46), all other numbers $N_{E_6}(T_1, \dots, T_d)$ being zero. If one substitutes accordingly in (44), using the information from (48), then one obtains an expression which agrees with (53) after simplification. $\square$

18. The dual F-triangle. Armstrong [1, Sec. 4] defines the *dual F-triangle*, denoted here by $\tilde{F}^m_\Phi(x, y)$, as

$$\tilde{F}^m_\Phi(x, y) = (-1)^n F^m_\Phi(-1-x, -1-y),$$

where n is the rank of the root system Φ. He conjectures that the dual F-triangle can be expressed in form of a weighted bivariate generating function for the faces of $\Delta^m(\Phi)$

involving the *Fuss–Narayana numbers* $\mathrm{Nar}^m(\Phi,i)$, the latter enumerating all elements of rank i in the m-divisible non-crossing partition poset $NC^m(\Phi)$. For explicit formulae for the Fuss–Narayana numbers see [1, Theorem 2.3.6]. These numbers occur also as h-numbers in [10, Theorem 9.2]. (One has to reverse the ordering of the numbers to convert one sequence of numbers into the other.) In view of our proof below, Armstrong's conjecture becomes the following theorem.

Theorem DF. *For any finite root system Φ, we have*

$$\tilde{F}^m_\Phi(x,y)=\sum_{k,l\ge0}\frac{\mathrm{Nar}^m(\Phi,k+l)}{\mathrm{Nar}^1(\Phi,k+l)}f_{k,l}\,x^ky^l. \tag{54}$$

Proof. Clearly, for the exceptional root systems one can verify (54) routinely by using the explicit formulae for the refined face numbers, as given through the formulae for the F-triangle in Section 7, and the formulae for the Fuss–Narayana numbers in [1, 10].

To verify (54) for the root systems A_n, B_n and D_n, some work has to be done. However, the verifications in these types are very similar to each other so that we give below only the proof in type A_n, leaving the proofs for B_n and D_n to the reader.

By Theorem FA, in type A_n the left-hand side of (54) is equal to

$$\begin{aligned}
(-1)^n&\sum_{k,l\ge0}\frac{l+1}{k+l+1}\binom{n}{k+l}\binom{m(n+1)+k-1}{k}(-1-x)^k(-1-y)^l\\
&=\sum_{k,l,r,s\ge0}\frac{l+1}{k+l+1}\binom{n}{k+l}\binom{m(n+1)+k-1}{k}\binom{k}{r}\binom{l}{s}(-1)^{n+k+l}x^ry^s\\
&=\sum_{k,r,s\ge0}\frac{s+1}{n+1}\binom{n-s-1}{n-k-s}\binom{m(n+1)+k-1}{k}\binom{k}{r}(-1)^{n+k+s}x^ry^s\\
&=\sum_{k,r,s\ge0}\frac{s+1}{n+1}\binom{n-s-1}{n-k-s}\binom{m(n+1)+k-1}{k-r}\binom{m(n+1)+r-1}{r}(-1)^{n+k+s}x^ry^s\\
&=\sum_{r,s\ge0}\frac{s+1}{n+1}\binom{m(n+1)}{n-s-r}\binom{m(n+1)+r-1}{r}x^ry^s.
\end{aligned}$$

As earlier, for the evaluation of the sums over l and k we used special instances of the Chu–Vandermonde summation.

On the other hand, by Theorem FA and by [1, Theorem 2.3.6], the right-hand side of (54) is equal to

$$\sum_{k,l\ge0}\frac{\frac{1}{n+1}\binom{n+1}{k+l}\binom{m(n+1)}{n-k-l}}{\frac{1}{n+1}\binom{n+1}{k+l}\binom{n+1}{n-k-l}}\frac{l+1}{k+l+1}\binom{n}{k+l}\binom{m(n+1)+k-1}{k}x^ky^l,$$

which is exactly the same expression. $\square$

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