# Counting Lattice Paths with a Linear Boundary II: $q$-Ballot and $q$-Catalan Numbers

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**Abstract**

Wir besprechen zwei wichtige Spezialfälle der Resultate über die Abzählung von Gitterpunktwegen, die eine gegebene Gerade nicht berühren, welche in der eigenen Arbeit „Counting lattice paths with a linear boundary" (Sitzungsberichte der mathem.-naturwissenschaftl. Klasse, Abt. II, 198. Band, 1.—3. Heft, 1989, 87—107) enthalten sind. Im ersten Fall erhalten wir verschiedene Typen von $q$-ballot-Zahlen, für die wir schöne geschlossene Darstellungen angeben können. Im zweiten Fall ergeben sich die von J. Fürlinger und J. Hofbauer (J. Combin. Theory **A 40** (1985), 248—264) eingeführten $q$-Catalan-Zahlen. Wir ergänzen ihre Resultate um einige neue und lösen einige der von ihnen gestellten Probleme.

## 5. Ballot Numbers

In this section the case $r = 1$, $a = b = q$ shall be discussed. Because of $\alpha(w) + \beta(w) = \operatorname{maj} w$ this means that we are dealing with the problem of counting $L_+(n, 1, t)$ $(L'_+(n, 1, t))$ by (des, maj). Since $|L_+(n, 1, t)|$ is a ballot number (for the connection between lattice paths and Bertrand's ballot problem see [15, pp. 1–4]), we shall obtain extensions of ballot numbers.

As Fürlinger and Hofbauer pointed out in [6], the case $r = 1$, $a = b = q$ gains its importance from the fact that $q$-Lagrange inversion can be used to evaluate the $q$-ballot numbers $B_n(q, t, x)$ and $B'_n(q, t, x)$ explicitely from their generating functions. The needed $q$-Lagrangeformula due to Gessel and Stanton [8] and the author [12] reads ([12, Theorem 1.(A) with $\varphi(z) = x/(1+xz)$, $\Phi(z) = 1/(1+z)$, $\mu = 0$ and $\lambda = t-1$; see Example 3]): If $t \geqslant 1$ and

$$f(z) = \sum_{n=0}^{\infty} \frac{c_n z^n}{(1+q^{-n}z)\dots(1+q^{-1}z)(1+qxz)\dots(1+q^{n+t-1}xz)},$$

then for $n \geqslant 1$

$$c_n = \langle z^n \rangle f(z)(1+qxz)\dots(1+q^{n+t-2}xz)(1+q^{-1}z)\dots \dots(1+q^{-n+1}z)(1-q^{t-1}xz^2). \tag{5.1}$$

This leads to the following results:

**Theorem 5:** Let $t \geqslant 1$. If

$$B_n(q,t,x) = \sum_{w \in L_+(n,1,t)} x^{\operatorname{des} w} q^{\operatorname{maj} w} \tag{5.2}$$

then

$$\sum_{n=0}^{\infty} \frac{B_n(q,t,x)\, q^{-\binom{n}{2}} z^n}{(1+q^{-n}z)\dots(1+z)(1+qxz)\dots(1+q^{n+t-1}xz)} = 1 \tag{5.3}$$

and

$$B_n(q,t,x) = \sum_{k=0}^{n-1} x^k q^{k^2} \frac{(q^k[n+t-k-1] + [t-1][k])}{[n+t-1][n-k]} \begin{bmatrix} n+t-1 \\ k \end{bmatrix} \begin{bmatrix} n \\ k+1 \end{bmatrix}, \tag{5.4}$$

in particular for $x = 1$

$$B_n(q,t,1) = \frac{[t]}{[n+t]} \begin{bmatrix} 2n+t-1 \\ n \end{bmatrix}. \tag{5.5}$$

If

$$B'_n(q,t,x) = \sum_{w \in L'_+(n,1,t)} x^{\operatorname{des} w} q^{\operatorname{maj} w} \tag{5.6}$$

then

$$\sum_{n=0}^{\infty} \frac{B'_n(q,t,x)\, q^{-\binom{n+1}{2}} z^n}{(1+q^{-n}z)\dots(1+q^{-1}z)(1+xz)\dots(1+q^{n+t-1}xz)} = 1 \tag{5.7}$$



and

$$B'_n(q,t,x) = q^n \sum_{k=1}^{n} x^k q^{k^2-k} \frac{[t]}{[n]} \begin{bmatrix} n+t-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix}, \tag{5.8}$$

in particular for $x = 1$

$$B'_n(q,t,1) = q^n \frac{[t]}{[n+t]} \begin{bmatrix} 2n+t-1 \\ n \end{bmatrix}. \tag{5.9}$$

Proof: (5.3) and (5.7) are just rearrangements of (4.15) and (4.2) for the special case $r = 1$, $a = b = q$. Application of (5.1) with $f(z) = (1+z)$ to (5.3) yields

$$q^{-\binom{n}{2}} B_n(t,x) = \langle z^n \rangle (1+qxz)\dots(1+q^{n+t-2}xz)(1+z)\dots$$

$$\dots(1+q^{-n+1}z)(1-q^{t-1}xz^2) = \tag{5.10}$$

$$= \sum_{k=0}^{n} x^k q^{\binom{k+1}{2}} \begin{bmatrix} n+t-2 \\ k \end{bmatrix} q^{\binom{n-k}{2}-(n-1)(n-k)} \begin{bmatrix} n \\ n-k \end{bmatrix} -$$

$$- \sum_{k=0}^{n} x^k q^{\binom{k}{2}} \begin{bmatrix} n+t-2 \\ k-1 \end{bmatrix} q^{\binom{n-k-1}{2}-(n-1)(n-k-1)} \begin{bmatrix} n \\ n-k-1 \end{bmatrix} q^{t-1},$$

by (2.9). This identity reduces to

$$B_n(t,x) = \sum_{k=0}^{n} x^k q^{k^2} \left( \begin{bmatrix} n+t-2 \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} - q^{t-1} \begin{bmatrix} n+t-2 \\ k-1 \end{bmatrix} \begin{bmatrix} n \\ k+1 \end{bmatrix} \right) \tag{5.11}$$

and finally to (5.4), by short evaluation. (5.5) can be derived by $q$-Vandermonde-convolution ([4, identity (18)]) from (5.11) or directly by (5.10) utilizing (2.9) again.

Application of (5.1) with $f(z) = (1+xz)$ to (5.7) gives

$$q^{-\binom{n+1}{2}} B'_n(t,x) = \langle z^n \rangle (1+xz)\dots(1+q^{n+t-2}xz)(1+q^{-1}z)\dots$$

$$\dots(1+q^{-n+1}z)(1-q^{t-1}xz^2) = \tag{5.12}$$

$$= \sum_{k=0}^{n} x^k q^{\binom{k}{2}} \begin{bmatrix} n+t-1 \\ k \end{bmatrix} q^{\binom{n-k}{2}-(n-1)(n-k)} \begin{bmatrix} n-1 \\ n-k \end{bmatrix} -$$

$$- \sum_{k=0}^{n} x^k q^{\binom{k-1}{2}} \begin{bmatrix} n+t-1 \\ k-1 \end{bmatrix} q^{\binom{n-k-1}{2}-(n-1)(n-k-1)} \begin{bmatrix} n-1 \\ n-k-1 \end{bmatrix} q^{t-1}$$

or equivalently

$$B'_n(t,x) = q^n \sum_{k=1}^{n} x^k q^{k^2-k} \left( \begin{bmatrix} n+t-1 \\ k \end{bmatrix} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - \right. \tag{5.13}$$
$$\left. - q^t \begin{bmatrix} n+t-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ k \end{bmatrix} \right),$$

which can be transformed into (5.8). Again (5.9) is proved either by Vandermonde-convolution from (5.13) or by combining (5.12) and (2.9). □

Remark: The fact $B'_n(q,t,1) = q^n B_n(q,t,1)$, which is observed by comparison of (5.5) and (5.9), was explained before by the arguments which led to (4.23).

Although we were not able to find any generating function for $\overline{B}_n(t,x)$ and $\overline{B}'_n(t,x)$, respectively, nevertheless, it is possible to give expressions similar to (5.4), (5.5), (5.8) and (5.9) for these $q$-ballot numbers, which will be stated in Theorem 6 below. A proof for (5.20), using the recurrence relation (4.29) for $r=1$, $a=b=q$, can be found in [6, p. 257]. Moreover, in the cited paper Fürlinger and Hofbauer indicate how a combinatorial proof might be given. We have found such a proof, slight modification of which also helps to find the results for $\overline{B}_n(t,x)$.

As mentioned in section 2 each path $w$ of $L'_+(n,1,t)$ has to end with a horizontal step, i.e. has the form $w = w'0$, where $w'$ is a path from $(0,0)$ to $(n+t-1,n)$ not entering the line $x = y+t-1$. Let $l'_+(n,1,t)$ denote the set of all these paths $w'$. Analogously we call the set of all paths from $(0,0)$ to $(n, n+t-1)$ not entering the line $y = x+t-1$ $l_+(n,1,t)$. Since the last step of a path of $l'_+(n,1,t)$ has to be a horizontal one, we gain

$$\overline{B}'_n(t,x) = \overline{b}'_n(t,qx), \tag{5.14}$$

where we have set

$$\overline{b}'_n(t,x) = \sum_{w \in l'_+(n,1,t)} x^{\overline{\operatorname{des} w}} q^{\overline{\operatorname{maj} w}}.$$

Similarly, if

$$\overline{b}_n(t,x) = \sum_{w \in l_+(n,1,t)} x^{\overline{\operatorname{des} w}} q^{\overline{\operatorname{maj} w}},$$



then

$$\overline{B}_n(t,x) = \overline{b}_n(t,qx). \tag{5.15}$$

We remark that, if we would define $b_n(t,x)$ and $b'_n(t,x)$ analogously, we would get $B_n(t,x) = b_n(t,x)$ and $B'_n(t,x) = b'_n(t,x)$.

**Theorem 6:** Let $t \geqslant 1$. If

$$\overline{B}_n(q,t,x) = \sum_{w \in L_+(n,1,t)} x^{\overline{\operatorname{des} w}} q^{\overline{\operatorname{maj} w}} \tag{5.16}$$

then

$$\overline{B}_n(q,t,x) = q^n \sum_{k=1}^{n} (qx)^k q^{k^2-k} \frac{[t]}{[n]} \begin{bmatrix} n+t-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix}, \tag{5.17}$$

in particular for $x = 1/q$

$$\overline{B}_n(q,t,1/q) = q^n \frac{[t]}{[n+t]} \begin{bmatrix} 2n+t-1 \\ n \end{bmatrix}. \tag{5.18}$$

If

$$\overline{B}'_n(q,t,x) = \sum_{w \in L'_+(n,r,t)} x^{\overline{\operatorname{des} w}} q^{\overline{\operatorname{maj} w}} \tag{5.19}$$

then

$$\overline{B}'_n(q,t,x) = \sum_{k=0}^{n-1} (qx)^k q^{k^2+k} \cdot \tag{5.20}$$
$$\cdot \frac{([n+t-k-1] + q^{n-k+1}[t-1][k])}{[n+t-1][n-k]} \begin{bmatrix} n+t-1 \\ k \end{bmatrix} \begin{bmatrix} n \\ k+1 \end{bmatrix}$$

and in particular for $x = 1/q$

$$\overline{B}'_n(q,t,1/q) = \frac{(1+q^{n+1}[t-1])}{[n+t]} \begin{bmatrix} 2n+t-1 \\ n \end{bmatrix}. \tag{5.21}$$

Proof: We begin with the proof of (5.20), for which we proceed in several steps. The main tool is the correspondence $\bar{\pi}$ by which we consider paths from $(0,0)$ to $(n+t-1,n)$ as pairs $(A_1 \ldots A_k | B_1 \ldots B_k) \in P([0, n+t-2] | [1,n])$, where $k$ is the number of "reverse" descents.

First step: Each path of $l'_+(n,1,t)$ has as last step a horizontal one. Therefore $l'_+(n,1,t)$ is equal to the set-theoretic difference of the set of all paths from $(0,0)$ to $(n+t-1,n)$ ending with a 0 and the set of all paths from $(0,0)$ to $(n+t-1)$ ending with a 0 and entering the line $x=y+t-1$. Let us write (5.20) in the (combinatorially) more suggestive form

$$\overline{B}'_n(t,x) = \sum_{k=0}^{n} (qx)^k q^{k^2+k} \cdot \left( \begin{bmatrix} n+t-2 \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} - q \begin{bmatrix} n+t-2 \\ k-1 \end{bmatrix} \begin{bmatrix} n \\ k+1 \end{bmatrix} \right) \tag{5.22}$$

then we will see that the first of the above mentioned two sets generates the first expression on the right-hand side of (5.22), the latter the second.

Second step: For a fixed $k$ we claim that the weighted sum $\sum \overline{M}(w)$, where the summation is over all paths $w$ from $(0,0)$ to $(n+t-1,n)$ of the form $w=w'0$ with $\overline{\text{des}}\, w = k$, is equal to

$$x^k q^{k^2+k} \begin{bmatrix} n+t-2 \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix},$$

which, in view of (5.14), would explain the first expression on the right-hand side of (5.22). To see this, we observe that by means of $\bar{\pi}$ there is a weight-preserving bijection between all those paths and

$$P\left( \begin{matrix} [1,n+t-2] \\ k \end{matrix} \,\middle|\, \begin{matrix} [1,n] \\ k \end{matrix} \right),$$

where the weight on

$$P\left( \begin{matrix} [1,n+t-2] \\ k \end{matrix} \,\middle|\, \begin{matrix} [1,n] \\ k \end{matrix} \right)$$

is the restriction of $M$ to this set. But, according to (2.9), the weighted sum $\sum M(v)$ over all elements $v$ of

$$P\left( \begin{matrix} [1,n+t-2] \\ k \end{matrix} \,\middle|\, \begin{matrix} [1,n] \\ k \end{matrix} \right)$$

is just

$$x^k q^{\binom{k+1}{2}} \begin{bmatrix} n+t-2 \\ k \end{bmatrix} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix},$$

which was to prove.



Third step: Again in view of (5.14) it remains to show that the weighted sum $\sum q^{\overline{\text{maj}}\, w}$, summed up over all paths from $(0,0)$ to $(n+t-1,n)$ of the form $w=w'0$ with $\overline{\text{des}}\, w = k$ and entering the line $x=y+t-1$, is equal to

$$q^{k^2+k+1} \begin{bmatrix} n+t-2 \\ k-1 \end{bmatrix} \begin{bmatrix} n \\ k+1 \end{bmatrix}.$$

In the sequel we shall use the following convention: For a pair

$$(A_1 \dots A_k | B_1 \dots B_l) \in P\left( \begin{matrix} [a_1,a_2] \\ k \end{matrix} \,\middle|\, \begin{matrix} [b_1,b_2] \\ l \end{matrix} \right)$$

the weight function $m$ is defined by

$$m(A_1 \dots A_k | B_1 \dots B_l) = q^{A_1 + \dots + A_k + B_1 + \dots + B_l}.$$

Via the correspondence $\bar{\pi}$ we see that there is a weight preserving bijection between the paths above (with weight $\overline{\text{maj}}$) and the subset

$$P_-\left( \begin{matrix} [1,n+t-2] \\ k \end{matrix} \,\middle|\, \begin{matrix} [1,n] \\ k \end{matrix} \right) \text{ of } P\left( \begin{matrix} [1,n+t-2] \\ k \end{matrix} \,\middle|\, \begin{matrix} [1,n] \\ k \end{matrix} \right),$$

containing all pairs $(A_1 \dots A_k | B_1 \dots B_k)$ of

$$P\left( \begin{matrix} [1,n+t-2] \\ k \end{matrix} \,\middle|\, \begin{matrix} [1,n] \\ k \end{matrix} \right)$$

where there exists an $i$ with $A_i < B_i$ (with weight $m$).

What we are going to set up, is a weight preserving one-to-one correspondence between

$$P_-\left( \begin{matrix} [1,n+t-2] \\ k \end{matrix} \,\middle|\, \begin{matrix} [1,n] \\ k \end{matrix} \right) \text{ and } P\left( \begin{matrix} [1,n+t-2] \\ k-1 \end{matrix} \,\middle|\, \begin{matrix} [1,n] \\ k+1 \end{matrix} \right).$$

This would complete the proof of (5.22) (and hence (5.20)) because, by utilizing (2.9) once more, the weighted sum $\sum m(v)$ summed over all

$$w \in P\left( \begin{matrix} [1,n+t-2] \\ k-1 \end{matrix} \,\middle|\, \begin{matrix} [1,n] \\ k+1 \end{matrix} \right)$$

is equal to

$$q^{\binom{k}{2}}\begin{bmatrix} n+t-2 \\ k-1 \end{bmatrix} q^{\binom{k+2}{2}}\begin{bmatrix} n \\ k+1 \end{bmatrix},$$

which explains the second expression on the right-hand side of (5.22).

Fourth step: Here we construct a map

$$\tau\colon P_{-}\left(\begin{matrix}[1,n+t-2] \\ k\end{matrix}\,\middle|\,\begin{matrix}[1,n] \\ k\end{matrix}\right) \to P\left(\begin{matrix}[1,n+t-2] \\ k-1\end{matrix}\,\middle|\,\begin{matrix}[1,n] \\ k+1\end{matrix}\right).$$

For a pair

$$(X\,|\,Y)\in P\left(\begin{matrix}[1,a_2] \\ k_1\end{matrix}\,\middle|\,\begin{matrix}[1,b_2] \\ k_2\end{matrix}\right)$$

we introduce an integer valued function $k_{(X|Y)}$ acting on natural numbers by

$$k_{(X|Y)}(l) = \sum_{i=1}^{l} \chi(i\in X) - \sum_{i=1}^{l} \chi(i\in Y), \tag{5.23}$$

where $\chi(\mathscr{A}) = 1$ if $\mathscr{A}$ is true, $\chi(\mathscr{A}) = 0$ otherwise. Let

$$(A\,|\,B) = (A_1 \ldots A_k\,|\,B_1 \ldots B_k)\in P_{-}\left(\begin{matrix}[1,n+t-2] \\ k\end{matrix}\,\middle|\,\begin{matrix}[1,n] \\ k\end{matrix}\right).$$

Then among the elements of $A\setminus B$ (which means the set-difference $\{A_1,\ldots,A_k\}\setminus\{B_1,\ldots,B_k\}$) we look for the smallest $c_1$ with largest $k_{(A|B)}$-value; or, more precisely, $c_1$ satisfies

$$\begin{aligned} &\text{either (i) } k_{(A|B)}(d) < k_{(A|B)}(c_1) \\ &\text{or } \quad\ \text{(ii) } k_{(A|B)}(d) = k_{(A|B)}(c_1) \text{ and } c_1 \leqslant d \end{aligned} \tag{5.24}$$

for all $d\in A\setminus B$. $A\setminus B$ is non-empty because for the smallest $j$ with $A_j < B_j$ we have $A_i > A_{i-1} \geqslant B_{i-1}$ for all $i \leqslant j$ and so $A_j > B_i$ for $i<j$ and $A_j < B_i$ for $i \geqslant j$, which implies $A_j\in A\setminus B$. This assures us of the existence of $c_1$. Evidently, $k_{(A|B)}(A_j) = j-(j-1) = 1$, hence $k_{(A|B)}(c_1)\geqslant 1$.

Assuming $c_1 > n$ we estimate

$$\begin{aligned} k_{(A|B)}(c_1) &= \sum_{i=1}^{c_1} \chi(i\in A) - \sum_{i=1}^{c_1} \chi(i\in B) \\ &= \sum_{i=1}^{c_1} \chi(i\in A) - k \\ &\leqslant k-k \\ &\leqslant 0 \end{aligned}$$



in contradiction to $k_{(A|B)}(c_1)\geqslant 1$. Therefore $c_1 \leqslant n$, and the map

$$\tau\colon P_{-}\left(\begin{matrix}[1,n+t-2] \\ k\end{matrix}\,\middle|\,\begin{matrix}[1,n] \\ k\end{matrix}\right) \to P\left(\begin{matrix}[1,n+t-2] \\ k-1\end{matrix}\,\middle|\,\begin{matrix}[1,n] \\ k+1\end{matrix}\right)$$

can be defined by

$$\tau(A\,|\,B) = \tau(A\setminus\{c_1\}\,|\,B\cup\{c_1\}), \tag{5.25}$$

where $c_1$ is constructed satisfying (5.24). As $\tau$, by definition, is seen to be weight-preserving, we have to show that $\tau$ is bijective.

Fifth step: We are going to construct the inverse map

$$\bar\tau\colon P\left(\begin{matrix}[1,n+t-2] \\ k-1\end{matrix}\,\middle|\,\begin{matrix}[1,n] \\ k+1\end{matrix}\right) \to P_{-}\left(\begin{matrix}[1,n+t-2] \\ k\end{matrix}\,\middle|\,\begin{matrix}[1,n] \\ k\end{matrix}\right)$$

of $\tau$. Let

$$(\tilde A\,|\,\tilde B)\in P\left(\begin{matrix}[1,n+t-2] \\ k-1\end{matrix}\,\middle|\,\begin{matrix}[1,n] \\ k+1\end{matrix}\right).$$

If $|\tilde A\setminus\tilde B| = l$, then, obviously, $|\tilde B\setminus\tilde A| = l+2$ and therefore $\tilde B\setminus\tilde A$ is nonempty.

First case: $k_{(\tilde A|\tilde B)}(d)\leqslant -1$ for all $d\in\tilde A\setminus\tilde B$. Let

$$c_2 = \min\{e : e\in\tilde B\setminus\tilde A\}. \tag{5.26}$$

Supposing $d\in\tilde A\setminus\tilde B$ with $d < c_2$, we get

$$\begin{aligned} k_{(\tilde A|\tilde B)}(d) &= \sum_{i=1}^{d} \chi(i\in\tilde A) - \sum_{i=1}^{d} \chi(i\in\tilde B) \\ &= \sum_{i=1}^{d} \chi(i\in\tilde A\setminus\tilde B) - \sum_{i=1}^{d} \chi(i\in\tilde B\setminus\tilde A) \\ &= \sum_{i=1}^{d} \chi(i\in\tilde A\setminus\tilde B) \\ &\geqslant 0 \end{aligned}$$

in contradiction to $k_{(\tilde A|\tilde B)}(d)\leqslant -1$. Then we define

$$\bar\tau(\tilde A\,|\,\tilde B) = (\tilde A\cup\{c_2\}\,|\,\tilde B\setminus\{c_2\}) \tag{5.27}$$

with $c_2$ given by (5.26). Setting $A = \tilde{A} \cup \{c_2\}$ and $B = \tilde{B} \setminus \{c_2\}$ we obtain, regarding the consideration above,

$$\begin{aligned} k_{(A|B)}(c_2) &= \sum_{i=1}^{c_2} \chi(i \in A \setminus B) - \sum_{i=1}^{c_2} \chi(i \in B \setminus A) \\ &= 1 - 0 \\ &= 1 \end{aligned} \tag{5.28}$$

Considering the case $c_2 > n+t-2$, because of $n+t-1 \geqslant n \geqslant c_2$ we must have $n+t-1 = n = c_2$, which would imply $k_{(A|B)}(c_2) = k_{(A|B)}(n) = 0$ contradicting (5.28). Therefore $(A|B)$ is an element of

$$P\left(\begin{matrix}[1, n+t-2] \\ k\end{matrix}\middle|\begin{matrix}[1, n] \\ k\end{matrix}\right);$$

because of $k_{(A|B)}(c_2) > 0$, even of

$$P_-\left(\begin{matrix}[1, n+t-2] \\ k\end{matrix}\middle|\begin{matrix}[1, n] \\ k\end{matrix}\right).$$

Besides, for all $d \in A \setminus B = (\tilde{A} \setminus \tilde{B}) \cup \{c_2\}$, which as we saw above automatically entails $d \geqslant c_2$, we have

$$k_{(A|B)}(d) = k_{(\tilde{A}|\tilde{B})}(d) + 2 \leqslant 1 = k_{(A|B)}(c_2).$$

But this implies $\tau(\bar{\tau}(\tilde{A}|\tilde{B})) = (\tilde{A}|\tilde{B})$ by remembering (5.25), the definition of $\tau$.

Second case: There is at least one element in $\tilde{A} \setminus \tilde{B}$ of which the $k_{(\tilde{A}|\tilde{B})}$-value is nonnegative. Let $f$ be the uniquely determined element of $\tilde{A} \setminus \tilde{B}$ with the property

$$\begin{aligned} &\text{either (i) } k_{(\tilde{A}|\tilde{B})}(d) < k_{(\tilde{A}|\tilde{B})}(f) \\ &\text{or (ii) } k_{(\tilde{A}|\tilde{B})}(d) = k_{(\tilde{A}|\tilde{B})}(f) \text{ and } d \leqslant f. \end{aligned} \tag{5.29}$$

Therefore in particular $k_{(\tilde{A}|\tilde{B})}(f) \geqslant 0$. Let here

$$c_2 = \min\{e : e \in \tilde{B} \setminus \tilde{A} \text{ and } e > f\}. \tag{5.30}$$

(If there would not exist any element of $\tilde{B} \setminus \tilde{A}$ larger than $f$, then $k_{(\tilde{A}|\tilde{B})}(n+t-1) \geqslant k_{(\tilde{A}|\tilde{B})}(f) \geqslant 0$ in contradiction to the simple fact $k_{(\tilde{A}|\tilde{B})}(n+t-1) = -2$.)



Suppose there is a $d \in \tilde{A} \setminus \tilde{B}$ with $f < d < c_2$, then by (5.29) $k_{(\tilde{A}|\tilde{B})}(d) < k_{(\tilde{A}|\tilde{B})}(f)$, but

$$\begin{aligned} k_{(\tilde{A}|\tilde{B})}(d) &= k_{(\tilde{A}|\tilde{B})}(f) + \sum_{i=f+1}^{d} \chi(i \in \tilde{A} \setminus \tilde{B}) - \sum_{i=f+1}^{d} \chi(i \in \tilde{B} \setminus \tilde{A}) \\ &= k_{(\tilde{A}|\tilde{B})}(f) + \sum_{i=f+1}^{d} \chi(i \in \tilde{A} \setminus \tilde{B}) - 0 \\ &\geqslant k_{(\tilde{A}|\tilde{B})}(f), \end{aligned}$$

which is a contradiction. Let $\bar{\tau}$ be defined in this case by (5.27), with the $c_2$ of (5.30). If we set $A = \tilde{A} \cup \{c_2\}$ and $B = \tilde{B} \setminus \{c_2\}$ again, then, since there is no $d \in \tilde{A} \setminus \tilde{B}$ between $f$ and $c_2$, of course $k_{(A|B)}(c_2) = k_{(\tilde{A}|\tilde{B})}(f) + 1$. Thus, in particular, because of $k_{(\tilde{A}|\tilde{B})}(f) \geqslant 0$ we conclude $k_{(A|B)}(c_2) > 0$. Supposing $c_2 > n+t-2$ leads as in the first case to $n+t-1 = n = c_2$, so $k_{(A|B)}(c_2) = 0$, which contradicts $k_{(A|B)}(c_2) > 0$. As in the first case, these facts furnish that $\bar{\tau}$ maps $(\tilde{A}|\tilde{B})$ into

$$P_-\left(\begin{matrix}[1, n+t-2] \\ k\end{matrix}\middle|\begin{matrix}[1, n] \\ k\end{matrix}\right).$$

For all $d \in \tilde{A} \setminus \tilde{B}$ smaller than $c_2$ we have

$$k_{(A|B)}(d) = k_{(\tilde{A}|\tilde{B})}(d) \leqslant k_{(\tilde{A}|\tilde{B})}(f) < k_{(A|B)}(c_2).$$

If $d$ is an element of $\tilde{A} \setminus \tilde{B}$ larger than $c_2$, then by (5.29)

$$k_{(\tilde{A}|\tilde{B})}(d) < k_{(\tilde{A}|\tilde{B})}(f),$$

thus

$$k_{(A|B)}(d) = k_{(\tilde{A}|\tilde{B})}(d) + 2 \leqslant 1 + k_{(\tilde{A}|\tilde{B})}(f) = k_{(A|B)}(c_2).$$

Regarding the definition of $\tau$, (5.25), these facts imply $\tau(\bar{\tau}(\tilde{A}|\tilde{B})) = (\tilde{A}|\tilde{B})$, which completes the proof of (5.20).

Proof of (5.17): The form of (5.17) we need for a combinatorial interpretation is

$$\overline{B}_n(q, t, x) = \sum_{k=0}^{n} (qx)^k q^{k^2} \cdot \left(\begin{bmatrix} n+t-1 \\ k \end{bmatrix}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - \begin{bmatrix} n+t-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ k \end{bmatrix}\right). \tag{5.31}$$

In order to modify the above proof of (5.20) that it suffices to show (5.31), first observe that each path of $l_+(n,1,t)$ ends with a vertical step, hence, by $\bar\pi$, corresponds to a pair

$$(A_1\dots A_k|B_1\dots B_k)\in P\begin{pmatrix}[0,n-1] & [1,n+t-1]\\ k & k\end{pmatrix}$$

with $A_1=0$. There is a weight-preserving one-to-one correspondence between the set of all paths from $(0,0)$ to $(n,n+t-1)$ ending with a vertical step and

$$P\begin{pmatrix}[1,n-1] & [1,n+t-1]\\ k-1 & k\end{pmatrix},$$

achieved by $\bar\pi$ and dropping the zero in the left block. Then almost word by word the arguments of the proof of (5.20) show that the weighted sum $\sum q^{\overline{\operatorname{maj}} w}$, summed up over all paths $w$ from $(0,0)$ to $(n,n+t-1)$ of the form $w=w'1$, is equal to

$$q^{\binom{k}{2}}\begin{bmatrix}n-1\\k-1\end{bmatrix}q^{\binom{k+1}{2}}\begin{bmatrix}n+t-1\\k\end{bmatrix}$$

and the weighted sum $\sum q^{\overline{\operatorname{maj}} w}$, summed up over all paths $w$ with the additional condition that they enter the line $y=x+t-1$, is equal to

$$q^{\binom{k+1}{2}}\begin{bmatrix}n-1\\k\end{bmatrix}q^{\binom{k}{2}}\begin{bmatrix}n+t-1\\k-1\end{bmatrix},$$

which in view of (5.15) yields (5.31).

Proofs of (5.18) and (5.21): For (5.18) apply $q$-Vandermonde convolution ([4, identity (18)]) to (5.17) with $x=1/q$, in order to prove (5.21) we write (5.20) in the form

$$\overline{B}'_n(q,t,x)=\sum_{k=0}^{n}(qx)^k q^{k^2}\cdot$$
$$\cdot\left(\begin{bmatrix}n+t-1\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}-\begin{bmatrix}n+t-2\\k-1\end{bmatrix}\begin{bmatrix}n+1\\k+1\end{bmatrix}\right),\qquad(5.32)$$

for $x=1/q$ apply $q$-Vandermonde convolution to gain

$$\overline{B}'_n(q,t,1/q)=\begin{bmatrix}2n+t-1\\n\end{bmatrix}-q\begin{bmatrix}2n+t-1\\n-1\end{bmatrix},\qquad(5.33)$$

which reduces to (5.21). □



Remarks: (1) Similar arguments to those in the proof of (5.20) are possible to give the expressions (5.4) and (5.8) via the forms (5.11) and (5.13), respectively, combinatorial interpretations.

(2) It is interesting to note that the results (5.5), (5.9), (5.18), and (5.21) are by-products of closed expressions of the generating functions for reverse plane partitions, for column-strict reverse plane partitions, for column-strict plane partitions, and plane partitions, respectively, in that order, with the fixed shape $(n+t-1,n)$. (For terminology and theory of plane partitions we refer the reader to [17, 18, 19]). In fact, in [18, Corollary 5.3] Stanley proves for $P$ being a partially ordered set with $|P|=p$, $\omega$ a labeling of $P$, that the generating function for, what he calls, $(P,\omega)$-partitions can be written

$$U(P,\omega;q)=\frac{W(P,\omega;q)}{(1-q)(1-q^2)\dots(1-q^p)},\qquad(5.34)$$

where ([18, Corollary 7.2])

$$W(P,\omega;q)=\sum_\sigma q^{\operatorname{ind}\sigma},$$

and the sum is over all permutations $\sigma$ of the $\omega$-separator of $P$. Let $P_0$ be the set $\{(i,j):1\leqslant i\leqslant 2$, if $i=1$ then $1\leqslant j\leqslant n+t-1$, and if $i=2$ then $1\leqslant j\leqslant n\}$. Let $P_0$ be partially ordered by

$$(i_1,j_1)\leqslant(i_2,j_2)\text{ if and only if } i_1\geqslant i_2 \text{ and } j_1\geqslant j_2$$

then, if $\omega$ is a natural labeling ([18, pp. 4, 5]), the $(P_0,\omega)$-partitions are reverse plane partitions of shape $(n+t-1,n)$. There is no difficulty to show that in this case

$$W(P_0,\omega;q)=\sum_{w\in L_+(n,1,t)}q^{\operatorname{maj} w}=B_n(q,t,1),$$

hence, by (5.34), and because of $|P|=(n+t-1)+n=2n+t-1$,

$$B_n(q,t,1)=(1-q)(1-q^2)\dots(1-q^{2n+t-1})\,U(P_0,\omega;q).\qquad(5.35)$$

The generating function for reverse plane partitions with a fixed shape is a well known result of Stanley [19, Proposition 18.3], it is equal to $1/(1-q^{d_1})\dots(1-q^{d_p})$, where the $d_i$'s are the hook-lengths of the shape.

For the shape $(n+t-1, n)$ the hook-lengths turn out to be $1, 2, \ldots, t-1, t+1, \ldots, n+t, 1, 2, \ldots, n$, therefore, by (5.35),

$$B_n(q, t, 1) = \frac{(1-q)\ldots(1-q^{2n+t-1})\cdot(1-q^t)}{(1-q)\ldots(1-q^n)(1-q)\ldots(1-q^{n+t})},$$

which is just (5.5).

In the remaining three cases the procedure is almost the same, the generating functions for column-strict reverse plane partitions ([19, Proposition 18.4]) and for column-strict plane partitions ([19, Theorem 15.3 for $m=\infty$]) being due to Stanley. The generating function for plane partitions was evaluated by MacMahon [13, section X, ch. 1], in his context paths of $l'_+(n,1,t)$ are called "two-element lattice permutations" ([13, p. 214]). Besides, by considering plane partitions with restricted part magnitude, he is led to study "sublattice permutations". In this connection he states (5.20) ([14, p. 1429]), but without proof.

(3) Comparing (5.8) and (5.17), we see that there must be a weight-preserving bijection $\psi$ between $l'_+(n,1,t)$ and $l_+(n,1,t)$, meaning $\overline{\operatorname{des}}\,\psi(w) = \operatorname{des} w$ and $\overline{\operatorname{maj}}\,\psi(w) = \operatorname{maj} w$ for all $w \in l'_+(n,1,t)$. So far I was not able to find any.

(4) The form (5.32) has a combinatorial meaning, too.

$$q^{k^2}\begin{bmatrix} n+t-1 \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix}$$

counts all paths $w$ from the origin to $(n+t-1, n)$ with $\overline{\operatorname{des}}\, w = k$ with respect to $\overline{\operatorname{maj}}$,

$$q^{k^2}\begin{bmatrix} n+t-2 \\ k-1 \end{bmatrix}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$$

counts all those paths with the additional condition that they enter the line $x = y+t-1$. The latter fact is seen by extending the bijection $\tau$ to a bijection $\tau'$:

$$P_-\left(\begin{matrix}[0, n+t-2] \\ k\end{matrix}\,\middle|\,\begin{matrix}[1, n] \\ k\end{matrix}\right) \to P\left(\begin{matrix}[1, n+t-2] \\ k-1\end{matrix}\,\middle|\,\begin{matrix}[0, n] \\ k+1\end{matrix}\right),$$

where the restriction of $\tau'$ to the subset

$$P_-\left(\begin{matrix}[1, n+t-2] \\ k\end{matrix}\,\middle|\,\begin{matrix}[1, n] \\ k\end{matrix}\right)$$



is identical with $\tau$ and

$$\tau'(0A_2 \ldots A_k | B_1 \ldots B_k) = (A_2 \ldots A_k | 0B_1 \ldots B_k).$$

(5) For an elegant combinatorial proof of (5.21) in the form (5.33) we refer the reader to [6, p. 255]. The same method allows us to count the set of all paths from the origin to $(l, m)$ not touching the line $y = x+t$, say $L(l, m: t)$, by means of maj and $\overline{\operatorname{maj}}$. The result for $q=1$ is the well known standard example for application of the reflection principle (see [15, p. 3]).

**Theorem 7:** Let $t$ be a nonnegative integer. If $m < l+t$,

$$\sum_{w \in L(l,m:t)} q^{\operatorname{maj} w} = \begin{bmatrix} l+m \\ l \end{bmatrix} - q^t \begin{bmatrix} l+m \\ l+t \end{bmatrix} \tag{5.36}$$

and

$$\sum_{w \in L(l,m:t)} q^{\overline{\operatorname{maj}}\, w} = \begin{bmatrix} l+m \\ l \end{bmatrix} - \begin{bmatrix} l+m \\ l+t \end{bmatrix}. \tag{5.37}$$

If $m > l-t$,

$$\sum_{w \in L(l,m:-t)} q^{\operatorname{maj} w} = \begin{bmatrix} l+m \\ l \end{bmatrix} - \begin{bmatrix} l+m \\ l-t \end{bmatrix} \tag{5.38}$$

and

$$\sum_{w \in L(l,m:-t)} q^{\overline{\operatorname{maj}}\, w} = \begin{bmatrix} l+m \\ l \end{bmatrix} - q^{m-l+t} \begin{bmatrix} l+m \\ l-t \end{bmatrix}. \tag{5.39}$$

Sketch of Proof: To obtain (5.36) we use the well-known result ([5, p. 266; 13, p. 206]) that the weighted sum $\sum q^{\operatorname{maj} w}$, where the summation is over all words $w$ with $l$ 0's and $m$ 1's (i.e. all paths from $(0,0)$ to $(l,m)$), equals $\begin{bmatrix} l+m \\ l \end{bmatrix}$. By $t$-fold iteration of the procedure described in [6, p. 255], we gain a bijection $\varphi$ mapping the paths from $(0,0)$ to $(l,m)$ touching the line $y = x+t$ onto the set of all paths from $(0,0)$ to $(l+t, m-t)$, with the property $\operatorname{maj}\varphi(w) = \operatorname{maj} w - t$. This explains the second expression on the right-hand side of (5.36).

To prove (5.37) we modify Fürlinger and Hofbauer's technique in the way that we determine the last of the deepest points, coded

$\ldots 100\ldots$, of a path touching $y = x + t$, which we change into $\ldots 110\ldots$ $(l - m + t)$-fold iteration of this process yields a bijection $\Phi$ between the paths from $(0,0)$ to $(l,m)$ touching $y = x + t$ and all paths from $(0,0)$ to $(m - t, l + t)$, with $\operatorname{maj}\Phi(w) = \operatorname{maj} w$, which explains the second expression on the right-hand side of (5.37).

(5.38) and (5.39) can be deduced from (5.37) and (5.36), respectively, by reflection at the main diagonal and reversing the paths. □

Needless to say that (5.5), (5.9), (5.18), and (5.21) are special cases of (5.36), (5.38), (5.37), and (5.39), in this order.

(6) Trying to count $L(l, m:t)$ with respect to maj and des at once, a recurrence relation for the numbers

$$H(l, m:t, x, q) = \sum_{w\in L(l,m:t)} x^{\operatorname{des} w} q^{\operatorname{maj} w} \tag{5.40}$$

can be derived in the usual manner:

$$\begin{aligned} H(l, m:t) &= H(l, m-1:t) + H(l-1, m:t) + \\ &+ (q^{l+m-1}x - 1)\, H(l-1, m-1:t) \end{aligned} \tag{5.41}$$

for $m < l + t - 1$ and $H(l, l+t-1:t) = H(l, l+t-2:t)$, with the initial conditions $H(0, m:t) = H(l, 0:t) = 1$, where we have written $H(l, m:t)$ instead of $H(l, m:t, x, q)$, for short. Let

$$H_k(l, m:t, q) = \sum_{\substack{w\in L(l,m:t)\\ \operatorname{des} w = k}} q^{\operatorname{maj} w}, \tag{5.42}$$

shortly $H_k(l, m:t)$, then (5.41) implies

$$\begin{aligned} H_k(l, m:t) &= H_k(l, m-1:t) + H_k(l-1, m:t) + \\ &+ q^{l+m-1} H_{k-1}(l-1, m-1:t) - H_k(l-1, m-1:t) \end{aligned} \tag{5.43}$$

for $m < l + t - 1$ and $H_k(l, l+t-1:t) = H_k(l, l+t-2:t)$, with the initial conditions $H_k(0, m:t) = H_k(l, 0:t) = \delta_{k0}$.

The formula (5.32) gives an explicit expression for $H_k(n + t - 1, n, 1)$, which is of the form

$$H_k(l, m:t, q) = \sum_{k_1+k_2=s_1+s_2} \cdot q^{k^2} \begin{bmatrix} l + s_1 \\ k + k_1 \end{bmatrix} \begin{bmatrix} m + s_2 \\ k + k_2 \end{bmatrix} f(s_1, s_2, k_1, k_2, t). \tag{5.44}$$



Indeed the numbers

$$q^{k^2} \begin{bmatrix} l + s_1 \\ k + k_1 \end{bmatrix} \begin{bmatrix} m + s_2 \\ k + k_2 \end{bmatrix},$$

subject to the condition $k_1 + k_2 = s_1 + s_2$, satisfy the recurrence relation (5.43). We conjecture that it is possible to determine the constants $f(s_1, s_2, k_1, k_2, t)$ in a proper manner, in order to establish (5.44) for arbitrary $l$ and $m$.

## 6. Catalan Numbers

In this section $r$ and $t$ are set equal to one, i.e. we deal with the numbers $C_n(x, a, b)$, etc. Clearly we have $\overline{C'_n}(x, a, b) = C_n(ax, a, b)$ and $\overline{C_n}(x, a, b) = C'_n(bx, a, b)$. Moreover we get by (4.21)

$$C'_n(x, a, b) = b^n x C_n(x, b, a) \qquad \text{for } n \geqslant 1,$$

thus we can restrict ourselves to the study of the numbers $C_n(x, a, b)$. A great deal of that was done in [6, Theorem on p. 258], where a generating function and three recurrence relations are given for these $q$-Catalan numbers. It is pointed out there that all known $q$-Catalan numbers are covered by the numbers $C_n(x, a, b)$, we refer the reader to [6] for details and references.

Fürlinger und Hofbauer only have combinatorial proofs for two of the recurrence relations and proofs by generating function for two of the recurrence relations, respectively. We in turn shall add three further recurrence relations ((6.5), (6.7), (6.8) below) and derive all recurrence relations as well by combinatorial means as by the generating function.

**Theorem 8:** The following statements are equivalent:

$$C_n(x, a, b) = \sum_{w\in L_+(n,1,1)} x^{\operatorname{des} w} a^{\alpha(w)} b^{\beta(w)}; \tag{6.1}$$

$$\sum_{n=0}^{\infty} \frac{C_n(x, a, b)\, a^n z^n}{(1+z)\ldots(a^n+z)(1+bxz)\ldots(1+b^n xz)} = 1; \tag{6.2}$$

$$C_{n+1}(x) = C_n(ax) + x \sum_{k=0}^{n-1} (ab)^{k+1} C_k(ax)\, C_{n-k}((ab)^{k+1}x),\ C_0 = 1; \tag{6.3}$$

$$C_{n+1}(x) = C_n(ax) + x \sum_{k=1}^{n} (ab)^k C_k(x)\, C_{n-k}(a^{k+1}b^k x),\ C_0 = 1; \tag{6.4}$$

$$C_{n+1}(x) = C_n(abx) + bx \sum_{k=1}^{n} a^k C_k(bx) C_{n-k}((ab)^{k+1}x),\ C_0 = 1; \tag{6.5}$$

$$C_{n+1}(x) = C_n(abx) + bx \sum_{k=0}^{n-1} a^{k+1} C_k(abx) C_{n-k}((ab)^{k+1}x),\ C_0 = 1; \tag{6.6}$$

$$C_{n+1}(x) = C_n(x) + a^n x \sum_{k=0}^{n-1} b^{k+1} C_k(x) C_{n-k}(a^k b^{k+1}x),\ C_0 = 1; \tag{6.7}$$

$$C_{n+1}(x) = C_n(x) + a^n x \sum_{k=1}^{n} b^k C_k(x) C_{n-k}(a^k b^k x),\ C_0 = 1. \tag{6.8}$$

Proof: (A) Combinatorial proofs. Again we utilize the correspondence $\pi$ between $L_+(n,1,1)$ and $P_1([0,n-1]|[1,n])$, remembering (3.1). Since, by definition for a pair $(A_1 \ldots A_k | B_1 \ldots B_k) \in P_1([0,n-1]|[1,n])$, there must hold $A_1 + 1 > B_1$ and $A_k + 1 > B_k$, we obtain $A_1 \geqslant 1$ and $B_k \leqslant n-1$, hence $P_1([0,n-1]|[1,n]) = P_1([1,n-1]|[1,n-1])$, for which we will write $P_1(n)$ for short.

(6.1) $\Leftrightarrow$ (6.2). This is Theorem 3, (4.14) $\Leftrightarrow$ (4.15).

(6.1) $\Leftrightarrow$ (6.3) and (6.1) $\Leftrightarrow$ (6.4) where done in [6, p. 259]. Moreover, (6.3) is the special case of (4.17) with $r = t_1 = t_2 = 1$. This is established by respecting

$$C_{n+1}(x,a,b) = G_n(1,2,ax,a,b)$$

and

$$C_n(x,a,b) = G_n(1,1,x,a,b).$$

(6.1) $\Leftrightarrow$ (6.5). The method is similar to that we applied to prove (4.6), in particular we remind the reader of what we understood by lhs- and rhs-symbols.

Let $(A|B) = (A_1 \ldots A_l | B_1 \ldots B_l) \in P_1(n+1)$. The left-hand side of (6.5) can be written $\sum M(v)$, where the sum is over all elements $v$ of $P_1(n+1)$. In this sense, $(A|B)$ is called a lhs-symbol, where the corresponding contribution to the left-hand side is equal to $M(A|B)$. Similarly, we have two kinds of rhs-symbols, on the one hand the elements of $P_1(n)$, on the other triples $((C|D), k, (E|F))$, where $(C|D) \in P_1(k)$ and $(E|F) \in P_1(n-k)$, $k \in \{1, \ldots, n\}$. We are going to define a map $\rho_1$ from lhs- to rhs-symbols, which will turn out to be a bijection. Moreover, there will hold proper weight conditions so that (6.5) can be established.



Let $(A_1 \ldots A_l | B_1 \ldots B_l) \in P_1(n+1)$, i.e. a lhs-symbol, then we set

(i) for $B_1 \geqslant 2$: $\rho_1(A_1 \ldots A_l | B_1 \ldots B_l) =$

$= ((A_1 - 1) \ldots (A_l - 1) | (B_1 - 1) \ldots (B_l - 1))$,

(ii) for $B_1 = 1$: $\rho_1(A_1 \ldots A_l | B_1 \ldots B_l) =$

$= ((A_1 \ldots A_{j-1} | (B_2 - 1) \ldots (B_j - 1))$,

$A_j, ((A_{j+1} - A_j - 1) \ldots$

$\ldots (A_l - A_j - 1) | (B_{j+1} - A_j - 1) \ldots (B_l - A_j - 1))$,

where $j$ is minimal with $A_j < B_{j+1} - 1$ (if there is none, then $j = l$).

Obviously $\rho_1$ maps the pairs $(A|B)$ subject to $B_1 \geqslant 2$ into $P_1(n)$ with

$$\begin{aligned} \operatorname{des}(A|B) &= \operatorname{des}\rho_1(A|B) \\ \alpha(A|B) &= \alpha(\rho_1(A|B)) + \operatorname{des}(\rho_1(A|B)) \\ \beta(A|B) &= \beta(\rho_1(A|B)) + \operatorname{des}(\rho_1(A|B)). \end{aligned} \tag{6.9}$$

The pairs $(A|B)$ with $B_1 = 1$, by $\rho_1$, yield a triple $((C|D), A_j, (E|F))$, where $(C|D) \in P_1(A_j)$ and $(E|F) \in P_1(n - A_j)$, besides,

$$\begin{aligned} \operatorname{des}(A|B) &= \operatorname{des}(C|D) + \operatorname{des}(E|F) + 1 \\ \alpha(A|B) &= \alpha(C|D) + \alpha(E|F) + A_j + (A_j + 1)\operatorname{des}(E|F) \\ \beta(A|B) &= \beta(C|D) + \beta(E|F) + \operatorname{des}(C|D) + (A_j + 1)\operatorname{des}(E|F) + 1. \end{aligned} \tag{6.10}$$

Evidently the inverse map of $\rho_1$ is given by

(i) Let $(A_1 \ldots A_l | B_1 \ldots B_l)$ be an element of $P_1(n)$ then

$\bar\rho_1(A_1 \ldots A_l | B_1 \ldots B_l) =$

$= ((A_1 + 1) \ldots (A_l + 1) | (B_1 + 1) \ldots (B_l + 1))$.

(ii) For $(C_1 \ldots C_{j-1} | D_1 \ldots D_{j-1}) \in P_1(k)$

and $(E_1 \ldots E_i | F_1 \ldots F_i) \in P_1(n-k)$

set $\rho_1((C|D), k, (E|F)) =$

$= (C_1 \ldots C_{j-1}\, k\, (E_1 + k + 1) \ldots$

$\ldots (E_i + k + 1) | 1\, (D_1 + 1) \ldots$

$\ldots (D_{j-1} + 1)(F_1 + k + 1) \ldots (F_i + k + 1))$.

Therefore, regarding (6.9) and (6.10),

$$\sum_{v\in P_1(n+1)} x^{\operatorname{des} v} a^{\alpha(v)} b^{\beta(v)} =$$

$$= \sum_{v'\in P_1(n)} x^{\operatorname{des} v'} a^{\alpha(v')+\operatorname{des} v'} b^{\beta(v')+\operatorname{des} v'} +$$

$$+ \sum_{k=1}^{n} \sum_{v''\in P_1(k)} \sum_{v'''\in P_1(n-k)} \cdot$$

$$\cdot x^{\operatorname{des} v''+\operatorname{des} v'''+1} a^{\alpha(v'')+\alpha(v''')+k+(k+1)\operatorname{des} v'''} \cdot$$

$$\cdot b^{\beta(v'')+\beta(v''')+\operatorname{des} v''+(k+1)\operatorname{des} v'''+1},$$

which reduces to (6.5).

For the remaining recurrence relations we shall only state the map from lhs- to rhs-symbols and leave the details to the reader. Let $(A_1 \ldots A_l | B_1 \ldots B_l)$ be an element of $P_1(n+1)$.

(6.1) ⇔ (6.6).

(i) For $B_1 \geqslant 2$ let $\rho_2(A_1 \ldots A_l | B_1 \ldots B_l) =$

$= ((A_1 - 1) \ldots (A_l - 1) | (B_1 - 1) \ldots (B_l - 1))$.

(ii) For $B_1 = 1$ let $\rho_2(A_1 \ldots A_l | B_1 \ldots B_l) =$

$= ((A_1 - 1) \ldots (A_{j-1} - 1) | (B_2 - 1) \ldots (B_j - 1)), A_j,$

$(A_{j+1} - A_j) \ldots (A_l - A_j) | (B_{j+1} - A_j) \ldots (B_l - A_j)),$

where $j$ is minimal with $A_j < B_{j+1}$ (if there is none, then $j = l$).

(6.1) ⇔ (6.7).

(i) For $A_l < n$ let $\rho_3(A_1 \ldots A_l | B_1 \ldots B_l) =$

$= (A_1 \ldots A_l | B_1 \ldots B_l)$.

(ii) For $A_l = n$ let $\rho_3(A_1 \ldots A_l | B_1 \ldots B_l) =$

$= ((A_1 \ldots A_j | B_1 \ldots B_j), B_{j+1}, ((A_{j+1} - B_{j+1} + 1) \ldots$

$\ldots (A_{l-1} - B_{j+1} + 1) | (B_{j+2} - B_{j+1}) \ldots (B_l - B_{j+1})),$

where $j$ is maximal with $A_j < B_{j+1} - 1$ (if there is none, then $j = 0$).

(6.1) ⇔ (6.8).

(i) For $A_l < n$ let $\rho_4(A_1 \ldots A_l | B_1 \ldots B_l) =$

$= (A_1 \ldots A_l | B_1 \ldots B_l)$.



(ii) For $A_l = n$ let $\rho_4(A_1 \ldots A_l | B_1 \ldots B_l) =$

$= ((A_1 \ldots A_j | B_1 \ldots B_j), B_{j+1}, ((A_{j+1} - B_{j+1}) \ldots$

$\ldots (A_{l-1} - B_{j+1}) | (B_{j+2} - B_{j+1}) \ldots (B_l - B_{j+1})),$

where $j$ is maximal with $A_j < B_{j+1}$ (if there is none, then $j = 0$).

(B) Proofs by the generating function (6.2). It is convenient to write (6.2) by (2.9) in the form

$$\sum_{n=0}^{\infty} \frac{C_n(x)\, a^{-\binom{n}{2}} z^n}{(-z; a^{-1})_{n+1} (-bxz; b)_n} = 1 \tag{6.11}$$

which, because of $C_0(x) = 1$, is equivalent with

$$\sum_{n=1}^{\infty} \frac{C_n(x)\, a^{-\binom{n}{2}} z^n}{(-a^{-1} z; a^{-1})_n (-bxz; b)_n} = z. \tag{6.12}$$

(6.2) ⇔ (6.3) and (6.2) ⇔ (6.6) were done in [4, p. 259]. The difficulty we have to encounter in proving the remaining recurrence relations by generating function is that (with the exception of (6.5)) the lhs- and rhs-generating functions cannot be evaluated explicitely. The technique is, after finding the "right" powers, to convert the rhs-generating function into the lhs-generating function by repeated use of (6.11) and (6.12).

(6.2) ⇔ (6.4).

$$\sum_{n=0}^{\infty} \frac{\left[C_n(ax) + x \sum_{k=1}^{n} (ab)^k C_k(x)\, C_{n-k}(a^{k+1} b^k x)\right] a^{-\binom{n+1}{2}} z^{n+1}}{(-a^{-1} z; a^{-1})_{n+1} (-xz; b)_{n+1}} =$$

$$= \frac{z}{1 + xz} \sum_{n=0}^{\infty} \frac{C_n(ax)\, a^{-\binom{n}{2}} (z/a)^n}{(-a^{-1} z; a^{-1})_{n+1} (-bxz; b)_n} +$$

$$+ x \sum_{k=1}^{\infty} \frac{b^k C_k(x)\, a^{-\binom{k}{2}} z^{k+1}}{(-a^{-1} z; a^{-1})_{k+1} (-xz; b)_{k+1}} \cdot$$

$$\cdot \sum_{j=0}^{\infty} \frac{C_j(a^{k+1} b^k x)\, a^{-\binom{j}{2}-j(k+1)} z^j}{(-a^{-k-2} z; a^{-1})_j\, (-b^{k+1} xz; b)_j} =$$

$$= \frac{z}{1+xz} + x \sum_{k=1}^{\infty} \frac{b^k C_k(x)\, a^{-\binom{k}{2}} z^{k+1}}{(-a^{-1} z; a^{-1})_{k+1} (-xz; b)_{k+1}} (1 + z/a^{k+1})$$

by (6.11) and (6.12). Renewed use of (6.12) turns this into

$$\sum_{k=1}^{\infty} \frac{C_k(x)\, a^{-\binom{k}{2}} z^k}{(-a^{-1} z; a^{-1})_k (-xz; b)_{k+1}} +$$

$$+ \sum_{k=1}^{\infty} \frac{(b^k xz)\, C_k(x)\, a^{-\binom{k}{2}} z^k}{(-a^{-1} z; a^{-1})_k (-xz; b)_{k+1}},$$

which simplifies to the corresponding generating function for the left-hand side of (6.4).

(6.2) $\Leftrightarrow$ (6.5).

$$\sum_{n=0}^{\infty} \frac{[C_n(abx) + bx \sum_{k=1}^{n} a^k C_k(bx)\, C_{n-k}((ab)^{k+1} x)]\, a^{-\binom{n+1}{2}} z^{n+1}}{(-a^{-1} z; a^{-1})_{n+1} (-bxz; b)_{n+1}} =$$

$$= \frac{z}{1+bxz} \sum_{n=0}^{\infty} \frac{C_n(abx)\, a^{-\binom{n}{2}} (z/a)^n}{(-a^{-1} z; a^{-1})_{n+1} (-b^2 xz; b)_n} +$$

$$+ bx \sum_{k=1}^{\infty} \frac{C_k(bx)\, a^{-\binom{k}{2}} z^{k+1}}{(-a^{-1} z; a^{-1})_k (-bxz; b)_{k+1}} \cdot$$

$$\cdot \sum_{j=0}^{\infty} \frac{C_j((ab)^{k+1} z)\, a^{-\binom{j}{2}-j(k+1)} z^j}{(-a^{-k-1} z; a^{-1})_{j+1} (-b^{k+2} xz; b)_j} =$$

$$= \frac{z}{1+bxz} + bx \sum_{k=1}^{\infty} \frac{C_k(bx)\, a^{-\binom{k}{2}} z^{k+1}}{(-a^{-1} z; a^{-1})_k (-bxz; b)_{k+1}},$$



by two-fold use of (6.11). Application of (6.12) turns this expression into

$$\frac{z}{1+bxz} + bxz \frac{z}{1+bxz},$$

which is equal to $z$, and by (6.12) to

$$\sum_{n=0}^{\infty} \frac{C_{n+1}(x)\, a^{-\binom{n+1}{2}} z^{n+1}}{(-a^{-1} z; a^{-1})_{n+1} (-bxz; b)_{n+1}},$$

the generating function for the left-hand side of (6.5).

(6.2) $\Leftrightarrow$ (6.7).

$$\sum_{n=0}^{\infty} \frac{[C_n(x) + a^n x \sum_{k=0}^{n-1} b^{k+1} C_k(x)\, C_{n-k}(a^k b^{k+1} x)]\, a^{-\binom{n+1}{2}} z^{n+1}}{(-z; a^{-1})_{n+1} (-bxz; b)_{n+1}} =$$

$$= \sum_{k=0}^{\infty} \frac{C_k(x)\, a^{-\binom{k+1}{2}} z^{k+1}}{(-z; a^{-1})_{k+1} (-bxz; b)_{k+1}} +$$

$$+ x \sum_{k=0}^{\infty} \frac{b^{k+1} C_k(x)\, a^{-\binom{k}{2}} z^{k+1}}{(-z; a^{-1})_{k+1} (-bxz; b)_{k+1}} \cdot$$

$$\cdot \sum_{j=1}^{\infty} \frac{C_j(a^k b^{k+1} x)\, a^{-\binom{j}{2}-jk} z^j}{(-a^{-k-1} z; a^{-1})_j (-b^{k+2} xz; b)_j} =$$

$$= \sum_{k=0}^{\infty} \frac{C_k(x)\, a^{-\binom{k+1}{2}} z^{k+1}}{(-z; a^{-1})_{k+1} (-bxz; b)_{k+1}} (1 + b^{k+1} x a^k (z/a^k)) =$$

$$= \sum_{k=0}^{\infty} \frac{C_k(x)\, a^{-\binom{k+1}{2}} z^{k+1}}{(-z; a^{-1})_{k+1} (-bxz; b)_k},$$

the step before the last being due to (6.12). Continuing we get

$$\sum_{k=0}^{\infty}\frac{C_k(x)\,a^{-\binom{k+1}{2}}z^{k+1}}{(-z;a^{-1})_{k+1}(-bxz;b)_k}=$$

$$=\frac{z}{1+z}+\sum_{n=1}^{\infty}\frac{C_n(x)\,a^{-\binom{n+1}{2}}z^{n+1}}{(-z;a^{-1})_{n+1}(-bxz;b)_n}=$$

$$=\sum_{n=1}^{\infty}\frac{C_n(x)\,a^{-\binom{n}{2}}z^{n}}{(-z;a^{-1})_{n+1}(-bxz;b)_n}+$$

$$+\sum_{n=1}^{\infty}\frac{C_n(x)\,a^{-\binom{n+1}{2}}z^{n+1}}{(-z;a^{-1})_{n+1}(-bxz;b)_n},$$

by (6.12), which, after simplifying by the factor $(1+z/a^n)$ and index transformation, turns into the corresponding generating function for the left-hand side of (6.7).

(6.2) ⇔ (6.8).

$$\sum_{n=0}^{\infty}\frac{[C_n(x)+a^n x\sum_{k=1}^{n}b^k C_k(x)\,C_{n-k}(a^k b^k x)]\,a^{-\binom{n+1}{2}}z^{n+1}}{(-z;a^{-1})_{n+1}(-xz;b)_{n+1}}=$$

$$=\sum_{k=0}^{\infty}\frac{C_k(x)\,a^{-\binom{k+1}{2}}z^{k+1}}{(-z;a^{-1})_{k+1}(-xz;b)_{k+1}}+$$

$$+x\sum_{k=1}^{\infty}\frac{b^k C_k(x)\,a^{-\binom{k}{2}}z^{k+1}}{(-z;a^{-1})_k(-xz;b)_{k+1}}\cdot$$

$$\cdot\sum_{j=0}^{\infty}\frac{C_j(a^k b^k x)\,a^{-\binom{j}{2}-jk}z^j}{(-a^{-k}z;a^{-1})_{j+1}(-b^{k+1}xz;b)_j}=$$



$$=\frac{z}{(1+z)(1+xz)}+\sum_{k=1}^{\infty}\frac{C_k(x)\,a^{-\binom{k+1}{2}}z^{k+1}}{(-z;a^{-1})_{k+1}(-xz;b)_{k+1}}+$$

$$+\sum_{k=1}^{\infty}\frac{b^k xz\,C_k(x)\,a^{-\binom{k}{2}}z^k}{(-z;a^{-1})_k(-xz;b)_{k+1}}$$

by (6.11). Applying (6.12) to the left-most expression, we see that the last line equals

$$\sum_{k=1}^{\infty}\frac{C_k(x)\,a^{-\binom{k}{2}}z^k}{(-z;a^{-1})_{k+1}(-xz;b)_{k+1}}\,(1+z/a^k+b^k xz\,(1+z/a^k)).$$

After simplifying by $(1+z/a^k)(1+b^k xz)$ and index-transformation, we obtain the corresponding generating function of the left-hand side of (6.8). □

The problem to prove the equivalence of two of the recurrence relations (5.3)–(5.8) directly remains open. At least, by introducing three specified bijections on $l_+(n,1,1)$ (which by $\pi$ is isomorphic to $P_1(n)$), some surprising connections are laid open.

Let $\lambda_1: l_+(n,1,1)\to l_+(n,1,1)$ be defined by

$$\lambda_1(w_1 w_2\dots w_{2n})=(1-w_{2n})\dots(1-w_2)(1-w_1). \tag{6.13}$$

That is, $\lambda_1$ reverses $w=w_1\dots w_{2n}$ and exchanges 0 and 1. Obviously $\lambda_1$ is a bijection on $l_+(n,1,1)$ with

$$\operatorname{des}\lambda_1(w)=\operatorname{des}w$$
$$\alpha(\lambda_1(w))=n\cdot\operatorname{des}w-\beta(w)$$
$$\beta(\lambda_1(w))=n\cdot\operatorname{des}w-\alpha(w)$$

Hence

$$C_n(x,a,b)=\sum_{w\in l_+(n,1,1)}x^{\operatorname{des}w}a^{\alpha(w)}b^{\beta(w)}$$
$$=\sum_{w\in l_+(n,1,1)}x^{\operatorname{des}\lambda_1(w)}a^{\alpha(\lambda_1(w))}b^{\beta(\lambda_1(w))}$$
$$=C_n(a^n b^n x,1/b,1/a). \tag{6.14}$$

Another bijection on $l_+(n,1,1)$, defined via

$$P_1(n) = P_1([0,n-1] | [0,n-1]),$$

is given by

$$\lambda_2(A|B) = ([0,n-1] \setminus B | [0,n-1] \setminus A), \tag{6.15}$$

where $(A|B) \in P_1(n)$. $\lambda_2$ satisfies

$$\operatorname{des} \lambda_2(w) = n - 1 - \operatorname{des} w$$

$$\alpha(\lambda_2(w)) = \binom{n}{2} - \beta(w)$$

$$\beta(\lambda_2(w)) = \binom{n}{2} - \alpha(w)$$

thus we obtain another "transfer formula" for $C_n(x,a,b)$:

$$C_n(x,a,b) = (ab)^{\binom{n}{2}} x^{n-1} C_n(1/x, 1/b, 1/a) \text{ for } n \geqslant 1. \tag{6.16}$$

The composition of $\lambda_1$ and $\lambda_2$ yields

$$C_n(x,a,b) = (ab)^{\binom{n}{2}} x^{n-1} C_n(1/a^n b^n x, a, b) \text{ for } n \geqslant 1. \tag{6.17}$$

Now we demonstrate that combining (6.3) with (6.14) implies (6.4). By (6.14) we have

$$C_{n+1}(x,a,b) = C_{n+1}(a^{n+1} b^{n+1} x, 1/b, 1/a).$$

Applying (6.3) to the right-hand side of this equation, we get

$$C_{n+1}(x,a,b) = C_n(a^{n+1} b^n x, 1/b, 1/a) + $$
$$+ a^{n+1} b^{n+1} x \sum_{k=0}^{n-1} (ab)^{-k-1} C_k(a^{n+1} b^n x, 1/b, 1/a) \cdot$$
$$\cdot C_{n-k}((ab)^{n-k} x, 1/b, 1/a)$$

and, retransforming by (6.14), finally

$$C_{n+1}(x,a,b) = C_n(ax,a,b) + x \sum_{k=0}^{n-1} (ab)^{n-k} \cdot$$
$$\cdot C_k(a^{n+1-k} b^{n-k} x, a, b)\, C_{n-k}(x,a,b),$$



which indeed save to index transformation is (6.4). The table below lists the results of all possible combinations of (6.3)–(6.8) with (6.14), (6.16), (6.17), respectively.

| | (6.3) | (6.4) | (6.5) | (6.6) | (6.7) | (6.8) |
|---|---|---|---|---|---|---|
| (6.14) | (6.4) | (6.3) | (6.7) | (6.8) | (6.5) | (6.6) |
| (6.16) | (6.5) | (6.7) | (6.3) | (6.6) | (6.4) | (6.8) |
| (6.17) | (6.7) | (6.5) | (6.4) | (6.8) | (6.3) | (6.6) |

Studying this table we discover the remarkable fact that in this respect there are two classes of recurrence relations with different cardinality, namely {(6.3), (6.4), (6.5), (6.7)} and {(6.6), (6.8)}, where the elements of the same class are related by (6.14), (6.16) or (6.17). Perhaps there are more transfer formulas for the Catalan numbers $C_n(x,a,b)$ which convert this strange asymmetric situation into a symmetric one.

## 7. Counting $L_+(n,r,t)$ by the Inversion Statistics

For a word $w = w_1 \dots w_\mu$ the statistics inv is defined by

$$\operatorname{inv} w = \sum_{i<j} \chi(w_i > w_j).$$

Carlitz, Riordan and Scoville [1, 2, 3] extensively study $q$-analogues of ballot numbers. Fürlinger and Hofbauer [6] show that counting "Catalan-paths" by inversions essentially yields the same $q$-Catalan numbers. We extend their results in counting paths of $L_+(n,r,t)$ by means of the inversion statistics. The next theorem summarizes the analogues for (1.2)–(1.4), the proof is left to the reader.

**Theorem 9:** If

$$g_n(r,t,q) = \sum_{w \in L_+(n,r,t)} q^{\operatorname{inv} w}, \tag{7.1}$$

then

$$\sum_{n=0}^{\infty} \frac{g_n(t)\, q^{\binom{n}{2}} z^n}{(1+z)\dots(1+q^{(r+1)n+t-1} z)} = 1; \tag{7.2}$$

$$g_n(t) = g_n(t-1)\, q^n + g_{n-1}(t+r) \tag{7.3}$$

with $g_0(t) = 1$ and $g_n(0) = \delta_{n0}$;

$$g_n(t_1 + t_2) = \sum_{k=0}^{n} q^{(rk+t_1)(n-k)} g_k(t_1)\, g_{n-k}(t_2); \tag{7.4}$$

$$\begin{bmatrix}(r+1)n + t_1 + t_2 \\ n\end{bmatrix} = \sum_{k=0}^{n} q^{(rk+t_1)(n-k)} g_k(t_1) \begin{bmatrix}(r+1)(n-k) + t_2 \\ n-k\end{bmatrix}. \tag{7.5}$$

□

The generating function identity (7.2) for $r = t = 1$ was given in [6, identity (2.5)]. Comparing the recurrence relation (7.3) with identity (6.6) of [1], we get for the $q$-ballot numbers $b(n, k, q)$ of [1]:

$$g_n(1, t, q) = q^{\binom{n}{2} - n(t-1)}\, b(n + t - 1, n, 1/q). \tag{7.6}$$

The convolution identity (7.4) extends identity (2.2) of [6].

Counting $L'_+(n, r, t)$ by inversions does not yield a really "different" $q$-analogue of the Gould numbers. In fact, the numbers

$$g'_n(r, t, q) = \sum_{w \in L'_+(n,r,t)} q^{\operatorname{inv} w}$$

can be expressed in terms of $g_n(r, t, q)$ by

$$g'_n(r, t, q) = q^{n(rn+t)} g_n(r, t, 1/q).$$

It would be interesting if there is a joint generalization of Theorems 3 and 9, counting $L_+(n, r, t)$ by (des, α, β, inv) at the same time; in particular: Is it possible to find a generating function for the numbers $\Gamma_n = \sum_{w \in L_+(n,r,t)} x^{\operatorname{des} w} a^{\alpha(w)} b^{\beta(w)} q^{\operatorname{inv} w}$ or at least for $\gamma_n = \sum_{w \in L_+(n,r,t)} x^{\operatorname{des} w} p^{\operatorname{maj} w} q^{\operatorname{inv} w}$? In counting permutations by (des, maj, inv) one secceeds in finding generating functions (see [7] and the references cited there). The technique Garsia and Gessel apply in [7] is related to that discussed in Remark (2) after Theorem 6, it also uses partitions (in fact these techniques trace back to MacMahon [13]). Maybe these methods can be modified to be applicable on $\gamma_n$ or even $\Gamma_n$.

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