A Systematic List of Two- and Three-term Contiguous Relations for Basic Hypergeometric Series

C. Krattenthaler

Institut für Mathematik der Universität Wien,
Strudlhofgasse 4, A-1090 Wien, Austria.
e-mail: KRATT@Pap.Univie.Ac.At

Contiguous relations base on simply two-, three-, four-term, ... identities of certain products. Usually these can be derived from special cases of terminating basic hypergeometric summation or transformation formulas. In what follows, the symbols C$\langle$*number*$\rangle$ refer to replacements rules of the *Mathematica* package HYPQ which is available by anonymous ftp at `Pap.Univie.Ac.At` (cf. [2, 3]).

1. A two-term contiguous relation. This relation results from splitting the first term of a basic hypergeometric series from the rest. Recall that

$$ {}_r\phi_s\left[\begin{matrix}(A)\\(B)\end{matrix};q,z\right]=\sum_{k=0}^{\infty}\frac{(A)_k}{(q)_k(B)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}z^k, $$

where (A) stands for a sequence $A_1,A_2,\dots,A_r$ and (B) stands for a sequence $B_1,B_2,\dots,B_s$. $(A)_k$ is understood as an abbreviation for $\prod_{i=1}^{r}A_i$, etc. In the sequel (A) and (B) also may be shorter sequences, however the length will always be clear from the context.

With these conventions we have (C01, C02)

$$ {}_r\phi_s\left[\begin{matrix}(A)\\(B)\end{matrix};q,z\right]=1+\frac{(-1)^{1+s-r}z}{(1-q)}\frac{\prod_{i=1}^{r}(1-A_i)}{\prod_{i=1}^{s}(1-B_i)}{}_{r+1}\phi_{s+1}\left[\begin{matrix}q,(qA)\\q^2,(qB)\end{matrix};q,q^{1+s-r}z\right] \tag{1.1} $$

2. Simple three-term contiguous relations. Take the identity $1-q^k=(1-q^k)$. When multiplying this identity by $z^k(A)_k/(B)_k(q)_k$ and summing over all k, from this identity we get (C11–C13)

$$ {}_r\phi_s\left[\begin{matrix}(A)\\(B)\end{matrix};q,z\right]-{}_r\phi_s\left[\begin{matrix}(A)\\(B)\end{matrix};q,qz\right]=(-1)^{1+s-r}z\frac{\prod_{i=1}^{r}(1-A_i)}{\prod_{i=1}^{s}(1-B_i)}{}_r\phi_s\left[\begin{matrix}(qA)\\(qB)\end{matrix};q,q^{1+s-r}z\right] \tag{2.1} $$

Next take the identity $(1-aq^k)-(1-a)=a(1-q^k)$. This gives (C14, C15, C16, [1, Ex. I.9 (ii)])

$$
{}_r\phi_s\left[\begin{matrix} aq,(A) \\ (B)\end{matrix};q,z\right]-{}_r\phi_s\left[\begin{matrix} a,(A) \\ (B)\end{matrix};q,z\right]
=(-1)^{1+s-r}az\frac{\prod_{i=1}^{r-1}(1-A_i)}{\prod_{i=1}^{s}(1-B_i)}\,{}_r\phi_s\left[\begin{matrix} aq,(Aq) \\ (Bq)\end{matrix};q,q^{1+s-r}z\right]
\tag{2.2}
$$

when multiplying by $z^k(aq)_{k-1}(A)_k/(q)_k(B)_k$ and summing over k.

When multiplying by $z^k(A)_k/(q)_k(a)_{k+1}(B)_k$ and summing over k we get (C20–C22, [1, Ex. I.9 (i)])

$$
{}_r\phi_s\left[\begin{matrix} (A) \\ a,(B)\end{matrix};q,z\right]-{}_r\phi_s\left[\begin{matrix} (A) \\ aq,(B)\end{matrix};q,z\right]
=\frac{(-1)^{1+s-r}az}{(1-a)(1-aq)}\frac{\prod_{i=1}^{r}(1-A_i)}{\prod_{i=1}^{s-1}(1-B_i)}\,{}_r\phi_s\left[\begin{matrix} (Aq) \\ aq^2,(Bq)\end{matrix};q,q^{1+s-r}z\right]
\tag{2.3}
$$

Now consider the similar identity $(1-aq^k)-(1-a)q^k=1-q^k$. From this we get (C17, C18, C19)

$$
{}_r\phi_s\left[\begin{matrix} aq,(A) \\ (B)\end{matrix};q,z\right]-{}_r\phi_s\left[\begin{matrix} a,(A) \\ (B)\end{matrix};q,qz\right]
=(-1)^{1+s-r}z\frac{\prod_{i=1}^{r-1}(1-A_i)}{\prod_{i=1}^{s}(1-B_i)}\,{}_r\phi_s\left[\begin{matrix} aq,(Aq) \\ (Bq)\end{matrix};q,q^{1+s-r}z\right]
\tag{2.4}
$$

and (C23–C25)

$$
{}_r\phi_s\left[\begin{matrix} (A) \\ a,(B)\end{matrix};q,z\right]-{}_r\phi_s\left[\begin{matrix} (A) \\ aq,(B)\end{matrix};q,qz\right]
=\frac{(-1)^{1+s-r}z}{(1-a)(1-aq)}\frac{\prod_{i=1}^{r}(1-A_i)}{\prod_{i=1}^{s-1}(1-B_i)}\,{}_r\phi_s\left[\begin{matrix} (Aq) \\ aq^2,(Bq)\end{matrix};q,q^{1+s-r}z\right]
\tag{2.5}
$$

Next consider $(1-aq^k)-(1-bq^k)=q^k(b-a)$. Multiplication by $(a)_k(b)_k(A)_k/(B)_k$ gives (C26–C27)

$$
(1-a)\,{}_r\phi_s\left[\begin{matrix} aq,b,(A) \\ (B)\end{matrix};q,z\right]-(1-b)\,{}_r\phi_s\left[\begin{matrix} a,bq,(A) \\ (B)\end{matrix};q,z\right]
=(b-a)\,{}_r\phi_s\left[\begin{matrix} a,b,(A) \\ (B)\end{matrix};q,qz\right].
\tag{2.6}
$$

Multiplication by $(a)_k(A)_k/(bq)_k(B)_k$ gives (C34–C36)

$$(1-a)\,{}_r\phi_s\left[\begin{matrix} aq,(A) \\ bq,(B) \end{matrix};q,z\right] - (1-b)\,{}_r\phi_s\left[\begin{matrix} a,(A) \\ b,(B) \end{matrix};q,z\right] = (b-a)\,{}_r\phi_s\left[\begin{matrix} a,(A) \\ bq,(B) \end{matrix};q,qz\right]. \quad (2.7)$$

Finally, multiplication by $(A)_k/(aq)_k(bq)_k(B)_k$ gives (C45–C46)

$$(1-a)\,{}_r\phi_s\left[\begin{matrix} (A) \\ a,bq,(B) \end{matrix};q,z\right] - (1-b)\,{}_r\phi_s\left[\begin{matrix} (A) \\ aq,b,(B) \end{matrix};q,z\right] = (b-a)\,{}_r\phi_s\left[\begin{matrix} (A) \\ aq,bq,(B) \end{matrix};q,z\right]. \quad (2.8)$$

Now consider $(1-aq^k)\,b-(1-bq^k)\,a=b-a$. By similar considerations we obtain (C28–C29, [1, Ex. I.10 (i)])

$$b(1-a)\,{}_r\phi_s\left[\begin{matrix} aq,b,(A) \\ (B) \end{matrix};q,z\right] - a(1-b)\,{}_r\phi_s\left[\begin{matrix} a,bq,(A) \\ (B) \end{matrix};q,z\right] = (b-a)\,{}_r\phi_s\left[\begin{matrix} a,b,(A) \\ (B) \end{matrix};q,z\right]. \quad (2.9)$$

and (C37–C39)

$$b(1-a)\,{}_r\phi_s\left[\begin{matrix} aq,(A) \\ bq,(B) \end{matrix};q,z\right] - a(1-b)\,{}_r\phi_s\left[\begin{matrix} a,(A) \\ b,(B) \end{matrix};q,z\right] = (b-a)\,{}_r\phi_s\left[\begin{matrix} a,(A) \\ bq,(B) \end{matrix};q,z\right]. \quad (2.10)$$

and (C47–C48)

$$b(1-a)\,{}_r\phi_s\left[\begin{matrix} (A) \\ a,bq,(B) \end{matrix};q,z\right] - a(1-b)\,{}_r\phi_s\left[\begin{matrix} (A) \\ aq,b,(B) \end{matrix};q,z\right] = (b-a)\,{}_r\phi_s\left[\begin{matrix} (A) \\ aq,bq,(B) \end{matrix};q,z\right]. \quad (2.11)$$

3. Contiguous relations resulting from the $n=1$ case of the q–Saalschütz sum. Set $n=1$ in the q–Saalschütz [1, (1.7.2); Appendix (II.12)] sum. The resulting identity is equivalent with

$$(1-C)(1-C/AB)+\frac{C}{AB}(1-A)(1-B)=(1-C/A)(1-C/B). \quad (3.1)$$

Now let us take $A=a, B=1/bq^k, C=1/q^k$. After some simplification (3.1) reduces to

$$(1-aq^k)(1-b)-(1-a)(1-bq^k)=(a-b)(1-q^k).$$

From this we obtain the contiguous relations (C30–C31, [1, Ex. I.9 (iv)])

$$
{}_r\phi_s\left[\begin{matrix} aq, b, (A) \\ (B) \end{matrix}; q, z\right] - {}_r\phi_s\left[\begin{matrix} a, bq, (A) \\ (B) \end{matrix}; q, z\right]
$$

$$
= (-1)^{1+s-r} z(a-b) \frac{\prod_{i=1}^{r-2}(1-A_i)}{\prod_{i=1}^{s}(1-B_i)} {}_r\phi_s\left[\begin{matrix} aq, bq, (Aq) \\ (Bq) \end{matrix}; q, q^{1+s-r}z\right]. \quad (3.2)
$$

when multiplying by $z^k (a)_{k-1} (b)_{k-1} (A)_k/(B)_k$, and (C40–C42, [1, Ex. I.9 (iii)])

$$
{}_r\phi_s\left[\begin{matrix} aq, (A) \\ bq, (B) \end{matrix}; q, z\right] - {}_r\phi_s\left[\begin{matrix} a, (A) \\ b, (B) \end{matrix}; q, z\right]
$$

$$
= \frac{(-1)^{1+s-r} z(a-b)}{(1-b)(1-bq)} \frac{\prod_{i=1}^{r-1}(1-A_i)}{\prod_{i=1}^{s-1}(1-B_i)} {}_r\phi_s\left[\begin{matrix} aq, (Aq) \\ bq^2, (Bq) \end{matrix}; q, q^{1+s-r}z\right]. \quad (3.3)
$$

when multiplying by $z^k (a)_{k-1} (A)_k/(b)_{k+1} (B)_k$, and (C49–C50)

$$
{}_r\phi_s\left[\begin{matrix} (A) \\ a, bq, (B) \end{matrix}; q, z\right] - {}_r\phi_s\left[\begin{matrix} (A) \\ aq, b, (B) \end{matrix}; q, z\right]
$$

$$
= \frac{(-1)^{1+s-r} z(a-b)}{(1-a)(1-aq)(1-b)(1-bq)} \frac{\prod_{i=1}^{r}(1-A_i)}{\prod_{i=1}^{s-2}(1-B_i)} {}_r\phi_s\left[\begin{matrix} (Aq) \\ aq^2, bq^2, (Bq) \end{matrix}; q, q^{1+s-r}z\right]. \quad (3.4)
$$

Next in (3.1) set $A = a, B = b, C = q^{-k}$. Then (3.1) reduces to

$$
(1-aq^k)(1-bq^k) - q^k(1-a)(1-b) = (1-abq^k)(1-q^k).
$$

In a similar manner as above, from this we get the contiguous relations (C32–C33, C73)

$$
{}_r\phi_s\left[\begin{matrix} aq, bq, (A) \\ (B) \end{matrix}; q, z\right] - {}_r\phi_s\left[\begin{matrix} a, b, (A) \\ (B) \end{matrix}; q, qz\right]
$$

$$
= (-1)^{1+s-r} z(1-abq) \frac{\prod_{i=1}^{r-2}(1-A_i)}{\prod_{i=1}^{s}(1-B_i)} {}_{r+1}\phi_{s+1}\left[\begin{matrix} aq, bq, abq^2, (Aq) \\ abq, (Bq) \end{matrix}; q, q^{1+s-r}z\right]. \quad (3.5)
$$

and (C43–C44, C88, [5, (6.5)])

$$
{}_r\phi_s\left[\begin{matrix} aq, (A) \\ b, (B) \end{matrix}; q, z\right] - {}_r\phi_s\left[\begin{matrix} a, (A) \\ bq, (B) \end{matrix}; q, qz\right]
$$

$$
= \frac{(-1)^{1+s-r} z(1-abq)}{(1-b)(1-bq)} \frac{\prod_{i=1}^{r-1}(1-A_i)}{\prod_{i=1}^{s-1}(1-B_i)} {}_{r+1}\phi_{s+1}\left[\begin{matrix} aq, abq^2, (Aq) \\ bq^2, abq, (Bq) \end{matrix}; q, q^{1+s-r}z\right]. \quad (3.6)
$$

and (C51–C52, C98)

$$
{}_r\phi_s\left[\begin{matrix}(A)\\ a,b,(B)\end{matrix};q,z\right]-{}_r\phi_s\left[\begin{matrix}(A)\\ aq,bq,(B)\end{matrix};q,qz\right]=\frac{(-1)^{1+s-r}z(1-abq)}{(1-a)(1-aq)(1-b)(1-bq)}
$$
$$
\frac{\prod\limits_{i=1}^{r}(1-A_i)}{\prod\limits_{i=1}^{s-2}(1-B_i)}\,{}_{r+1}\phi_{s+1}\left[\begin{matrix}abq^2,(Aq)\\ aq^2,bq^2,abq,(Bq)\end{matrix};q,q^{1+s-r}z\right]. \quad (3.7)
$$

Now in (3.1) we set $A=aq^k, B=c/b, C=cq^k$. After some simplification we obtain

$$
(1-cq^k)(a-b)+(1-aq^k)(b-c)+(1-bq^k)(c-a)=0.
$$

From this identity we obtain the contiguous relations (C53)

$$
{}_r\phi_s\left[\begin{matrix}a,b,cq,(A)\\ (B)\end{matrix};q,z\right]\frac{(a-b)}{(1-a)(1-b)}+{}_r\phi_s\left[\begin{matrix}aq,b,c,(A)\\ (B)\end{matrix};q,z\right]\frac{(b-c)}{(1-b)(1-c)}
$$
$$
+{}_r\phi_s\left[\begin{matrix}a,bq,c,(A)\\ (B)\end{matrix};q,z\right]\frac{(c-a)}{(1-c)(1-a)}=0 \quad (3.8)
$$

and (C54–C55)

$$
{}_r\phi_s\left[\begin{matrix}a,b,(A)\\ c,(B)\end{matrix};q,z\right]\frac{(a-b)}{(1-a)(1-b)}+{}_r\phi_s\left[\begin{matrix}aq,b,(A)\\ cq,(B)\end{matrix};q,z\right]\frac{(b-c)}{(1-b)(1-c)}
$$
$$
+{}_r\phi_s\left[\begin{matrix}a,bq,(A)\\ cq,(B)\end{matrix};q,z\right]\frac{(c-a)}{(1-c)(1-a)}=0 \quad (3.9)
$$

and (C56–C57)

$$
{}_r\phi_s\left[\begin{matrix}a,(A)\\ bq,c,(B)\end{matrix};q,z\right]\frac{(a-b)}{(1-a)(1-b)}+{}_r\phi_s\left[\begin{matrix}aq,(A)\\ bq,cq,(B)\end{matrix};q,z\right]\frac{(b-c)}{(1-b)(1-c)}
$$
$$
+{}_r\phi_s\left[\begin{matrix}a,(A)\\ b,cq,(B)\end{matrix};q,z\right]\frac{(c-a)}{(1-c)(1-a)}=0 \quad (3.10)
$$

and (C58)

$$
{}_r\phi_s\left[\begin{matrix}(A)\\ aq,bq,c,(B)\end{matrix};q,z\right]\frac{(a-b)}{(1-a)(1-b)}+{}_r\phi_s\left[\begin{matrix}(A)\\ a,bq,cq,(B)\end{matrix};q,z\right]\frac{(b-c)}{(1-b)(1-c)}
$$
$$
+{}_r\phi_s\left[\begin{matrix}(A)\\ aq,b,cq,(B)\end{matrix};q,z\right]\frac{(c-a)}{(1-c)(1-a)}=0 \quad (3.11)
$$

Now we set $A=b/a, B=ab/c, C=bq^k$ in (3.1) to finally get

$$
(1-aq^k)(1-cq^k/a)-(1-bq^k)(1-cq^k/b)=q^k(b-a)(1-c/ab).
$$

From this we derive the contiguous relations (C59–C60)

$$
\begin{aligned}
{}_r\phi_s\left[\begin{matrix} aq, cq/a, b, c/b, (A) \\ (B) \end{matrix}; q, z\right](1-a)(1-c/a) \\
- {}_r\phi_s\left[\begin{matrix} a, c/a, bq, cq/b, (A) \\ (B) \end{matrix}; q, z\right](1-b)(1-c/b) \\
= {}_r\phi_s\left[\begin{matrix} a, c/a, b, c/b, (A) \\ (B) \end{matrix}; q, qz\right](b-a)(1-c/ab)
\end{aligned} \tag{3.12}
$$

and (C64–C66)

$$
\begin{aligned}
{}_r\phi_s\left[\begin{matrix} aq, cq/a, b, (A) \\ cq/b, (B) \end{matrix}; q, z\right](1-a)(1-c/a) - {}_r\phi_s\left[\begin{matrix} a, c/a, bq, (A) \\ c/b, (B) \end{matrix}; q, z\right](1-b)(1-c/b) \\
= {}_r\phi_s\left[\begin{matrix} a, c/a, b, (A) \\ cq/b, (B) \end{matrix}; q, qz\right](b-a)(1-c/ab)
\end{aligned} \tag{3.13}
$$

and (C74–C75, [5, (5.4)])

$$
\begin{aligned}
{}_r\phi_s\left[\begin{matrix} aq, b, (A) \\ c/a, cq/b, (B) \end{matrix}; q, z\right](1-a)(1-c/a) - {}_r\phi_s\left[\begin{matrix} a, bq, (A) \\ cq/a, c/b, (B) \end{matrix}; q, z\right](1-b)(1-c/b) \\
= {}_r\phi_s\left[\begin{matrix} a, b, (A) \\ cq/a, cq/b, (B) \end{matrix}; q, qz\right](b-a)(1-c/ab)
\end{aligned} \tag{3.14}
$$

and (C80–C82)

$$
\begin{aligned}
{}_r\phi_s\left[\begin{matrix} aq, cq/a, (A) \\ bq, cq/b, (B) \end{matrix}; q, z\right](1-a)(1-c/a) - {}_r\phi_s\left[\begin{matrix} a, c/a, (A) \\ b, c/b, (B) \end{matrix}; q, z\right](1-b)(1-c/b) \\
= {}_r\phi_s\left[\begin{matrix} a, c/a, (A) \\ bq, cq/b, (B) \end{matrix}; q, qz\right](b-a)(1-c/ab)
\end{aligned} \tag{3.15}
$$

and (C89–C91)

$$
\begin{aligned}
{}_r\phi_s\left[\begin{matrix} aq, (A) \\ c/a, bq, cq/b, (B) \end{matrix}; q, z\right](1-a)(1-c/a) \\
- {}_r\phi_s\left[\begin{matrix} a, (A) \\ cq/a, b, c/b, (B) \end{matrix}; q, z\right](1-b)(1-c/b) \\
= {}_r\phi_s\left[\begin{matrix} a, (A) \\ cq/a, bq, cq/b, (B) \end{matrix}; q, qz\right](b-a)(1-c/ab)
\end{aligned} \tag{3.16}
$$

and (C99–C100)

$$
\begin{aligned}
{}_r\phi_s\left[\begin{matrix} (A) \\ a, c/a, bq, cq/b, (B) \end{matrix}; q, z\right](1-a)(1-c/a) \\
- {}_r\phi_s\left[\begin{matrix} (A) \\ aq, cq/a, b, c/b, (B) \end{matrix}; q, z\right](1-b)(1-c/b) \\
= {}_r\phi_s\left[\begin{matrix} (A) \\ aq, cq/a, bq, cq/b, (B) \end{matrix}; q, qz\right](b-a)(1-c/ab)
\end{aligned} \tag{3.17}
$$

Finally, in (3.1) we set $A = aq^k, B = bq^k, C = cq^{2k}$. This leads to

$$(1 - cq^k/a)(1 - cq^k/b) - \frac{c}{ab}(1 - aq^k)(1 - bq^k) = (1 - c/ab)(1 - cq^{2k}).$$

From this we obtain the contiguous relations (C61, C110)

$$\begin{aligned}
&{}_r\phi_s\left[\begin{matrix} a, cq/a, b, cq/b, (A) \\ (B) \end{matrix}; q, z\right](1 - c/a)(1 - c/b) \\
&\qquad - {}_r\phi_s\left[\begin{matrix} aq, c/a, bq, c/b, (A) \\ (B) \end{matrix}; q, z\right]\frac{(1 - a)(1 - b)c}{ab} \\
&\quad = {}_{r+2}\phi_{s+2}\left[\begin{matrix} q\sqrt{c}, -q\sqrt{c}, a, c/a, b, c/b, (A) \\ \sqrt{c}, -\sqrt{c}, (B) \end{matrix}; q, z\right](1 - c)(1 - c/ab)
\end{aligned} \tag{3.18}$$

and (C67–C68, C112)

$$\begin{aligned}
&{}_r\phi_s\left[\begin{matrix} a, cq/a, b, (A) \\ c/b, (B) \end{matrix}; q, z\right](1 - c/a)(1 - c/b) \\
&\qquad - {}_r\phi_s\left[\begin{matrix} aq, c/a, bq, (A) \\ cq/b, (B) \end{matrix}; q, z\right]\frac{(1 - a)(1 - b)c}{ab} \\
&\quad = {}_{r+2}\phi_{s+2}\left[\begin{matrix} q\sqrt{c}, -q\sqrt{c}, a, c/a, b, (A) \\ \sqrt{c}, -\sqrt{c}, cq/b, (B) \end{matrix}; q, z\right](1 - c)(1 - c/ab)
\end{aligned} \tag{3.19}$$

and (C76–C77, C114, [4, proof of (3.10)])

$$\begin{aligned}
&{}_r\phi_s\left[\begin{matrix} a, b, (A) \\ c/a, c/b, (B) \end{matrix}; q, z\right](1 - c/a)(1 - c/b) \\
&\qquad - {}_r\phi_s\left[\begin{matrix} aq, bq, (A) \\ cq/a, cq/b, (B) \end{matrix}; q, z\right]\frac{(1 - a)(1 - b)c}{ab} \\
&\quad = {}_{r+2}\phi_{s+2}\left[\begin{matrix} q\sqrt{c}, -q\sqrt{c}, a, b, (A) \\ \sqrt{c}, -\sqrt{c}, cq/a, cq/b, (B) \end{matrix}; q, z\right](1 - c)(1 - c/ab)
\end{aligned} \tag{3.20}$$

and (C83, C115)

$$\begin{aligned}
&{}_r\phi_s\left[\begin{matrix} a, cq/a, (A) \\ bq, c/b, (B) \end{matrix}; q, z\right](1 - c/a)(1 - c/b) \\
&\qquad - {}_r\phi_s\left[\begin{matrix} aq, c/a, (A) \\ b, cq/b, (B) \end{matrix}; q, z\right]\frac{(1 - a)(1 - b)c}{ab} \\
&\quad = {}_{r+2}\phi_{s+2}\left[\begin{matrix} q\sqrt{c}, -q\sqrt{c}, a, c/a, (A) \\ \sqrt{c}, -\sqrt{c}, bq, cq/b, (B) \end{matrix}; q, z\right](1 - c)(1 - c/ab)
\end{aligned} \tag{3.21}$$

and (C92–C93, C118)

$$\begin{aligned}
&{}_r\phi_s\left[\begin{matrix} a, (A) \\ c/a, bq, c/b, (B) \end{matrix}; q, z\right](1 - c/a)(1 - c/b) \\
&\qquad - {}_r\phi_s\left[\begin{matrix} aq, (A) \\ cq/a, b, cq/b, (B) \end{matrix}; q, z\right]\frac{(1 - a)(1 - b)c}{ab} \\
&\quad = {}_{r+2}\phi_{s+2}\left[\begin{matrix} q\sqrt{c}, -q\sqrt{c}, a, (A) \\ \sqrt{c}, -\sqrt{c}, cq/a, bq, cq/b, (B) \end{matrix}; q, z\right](1 - c)(1 - c/ab)
\end{aligned} \tag{3.22}$$

and (C101, C120)

$$
{}_r\phi_s\left[\begin{matrix}(A)\\ aq, c/a, bq, c/b, (B)\end{matrix}; q, z\right](1-c/a)(1-c/b)
- {}_r\phi_s\left[\begin{matrix}(A)\\ a, cq/a, b, cq/b, (B)\end{matrix}; q, z\right]\frac{(1-a)(1-b)c}{ab}
= {}_{r+2}\phi_{s+2}\left[\begin{matrix}q\sqrt{c}, -q\sqrt{c}, (A)\\ \sqrt{c}, -\sqrt{c}, aq, cq/a, bq, cq/b, (B)\end{matrix}; q, z\right](1-c)(1-c/ab) \tag{3.23}
$$

4. Contiguous relations resulting from the $n=1$ case of Jackson's sum. Set $n=1$ in Jackson's [1, (2.6.2); Appendix (II.22)] ${}_8\phi_7$-sum. In the resulting identity replace a by $CD/q, b$ by A, c by $C/A, d$ by BD. After some simplification this yields

$$
(1-A)(1-C/A)(1-BD)(1-CD/B)-(1-B)(1-C/B)(1-AD)(1-CD/A)
= (B-A)(1-C/AB)(1-D)(1-CD). \tag{4.1}
$$

Now we set $A=a, B=b, C=c$, and $D=q^k$ to get

$$
(1-a)(1-c/a)(1-bq^k)(1-cq^k/b)-(1-b)(1-c/b)(1-q^k)(1-cq^k/a)
= (b-a)(1-c/ab)(1-q^k)(1-cq^k).
$$

From this identitiy we derive the contiguous relations (C62, C104)

$$
{}_r\phi_s\left[\begin{matrix}a, c/a, bq, cq/b, (A)\\ (B)\end{matrix}; q, z\right] - {}_r\phi_s\left[\begin{matrix}aq, cq/a, b, c/b, (A)\\ (B)\end{matrix}; q, z\right]
= (-1)^{1+s-r}z(b-a)(1-c/ab)(1-cq)\frac{\prod_{i=1}^{r-4}(1-A_i)}{\prod_{i=1}^{s}(1-B_i)}
{}_{r+1}\phi_{s+1}\left[\begin{matrix}aq, cq/a, bq, cq/b, cq^2, (Aq)\\ cq, (Bq)\end{matrix}; q, q^{1+s-r}z\right] \tag{4.2}
$$

and (C69–C70, C105)

$$
{}_r\phi_s\left[\begin{matrix}a, c/a, bq, (A)\\ c/b, (B)\end{matrix}; q, z\right] - {}_r\phi_s\left[\begin{matrix}aq, cq/a, b, (A)\\ cq/b, (B)\end{matrix}; q, z\right]
= (-1)^{1+s-r}z\frac{(b-a)(1-c/ab)(1-cq)}{(1-c/b)(1-cq/b)}\frac{\prod_{i=1}^{r-3}(1-A_i)}{\prod_{i=1}^{s-1}(1-B_i)}
{}_{r+1}\phi_{s+1}\left[\begin{matrix}aq, cq/a, bq, cq^2, (Aq)\\ cq^2/b, cq, (Bq)\end{matrix}; q, q^{1+s-r}z\right] \tag{4.3}
$$

and (C84–C85, C107)

$$
{}_r\phi_s\left[\begin{matrix} a, c/a, (A) \\ b, c/b, (B) \end{matrix}; q, z\right] - {}_r\phi_s\left[\begin{matrix} aq, cq/a, (A) \\ bq, cq/b, (B) \end{matrix}; q, z\right]
$$

$$
= (-1)^{1+s-r} z \frac{(b-a)(1-c/ab)(1-cq)}{(1-b)(1-bq)(1-c/b)(1-cq/b)} \frac{\prod_{i=1}^{r-2}(1-A_i)}{\prod_{i=1}^{s-2}(1-B_i)}
$$

$$
{}_{r+1}\phi_{s+1}\left[\begin{matrix} aq, cq/a, cq^2, (Aq) \\ bq^2, cq^2/b, cq, (Bq) \end{matrix}; q, q^{1+s-r}z\right] \quad (4.4)
$$

and (C78, C106, [5, (5.6)])

$$
{}_r\phi_s\left[\begin{matrix} a, bq, (A) \\ cq/a, c/b, (B) \end{matrix}; q, z\right] - {}_r\phi_s\left[\begin{matrix} aq, b, (A) \\ c/a, cq/b, (B) \end{matrix}; q, z\right]
$$

$$
= (-1)^{1+s-r} z \frac{(b-a)(1-c/ab)(1-cq)}{(1-c/a)(1-cq/a)(1-c/b)(1-cq/b)} \frac{\prod_{i=1}^{r-2}(1-A_i)}{\prod_{i=1}^{s-2}(1-B_i)}
$$

$$
{}_{r+1}\phi_{s+1}\left[\begin{matrix} aq, bq, cq^2, (Aq) \\ cq^2/a, cq^2/b, cq, (Bq) \end{matrix}; q, q^{1+s-r}z\right] \quad (4.5)
$$

and (C94–C95, C108)

$$
{}_r\phi_s\left[\begin{matrix} a, (A) \\ cq/a, b, c/b, (B) \end{matrix}; q, z\right] - {}_r\phi_s\left[\begin{matrix} aq, (A) \\ c/a, bq, cq/b, (B) \end{matrix}; q, z\right]
$$

$$
= (-1)^{1+s-r} z \frac{(b-a)(1-c/ab)(1-cq)}{(1-c/a)(1-cq/a)(1-b)(1-bq)(1-c/b)(1-cq/b)} \frac{\prod_{i=1}^{r-1}(1-A_i)}{\prod_{i=1}^{s-3}(1-B_i)}
$$

$$
{}_{r+1}\phi_{s+1}\left[\begin{matrix} aq, cq^2, (Aq) \\ cq^2/a, bq^2, cq^2/b, cq, (Bq) \end{matrix}; q, q^{1+s-r}z\right] \quad (4.6)
$$

and (C102, C109)

$$
{}_r\phi_s\left[\begin{matrix} (A) \\ aq, cq/a, b, c/b, (B) \end{matrix}; q, z\right] - {}_r\phi_s\left[\begin{matrix} (A) \\ a, c/a, bq, cq/b, (B) \end{matrix}; q, z\right]
$$

$$
= (-1)^{1+s-r} z \frac{(b-a)(1-c/ab)(1-cq)}{(1-a)(1-aq)(1-c/a)(1-cq/a)(1-b)(1-bq)(1-c/b)(1-cq/b)}
$$

$$
\frac{\prod_{i=1}^{r}(1-A_i)}{\prod_{i=1}^{s-4}(1-B_i)} {}_{r+1}\phi_{s+1}\left[\begin{matrix} cq^2, (Aq) \\ aq^2, cq^2/a, bq^2, cq^2/b, cq, (Bq) \end{matrix}; q, q^{1+s-r}z\right] \quad (4.7)
$$

Next in (4.1) set $A = aq^k, B = bq^k, C = q^k ab/d, D = c/ab$. This results into

$$(1-aq^k)(1-cq^k/a)(1-b/d)(1-c/bd) - (1-bq^k)(1-cq^k/b)(1-a/d)(1-c/ad) \\ = \frac{1}{d}(a-b)(1-c/ab)(1-q^k d)(1-cq^k/d).$$

From this identity we obtain the contiguous relations (C63, C111)

$$\begin{aligned} &{}_r\phi_s\left[\begin{matrix} aq, cq/a, b, c/b, (A) \\ (B) \end{matrix}; q, z\right] \frac{(1-b/d)(1-c/bd)}{(1-b)(1-c/b)} \\ &\qquad - {}_r\phi_s\left[\begin{matrix} a, c/a, bq, cq/b, (A) \\ (B) \end{matrix}; q, z\right] \frac{(1-a/d)(1-c/ad)}{(1-a)(1-c/a)} \\ &= {}_{r+2}\phi_{s+2}\left[\begin{matrix} a, c/a, b, c/b, dq, cq/d, (A) \\ d, c/d, (B) \end{matrix}; q, z\right] \frac{(a-b)(1-c/ab)(1-d)(1-c/d)}{d(1-a)(1-c/a)(1-b)(1-c/b)} \end{aligned} \tag{4.8}$$

and (C71–C72, C113)

$$\begin{aligned} &{}_r\phi_s\left[\begin{matrix} aq, cq/a, b, (A) \\ cq/b, (B) \end{matrix}; q, z\right] \frac{(1-b/d)(1-c/bd)}{(1-b)(1-c/b)} \\ &\qquad - {}_r\phi_s\left[\begin{matrix} a, c/a, bq, (A) \\ c/b, (B) \end{matrix}; q, z\right] \frac{(1-a/d)(1-c/ad)}{(1-a)(1-c/a)} \\ &= {}_{r+2}\phi_{s+2}\left[\begin{matrix} a, c/a, b, dq, cq/d, (A) \\ cq/b, d, c/d, (B) \end{matrix}; q, z\right] \frac{(a-b)(1-c/ab)(1-d)(1-c/d)}{d(1-a)(1-c/a)(1-b)(1-c/b)} \end{aligned} \tag{4.9}$$

and (C86–C87, C117)

$$\begin{aligned} &{}_r\phi_s\left[\begin{matrix} aq, cq/a, (A) \\ bq, cq/b, (B) \end{matrix}; q, z\right] \frac{(1-b/d)(1-c/bd)}{(1-b)(1-c/b)} \\ &\qquad - {}_r\phi_s\left[\begin{matrix} a, c/a, (A) \\ b, c/b, (B) \end{matrix}; q, z\right] \frac{(1-a/d)(1-c/ad)}{(1-a)(1-c/a)} \\ &= {}_{r+2}\phi_{s+2}\left[\begin{matrix} a, c/a, dq, cq/d, (A) \\ bq, cq/b, d, c/d, (B) \end{matrix}; q, z\right] \frac{(a-b)(1-c/ab)(1-d)(1-c/d)}{d(1-a)(1-c/a)(1-b)(1-c/b)} \end{aligned} \tag{4.10}$$

and (C79, C116)

$$\begin{aligned} &{}_r\phi_s\left[\begin{matrix} aq, b, (A) \\ c/a, cq/b, (B) \end{matrix}; q, z\right] \frac{(1-b/d)(1-c/bd)}{(1-b)(1-c/b)} \\ &\qquad - {}_r\phi_s\left[\begin{matrix} a, bq, (A) \\ cq/a, c/b, (B) \end{matrix}; q, z\right] \frac{(1-a/d)(1-c/ad)}{(1-a)(1-c/a)} \\ &= {}_{r+2}\phi_{s+2}\left[\begin{matrix} a, b, dq, cq/d, (A) \\ cq/a, cq/b, d, c/d, (B) \end{matrix}; q, z\right] \frac{(a-b)(1-c/ab)(1-d)(1-c/d)}{d(1-a)(1-c/a)(1-b)(1-c/b)} \end{aligned} \tag{4.11}$$

and (C96–C97, C119)

$$
{}_{r}\phi_{s}\left[\begin{matrix} aq, (A) \\ c/a, bq, cq/b, (B) \end{matrix}; q, z\right] \frac{(1-b/d)(1-c/bd)}{(1-b)(1-c/b)}
$$
$$
- {}_{r}\phi_{s}\left[\begin{matrix} a, (A) \\ cq/a, b, c/b, (B) \end{matrix}; q, z\right] \frac{(1-a/d)(1-c/ad)}{(1-a)(1-c/a)}
$$
$$
= {}_{r+2}\phi_{s+2}\left[\begin{matrix} a, dq, cq/d, (A) \\ cq/a, bq, cq/b, d, c/d, (B) \end{matrix}; q, z\right] \frac{(a-b)(1-c/ab)(1-d)(1-c/d)}{d(1-a)(1-c/a)(1-b)(1-c/b)} \tag{4.12}
$$

and (C103, C121)

$$
{}_{r}\phi_{s}\left[\begin{matrix} (A) \\ a, c/a, bq, cq/b, (B) \end{matrix}; q, z\right] \frac{(1-b/d)(1-c/bd)}{(1-b)(1-c/b)}
$$
$$
- {}_{r}\phi_{s}\left[\begin{matrix} (A) \\ aq, cq/a, b, c/b, (B) \end{matrix}; q, z\right] \frac{(1-a/d)(1-c/ad)}{(1-a)(1-c/a)}
$$
$$
= {}_{r+2}\phi_{s+2}\left[\begin{matrix} dq, cq/d, (A) \\ aq, cq/a, bq, cq/b, d, c/d, (B) \end{matrix}; q, z\right] \frac{(a-b)(1-c/ab)(1-d)(1-c/d)}{d(1-a)(1-c/a)(1-b)(1-c/b)} \tag{4.13}
$$

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