Counting Lattice Paths with a Linear Boundary I.

By

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Abstract

Gitterpunktwege in der Ebene, die eine gegebene Gerade nicht berühren, werden mittels vier verschiedener Statistiken, die von J. Fürlinger und J. Hofbauer (J. Combin. Theory A 40 (1985), 248—264) eingeführt wurden und die bekannte Descent-major-Statistik verallgemeinern, abgezählt. Wir geben erzeugende Funktionen, Rekurrenzrelationen und Konvolutionsidentitäten für die resultierenden Zahlen, die Verallgemeinerungen der bekannten Gould-Zahlen $A_n(a,b) = \frac{a}{a+bn}\binom{a+bn}{n}$ sind.

1. Introduction

In this paper we consider lattice paths in the plane consisting of unit horizontal and vertical steps in the positive direction, which in the sequel we shall call paths for short. For the nonnegative integers n, r, t let $L_+(n,r,t)$ denote the set of paths from the origin to $(n, rn+t)$ not touching the line $y = rx + t$ except at its final point. For instance, the path in Fig. 1 below is an element of $L_+(2,2,3)$.

The number of these lattice paths is seen to be ([15, p. 9; 5, p. 80])

$$|L_+(n,r,t)| = \frac{t}{t+(r+1)n}\binom{t+(r+1)n}{n}. \tag{1.1}$$

($|A|$ stands for the cardinality of the set A.) By use of Lagrange inversion (see e.g. [16, section 4.5]) we get the expansion

$$\sum_{n=0}^{\infty} |L_+(n,r,t)| \frac{z^n}{(1+z)^{(r+1)n+t}} = 1. \tag{1.2}$$

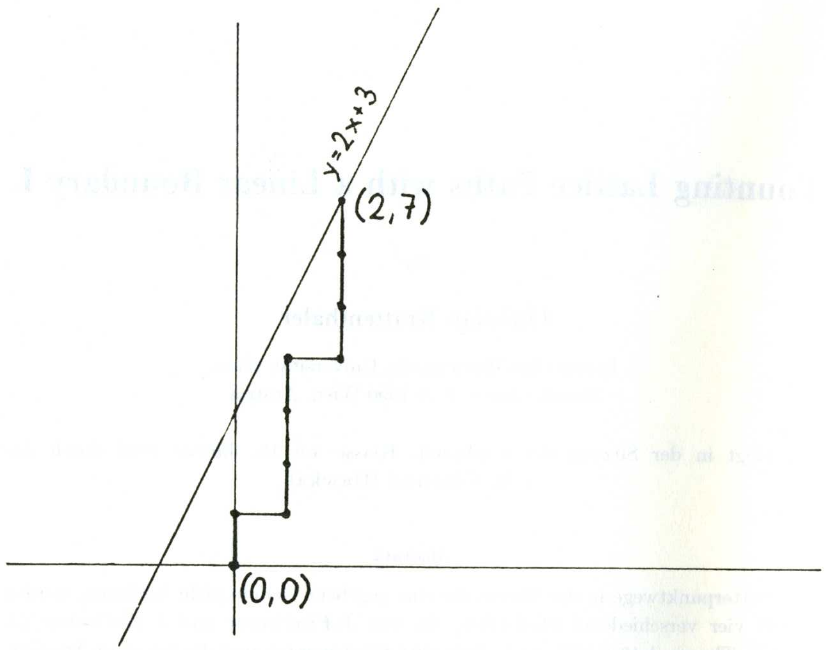

Fig. 1

There are two celebrated convolution identities involving the Gould numbers given in (1.1), the generalized Vandermonde convolution formulas

$$\sum_{k=0}^{n} \frac{a}{a+bk}\binom{a+bk}{k}\frac{c}{c+b(n-k)}\binom{c+b(n-k)}{n-k} = \frac{a+c}{a+c+bn}\binom{a+c+bn}{n}, \tag{1.3}$$

due to Rothe, and Gould's

$$\sum_{k=0}^{n} \frac{a}{a+bk}\binom{a+bk}{k}\binom{c+b(n-k)}{n-k} = \binom{a+c+bn}{n} \tag{1.4}$$

(see [9, 10] for references).

Mohanty [15] also considers the "dual" problem. Let $L'_+(n,r,t)$ denote the set of all paths from the origin to $(rn+t,n)$ not touching the

line $x = ry + t$ except at its final point. Reflection at the main diagonal $x = y$ turns each element of $L_+(n,r,t)$ into an element of $L'_+(n,r,t)$, and vice versa. Hence (1.1) and (1.2) hold for $|L'_+(n,r,t)|$, too.

We are going to count $L_+(n,r,t)$ and $L'_+(n,r,t)$ by means of two three-variate statistics introduced in (2.1) and (2.2), respectively. (Considering the bijection between $L_+(n,r,t)$ and $L'_+(n,r,t)$ in fact we are counting $L_+(n,r,t)$ by four different three-variate statistics.) We shall give a parallel discussion of the resulting generalized Gould numbers $G_n(r,t,x,a,b)$, $G'_n(r,t,x,a,b)$, $\bar{G}_n(r,t,x,a,b)$ and $\bar{G}'_n(r,t,x,a,b)$ defined in section 2.

In Theorems 2 and 3 we obtain the analogues of (1.2)–(1.4) for G'_n and G_n, respectively, Theorem 2 being due to Hofbauer [11]. For $\bar{G}_n$ and $\bar{G}'_n$ we do not have generating functions, the corresponding analogues of the convolution identities (1.3) and (1.4) are stated in Theorem 4.

In section 5 we direct our attention to the important special case $r = 1$, $a = b = q$. What is obtained are q-extensions of ballot numbers ([2, 16]). In this case, by q-Lagrange inversion and combinatorial arguments, even closed expressions, which generalize (1.1), can be given for these numbers. This is done in Theorems 5 and 6.

Section 6 deals with the generalized Catalan numbers introduced by Fürlinger and Hofbauer [6], which in our context is the special case $r = t = 1$. We add further results and answer some questions put in the cited paper.

In the concluding section 7 we touch briefly the results obtained by counting $L_+(n,r,t)$ by inversions, which extend some results of Carlitz, Riordan and Scoville [1, 2, 3] and Fürlinger and Hofbauer [6], which treat another q-analog of ballot numbers.

It is our intention to enveil the combinatorial significance of the identities involving the generalized Gould numbers G_n, G'_n, $\bar{G}_n$, $\bar{G}'_n$. Moreover we oppose combinatorial proofs to proofs by means of generating functions (if available). Generally speaking, in our context proofs by generating functions, once obtained, are easy to look through (unlike some of the combinatorial proofs) and much shorter than combinatorial ones, but ordinarily it is a little "miracle" that they run through, they do not explain why, e.g., exactly this power of q has to be taken as constant, etc. We could say that proofs by generating functions in our context have a describing, on the contrary combinatorial ones an explaining function.

2. Notation

For the nonnegative integers (n, r, t) we write $L(n, r, t)$ for the set of all paths from $(0, 0)$ to $(n, rn + t)$, $L'(n, r, t)$ for the set of all paths from $(0, 0)$ to $(rn + t, n)$. $L_+(n, r, t)$ and $L'_+(n, r, t)$ stand for the set of paths introduced in the introduction. By convention $L_+(n, r, 0)$ and $L'_+(n, r, 0)$ are empty, with the exception of the case $n = 0$, where $L_+(0, r, 0)$, $L'_+(0, r, 0)$, $L(0, r, 0)$, $L'(0, r, 0)$ contain the empty path as single element. Though obvious, we mention that each path of $L_+(n, r, t)$ ($L'_+(n, r, t)$) has to end with a vertical (horizontal) step, thus we could also consider the problem of counting all paths from $(0, 0)$ to $(n, rn + t - 1)$ ($(rn + t - 1, n)$ resp.) not entering the line $y = rx + t - 1$ ($x = ry + t - 1$ resp.). Sometimes we shall use the notation $l_+(n, r, t)$ ($l'_+(n, r, t)$) for this set of paths.

In the sequel each path from $(0, 0)$ to (l, m) will be identified with a word $w = w_1 w_2 \dots w_{l+m}$ by proceeding along the path, starting from the origin, and writing a 0 for each horizontal and a 1 for each vertical step. For example the path in Fig. 1 in this sense is written $w_0 = 101110111$. On a 0-1-word w we define the "down-set" $D(w)$ to be

$$D(w) = \{i : w_i > w_{i+1},\ 1 \leqslant i \leqslant l + m - 1\},$$

and the three statistics

$$\begin{aligned} \operatorname{des} w &= |D(w)| \\ \alpha(w) &= \sum_{i \in D(w)} |\{j \leqslant i : w_j = 0\}| \\ \beta(w) &= \sum_{i \in D(w)} |\{j \leqslant i : w_j = 1\}|. \end{aligned} \tag{2.1}$$

The classical major index of w

$$\operatorname{maj} w = \sum_{i \in D(w)} i$$

obviously satisfies $\operatorname{maj} w = \alpha(w) + \beta(w)$, thus α and β refine the major statistics. Counting des, α, β, maj in w "from the back" of w, leads us to the definitions

$$\bar{D}(w) = \{i : w_i > w_{i-1},\ 2 \leqslant i \leqslant l + m\}$$

and

$$\begin{aligned} \overline{\operatorname{des}}\, w &= |\bar{D}(w)| \\ \bar{\alpha}(w) &= \sum_{i \in \bar{D}(w)} |\{j \geqslant i : w_j = 0\}| \\ \bar{\beta}(w) &= \sum_{i \in \bar{D}(w)} |\{j \geqslant i, w_j = 1\}|. \end{aligned} \tag{2.2}$$

$\overline{\operatorname{maj}}$ is defined by

$$\overline{\operatorname{maj}}\, w = \sum_{i \in \bar{D}(w)} (l(w) - i + 1),$$

where $l(w)$ denotes the length of w, and is equal to $\bar{\alpha} + \bar{\beta}$. For instance for the example in Fig. 1, identified with the word $w_0 = 101110111$, we have: $D(w_0) = \{1, 5\}$, $\operatorname{des} w_0 = 2$, $\alpha(w_0) = 1$, $\beta(w_0) = 5$, $\operatorname{maj} w_0 = 6$, $\bar{D}(w_0) = \{3, 7\}$, $\overline{\operatorname{des}}\, w_0 = 2$, $\bar{\alpha}(w_0) = 1$, $\bar{\beta}(w_0) = 9$, $\overline{\operatorname{maj}}\, w_0 = 10$.

With the help of (2.1) and (2.2) we define the weight functions M and $\bar{M}$ acting on 0-1-words (and thus on paths):

$$M(w) = x^{\operatorname{des} w} a^{\alpha(w)} b^{\beta(w)},$$

$$\bar{M}(w) = x^{\overline{\operatorname{des}}\, w} a^{\bar{\alpha}(w)} b^{\bar{\beta}(w)}.$$

Counting $L_+(n, r, t)$ and $L'_+(n, r, t)$ by means of M and $\bar{M}$, respectively, gives the four generalized Gould numbers

$$G_n(r, t, x, a, b) = \sum_{w \in L_+(n,r,t)} M(w) \tag{2.3}$$

$$G'_n(r, t, x, a, b) = \sum_{w \in L'_+(n,r,t)} M(w) \tag{2.4}$$

$$\bar{G}_n(r, t, x, a, b) = \sum_{w \in L_+(n,r,t)} \bar{M}(w) \tag{2.5}$$

$$\bar{G}'_n(r, t, x, a, b) = \sum_{w \in L'_+(n,r,t)} \bar{M}(w). \tag{2.6}$$

As counting $L(n, r, t)$ by M and $\bar{M}$, respectively, yields the same result (the same valid for $L'(n, r, t)$), it suffices to define

$$A_n(r, t, x, a, b) = \sum_{w \in L(n,r,t)} M(w) \tag{2.7}$$

and

$$A'_n(r, t, x, a, b) = \sum_{w \in L'(n,r,t)} M(w). \tag{2.8}$$

In the special case $r = 1$, $a = b = q$ we shall write

$$G_n(1, t, x, q, q) = B_n(q, t, x) \text{ etc.},$$

in the case $r = t = 1$

$$G_n(1, 1, x, a, b) = C_n(x, a, b) \text{ etc.}$$

Moreover, we shall frequently use the following conventions: $P([a_1, a_2] \| [b_1, b_2])$ denotes the set of all pairs $(A_1 \dots A_k | B_1 \dots B_k)$, where $a_1 \leqslant A_i \leqslant a_2$ and $b_1 \leqslant B_i \leqslant b_2$ for $1 \leqslant i \leqslant k$, k a nonnegative integer (by convention for $k = 0$ the empty symbol $(|)$ is element of $P([a_1, a_2] \| [b_1, b_2])$), and the A_i's and B_i's are in increasing order. $(A|B)$ may also be considered as a pair of subsets of $[a_1, a_2]$ and $[b_1, b_2]$ with the same cardinality, where $[a_1, a_2]$ denotes the set of all integers not smaller than a_1 and not larger than a_2. Besides the elements of such a subset are considered to be ordered from bottom to top. $P_r([a_1, a_2] \| [b_1, b_2])$ denotes the subset of $P([a_1, a_2] \| [b_1, b_2])$ with the additional condition $rA_i + t > B_i$ for $1 \leqslant i \leqslant k$; $P'_r([a_1, a_2] \| [b_1, b_2])$, that with $A_i < rB_{i-1} + t$ for $1 \leqslant i \leqslant k$ and $B_k = n$ where B_0 is set equal to zero. Finally for nonnegative integers k, l

$$P\left(\begin{array}{c|c} [a_1, a_2] & [b_1, b_2] \\ k & l \end{array}\right)$$

stands for the set of all pairs $(A_1 \dots A_k | B_1 \dots B_l)$ with $a_1 \leqslant A_i \leqslant a_2$ and $b_1 \leqslant B_i \leqslant b_2$ where again the A_i's and B_i's, respectively, are in increasing order.

Given a pair $(A_1 \dots A_k | B_1 \dots B_k)$, we set

$$\begin{aligned} \operatorname{des}(A_1 \dots A_k | B_1 \dots B_k) &= k \\ \alpha(A_1 \dots A_k | B_1 \dots B_k) &= A_1 + \dots + A_k \\ \beta(A_1 \dots A_k | B_1 \dots B_k) &= B_1 + \dots + B_k \\ \operatorname{maj} &= \alpha + \beta \end{aligned}$$

and

$$\begin{aligned} M(A_1 \dots A_k | B_1 \dots B_k) &= x^k a^{A_1 + \dots + A_k} b^{B_1 + \dots + B_k} \\ &= x^{\operatorname{des}(A|B)} a^{\alpha(A|B)} b^{\beta(A|B)}. \end{aligned}$$

We adopt the usual q-notations $[n] = (q^n - 1)/(q - 1)$,

$$[n]! = [n][n-1] \dots [1], \; [0]! = 1,$$

$$\begin{aligned} (z; q)_n &= (1 - z)(1 - qz) \dots (1 - q^{n-1} z) = \\ &= \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} z^k, \end{aligned} \tag{2.9}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

whenever $0 \leqslant k \leqslant n$, zero otherwise.

3. Preliminaries

Our main tool for combinatorial interpretations is the correspondence π between paths from $(0, 0)$ to (l, m) and elements of $P([0, l-1] \| [1, m])$. This is established by writing at each descent in the path w the number of 0's below into the left "block" and the number of 1's below into the right block. More precisely,

$$\pi(w) = (A_1 \dots A_k | B_1 \dots B_k),$$

where $k = \operatorname{des} w$, and if $D(w) = \{i_1, i_2 \dots, i_k\}$ where the i_ν's are in increasing order, then $A_\nu = |\{j \leqslant i_\nu : w_j = 0\}|$ and $B_\nu = |\{j \leqslant i_\nu : w_j = 1\}|$. (E.g. $\pi(w_0) = (01|14)$ for the path in Fig. 1 being represented by $w_0 = 101110111$.) Similarly, we introduce the correspondence $\bar{\pi}$ by

$$\bar{\pi}(w) = (A_1 \dots A_k | B_1 \dots B_k),$$

where $k = \overline{\operatorname{des}}\, w$, and if $\bar{D}(w) = \{i_1, \dots, i_k\}$ where the i_ν's are in decreasing order, then $A_\nu = \{j \geqslant i_\nu : w_j = 0\}$ and $B_\nu = \{j \geqslant i_\nu : w_j = 1\}$ (e.g. $\bar{\pi}(w_0) = (01|36)$).

Obviously these correspondences are weight-preserving i.e.

$$\begin{aligned} \operatorname{des}(\pi(w)) &= \operatorname{des}(w) \\ \alpha(\pi(w)) &= \alpha(w) \\ \beta(\pi(w)) &= \beta(w) \\ M(\pi(w)) &= M(w) \end{aligned}$$

and

$$\begin{aligned} \operatorname{des}(\bar\pi(w)) &= \overline{\operatorname{des} w} \\ \alpha(\bar\pi(w)) &= \bar\alpha(w) \\ \beta(\bar\pi(w)) &= \bar\beta(w) \\ M(\bar\pi(w)) &= \bar M(w). \end{aligned}$$

Another fact being clear from the definition of π is that π provides the following (weight-preserving) bijections:

$$L_+(n,r,t) \overset{\pi}{\leftrightarrow} P_r([0,n-1]|[1,rn+t-1]), \tag{3.1}$$

$$L'_+(n,r,t) \overset{\pi}{\leftrightarrow} P'_r([0,rn+t-1]|[1,n]). \tag{3.2}$$

The term $(rn+t-1)$ in (3.1) results from the fact that each path of $L_+(n,r,t)$ has to end with a vertical step.

As first application of the correspondence π we derive the following Proposition.

Proposition 1: For n, r, t being nonnegative integers there holds

$$\begin{aligned} A_n(r,t,x,a,b) &= \langle z^n\rangle (1+z)(a+z)\dots \\ &\dots(a^{n-1}+z)(1+bxz)(1+b^2xz)\dots(1+b^{rn+t}xz) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} A'_n(r,t,x,a,b) &= \langle z^n\rangle (1+xz)(1+axz)\dots \\ &\dots(1+a^{rn+t-1}xz)(b+z)\dots(b^n+z), \end{aligned} \tag{3.4}$$

where $\langle z^n\rangle$ denotes the coefficient of z^n.

Proof: To obtain z^n on the right-hand side of (3.3) we have to select k factors of the form $(1+b^jxz)$, say $(1+b^{B_1}xz)$, $(1+b^{B_2}xz), \dots, (1+b^{B_k}xz)$, and $(n-k)$ factors of the form (a^i+z), where we take z from. As contribution of this choice we get $x^k a^{A_1+\dots+A_k} b^{B_1+\dots+B_k}$, where $(a^{A_i}+z)$ are the remaining factors of this form. We consider the A_i's and B_i's, respectively, to be in increasing order. But, since $(A_1\dots A_k|B_1\dots B_k) \in P([0,n-1]|[1,rn+t]$ the right-hand side of (3.1) equals

$$\sum_{v\in P([0,n-1]|[1,rn+t])} M(v),$$

which, as π is a weight-preserving bijection between $L(n,r,t)$ and $P([0,n-1]|[1,rn+t])$, is the same as $A_n(r,t,x,a,b)$. Similar arguments furnish (3.4). □

4. General case: Generating function, recurrence relation and convolution formulas

First we state the results which generalize (1.2)–(1.4) involving $G'_n(r,t,x,a,b)$. If no confusion is possible we will write $G'_n(t,x)$ instead of $G'_n(r,t,x,a,b)$, the same concerning $G_n(t,x)$, $A_n(t,x)$ etc.

Theorem 2: The following three statements are equivalent:

$$G'_n(r,t,x,a,b) = \sum_{w\in L'_+(n,r,t)} x^{\operatorname{des} w} a^{\alpha(w)} b^{\beta(w)}; \tag{4.1}$$

$$\sum_{n=0}^{\infty} \frac{G'_n(t,x)\,z^n}{(1+xz)\dots(1+a^{rn+t-1}xz)(b+z)\dots(b^n+z)} = 1; \tag{4.2}$$

$$\begin{aligned} G'_n(t,x) &= G'_n(t-1,ax) + G'_{n-1}(t+r,bx) + \\ &+ (bx-1)\,G'_{n-1}(t+r-1,abx) \end{aligned} \tag{4.3}$$

for $t \geqslant 1$ with $G'_0(t,x)=1$ and $G'_n(0,x)=\delta_{n0}$ where d_{nk} is the Kronecker-delta.

Moreover there hold the convolution identities

$$G'_n(t_1+t_2,x) = \sum_{k=0}^{n} G'_k(t_1,x)\,G'_{n-k}(t_2,a^{rk+t_1}b^kx) \tag{4.4}$$

and

$$A'_n(t_1+t_2,x) = \sum_{k=0}^{n} G'_k(t_1,x)\,A'_{n-k}(t_2,a^{rk+t_1}b^kx). \tag{4.5}$$

Proof: (A) Combinatorial proofs.

(4.1) ⇔ (4.2). We use the correspondence π.

First step: We prove

$$\begin{aligned} &\langle z^s\rangle \sum_{n=0}^{s} G'_n(t,x)z^n(1+a^{rn+t}xz)\dots \\ &\dots(1+a^{rs+t-1}xz)(b^{n+1}+z)\dots(b^s+z) = \\ &= \langle z^s\rangle (1+xz)\dots(1+a^{rs+t-1}xz)(b+z)\dots(b^s+z). \end{aligned} \tag{4.6}$$

To obtain z^s on the right-hand side of (4.6), we have to select k factors of the form $(1+a^ixz)$ and $(s-k)$ factors of the form (b^j+z), where we take z form. As contribution we gain $x^k a^{A_1+\dots+A_k} b^{B_1+\dots+B_k}$, where

$(A|B)=(A_1\ldots A_k|B_1\ldots B_k)\in P([0,rs+t-1]|[1,s])$. Each monomial appearing on the right-hand side of (4.6) can be represented by a pair $(A|B)\in P([0,rs+t-1]|[1,s])$, to which we shall refer as "right-hand side symbol" (rhs-symbol). Evidently $M(A|B)$ gives the value of the corresponding contribution.

Similarly, each of the monomials on the left-hand side can be symbolized by a pair $((C|D),\ (E|F))$, where, regarding (3.2), $(C|D)\in P'_r([0,rn+t-1]|[1,n])$ and $(E|F)\in P([rn+t,rs+t-1]|[n+1,s])$, where the value of the corresponding contribution is given by $M((C|D),\ (E|F))=M(C|D)\cdot M(E|F)$. The index variable n is recognized by the last entry of D, which has to be equal to n, since each path of $L'_+(n,r,t)$ ends with a horizontal step coded by 0. We will call $((C|D),\ (E|F))$ a left-hand side symbol (lhs-symbol).

We shall construct a weight-preserving bijection between the rhs- and lhs-symbols.

Given $(A|B)=(A_1\ldots A_k|B_1\ldots B_k)$ a rhs-symbol let j be minimal with $A_j\geqslant rB_{j-1}+t$ (if there is none, then $j=k+1$). Again by convention $B_0=0$. Then we define

$$\sigma_1(A_1\ldots A_k|B_1\ldots B_k)=$$
$$=((A_1\ldots A_{j-1}|B_1\ldots B_{j-1}),\ (A_j\ldots A_k|B_j\ldots B_k)).$$

Obviously

$$(A_1\ldots A_{j-1}|B_1\ldots B_{j-1})\in P'_r([0,rB_{j-1}+t-1]|[1,B_{j-1}])$$

and

$$(A_j\ldots A_k|B_j\ldots B_k)\in P([rB_{j-1}+t,rs+t-1]|[B_{j-1}+1,s]).$$

Therefore $\sigma_1(A|B)$ belongs to the lhs-symbols. Moreover, evidently $M(\sigma_1(A|B))=M(A|B)$, which implies that σ_1 is weight-preserving. It is easy to see that, if $((C|D),\ (E|F))$ is a lhs-symbol, then

$$\bar\sigma_1((C|D),\ (E|F))=(C\cup E|D\cup F)$$

is the inverse map for σ_1 ($C\cup E$ stands for the block which arises by putting the blocks C and E together, one after the other). Thus (4.6) is verified.

Second step: We prove

$$\langle z^k\rangle\sum_{n=0}^{s}G'_n(t,x)z^n(1+a^{rn+t}xz)\ldots$$
$$\ldots(1+a^{rs+t-1}xz)(b^{n+1}+z)\ldots(b^s+z)= \tag{4.7}$$
$$=\langle z^k\rangle(1+xz)\ldots(1+a^{rs+t-1}xz)(b+z)\ldots(b^s+z)$$

for $0\leqslant k\leqslant s$ by induction. The case $s=0$ is trivial. Suppose (4.7) is true for s. By multiplication with

$$(1+a^{rs+t}xz)\ldots(1+a^{r(s+1)+t-1}xz)(b^{s+1}+z)$$

in (4.7) we get

$$\langle z^k\rangle\sum_{n=0}^{s+1}G'_n(t,x)z^n(1+a^{rn+t}xz)\ldots$$
$$\ldots(1+a^{r(s+1)+t-1}xz)(b^{n+1}+z)\ldots(b^{s+1}+z)= \tag{4.8}$$
$$=\langle z^k\rangle(1+xz)\ldots(1+a^{r(s+1)+t-1}xz)(b+z)\ldots(b^{s+1}+z)$$

for $0\leqslant k\leqslant s$ because of induction hypothesis. ($n=s+1$ does not make any contribution to the left-hand side of (4.8).) (4.6), with s replaced by $(s+1)$, then proves (4.8) for $k=s+1$.

Third step: Division of $(1+xz)\ldots(1+a^{rs+t-1}xz)(b+z)\ldots(b^s+z)$ in (4.7) leads to

$$\langle z^k\rangle\sum_{n=0}^{s}\frac{G'_n(t,x)z^n}{(1+xz)\ldots(1+a^{rn+t-1}xz)(b+z)\ldots(b^n+z)}=\langle z^k\rangle 1$$

for $0\leqslant k\leqslant s$, hence also

$$\langle z^k\rangle\sum_{n=0}^{\infty}\frac{G'_n(t,x)z^n}{(1+xz)\ldots(1+a^{rn+t-1}xz)(b+z)\ldots(b^n+z)}=\langle z^k\rangle 1.$$

As the last identity is valid for all k, the proof of (4.2) is complete.

(4.1) $\Leftrightarrow$ (4.3). A path $w\in L'_+(n,r,t)$ has either the form $0w'$ or $1w''$ with $w'\in L'_+(n,r,t-1)$ and $w''\in L'_+(n-1,r,t+r)$. In the first case we have

$$\operatorname{des} w=\operatorname{des} w'$$
$$\alpha(w)=\alpha(w')+\operatorname{des} w'$$
$$\beta(w)=\beta(w').$$

This gives the contribution $G'_n(t-1, ax)$ on the right-hand side of (4.3) The second case is a little bit more delicate. For w'' with 1 as first letter we have

$$\begin{aligned} \operatorname{des} w &= \operatorname{des} w'' \\ \alpha(w) &= \alpha(w'') \\ \beta(w) &= \beta(w'') + \operatorname{des} w''. \end{aligned} \tag{4.9}$$

If $w'' = 0w'''$, where $w''' \in L'_+(n-1, r, t+r-1)$, then

$$\begin{aligned} \operatorname{des} w &= \operatorname{des} w'' + 1 = \operatorname{des} w''' + 1 \\ \alpha(w) &= \alpha(w'') = \alpha(w''') + \operatorname{des} w''' \\ \beta(w) &= \beta(w'') + \operatorname{des} w'' + 1 = \beta(w''') + \operatorname{des} w''' + 1. \end{aligned}$$

Therefore, if we write $G'_{n-1}(t+r, bx)$—the expression which corresponds to (4.9), supposing to be valid for all $w'' \in L'_+(n-1, r, t+r)$—on the right-hand side of (4.3), then, in counting the words of the form $10w'''$, the factor bx was missing. Consequently $(bx-1)\,G'_{n-1}(t+r-1, abx)$ has to be added to get the right recurrence relation. The initial conditions are easily checked directly.

(4.1) $\Rightarrow$ (4.4). Let a path $w \in L'_+(n, r, t_1+t_2)$ touch the line $x = ry + t_1$ at $(rk+t_1, k)$ for the first time. Therefore w can be decomposed $w'w''$, where $w' \in L'_+(k, r, t_1)$ and $w'' \in L'_+(n-k, r, t_2)$. As the last step in w' has to be a horizontal one, there holds

$$\begin{aligned} \operatorname{des} w &= \operatorname{des} w' + \operatorname{des} w'' \\ \alpha(w) &= \alpha(w') + \alpha(w'') + (rk+t_1)\operatorname{des} w'' \\ \beta(w) &= \beta(w') + \beta(w'') + k \operatorname{des} w'', \end{aligned}$$

from which (4.4) immediately follows.

(4.1) $\Rightarrow$ (4.5). This is established almost word by word as in the proof of (4.1) $\Rightarrow$ (4.4).

(B) Proofs by generating function.

(4.2) $\Leftrightarrow$ (4.3).

$$\sum_{n=0}^{\infty} \frac{[G'_n(t-1, ax) + G'_{n-1}(t+r, bx) + (bx-1)\,G'_{n-1}(t+r-1, abx)]z^n}{(1+xz)\dots(1+a^{rn+t-1}xz)(b+z)\dots(b^n+z)} =$$

$$= \frac{1}{1+xz} \sum_{n=0}^{\infty} \frac{G'_n(t-1, ax)z^n}{(1+axz)\dots(1+a^{rn+t-1}xz)(b+z)\dots(b^n+z)} +$$

$$+ \frac{z}{b+z} \sum_{n=0}^{\infty} \frac{G'_{n-1}(t+r, bx)(z/b)^{n-1}}{(1+xz)\dots(1+a^{rn+t-1}xz)(b+z/b)\dots(b^{n-1}+z/b)} +$$

$$+ \frac{(bx-1)z}{(1+xz)(b+z)} \sum_{n=1}^{\infty} \frac{G'_{n-1}(t+r-1, abx)(z/b)^{n-1}}{(1+axz)\dots(1+a^{rn+t-1}xz)(b+z/b)\dots(b^{n-1}+z/b)} =$$

$$= \frac{1}{1+xz} + \frac{z}{b+z} + \frac{(bx-1)z}{(1+xz)(b+z)} =$$

$$= 1$$

by frequent use of (4.2). Comparing the coefficients of $z^n/(1+xz)\dots(1+a^{rn+t-1}xz)(b+z)\dots(b^n+z)$ with (4.2) furnishes the recurrence relation.

(4.2) $\Rightarrow$ (4.4).

$$\sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} G'_k(t_1, x)\,G'_{n-k}(t_2, a^{rk+t_1}b^k x)z^n}{(1+xz)\dots(1+a^{rn+t_1+t_2-1}xz)(b+z)\dots(b^n+z)} =$$

$$= \sum_{k=0}^{\infty} \frac{G'_k(t_1, x)z^k}{(1+xz)\dots(1+a^{rk+t_1-1}xz)(b+z)\dots(b^k+z)}\,\cdot$$

$$\cdot \sum_{j=0}^{\infty} \frac{G'_j(t_2, a^{rk+t_1}b^k x)(z/b^k)^j}{(1+a^{rk+t_1}xz)\dots(1+a^{rk+rj+t_1+t_2-1}xz)(b+z/b^k)\dots(b^j+z/b^k)} =$$

$$= \sum_{k=0}^{\infty} \frac{G'_k(t_1, x)z^k \cdot 1}{(1+xz)\dots(1+a^{rk+t_1-1}xz)(b+z)\dots(b^k+z)} =$$

$$= 1$$

by two-fold use of (4.2). Again comparing the coefficients with (4.2), where t is replaced by (t_1+t_2), leads to the equation (4.4).

(4.2) $\Rightarrow$ (4.5). Multiplicating both sides of (4.2) for $t = t_1$ by $(1+xz)\dots(1+a^{rs+t_1+t_2-1}xz)(b+z)\dots(b^s+z)$ and equating the coefficient of z^s give

$$\langle z^s\rangle \sum_{n=0}^{s} G'_n(t_1, x)z^n(1+a^{rn+t_1}xz)\dots$$

$$\dots(1+a^{rs+t_1+t_2-1}xz)(b^{n+1}+z)\dots(b^s+z) =$$

$$= \langle z^s\rangle (1+xz)\dots(1+a^{rs+t_1+t_2-1}xz)(b+z)\dots(b^s+z).$$

By remembering Proposition 1. (3.4) we get (4.5). $\square$

Remark: The proof (4.1) $\Leftrightarrow$ (4.2) does not give a direct combinatorial interpretation of (4.2). The coefficient of each z^m is a (finite) sum of monomials in x, a, b, each of which can be represented by a symbol proceeding similar to the above proof (4.1) $\Leftrightarrow$ (4.2). We conjecture that there is a "killer involution" in the sense of Zeilberger [20] on these symbols (with the exception of $m=0$), which would directly prove that the coefficient of z^m on both sides of (4.2) is equal to δ_{m0}.

An interesting feature arises when opposing the equations (3.4) and (4.2). Let us recall the situation for $x=a=b=1$. On the one hand we have

$$|L'(n,r,t)| = \langle z^n\rangle (1+z)^{(r+1)n+t} \tag{4.10}$$

and

$$|L'_+(n,r,t)| = \langle z^n\rangle (1+z)^{(r+1)n+t-1}(1-rz), \tag{4.11}$$

on the other, by Lagrange inversion ([16, section 4.5]), we obtain the generating functions

$$\sum_{n=0}^{\infty} |L'(n,r,t)| \frac{z^n}{(1+z)^{(r+1)n+t+1}}(1-rz) = 1 \tag{4.12}$$

and

$$\sum_{n=0}^{\infty} |L'_+(n,r,t)| \frac{z^n}{(1+z)^{(r+1)n+t}} = 1. \tag{4.13}$$

Obviously (3.4) is the analogue for (4.10), (4.2) that for (4.13). Because of lack of a Lagrange inversion formula in the general case, I was not able to find analogues for (4.11) or (4.12). An exception is the case $r=1$, $a=b=q$, where a Lagrange formula exists, which can be applied to get analogues for (4.11) and (4.12) in this special case. As mentioned in the introduction even closed expressions can be given, which will be done in section 5.

The next Theorem deals with the "dual" problem of counting $L_+(n,r,t)$ by (des, α, β).

Theorem 3: The following statements are equivalent:

$$G_n(r,t,x,a,b) = \sum_{w\in L_+(n,r,t)} x^{\operatorname{des} w} a^{\alpha(w)} b^{\beta(w)}; \tag{4.14}$$

$$\sum_{n=0}^{\infty} \frac{G_n(t,x)a^n z^n}{(1+z)\dots(a^n+z)(1+bxz)\dots(1+b^{rn+t-1}xz)} = 1 \quad t\geqslant 1; \tag{4.15}$$

$$G_n(t,x) = G_n(t-1,bx) + \\ + G_{n-1}(t+r,ax) + (bx-1)G_{n-1}(t+r-1,abx) \tag{4.16}$$

for $t\geqslant 2$ and $G_n(1,x) = G_{n-1}(1+r,ax)$ with the initial conditions $G_0(t,x)=1$ and $G_n(0,x)=\delta_{n0}$.

Moreover for $t_1,t_2\geqslant 1$ there hold the convolution identities

$$G_n(t_1+t_2,x) = \sum_{k=0}^{n} G_k(t_1,x)\,[G_{n-k}(t_2,a^k b^{rk+t_1}x) + \\ + (a^k b^{rk+t_1}x-1)\,G_{n-k-1}(t_1+r,a^{k+1}b^{rk+t_1}x)] \tag{4.17}$$

and

$$A_n(t_1+t_2,x) = \sum_{k=0}^{n} G_k(t_1,x)\,[A_{n-k}(t_2,a^k b^{rk+t_1}x) + \\ + (a^k b^{rk+t_1}x-1)\,A_{n-k-1}(t_2+r,a^{k+1}b^{rk+t_1}x)] \tag{4.18}$$

where for convenience $A_{-1}(t,x) = G_{-1}(t,x) = 0$.

Proof: We proceed quite analogously to the proof of Theorem 2, in particular we adopt the terms "lhs-symbol" and "rhs-symbol".

(A) Combinatorial Proofs.

(4.14) $\Leftrightarrow$ (4.15). First step. Provided $t\geqslant 1$ we prove

$$\langle z^s\rangle \sum_{n=0}^{s} G_n(t,x)a^n z^n (a^{n+1}+z)\dots \\ \dots(a^s+z)(1+b^{rn+t}xz)\dots(1+b^{rs+t-1}xz) = \\ = \langle z^s\rangle (1+z)\dots(a^s+z)(1+bxz)\dots(1+b^{rs+t-1}xz). \tag{4.19}$$

Here the rhs-contributions can be represented by pairs $(A_1\dots A_{k+1}|B_1\dots B_k)$, where $0\leqslant A_i\leqslant s$ for $1\leqslant i\leqslant k+1$ and $1\leqslant B_i\leqslant rs+t-1$ for $1\leqslant i\leqslant k$ and k nonnegative, to which we will refer as rhs-symbols. The weight $M(A_1\dots A_{k+1}|B_1\dots B_k) = x^k a^{A_1+\dots+A_{k+1}} b^{B_1+\dots+B_k}$ gives the value of the corresponding contribution. The lhs-contributions are symbolized by triples $((C|D),n,(E|F))$, where, in view of (3.1), $(C|D)\in P_r([0,n-1]|[1,rn+t-1])$ and $(E|F)\in P([n+1,s]|[rn+t,rs+t-1])$ with weight $M((C|D),n,(E|F)) = M(C|D)a^n M(E|F)$.

Again we shall give a weight preserving bijection between the lhs- and rhs-symbols of (4.19), which would assure us of (4.19).

For a rhs-symbol $(A|B) = (A_1 \dots A_{k+1}|B_1 \dots B_k)$ let j be minimal with $rA_j + t \leqslant B_j$. Let σ_2 act on $(A|B)$ as follows:

$$\sigma_2(A_1 \dots A_{k+1}|B_1 \dots B_k) = $$
$$= ((A_1 \dots A_{j-1}|B_1 \dots B_{j-1}), A_j, (A_{j+1} \dots A_{k+1}|B_j \dots B_k)).$$

Obviously

$$(A_1 \dots A_{j-1}|B_1 \dots B_{j-1}) \in P_r([0, A_j - 1]|[1, rA_j + t - 1])$$

and

$$(A_{j+1} \dots A_{k+1}|B_j \dots B_k) \in P([A_j + 1, s]|[rA_j + t, rs + t - 1]).$$

Therefore $\sigma_2(A|B)$ belongs to the lhs-symbols. Besides, $M(\sigma_2(A|B)) = M(A|B)$, hence σ_2 is weight-preserving. Evidently, if $((C|D), n, (E|F))$ is a lhs-symbol, by

$$\bar{\sigma}_2((C|D), n, (E|F)) = (C \cup \{n\} \cup E|D \cup F)$$

the inverse map for σ_2 is given.

The second and third step essentially are the same as those in the proof of Theorem 2, (4.1) $\Leftrightarrow$ (4.2).

(4.14) $\Leftrightarrow$ (4.16). The proof of Theorem 2, (4.1) $\Leftrightarrow$ (4.3) is transferred in an obvious manner.

(4.14) $\Rightarrow$ (4.17). This time a path $w \in L_+(n, r, t_1 + t_2)$ is decomposed $w'w''$, where $w' \in L_+(k, r, t_1)$, $w'' \in L_+(n-k, r, t_2)$. Similarly to the proof of (4.1) $\Leftrightarrow$ (4.3) in Theorem 2, we have to distinguish between those w'' which start with a 1 and those with 0 as first letter. The latter class of words makes it necessary to add the term $(a^k b^{rk+t_1} - 1)\, G_{n-k-1}(t_2 + r, a^{k+1} b^{rk+t_1} x)$ on the right-hand side of (4.17).

(4.14) $\Rightarrow$ (4.18) is established almost word by word as before (4.14) $\Rightarrow$ (4.17).

(B) Proofs by generating functions.

(4.15) $\Leftrightarrow$ (4.16). If $t \geqslant 2$ we have by repeated use of (4.15)

$$\sum_{n=0}^{\infty} \frac{[G_n(t-1, bx) + G_{n-1}(t+r, ax) + (bx-1)\, G_{n-1}(t+r-1, abx)]\, a^n z^n}{(1+z)\dots(a^n+z)(1+bxz)\dots(1+b^{rn+t-1}xz)} =$$

$$= \frac{1}{1+bxz} \sum_{n=0}^{\infty} \frac{G_n(t-1, bx)\, a^n z^n}{(1+z)\dots(a^n+z)(1+b^2xz)\dots(1+b^{rn+t-1}xz)} +$$
$$+ \frac{z}{1+z} \sum_{n=1}^{\infty} \frac{G_{n-1}(t+r, ax)\, a^{n-1}(z/a)^{n-1}}{(1+z/a)\dots(a^{n-1}+z/a)(1+bxz)\dots(1+b^{rn+t-1}xz)} +$$
$$+ \frac{(bx-1)\,z}{(1+z)(1+bxz)} \sum_{n=1}^{\infty} \frac{G_{n-1}(t+r-1, abx)\, a^{n-1}(z/a)^{n-1}}{(1+z/a)\dots(a^{n-1}+z/a)(1+b^2xz)\dots(1+b^{rn+t-1}xz)} =$$
$$= \frac{1}{1+bxz} + \frac{z}{1+z} + \frac{(bx-1)\,z}{(1+z)(1+bxz)} =$$
$$= 1.$$

Comparing the coefficients of $a^n z^n/(1+z)\dots(a^n+z)(1+bxz)\dots(1+b^{rn+t-1}xz)$ with (4.15) we get the recurrence relation for $t \geqslant 2$. For $t = 1$ we start with

$$\sum_{n=1}^{\infty} \frac{G_{n-1}(1+r, ax)\, a^n z^n}{(1+z)\dots(a^n+z)(1+bxz)\dots(1+b^{rn}xz)} =$$
$$= \frac{z}{1+z} \sum_{n=1}^{\infty} \frac{G_{n-1}(1+r, ax)\, a^{n-1}(z/a)^{n-1}}{(1+z/a)\dots(a^{n-1}+z/a)(1+bxz)\dots(1+b^{rn}xz)} =$$
$$= \frac{z}{1+z} =$$
$$= 1 - \frac{1}{1+z} =$$
$$= \sum_{n=1}^{\infty} \frac{G_n(1, x)\, a^n z^n}{(1+z)\dots(a^n+z)(1+bxz)\dots(1+b^{rn}xz)}.$$

Again only comparing of coefficients has to be done.

(4.15) $\Rightarrow$ (4.17).

$$\sum_{n=0}^{\infty} \frac{a^n z^n G_k(t_1, x)\,[G_{n-k}(t_2, a^k b^{rk+t_1} x) + (a^k b^{rk+t_1} x - 1)\, G_{n-k-1}(t_2 + r, a^{k+1} b^{rk+t_1} x)]}{(1+z)\dots(a^n+z)(1+bxz)\dots(1+b^{rn+t_1+t_2}xz)} =$$
$$= \sum_{k=0}^{\infty} \frac{G_k(t_1, x)\, a^k z^k}{(1+z)\dots(a^k+z)(1+bxz)\dots(1+b^{rk+t_1-1}xz)}\,.$$
$$\cdot \sum_{j=0}^{\infty} \frac{a^j z^j\,[G_j(t_2, a^k b^{rk+t_1} x) + (a^k b^{rk+t_1} x - 1)\, G_{j-1}(t_2 + r, a^{k+1} b^{rk+t_1} x)]}{(a^{k+1}+z)\dots(a^{k+j}+z)(1+b^{rk+t_1}xz)\dots(1+b^{r(k+j)+t_1+t_2-1}xz)} =$$

$$=\sum_{k=0}^{\infty}\frac{G_k(t_1,x)a^k z^k}{(1+z)\dots(a^k+z)(1+bxz)\dots(1+b^{rk+t_1-1}xz)}\cdot$$

$$\left(\frac{1+z/a^k}{1+b^{rk+t_1}xz}+\frac{(a^k b^{rk+t_1}x-1)z/a^k}{1+b^{rk+t_1}xz}\right)=$$

$$=\sum_{k=0}^{\infty}\frac{G_k(t_1,x)a^k z^k\cdot 1}{(1+z)\dots(a^k+z)(1+bxz)\dots(1+b^{rk+t_1-1}xz)}=$$

$$=1$$

by repeated use of (4.15), as desired.

(4.15) $\Rightarrow$ (4.18). Multiplication of (4.15) with $t=t_1$ by $(1+z)\dots(a^{s-1}+z)(1+bxz)\dots(1+b^{rs+t_1+t_2}xz)$ and equating coefficients of z^s yield

$$\langle z^s\rangle\sum_{n=0}^{s}G_n(t_1,x)a^n z^n(a^{n+1}+z)\dots$$
$$\dots(a^{s-1}+z)(1+b^{rn+t_1}xz)\dots(1+b^{rs+t_1+t_2}xz)=$$
$$=\langle z^s\rangle(1+z)\dots(a^{s-1}+z)(1+bxz)\dots(1+b^{rs+t_1+t_2}xz).$$

Little arrangement turns this equation into

$$\langle z^s\rangle\sum_{n=0}^{s}G_n(t_1,x)z^n(a^n+z)(a^{n+1}+z)\dots$$
$$\dots(a^{s-1}+z)(1+b^{rn+t_1+1}xz)\dots(1+b^{rs+t_1+t_2}xz)+$$
$$+\langle z^s\rangle\sum_{n=0}^{s}G_n(t_1,x)z^{n+1}(a^n b^{rn+t_1}x-1)(a^{n+1}+z)\dots$$
$$\dots(a^{s-1}+z)(1+b^{rn+t_1+1}xz)(1+b^{rs+t_1+t_2}xz)=$$
$$=\langle z^s\rangle(1+z)\dots(a^{s-1}+z)(1+bxz)\dots(1+b^{rs+t_1+t_2}xz),$$

which by (3.3) is just (4.18).

□

Remark: In general the numbers $G_n(t,x)$ and $G'_n(t,x)$ are completely different extensions of the classical Gould numbers. Only in two special cases I succeeded in finding a relation between them, again utilizing the correspondence π.

For $t=1$ each path of $L'_+(n,r,1)$ via π can be identified with a pair $(A|B)=(A_1\dots A_k|B_1\dots B_k)\in P'_r([0,rn]|[1,n])$. Let

$$\vartheta_1(A_1\dots A_k|B_1\dots B_k)=(B_1\dots B_{k-1}|A_2\dots A_k).$$

Then we observe $A_1=0$ and $B_k=n$ since each path of $G'_n(1,x)$ has to start with a vertical and to end with a horizontal step, hence

$$M(A|B)=b^n x\tilde{M}(\vartheta_1(A|B)),\tag{4.20}$$

where $\tilde{M}$ means the weight function arising when exchanging a and b. Moreover, by definition $A_i<rB_{i-1}+1$, therefore $(B_1\dots B_{k-1}|A_2\dots A_k)$ is an element of $P_r([1,n-1]|[1,rn])=P_r([0,n-1]|[1,rn])$ since, supposing $(C_1\dots C_k|D_1\dots D_k)\in P_r([0,n-1]|[1,rn])$, by the condition $rC_1+1>D_1$ we get for $r\geqslant 1$ $C_1>0$ because of $D_1>0$, the case $r=0$ being clear from the start. In addition to that, ϑ_1 is seen to be a bijection between $P'_r([0,rn]|[1,n])$ and $P_r([0,n-1]|[1,rn])$. In view of (4.20) this implies

$$G'_n(r,1,x,a,b)=b^n xG_n(r,1,x,b,a).\tag{4.21}$$

The second case is $x=1$, where we define for

$$(A_1\dots A_k|B_1\dots B_k)\in P'_r([0,rn+t-1]|[1,n])$$

$$\vartheta_2(A_1\dots A_k|B_1\dots B_k)=\begin{cases}(B_1\dots B_{k-1}|A_2\dots A_k) & \text{for } A_1=0\\ (0B_1\dots B_{k-1}|A_1\dots A_k) & \text{for } A_1\neq 0.\end{cases}$$

Again we observe $B_k=n$, hence, because of $x=1$,

$$M(A|B)=b^n\tilde{M}(\vartheta_2(A|B)).\tag{4.22}$$

As ϑ_2 turns out to be a bijection between $P'_r([0,rn+t-1]|[1,n])$ and $P_r([0,n-1]|[1,rn+t-1])$, by (4.22) we obtain

$$G'_n(r,t,1,a,b)=b^n G_n(r,t,1,b,a).\tag{4.23}$$

Next we turn to the reverse statistic $\bar{M}$. I was not able to find any generating function for the resulting numbers $\bar{G}_n(r,t,x,a,b)$ or $\bar{G}'_n(r,t,x,a,b)$, respectively. Still we succeed in finding analogues of the recurrence relation and convolution formulas. These facts are subsumed in

Theorem 4: For

$$\bar{G}_n(r,t,x,a,b)=\sum_{w\in L_+(n,r,t)} x^{\overline{\operatorname{des}}\,w} a^{\bar{\alpha}(w)} b^{\beta(w)} \tag{4.24}$$

there holds

$$\begin{aligned}\bar{G}_n(t,x)&=\bar{G}_n(t-1,x)+\bar{G}_{n-1}(t+r,x)+\\&+(a^{n-1}b^{rn+t}x-1)\,\bar{G}_{n-1}(t+r-1,x)\end{aligned} \tag{4.25}$$

for $t\geqslant 1$ with $\bar{G}_0(t,x)=1$ and $\bar{G}_n(0,x)=\delta_{n0}$,

$$\bar{G}_n(t_1+t_2,x)=\sum_{k=0}^{n}\bar{G}_k(t_1,x)\,\bar{G}_{n-k}(t_2,a^k b^{rk+t_1}x), \tag{4.26}$$

$$A_n(t_1+t_2,x)=\sum_{k=0}^{n}A_k(t_1,x)\,\bar{G}_{n-k}(t_2,a^k b^{rk+t_1}x). \tag{4.27}$$

If

$$\bar{G}'_n(t,x)=\sum_{w\in L'_+(n,r,t)} x^{\overline{\operatorname{des}}\,w} a^{\bar{\alpha}(w)} b^{\beta(w)} \tag{4.28}$$

then

$$\begin{aligned}\bar{G}'_n(t,x)&=\bar{G}'_n(t-1,x)+\bar{G}'_{n-1}(t+r,x)+\\&+(a^{rn+t-1}b^{n}x-1)\,\bar{G}'_{n-1}(t+r-1,x)\end{aligned} \tag{4.29}$$

for $t\geqslant 2$ and $\bar{G}'_n(1,x)=\bar{G}'_{n-1}(1+r,x)$ with the initial conditions $\bar{G}'_0(t,x)=1$ and $\bar{G}'_n(0,x)=\delta_{n0}$, and for $t_1,t_2\geqslant 1$

$$\begin{aligned}\bar{G}'_n(t_1+t_2,x)&=\sum_{k=0}^{n}[\bar{G}'_k(t_1,x)+\\&+(a^{rk+t_1}b^k x-1)\,\bar{G}'_{k-1}(t_1+r,x)]\,\bar{G}'_{n-k}(t_2,a^{rk+t_1}b^k x)\end{aligned} \tag{4.30}$$

$$\begin{aligned}A'_n(t_1+t_2,x)&=\sum_{k=0}^{n}[A'_k(t_1,x)+\\&+(a^{rk+t_1}b^k x-1)\,A'_{k-1}(t_1+r,x)]\,\bar{G}'_{n-k}(t_2,a^{rk+t_1}b^k x)\end{aligned} \tag{4.31}$$

where $A'_{-1}(t,x)=\bar{G}'_{-1}(t,x)=0$.

Proof: We only have to transfer the arguments of the proofs of Theorem 2 and 3. We leave the details to the reader. □

(to be continued)

References

[1] Carlitz, L.: Sequences, paths, ballot numbers. Fibonacci Quart. **10** (1972), 531–549.

[2] Carlitz, L., and J. Riordan: Two element lattice permutation numbers and their q-generalization. Duke J. Math. **31** (1964), 371–388.

[3] Carlitz, L., and R. Scoville: A note on weighted sequences. Fibonacci Quart. **13** (1975), 303–306.

[4] Cigler, J.: Operatormethoden für q-Identitäten. Monatsh. Math. **88** (1979), 87–105.

[5] Comtet, L.: Advanced combinatorics. Reidel, Boston 1974.

[6] Fürlinger, J., and J. Hofbauer: q-Catalan numbers. J. Combin. Theory **A40** (1985), 248–264.

[7] Garsia, A. M., and I. Gessel: Permutation statistics and partitions. Adv. in Math. **31** (1979), 288–305.

[8] Gessel, I., and D. Stanton: Application of q-Lagrange inversion to basic hypergeometric series. Trans. Amer. Math. Soc. **277** (1983), 173–201.

[9] Gould, H. W.: Some generalizations of Vandermonde's convolution. Amer. Math. Monthly **63** (1956), 84–91.

[10] Gould, H. W.: Final analysis of Vandermonde's convolution. Amer. Math. Monthly **64** (1957), 409–415.

[11] Hofbauer, J.: private communication.

[12] Krattenthaler, C.: A new q-Lagrange formula and some applications. Proc. Amer. Math. Soc. **90** (1984), 338–344.

[13] MacMahon, P. A.: Combinatory Analysis II. Cambridge Univ. Press, London 1915–1916. Reprinted, Chelsea, New York 1960.

[14] MacMahon, P. A.: Collected papers: Combinatorics, Vol. I. MIT Press, Cambridge, Mass. 1978.

[15] Mohanty, S. G.: Lattice path counting and applications. Academic Press, New York 1979.

[16] Riordan, J.: Combinatorial Identities. Wiley, New York 1968.

[17] Stanley, R. P.: Ordered structures and partitions, thesis. Harvard University (January 1971).

[18] Stanley, R. P.: Ordered structures and partitions. Mem. Amer. Math. Soc. **119** (1972).

[19] Stanley, R. P.: Theory and applications of plane partitions, Part I. Studies Appl. Math. **50** (1971), 167–188; Part II. Studies Appl. Math. **50** (1971), 259–279.

[20] Zeilberger, D.: A combinatorial approach to matrix algebra. Discrete Math. **56** (1985), 61–72.

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