ADVANCED DETERMINANT CALCULUS

C. KRATTENTHALER[†]

Institut für Mathematik der Universität Wien,
Strudlhofgasse 4, A-1090 Wien, Austria.
E-mail: `kratt@pap.univie.ac.at`
WWW: `http://radon.mat.univie.ac.at/People/kratt`

Dedicated to the pioneer of determinant evaluations (among many other things),
George Andrews

Abstract. The purpose of this article is threefold. First, it provides the reader with a few useful and efficient tools which should enable her/him to evaluate nontrivial determinants for the case such a determinant should appear in her/his research. Second, it lists a number of such determinants that have been already evaluated, together with explanations which tell in which contexts they have appeared. Third, it points out references where further such determinant evaluations can be found.

1. Introduction

Imagine, you are working on a problem. As things develop it turns out that, in order to solve your problem, you need to evaluate a certain determinant. Maybe your determinant is

$$\det_{1\le i,j,\le n}\left(\frac{1}{i+j}\right), \tag{1.1}$$

or

$$\det_{1\le i,j\le n}\left(\binom{a+b}{a-i+j}\right), \tag{1.2}$$

or it is possibly

$$\det_{0\le i,j\le n-1}\left(\binom{\mu+i+j}{2i-j}\right), \tag{1.3}$$

2000 *Mathematics Subject Classification.* Primary 05A19; Secondary 05A10 05A15 05A17 05A18 05A30 05E10 05E15 11B68 11B73 11C20 15A15 33C45 33D45.

Key words and phrases. Determinants, Vandermonde determinant, Cauchy's double alternant, Pfaffian, discrete Wronskian, Hankel determinants, orthogonal polynomials, Chebyshev polynomials, Meixner polynomials, Meixner–Pollaczek polynomials, Hermite polynomials, Charlier polynomials, Laguerre polynomials, Legendre polynomials, ultraspherical polynomials, continuous Hahn polynomials, continued fractions, binomial coefficient, Genocchi numbers, Bernoulli numbers, Stirling numbers, Bell numbers, Euler numbers, divided difference, interpolation, plane partitions, tableaux, rhombus tilings, lozenge tilings, alternating sign matrices, noncrossing partitions, perfect matchings, permutations, inversion number, major index, descent algebra, noncommutative symmetric functions.

[†] Research partially supported by the Austrian Science Foundation FWF, grants P12094-MAT and P13190-MAT.

or maybe

$$\det_{1\le i,j\le n}\left(\binom{x+y+j}{x-i+2j}-\binom{x+y+j}{x+i+2j}\right). \tag{1.4}$$

Honestly, which ideas would you have? (Just to tell you that I do not ask for something impossible: Each of these four determinants can be evaluated in "closed form". If you want to see the solutions immediately, plus information where these determinants come from, then go to (2.7), (2.17)/(3.12), (2.19)/(3.30), respectively (3.47).)

Okay, let us try some row and column manipulations. Indeed, although it is not completely trivial (actually, it is quite a challenge), that would work for the first two determinants, (1.1) and (1.2), although I do not recommend that. However, I do not recommend *at all* that you try this with the latter two determinants, (1.3) and (1.4). I promise that you will fail. (The determinant (1.3) does not look much more complicated than (1.2). Yet, it is.)

So, what should we do instead?

Of course, let us look in the literature! Excellent idea. We may have the problem of not knowing where to start looking. Good starting points are certainly classics like [119], [120], [121], [127] and [178][1]. This will lead to the first success, as (1.1) does indeed turn up there (see [119, vol. III, p. 311]). Yes, you will also find evaluations for (1.2) (see e.g. [126]) and (1.3) (see [112, Theorem 7]) in the existing literature. But at the time of the writing you will not, to the best of my knowledge, find an evaluation of (1.4) in the literature.

The purpose of this article is threefold. First, I want to describe a few useful and efficient tools which should enable you to evaluate nontrivial determinants (see Section 2). Second, I provide a list containing a number of such determinants that have been already evaluated, together with explanations which tell in which contexts they have appeared (see Section 3). Third, even if you should not find your determinant in this list, I point out references where further such determinant evaluations can be found, maybe your determinant is there.

Most important of all is that I want to convince you that, today,

Evaluating determinants is not (okay: may not be) *difficult!*

When George Andrews, who must be rightly called *the pioneer of determinant evaluations*, in the seventies astounded the combinatorial community by his highly nontrivial determinant evaluations (solving difficult enumeration problems on plane partitions), it *was* really difficult. His method (see Section 2.6 for a description) required a good "guesser" and an excellent "hypergeometer" (both of which he was and is). While at that time especially to be the latter was quite a task, in the meantime *both* guessing and evaluating binomial and hypergeometric sums has been largely *trivialized*, as both can be done (most of the time) *completely automatically*. For guessing (see Appendix A)

[1] Turnbull's book [178] does in fact contain rather lots of very general identities satisfied by determinants, than determinant "evaluations" in the strict sense of the word. However, suitable specializations of these general identities do also yield "genuine" evaluations, see for example Appendix B. Since the value of this book may not be easy to appreciate because of heavy notation, we refer the reader to [102] for a clarification of the notation and a clear presentation of many such identities.

this is due to tools like `Superseeker`[2], `gfun` and `Mgfun`[3] [152, 24], and `Rate`[4] (which is by far the most primitive of the three, but it is the most effective in this context). For "hypergeometrics" this is due to the "WZ-machinery"[5] (see [130, 190, 194, 195, 196]). And even if you should meet a case where the WZ-machinery should exhaust your computer's capacity, then there are still computer algebra packages like HYP and HYPQ[6], or HYPERG[7], which make *you* an expert hypergeometer, as these packages comprise large parts of the present hypergeometric knowledge, and, thus, enable you to conveniently manipulate binomial and hypergeometric series (which George Andrews did largely by hand) on the computer. Moreover, as of today, there are a few new (perhaps just overlooked) insights which make life easier in many cases. It is these which form large parts of Section 2.

So, if you see a determinant, don't be frightened, evaluate it yourself!

2. Methods for the evaluation of determinants

In this section I describe a few useful methods and theorems which (may) help you to evaluate a determinant. As was mentioned already in the Introduction, it is always possible that simple-minded things like doing some *row* and/or *column operations*, or applying *Laplace expansion* may produce an (usually inductive) evaluation of a determinant. Therefore, you are of course advised to try such things first. What I am mainly addressing here, though, is the case where that first, "simple-minded" attempt failed. (Clearly, there is no point in addressing row and column operations, or Laplace expansion.)

Yet, we must of course start (in Section 2.1) with some standard determinants, such as the *Vandermonde determinant* or *Cauchy's double alternant*. These are of course well-known.

In Section 2.2 we continue with some *general determinant evaluations* that generalize the evaluation of the Vandermonde determinant, which are however apparently not equally well-known, although they should be. *In fact, I claim that about 80 % of the determinants that you meet in "real life," and which can apparently be evaluated, are a special case of just the very first of these* (Lemma 3; see in particular Theorem 26 and the subsequent remarks). Moreover, as is demonstrated in Section 2.2, it is pure routine to check whether a determinant is a special case of one of these general determinants. Thus, it can be really considered as a "method" to see if a determinant can be evaluated by one of the theorems in Section 2.2.

[2] the electronic version of the "Encyclopedia of Integer Sequences" [162, 161], written and developed by Neil Sloane and Simon Plouffe; see `http://www.research.att.com/~njas/sequences/ol.html`

[3] written by Bruno Salvy and Paul Zimmermann, respectively Frederic Chyzak; available from `http://pauillac.inria.fr/algo/libraries/libraries.html`

[4] written in *Mathematica* by the author; available from `http://radon.mat.univie.ac.at/People/kratt`; the *Maple* equivalent `GUESS` by François Béraud and Bruno Gauthier is available from `http://www-igm.univ-mlv.fr/~gauthier`

[5] *Maple* implementations written by Doron Zeilberger are available from `http://www.math.temple.edu/~zeilberg`, *Mathematica* implementations written by Peter Paule, Axel Riese, Markus Schorn, Kurt Wegschaider are available from `http://www.risc.uni-linz.ac.at/research/combinat/risc/software`

[6] written in *Mathematica* by the author; available from `http://radon.mat.univie.ac.at/People/kratt`

[7] written in *Maple* by Bruno Ghauthier; available from `http://www-igm.univ-mlv.fr/~gauthier`

The next method which I describe is the so-called *"condensation method"* (see Section 2.3), a method which allows to evaluate a determinant inductively (if the method works).

In Section 2.4, a method, which I call the *"identification of factors" method*, is described. This method has been extremely successful recently. It is based on a very simple idea, which comes from one of the standard proofs of the Vandermonde determinant evaluation (which is therefore described in Section 2.1).

The subject of Section 2.5 is a method which is based on finding one or more *differential or difference equations* for the matrix of which the determinant is to be evaluated.

Section 2.6 contains a short description of George Andrews' favourite method, which basically consists of *explicitly* doing the *LU-factorization* of the matrix of which the determinant is to be evaluated.

The remaining subsections in this section are conceived as a complement to the preceding. In Section 2.7 a special type of determinants is addressed, *Hankel determinants.* (These are determinants of the form $\det_{1\le i,j\le n}(a_{i+j})$, and are sometimes also called *persymmetric* or *Turánian determinants.*) As is explained there, you should expect that a Hankel determinant evaluation is to be found in the domain of *orthogonal polynomials* and *continued fractions.* Eventually, in Section 2.8 a few further, possibly useful results are exhibited.

Before we finally move into the subject, it must be pointed out that the methods of determinant evaluation as presented here are ordered according to the conditions a determinant must satisfy so that the method can be applied to it, from "stringent" to "less stringent". I. e., first come the methods which require that the matrix of which the determinant is to be taken satisfies a lot of conditions (usually: it contains a lot of parameters, at least, implicitly), and in the end comes the method (LU-factorization) which requires nothing. In fact, this order (of methods) is also the order in which I recommend that you try them on your determinant. That is, what I suggest is (and this is the rule I follow):

(0) First try some simple-minded things (*row* and *column operations*, *Laplace expansion*). Do not waste too much time. If you encounter a *Hankel-determinant* then see Section 2.7.
(1) If that fails, check whether your determinant is a special case of one of the *general determinants* in Sections 2.2 (and 2.1).
(2) If that fails, see if the *condensation method* (see Section 2.3) works. (If necessary, try to introduce more parameters into your determinant.)
(3) If that fails, try the *"identification of factors"* method (see Section 2.4). Alternatively, and in particular if your matrix of which you want to find the determinant is the matrix defining a system of differential or difference equations, try the *differential/difference equation method* of Section 2.5. (If necessary, try to introduce a parameter into your determinant.)
(4) If that fails, try to work out the *LU-factorization* of your determinant (see Section 2.6).
(5) If all that fails, then we are really in trouble. Perhaps you have to put more efforts into determinant manipulations (see suggestion (0))? Sometimes it is worthwile to interpret the matrix whose determinant you want to know as a linear map and try to find a basis on which this map acts triangularly, or even diagonally (this

requires that the eigenvalues of the matrix are "nice"; see [47, 48, 84, 93, 192] for examples where that worked). Otherwise, maybe something from Sections 2.8 or 3 helps?

A final remark: It was indicated that some of the methods require that your determinant contains (more or less) parameters. Therefore it is always a good idea to:

Introduce more parameters into your determinant!

(We address this in more detail in the last paragraph of Section 2.1.) The more parameters you can play with, the more likely you will be able to carry out the determinant evaluation. (Just to mention a few examples: The condensation method needs, at least, two parameters. The "identification of factors" method needs, at least, one parameter, as well as the differential/difference equation method in Section 2.5.)

2.1. A FEW STANDARD DETERMINANTS. Let us begin with a short proof of the *Vandermonde determinant evaluation*

$$\det_{1\le i,j\le n}\left(X_i^{j-1}\right)=\prod_{1\le i<j\le n}(X_j-X_i). \tag{2.1}$$

Although the following proof is well-known, it makes still sense to quickly go through it because, by extracting the essence of it, we will be able to build a very powerful method out of it (see Section 2.4).

If $X_{i_1}=X_{i_2}$ with $i_1\ne i_2$, then the Vandermonde determinant (2.1) certainly vanishes because in that case two rows of the determinant are identical. Hence, $(X_{i_1}-X_{i_2})$ divides the determinant as a polynomial in the X_i's. But that means that the complete product $\prod_{1\le i<j\le n}(X_j-X_i)$ (which is exactly the right-hand side of (2.1)) must divide the determinant.

On the other hand, the determinant is a polynomial in the X_i's of degree at most $\binom{n}{2}$. Combined with the previous observation, this implies that the determinant equals the right-hand side product times, possibly, some constant. To compute the constant, compare coefficients of $X_1^0X_2^1\cdots X_n^{n-1}$ on both sides of (2.1). This completes the proof of (2.1).

At this point, let us extract the essence of this proof as we will come back to it in Section 2.4. The basic steps are:

1. *Identification of factors*
2. *Determination of degree bound*
3. *Computation of the multiplicative constant.*

An immediate generalization of the Vandermonde determinant evaluation is given by the proposition below. It can be proved in just the same way as the above proof of the Vandermonde determinant evaluation itself.

Proposition 1. *Let $X_1,X_2,\dots,X_n$ be indeterminates. If $p_1,p_2,\dots,p_n$ are polynomials of the form $p_j(x)=a_jx^{j-1}+$ lower terms, then*

$$\det_{1\le i,j\le n}\left(p_j(X_i)\right)=a_1a_2\cdots a_n\prod_{1\le i<j\le n}(X_j-X_i). \tag{2.2}$$

□

The following variations of the Vandermonde determinant evaluation are equally easy to prove.

Lemma 2. *The following identities hold true:*

$$\det_{1\le i,j\le n}\left(X_i^j - X_i^{-j}\right) = (X_1\cdots X_n)^{-n} \prod_{1\le i<j\le n} \big((X_i-X_j)(1-X_iX_j)\big) \prod_{i=1}^{n}(X_i^2-1), \tag{2.3}$$

$$\det_{1\le i,j\le n}\left(X_i^{j-1/2} - X_i^{-(j-1/2)}\right) \\ = (X_1\cdots X_n)^{-n+1/2} \prod_{1\le i<j\le n} \big((X_i-X_j)(1-X_iX_j)\big) \prod_{i=1}^{n}(X_i-1), \tag{2.4}$$

$$\det_{1\le i,j\le n}\left(X_i^{j-1} + X_i^{-(j-1)}\right) = 2\cdot (X_1\cdots X_n)^{-n+1} \prod_{1\le i<j\le n} \big((X_i-X_j)(1-X_iX_j)\big), \tag{2.5}$$

$$\det_{1\le i,j\le n}\left(X_i^{j-1/2} + X_i^{-(j-1/2)}\right) \\ = (X_1\cdots X_n)^{-n+1/2} \prod_{1\le i<j\le n} \big((X_i-X_j)(1-X_iX_j)\big) \prod_{i=1}^{n}(X_i+1). \tag{2.6}$$

□

We remark that the evaluations (2.3), (2.4), (2.5) are basically the Weyl denominator factorizations of types C, B, D, respectively (cf. [52, Lemma 24.3, Ex. A.52, Ex. A.62, Ex. A.66]). For that reason they may be called the *"symplectic"*, the *"odd orthogonal"*, and the *"even orthogonal" Vandermonde determinant evaluation*, respectively.

If you encounter generalizations of such determinants of the form $\det_{1\le i,j\le n}(x_i^{\lambda_j})$ or $\det_{1\le i,j\le n}(x_i^{\lambda_j} - x_i^{-\lambda_j})$, etc., then you should be aware that what you encounter is basically *Schur functions*, *characters for the symplectic groups*, or *characters for the orthogonal groups* (consult [52, 105, 137] for more information on these matters; see in particular [105, Ch. I, (3.1)], [52, p. 403, (A.4)], [52, (24.18)], [52, (24.40) + first paragraph on p. 411], [137, Appendix A2], [52, (24.28)]). In this context, one has to also mention Okada's general results on evaluations of determinants and Pfaffians (see Section 2.8 for definition) in [124, Sec. 4] and [125, Sec. 5].

Another standard determinant evaluation is the evaluation of *Cauchy's double alternant* (see [119, vol. III, p. 311]),

$$\det_{1\le i,j\le n}\left(\frac{1}{X_i+Y_j}\right) = \frac{\prod_{1\le i<j\le n}(X_i-X_j)(Y_i-Y_j)}{\prod_{1\le i,j\le n}(X_i+Y_j)}. \tag{2.7}$$

Once you have seen the above proof of the Vandermonde determinant evaluation, you will immediately know how to prove this determinant evaluation.

On setting $X_i = i$ and $Y_i = i$, $i = 1, 2, \dots, n$, in (2.7), we obtain the evaluation of our first determinant in the Introduction, (1.1). For the evaluation of a mixture of Cauchy's double alternant and Vandermonde's determinant see [15, Lemma 2].

Whether or not you tried to evaluate (1.1) directly, here is an *important lesson to be learned* (it was already mentioned earlier): To evaluate (1.1) directly is quite difficult, whereas proving its generalization (2.7) is almost completely trivial. Therefore, it is always a good idea to try to *introduce more parameters into your determinant.* (That is, in a way such that the more general determinant still evaluates nicely.) More parameters mean that you have more objects at your disposal to play with.

The most stupid way to introduce parameters is to just write X_i instead of the row index i, or write Y_j instead of the column index j.[8] For the determinant (1.1) even both simultaneously was possible. For the determinant (1.2) either of the two (but not both) would work. On the contrary, there seems to be no nontrivial way to introduce more parameters in the determinant (1.4). This is an indication that the evaluation of this determinant is in a different category of difficulty of evaluation. (Also (1.3) belongs to this "different category". It is possible to introduce one more parameter, see (3.32), but it does not seem to be possible to introduce more.)

2.2. A GENERAL DETERMINANT LEMMA, PLUS VARIATIONS AND GENERALIZATIONS. In this section I present an apparently not so well-known determinant evaluation that generalizes Vandermonde's determinant, and some companions. As Lascoux pointed out to me, most of these determinant evaluations can be derived from the evaluation of a certain determinant of minors of a given matrix due to Turnbull [179, p. 505], see Appendix B. However, this (these) determinant evaluation(s) *deserve(s) to be better known.* Apart from the fact that there are numerous applications of it (them) which I am aware of, my proof is that I meet very often people who stumble across a special case of this (these) determinant evaluation(s), and then have a hard time to actually do the evaluation because, usually, their special case does not show the hidden general structure which is lurking behind. On the other hand, as I will demonstrate in a moment, if you know this (these) determinant evaluation(s) then it is a matter completely mechanical in nature to see whether it (they) is (are) applicable to your determinant or not. If one of them is applicable, you are immediately done.

The determinant evaluation of which I am talking is the determinant lemma from [85, Lemma 2.2] given below. Here, and in the following, empty products (like $(X_i + A_n)(X_i + A_{n-1}) \cdots (X_i + A_{j+1})$ for $j = n$) equal 1 by convention.

Lemma 3. *Let $X_1, \dots, X_n$, $A_2, \dots, A_n$, and $B_2, \dots, B_n$ be indeterminates. Then there holds*

$$\det_{1\le i,j\le n}\Big((X_i + A_n)(X_i + A_{n-1}) \cdots (X_i + A_{j+1})(X_i + B_j)(X_i + B_{j-1}) \cdots (X_i + B_2)\Big)$$
$$= \prod_{1\le i<j\le n}(X_i - X_j) \prod_{2\le i\le j\le n}(B_i - A_j). \tag{2.8}$$

□

[8] Other common examples of introducing more parameters are: Given that the (i,j)-entry of your determinant is a binomial such as $\binom{i+j}{2i-j}$, try $\binom{x+i+j}{2i-j}$ (that works; see (3.30)), or even $\binom{x+y+i+j}{y+2i-j}$ (that does not work; but see (1.2)), or $\binom{x+i+j}{2i-j} + \binom{y+i+j}{2i-j}$ (that works; see (3.32), and consult Lemma 19 and the remarks thereafter). However, sometimes parameters have to be introduced in an unexpected way, see (3.49). (The parameter x was introduced into a determinant of Bombieri, Hunt and van der Poorten, which is obtained by setting $x = 0$ in (3.49).)

Once you have guessed such a formula, it is easily proved. In the proof in [85] the determinant is reduced to a determinant of the form (2.2) by suitable column operations. Another proof, discovered by Amdeberhan (private communication), is by condensation, see Section 2.3. For a derivation from the above mentioned evaluation of a determinant of minors of a given matrix, due to Turnbull, see Appendix B.

Now let us see what the value of this formula is, by checking if it is of any use in the case of the second determinant in the Introduction, (1.2). The recipe that you should follow is:

1. *Take as many factors out of rows and/or columns of your determinant, so that all denominators are cleared.*
2. *Compare your result with the determinant in* (2.8). *If it matches, you have found the evaluation of your determinant.*

Okay, let us do so:

$$\begin{aligned}\det_{1\le i,j\le n}\left(\binom{a+b}{a-i+j}\right)&=\prod_{i=1}^{n}\frac{(a+b)!}{(a-i+n)!\,(b+i-1)!}\\ &\quad\times\det_{1\le i,j\le n}\big((a-i+n)(a-i+n-1)\cdots(a-i+j+1)\\ &\qquad\cdot(b+i-j+1)(b+i-j+2)\cdots(b+i-1)\big)\\ &=(-1)^{\binom n2}\prod_{i=1}^{n}\frac{(a+b)!}{(a-i+n)!\,(b+i-1)!}\\ &\quad\times\det_{1\le i,j\le n}\big((i-a-n)(i-a-n+1)\cdots(i-a-j-1)\\ &\qquad\cdot(i+b-j+1)(i+b-j+2)\cdots(i+b-1)\big).\end{aligned}$$

Now compare with the determinant in (2.8). Indeed, the determinant in the last line is just the special case $X_i=i$, $A_j=-a-j$, $B_j=b-j+1$. Thus, by (2.8), we have a result immediately. A particularly attractive way to write it is displayed in (2.17).

Applications of Lemma 3 are abundant, see Theorem 26 and the remarks accompanying it.

In [87, Lemma 7], a determinant evaluation is given which is closely related to Lemma 3. It was used there to establish enumeration results about *shifted plane partitions of trapezoidal shape.* It is the first result in the lemma below. It is "tailored" for the use in the context of q-enumeration. For plain enumeration, one would use the second result. This is a limit case of the first (replace X_i by q^{X_i}, A_j by $-q^{-A_j}$ and C by q^C in (2.9), divide both sides by $(1-q)^{n(n-1)}$, and then let $q\to 1$).

Lemma 4. *Let $X_1,X_2,\dots,X_n,A_2,\dots,A_n$ be indeterminates. Then there hold*

$$\det_{1\le i,j\le n}\big((C/X_i+A_n)(C/X_i+A_{n-1})\cdots(C/X_i+A_{j+1})\\ \cdot(X_i+A_n)(X_i+A_{n-1})\cdots(X_i+A_{j+1})\big)=\prod_{i=2}^{n}A_i^{i-1}\prod_{1\le i<j\le n}(X_i-X_j)(1-C/X_iX_j),\tag{2.9}$$

and

$$\det_{1\le i,j\le n}\Big((X_i-A_n-C)(X_i-A_{n-1}-C)\cdots(X_i-A_{j+1}-C)\\ \cdot(X_i+A_n)(X_i+A_{n-1})\cdots(X_i+A_{j+1})\Big)=\prod_{1\le i<j\le n}(X_j-X_i)(C-X_i-X_j). \quad (2.10)$$

□

(Both evaluations are in fact special cases in disguise of (2.2). Indeed, the (i,j)-entry of the determinant in (2.9) is a polynomial in X_i+C/X_i, while the (i,j)-entry of the determinant in (2.10) is a polynomial in $X_i-C/2$, both of degree $n-j$.)

The standard application of Lemma 4 is given in Theorem 27.

In [88, Lemma 34], a common generalization of Lemmas 3 and 4 was given. In order to have a convenient statement of this determinant evaluation, we define the *degree* of a *Laurent polynomial* $p(X)=\sum_{i=M}^{N}a_ix^i$, $M,N\in\mathbb{Z}$, $a_i\in\mathbb{R}$ and $a_N\ne 0$, to be $\deg p:=N$.

Lemma 5. *Let $X_1,X_2,\dots,X_n,A_2,A_3,\dots,A_n,C$ be indeterminates. If $p_0,p_1,\dots,p_{n-1}$ are Laurent polynomials with $\deg p_j\le j$ and $p_j(C/X)=p_j(X)$ for $j=0,1,\dots,n-1$, then*

$$\det_{1\le i,j\le n}\Big((X_i+A_n)(X_i+A_{n-1})\cdots(X_i+A_{j+1})\\ \cdot(C/X_i+A_n)(C/X_i+A_{n-1})\cdots(C/X_i+A_{j+1})\cdot p_{j-1}(X_i)\Big)\\ =\prod_{1\le i<j\le n}(X_i-X_j)(1-C/X_iX_j)\prod_{i=1}^{n}A_i^{i-1}\prod_{i=1}^{n}p_{i-1}(-A_i)\ . \quad (2.11)$$

□

Section 3 contains several determinant evaluations which are implied by the above determinant lemma, see Theorems 28, 30 and 31.

Lemma 3 does indeed come out of the above Lemma 5 by setting $C=0$ and

$$p_j(X)=\prod_{k=1}^{j}(B_{k+1}+X).$$

Obviously, Lemma 4 is the special case $p_j\equiv 1$, $j=0,1,\dots,n-1$. It is in fact worth stating the $C=0$ case of Lemma 5 separately.

Lemma 6. *Let $X_1,X_2,\dots,X_n,A_2,A_3,\dots,A_n$ be indeterminates. If $p_0,p_1,\dots,p_{n-1}$ are polynomials with $\deg p_j\le j$ for $j=0,1,\dots,n-1$, then*

$$\det_{1\le i,j\le n}\Big((X_i+A_n)(X_i+A_{n-1})\cdots(X_i+A_{j+1})\cdot p_{j-1}(X_i)\Big)\\ =\prod_{1\le i<j\le n}(X_i-X_j)\prod_{i=1}^{n}p_{i-1}(-A_i)\ . \quad (2.12)$$

□

Again, Lemma 5 is tailored for applications in q-enumeration. So, also here, it may be convenient to state the according limit case that is suitable for plain enumeration (and perhaps other applications).

Lemma 7. *Let $X_1, X_2, \dots, X_n, A_2, A_3, \dots, A_n, C$ be indeterminates. If $p_0, p_1, \dots, p_{n-1}$ are polynomials with $\deg p_j \le 2j$ and $p_j(C - X) = p_j(X)$ for $j = 0, 1, \dots, n-1$, then*

$$\det_{1\le i,j\le n} \big((X_i + A_n)(X_i + A_{n-1}) \cdots (X_i + A_{j+1}) \\ \cdot (X_i - A_n - C)(X_i - A_{n-1} - C) \cdots (X_i - A_{j+1} - C) \cdot p_{j-1}(X_i)\big) \\ = \prod_{1\le i<j\le n} (X_j - X_i)(C - X_i - X_j) \prod_{i=1}^{n} p_{i-1}(-A_i)\ . \quad (2.13)$$

□

In concluding, I want to mention that, now since more than ten years, I have a different common generalization of Lemmas 3 and 4 (with some overlap with Lemma 5) in my drawer, without ever having found use for it. Let us nevertheless state it here; maybe it is exactly the key to the solution of a problem of yours.

Lemma 8. *Let $X_1, \dots, X_n$, $A_2, \dots, A_n$, $B_2, \dots B_n$, $a_2, \dots, a_n$, $b_2, \dots b_n$, and C be indeterminates. Then there holds*

$$\det_{1\le i,j\le n} \left(\begin{cases} (X_i + A_n) \cdots (X_i + A_{j+1})(C/X_i + A_n) \cdots (C/X_i + A_{j+1}) & \\ (X_i + B_j) \cdots (X_i + B_2)(C/X_i + B_j) \cdots (C/X_i + B_2) & j < m \\ (X_i + a_n) \cdots (X_i + a_{j+1})(C/X_i + a_n) \cdots (C/X_i + a_{j+1}) & \\ (X_i + b_j) \cdots (X_i + b_2)(C/X_i + b_j) \cdots (C/X_i + b_2) & j \ge m \end{cases} \right) \\ = \prod_{1\le i<j\le n} (X_i - X_j)(1 - C/X_iX_j) \prod_{2\le i\le j\le m-1} (B_i - A_j)(1 - C/B_iA_j) \\ \times \prod_{i=2}^{m} \prod_{j=m}^{n} (b_i - A_j)(1 - C/b_iA_j) \prod_{m+1\le i\le j\le n} (b_i - a_j)(1 - C/b_ia_j) \\ \times \prod_{i=2}^{m} (A_i \cdots A_n) \prod_{i=m+1}^{n} (a_i \cdots a_n) \prod_{i=2}^{m-1} (B_2 \cdots B_i) \prod_{i=m}^{n} (b_2 \cdots b_i). \quad (2.14)$$

□

The limit case which goes with this determinant lemma is the following. (There is some overlap with Lemma 7.)

Lemma 9. *Let $X_1,\dots,X_n$, $A_2,\dots,A_n$, $B_2,\dots,B_n$, $a_2,\dots,a_n$, $b_2,\dots,b_n$, and C be indeterminates. Then there holds*

$$\det_{1\le i,j\le n}\left(\begin{cases}(X_i+A_n)\cdots(X_i+A_{j+1})(X_i-A_n-C)\cdots(X_i-A_{j+1}-C)\\ (X_i+B_j)\cdots(X_i+B_2)(X_i-B_j-C)\cdots(X_i-B_2-C) & j<m\\ (X_i+a_n)\cdots(X_i+a_{j+1})(X_i-a_n-C)\cdots(X_i-a_{j+1}-C)\\ (X_i+b_j)\cdots(X_i+b_2)(X_i-b_j-C)\cdots(X_i-b_2-C) & j\ge m\end{cases}\right)$$

$$=\prod_{1\le i<j\le n}(X_i-X_j)(C-X_i-X_j)\prod_{2\le i\le j\le m-1}(B_i-A_j)(B_i+A_j+C)$$

$$\times\prod_{i=2}^{m}\prod_{j=m}^{n}(b_i-A_j)(b_i+A_j+C)\prod_{m+1\le i\le j\le n}(b_i-a_j)(b_i+a_j+C). \tag{2.15}$$

□

If you are looking for more determinant evaluations of such a general type, then you may want to look at [156, Lemmas A.1 and A.11] and [158, Lemma A.1].

2.3. The condensation method. This is Doron Zeilberger's favourite method. It allows (sometimes) to establish an elegant, *effortless* inductive proof of a determinant evaluation, in which the only task is to guess the result correctly.

The method is often attributed to Charles Ludwig Dodgson [38], better known as Lewis Carroll. However, the identity on which it is based seems to be actually due to P. Desnanot (see [119, vol. I, pp. 140–142]; with the first rigorous proof being probably due to Jacobi, see [18, Ch. 4] and [79, Sec. 3]). This identity is the following.

Proposition 10. *Let A be an $n\times n$ matrix. Denote the submatrix of A in which rows $i_1,i_2,\dots,i_k$ and columns $j_1,j_2,\dots,j_k$ are omitted by $A^{j_1,j_2,\dots,j_k}_{i_1,i_2,\dots,i_k}$. Then there holds*

$$\det A\cdot\det A^{1,n}_{1,n}=\det A^1_1\cdot\det A^n_n-\det A^n_1\cdot\det A^1_n. \tag{2.16}$$

□

So, what is the point of this identity? Suppose you are given a family $(\det M_n)_{n\ge0}$ of determinants, M_n being an $n\times n$ matrix, $n=0,1,\dots$. Maybe $M_n=M_n(a,b)$ is the matrix underlying the determinant in (1.2). Suppose further that you have already worked out a conjecture for the evaluation of $\det M_n(a,b)$ (we did in fact already evaluate this determinant in Section 2.2, but let us ignore that for the moment),

$$\det M_n(a,b):=\det_{1\le i,j\le n}\left(\binom{a+b}{a-i+j}\right)\overset{?}{=}\prod_{i=1}^{n}\prod_{j=1}^{a}\prod_{k=1}^{b}\frac{i+j+k-1}{i+j+k-2}. \tag{2.17}$$

Then you have already proved your conjecture, once you observe that

$$\begin{aligned}\big(M_n(a,b)\big)^n_n&=M_{n-1}(a,b),\\ \big(M_n(a,b)\big)^1_1&=M_{n-1}(a,b),\\ \big(M_n(a,b)\big)^1_n&=M_{n-1}(a+1,b-1),\\ \big(M_n(a,b)\big)^n_1&=M_{n-1}(a-1,b+1),\\ \big(M_n(a,b)\big)^{1,n}_{1,n}&=M_{n-2}(a,b).\end{aligned} \tag{2.18}$$

For, because of (2.18), Desnanot's identity (2.16), with $A = M_n(a,b)$, gives a recurrence which expresses $\det M_n(a,b)$ in terms of quantities of the form $\det M_{n-1}(\,.\,)$ and $\det M_{n-2}(\,.\,)$. So, it just remains to check the conjecture (2.17) for $n = 0$ and $n = 1$, and that the right-hand side of (2.17) satisfies the same recurrence, because that completes a perfect induction with respect to n. (What we have described here is basically the contents of [197]. For a bijective proof of Proposition 10 see [200].)

Amdeberhan (private communication) discovered that in fact the determinant evaluation (2.8) itself (which we used to evaluate the determinant (1.2) for the first time) can be proved by condensation. The reader will easily figure out the details. Furthermore, the condensation method also proves the determinant evaluations (3.35) and (3.36). (Also this observation is due to Amdeberhan [2].) At another place, condensation was used by Eisenkölbl [41] in order to establish a conjecture by Propp [138, Problem 3] about the enumeration of rhombus tilings of a hexagon where some triangles along the border of the hexagon are missing.

The reader should observe that crucial for a successful application of the method is the existence of (at least) *two parameters* (in our example these are a and b), which help to still stay within the same family of matrices when we take minors of our original matrix (compare (2.18)). (See the last paragraph of Section 2.1 for a few hints of how to introduce more parameters into your determinant, in the case that you are short of parameters.) Obviously, aside from the fact that we need at least two parameters, we can hope for a success of condensation only if our determinant is of a special kind.

2.4. THE "IDENTIFICATION OF FACTORS" METHOD. This is the method that I find most convenient to work with, once you encounter a determinant that is not amenable to an evaluation using the previous recipes. It is best to explain this method along with an example. So, let us consider the determinant in (1.3). Here it is, together with its, at this point, unproven evaluation,

$$\det_{0\le i,j\le n-1}\left(\binom{\mu+i+j}{2i-j}\right)\\ = (-1)^{\chi(n\equiv 3 \mod 4)}2^{\binom{n-1}{2}}\prod_{i=1}^{n-1}\frac{(\mu+i+1)_{\lfloor (i+1)/2\rfloor}\left(-\mu-3n+i+\frac{3}{2}\right)_{\lfloor i/2\rfloor}}{(i)_i}, \tag{2.19}$$

where $\chi(\mathcal{A}) = 1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A}) = 0$ otherwise, and where the *shifted factorial* $(a)_k$ is defined by $(a)_k := a(a+1)\cdots(a+k-1)$, $k \ge 1$, and $(a)_0 := 1$.

As was already said in the Introduction, this determinant belongs to a different category of difficulty of evaluation, so that nothing what was presented so far will immediately work on that determinant.

Nevertheless, I claim that the procedure which we chose to evaluate the Vandermonde determinant works also with the above determinant. To wit:

1. *Identification of factors*
2. *Determination of degree bound*
3. *Computation of the multiplicative constant.*

You will say: 'A moment please! The reason that this procedure worked so smoothly for the Vandermonde determinant is that there are so many (to be precise: n) variables at our disposal. On the contrary, the determinant in (2.19) has exactly *one* (!) variable.'

Yet — and this is the point that I want to make here — it works, *in spite of having just one variable at our disposal*!.

What we want to prove in the first step is that the right-hand side of (2.19) divides the determinant. For example, we would like to prove that $(\mu+n)$ divides the determinant (actually, $(\mu+n)^{\lfloor (n+1)/3\rfloor}$, we will come to that in a moment). Equivalently, if we set $\mu=-n$ in the determinant, then it should vanish. How could we prove that? Well, if it vanishes then there must be a linear combination of the columns, or of the rows, that vanishes. So, let us find such a linear combination of columns or rows. Equivalently, for $\mu=-n$ we find a vector in the kernel of the matrix in (2.19), respectively its transpose. More generally (and this addresses that we actually want to prove that $(\mu+n)^{\lfloor (n+1)/3\rfloor}$ divides the determinant):

> *For proving that $(\mu+n)^E$ divides the determinant, we find E linear independent vectors in the kernel.*

(For a formal justification that this does indeed suffice, see Section 2 of [91], and in particular the Lemma in that section.)

Okay, how is this done in practice? You go to your computer, crank out these vectors in the kernel, for $n=1,2,3,\dots$, and try to make a guess what they are in general. To see how this works, let us do it in our example. What the computer gives is the following (we are using *Mathematica* here):

```
In[1]:= V[2]

Out[1]= {0, c[1]}

In[2]:= V[3]

Out[2]= {0, c[2], c[2]}

In[3]:= V[4]

Out[3]= {0, c[1], 2 c[1], c[1]}

In[4]:= V[5]

Out[4]= {0, c[1], 3 c[1], c[3], c[1]}

In[5]:= V[6]

Out[5]= {0, c[1], 4 c[1], 2 c[1] + c[4], c[4], c[1]}

In[6]:= V[7]

Out[6]= {0, c[1], 5 c[1], c[3], -10 c[1] + 2 c[3], -5 c[1] + c[3], c[1]}

In[7]:= V[8]

Out[7]= {0, c[1], 6 c[1], c[3], -25 c[1] + 3 c[3], c[5], -9 c[1] + c[3], c[1]}

In[8]:= V[9]

Out[8]= {0, c[1], 7 c[1], c[3], -49 c[1] + 4 c[3],

⟩     -28 c[1] + 2 c[3] + c[6], c[6], -14 c[1] + c[3], c[1]}

In[9]:= V[10]
```

```
Out[9]= {0, c[1], 8 c[1], c[3], -84 c[1] + 5 c[3], c[5],

>      196 c[1] - 10 c[3] + 2 c[5], 98 c[1] - 5 c[3] + c[5], -20 c[1] + c[3],

>      c[1]}

In[10]:= V[11]

Out[10]= {0, c[1], 9 c[1], c[3], -132 c[1] + 6 c[3], c[5],

>      648 c[1] - 25 c[3] + 3 c[5], c[7], 234 c[1] - 9 c[3] + c[5],

>      -27 c[1] + c[3], c[1]}
```

Here, $V[n]$ is the generic vector (depending on the indeterminates $c[i]$) in the kernel of the matrix in (2.19) with $\mu = -n$. For convenience, let us denote this matrix by M_n.

You do not have to stare at these data for long to see that, in particular,

the vector $(0, 1)$ is in the kernel of M_2,
the vector $(0, 1, 1)$ is in the kernel of M_3,
the vector $(0, 1, 2, 1)$ is in the kernel of M_4,
the vector $(0, 1, 3, 3, 1)$ is in the kernel of M_5 (set $c[1] = 1$ and $c[3] = 3$),
the vector $(0, 1, 4, 6, 4, 1)$ is in the kernel of M_6 (set $c[1] = 1$ and $c[4] = 4$), etc.

Apparently,

$$\left(0, \tbinom{n-2}{0}, \tbinom{n-2}{1}, \tbinom{n-2}{2}, \dots, \tbinom{n-2}{n-2}\right) \tag{2.20}$$

is in the kernel of M_n. That was easy! But we need more linear combinations. Take a closer look, and you will see that the pattern persists (set $c[1] = 0$ everywhere, etc.). It will take you no time to work out a full-fledged conjecture for $\lfloor (n+1)/3 \rfloor$ linear independent vectors in the kernel of M_n.

Of course, there remains something to be proved. We need to actually prove that our guessed vectors are indeed in the kernel. E.g., in order to prove that the vector (2.20) is in the kernel, we need to verify that

$$\sum_{j=1}^{n-1} \binom{n-2}{j-1} \binom{-n+i+j}{2i-j} = 0$$

for $i = 0, 1, \dots, n-1$. However, verifying binomial identities is pure routine today, by means of Zeilberger's algorithm [194, 196] (see Footnote 5 in the Introduction).

Next you perform the same game with the other factors of the right-hand side product of (2.19). This is not much more difficult. (See Section 3 of [91] for details. There, slightly different vectors are used.)

Thus, we would have finished the first step, "identification of factors," of our plan: We have proved that the right-hand side of (2.19) divides the determinant as a polynomial in μ.

The second step, "determination of degree bound," consists of determining the (maximal) degree in μ of determinant and conjectured result. As is easily seen, this is $\binom{n}{2}$ in each case.

The arguments thus far show that the determinant in (2.19) must equal the right-hand side times, possibly, some constant. To determine this constant in the third step, "computation of the multiplicative constant," one compares coefficients of $x^{\binom{n}{2}}$ on both sides of (2.19). This is an enjoyable exercise. (Consult [91] if you do not want to do it yourself.) Further successful applications of this procedure can be found in [27, 30, 42, 89, 90, 92, 94, 97, 132].

Having done that, let me point out that most of the individual steps in this sort of calculation can be done (almost) *automatically*. In detail, what did we do? We had to

1. *Guess* the result. (Indeed, without the result we could not have got started.)
2. *Guess* the vectors in the kernel.
3. Establish a *binomial* (*hypergeometric*) *identity*.
4. Determine a degree bound.
5. Compute a particular value or coefficient in order to determine the multiplicative constant.

As I explain in Appendix A, *guessing can be largely automatized*. It was already mentioned in the Introduction that *proving binomial* (*hypergeometric*) *identities can be done by the computer*, thanks to the *"WZ-machinery"* [130, 190, 194, 195, 196] (see Footnote 5). Computing the degree bound is (in most cases) so easy that no computer is needed. (You may use it if you want.) It is only the determination of the multiplicative constant (item 5 above) by means of a special evaluation of the determinant or the evaluation of a special coefficient (in our example we determined the coefficient of $\mu^{\binom{n}{2}}$) for which I am not able to offer a recipe so that things could be carried out on a computer.

The reader should notice that crucial for a successful application of the method is the existence of (at least) *one parameter* (in our example this is μ) to be able to apply the polynomiality arguments that are the "engine" of the method. If there is no parameter (such as in the determinant in Conjecture 49, or in the determinant (3.46) which would solve the problem of q-enumerating totally symmetric plane partitions), then we even cannot get started. (See the last paragraph of Section 2.1 for a few hints of how to introduce a parameter into your determinant, in the case that you are short of a parameter.)

On the other hand, a significant advantage of the "identification of factors method" is that not only is it capable of proving evaluations of the form

$$\det(M) = \text{CLOSED FORM},$$

(where CLOSED FORM means a product/quotient of "nice" factors, such as (2.19) or (2.17)), but also of proving evaluations of the form

$$\det(M) = (\text{CLOSED FORM}) \times (\text{UGLY POLYNOMIAL}), \tag{2.21}$$

where, of course, M is a matrix containing (at least) one parameter, μ say. Examples of such determinant evaluations are (3.38), (3.39), (3.45) or (3.48). (The UGLY POLYNOMIAL in (3.38), (3.39) and (3.48) is the respective sum on the right-hand side, which in neither case can be simplified).

How would one approach the proof of such an evaluation? For one part, we already know. "Identification of factors" enables us to show that (CLOSED FORM) divides $\det(M)$ as a polynomial in μ. Then, comparison of degrees in μ on both sides of (2.21) yields that (UGLY POLYNOMIAL) is a (at this point unknown) polynomial in

μ of some maximal degree, m say. How can we determine this polynomial? Nothing "simpler" than that: We find $m+1$ values e such that we are able to evaluate $\det(M)$ at $\mu = e$. If we then set $\mu = e$ in (2.21) and solve for (UGLY POLYNOMIAL), then we obtain evaluations of (UGLY POLYNOMIAL) at $m+1$ different values of μ. Clearly, this suffices to find (UGLY POLYNOMIAL), e.g., by Lagrange interpolation.

I put "simpler" in quotes, because it is here where the crux is: We may not be able to find *enough* such special evaluations of $\det(M)$. In fact, you may object: 'Why all these complications? If we should be able to find $m+1$ special values of μ for which we are able to evaluate $\det(M)$, then what prevents us from evaluating $\det(M)$ as a whole, for generic μ?' When I am talking of evaluating $\det(M)$ for $\mu = e$, then what I have in mind is that the evaluation of $\det(M)$ at $\mu = e$ is "nice" (i.e., gives a "closed form," with no "ugly" expression involved, such as in (2.21)), which is easier to identify (that is, to guess; see Appendix A) and in most cases easier to prove. By experience, such evaluations are rare. Therefore, the above described procedure will only work if the degree of (UGLY POLYNOMIAL) is not too large. (If you are just a bit short of evaluations, then finding other informations about (UGLY POLYNOMIAL), like the leading coefficient, may help to overcome the problem.)

To demonstrate this procedure by going through a concrete example is beyond the scope of this article. We refer the reader to [28, 43, 50, 51, 89, 90] for places where this procedure was successfully used to solve difficult enumeration problems on rhombus tilings, respectively prove a conjectured constant term identity.

2.5. A DIFFERENTIAL/DIFFERENCE EQUATION METHOD. In this section I outline a method for the evaluation of determinants, often used by Vitaly Tarasov and Alexander Varchenko, which, as the preceding method, also requires (at least) one parameter.

Suppose we are given a matrix $M = M(z)$, depending on the parameter z, of which we want to compute the determinant. Furthermore, suppose we know that M satisfies a differential equation of the form

$$\frac{d}{dz} M(z) = T(z) M(z), \tag{2.22}$$

where $T(z)$ is some other known matrix. Then, by elementary linear algebra, we obtain a differential equation for the determinant,

$$\frac{d}{dz} \det M(z) = \operatorname{Tr}(T(z)) \cdot \det M(z), \tag{2.23}$$

which is usually easy to solve. (In fact, the differential operator in (2.22) and (2.23) could be replaced by *any* operator. In particular, we could replace d/dz by the *difference* operator with respect to z, in which case (2.23) is usually easy to solve as well.)

Any method is best illustrated by an example. Let us try this method on the determinant (1.2). Right, we did already evaluate this determinant twice (see Sections 2.2 and 2.3), but let us pretend that we have forgotten all this.

Of course, application of the method to (1.2) itself does not seem to be extremely promising, because that would involve the differentiation of binomial coefficients. So, let us first take some factors out of the determinant (as we also did in Section 2.2),

$$\begin{aligned}\det_{1\le i,j\le n}\left(\binom{a+b}{a-i+j}\right)=\prod_{i=1}^{n}\frac{(a+b)!}{(a-i+n)!\,(b+i-1)!}\\ \times\det_{1\le i,j\le n}\Big((a-i+n)(a-i+n-1)\cdots(a-i+j+1)\\ \cdot(b+i-j+1)(b+i-j+2)\cdots(b+i-1)\Big).\end{aligned}$$

Let us denote the matrix underlying the determinant on the right-hand side of this equation by $M_n(a)$. In order to apply the above method, we have need for a matrix $T_n(a)$ such that

$$\frac{d}{da}M_n(a)=T_n(a)M_n(a). \tag{2.24}$$

Similar to the procedure of Section 2.6, the best idea is to go to the computer, crank out $T_n(a)$ for $n=1,2,3,4,\dots$, and, out of the data, make a guess for $T_n(a)$. Indeed, it suffices that I display $T_5(a)$,

$$\left(\begin{matrix}\frac{1}{1+a+b}+\frac{1}{2+a+b}+\frac{1}{3+a+b}+\frac{1}{4+a+b} & \frac{4}{4+a+b} & -\frac{6}{3+a+b}+\frac{6}{4+a+b}\\ 0 & \frac{1}{1+a+b}+\frac{1}{2+a+b}+\frac{1}{3+a+b} & \frac{3}{3+a+b}\\ 0 & 0 & \frac{1}{1+a+b}+\frac{1}{2+a+b}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{matrix}\right.$$

$$\left.\begin{matrix}\frac{4}{2+a+b}-\frac{8}{3+a+b}+\frac{4}{4+a+b} & -\frac{1}{1+a+b}+\frac{3}{2+a+b}-\frac{3}{3+a+b}+\frac{1}{4+a+b}\\ -\frac{3}{2+a+b}+\frac{3}{3+a+b} & \frac{1}{1+a+b}-\frac{2}{2+a+b}+\frac{1}{3+a+b}\\ \frac{2}{2+a+b} & -\frac{1}{1+a+b}+\frac{1}{2+a+b}\\ \frac{1}{1+a+b} & \frac{1}{1+a+b}\\ 0 & 0\end{matrix}\right)$$

(in this display, the first line contains columns $1,2,3$ of $T_5(a)$, while the second line contains the remaining columns), so that you are forced to conclude that, apparently, it must be true that

$$T_n(a)=\left(\binom{n-i}{j-i}\sum_{k=0}^{n-i-1}\binom{j-i-1}{k}\frac{(-1)^k}{a+b+n-i-k}\right)_{1\le i,j,\le n}.$$

That (2.24) holds with this choice of $T_n(a)$ is then easy to verify. Consequently, by means of (2.23), we have

$$\frac{d}{da}\det M_n(a)=\left(\sum_{\ell=1}^{n-1}\frac{n-\ell}{a+b+\ell}\right)\det M_n(a),$$

so that

$$M_n(a)=\text{constant}\cdot\prod_{\ell=1}^{n-1}(a+b+\ell)^{n-\ell}. \tag{2.25}$$

The constant is found to be $(-1)^{\binom{n}{2}}\prod_{\ell=0}^{n-1}\ell!$, e.g., by dividing both sides of (2.25) by $a^{\binom{n}{2}}$, letting a tend to infinity, and applying (2.2) to the remaining determinant.

More sophisticated applications of this method (actually, of a version for *systems* of *difference* operators) can be found in [175, Proof of Theorem 5.14] and [176, Proofs of Theorems 5.9, 5.10, 5.11], in the context of the *Knizhnik–Zamolodchikov equations*.

2.6. LU-FACTORIZATION. This is George Andrews' favourite method. Starting point is the well-known fact (see [53, p. 33ff]) that, given a square matrix M, there exists, under suitable, not very stringent conditions (in particular, these are satisfied if all top-left principal minors of M are nonzero), a unique lower triangular matrix L and a unique upper diagonal matrix U, the latter with all entries along the diagonal equal to 1, such that

$$M = L \cdot U. \tag{2.26}$$

This unique factorization of the matrix M is known as the L(ower triangular)U(pper triangular)-*factorization of* M, or as well as the *Gauß decomposition* of M.

Equivalently, for a square matrix M (satisfying these conditions) there exists a unique lower triangular matrix L and a unique upper triangular matrix U, the latter with all entries along the diagonal equal to 1, such that

$$M \cdot U = L. \tag{2.27}$$

Clearly, once you know L and U, the determinant of M is easily computed, as it equals the product of the diagonal entries of L.

Now, let us suppose that we are given a family $(M_n)_{n\ge 0}$ of matrices, where M_n is an $n \times n$ matrix, $n = 0, 1, \dots$, of which we want to compute the determinant. Maybe M_n is the determinant in (1.3). By the above, we know that (normally) there exist uniquely determined matrices L_n and U_n, $n = 0, 1, \dots$, L_n being lower triangular, U_n being upper triangular with all diagonal entries equal to 1, such that

$$M_n \cdot U_n = L_n. \tag{2.28}$$

However, we do not know what the matrices L_n and U_n are. What George Andrews does is that he goes to his computer, cranks out L_n and U_n for $n = 1, 2, 3, 4, \dots$ (this just amounts to solving a system of linear equations), and, out of the data, tries to guess what the coefficients of the matrices L_n and U_n are. Once he has worked out a guess, he somehow proves that his guessed matrices L_n and U_n do indeed satisfy (2.28).

This program is carried out in [10] for the family of determinants in (1.3). As it turns out, guessing is really easy, while the underlying hypergeometric identities which are needed for the proof of (2.28) are (from a hypergeometric viewpoint) quite interesting.

For a demonstration of the method of LU-factorization, we will content ourselves here with trying the method on the Vandermonde determinant. That is, let M_n be the determinant in (2.1). We go to the computer and crank out the matrices L_n and U_n for small values of n. For the purpose of guessing, it suffices that I just display the matrices L_5 and U_5. They are

$$L_5 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & (X_2 - X_1) & 0 \\ 1 & (X_3 - X_1) & (X_3 - X_1)(X_3 - X_2) \\ 1 & (X_4 - X_1) & (X_4 - X_1)(X_4 - X_2) \\ 1 & (X_5 - X_1) & (X_5 - X_1)(X_5 - X_2) \end{pmatrix.$$
$$\begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ (X_4 - X_1)(X_4 - X_2)(X_4 - X_3) & 0 \\ (X_5 - X_1)(X_5 - X_2)(X_5 - X_3) & (X_5 - X_1)(X_5 - X_2)(X_5 - X_3)(X_5 - X_4) \end{matrix}\right)$$

(in this display, the first line contains columns $1, 2, 3$ of L_5, while the second line contains the remaining columns), and

$$U_5 = \begin{pmatrix} 1 & -e_1(X_1) & e_2(X_1, X_2) & -e_3(X_1, X_2, X_3) & e_4(X_1, X_2, X_3, X_4) \\ 0 & 1 & -e_1(X_1, X_2) & e_2(X_1, X_2, X_3) & -e_3(X_1, X_2, X_3, X_4) \\ 0 & 0 & 1 & -e_1(X_1, X_2, X_3) & e_2(X_1, X_2, X_3, X_4) \\ 0 & 0 & 0 & 1 & -e_1(X_1, X_2, X_3, X_4) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $e_m(X_1, X_2, \dots, X_s) = \sum_{1 \le i_1 < \cdots < i_m \le s} X_{i_1} \cdots X_{i_m}$ denotes the *m-th elementary symmetric function*.

Having seen that, it will not take you for long to guess that, apparently, L_n is given by

$$L_n = \left(\prod_{k=1}^{j-1} (X_i - X_k) \right)_{1 \le i,j \le n},$$

and that U_n is given by

$$U_n = \big((-1)^{j-i} e_{j-i}(X_1, \dots, X_{j-1}) \big)_{1 \le i,j \le n},$$

where, of course, $e_m(X_1, \dots) := 0$ if $m < 0$. That (2.28) holds with these choices of L_n and U_n is easy to verify. Thus, the Vandermonde determinant equals the product of diagonal entries of L_n, which is exactly the product on the right-hand side of (2.1).

Applications of LU-factorization are abundant in the work of George Andrews [4, 5, 6, 7, 8, 10]. All of them concern solutions to difficult enumeration problems on various types of plane partitions. To mention another example, Aomoto and Kato [11, Theorem 3] computed the LU-factorization of a matrix which arose in the theory of q-difference equations, thus proving a conjecture by Mimachi [118].

Needless to say that this allows for variations. You may try to guess (2.26) directly (and not its variation (2.27)), or you may try to guess the U(pper triangular)L(ower triangular) *factorization*, or its variation in the style of (2.27). I am saying this because it may be easy to guess the form of one of these variations, while it can be very difficult to guess the form of another.

It should be observed that the way LU-factorization is used here in order to evaluate determinants is very much in the same spirit as "identification of factors" as described in the previous section. In both cases, the essential steps are to *first guess* something, and *then prove* the guess. Therefore, the remarks from the previous section about guessing and proving binomial (hypergeometric) identities apply here as well. In particular, for guessing you are once more referred to Appendix A.

It is important to note that, as opposed to "condensation" or "identification of factors," LU-factorization does not require any parameter. So, in principle, *it is applicable to any determinant* (which satisfies the aforementioned conditions). If there are limitations, then, from my experience, it is that the coefficients which have to be guessed in LU-factorization tend to be more complicated than in "identification of factors". That is, guessing (2.28) (or one of its variations) may sometimes be not so easy.

2.7. HANKEL DETERMINANTS. A *Hankel determinant* is a determinant of a matrix which has constant entries along antidiagonals, i.e., it is a determinant of the form

$$\det_{1\le i,j,\le n}(c_{i+j}).$$

If you encounter a Hankel determinant, which you think evaluates nicely, then expect the evaluation of your Hankel determinant to be found within the domain of *continued fractions* and *orthogonal polynomials*. In this section I explain what this connection is.

To make things concrete, let us suppose that we want to evaluate

$$\det_{0\le i,j\le n-1}(B_{i+j+2}), \tag{2.29}$$

where B_k denotes the k-th *Bernoulli number*. (The Bernoulli numbers are defined via their generating function, $\sum_{k=0}^{\infty} B_k z^k/k! = z/(e^z-1)$.) You have to try hard if you want to find an evaluation of (2.29) *explicitly* in the literature. Indeed, you can find it, hidden in Appendix A.5 of [108]. However, even if you are not able to discover this reference (which I would not have as well, unless the author of [108] would not have drawn my attention to it), there is a rather straight-forward way to find an evaluation of (2.29), which I outline below. It is based on the fact, and this is the main point of this section, that evaluations of Hankel determinants like (2.29) are, at least *implicitly*, in the literature on the theory of orthogonal polynomials and continued fractions, which is very accessible today.

So, let us review the relevant facts about orthogonal polynomials and continued fractions (see [76, 81, 128, 174, 186, 188] for more information on these topics).

We begin by citing the result, due to Heilermann, which makes the connection between Hankel determinants and continued fractions.

Theorem 11. (Cf. [188, Theorem 51.1] or [186, Corollaire 6, (19), on p. IV-17]). *Let $(\mu_k)_{k\ge0}$ be a sequence of numbers with generating function $\sum_{k=0}^{\infty}\mu_k x^k$ written in the form*

$$\sum_{k=0}^{\infty}\mu_k x^k = \cfrac{\mu_0}{1+a_0x-\cfrac{b_1x^2}{1+a_1x-\cfrac{b_2x^2}{1+a_2x-\cdots}}}\ . \tag{2.30}$$

Then the Hankel determinant $\det_{0\le i,j\le n-1}(\mu_{i+j})$ *equals* $\mu_0^n b_1^{n-1} b_2^{n-2}\cdots b_{n-2}^2 b_{n-1}$. □

(We remark that a continued fraction of the type as in (2.30) is called a *J-fraction*.)

Okay, that means we would have evaluated (2.29) once we are able to explicitly expand the generating function $\sum_{k=0}^{\infty} B_{k+2}x^k$ in terms of a continued fraction of the form of the right-hand side of (2.30). Using the tools explained in Appendix A, it is easy to work out a conjecture,

$$\sum_{k=0}^{\infty} B_{k+2}x^k = \cfrac{1/6}{1-\cfrac{b_1x^2}{1-\cfrac{b_2x^2}{1-\cdots}}}\quad , \tag{2.31}$$

where $b_i = -i(i+1)^2(i+2)/4(2i+1)(2i+3)$, $i = 1, 2, \dots$. If we would find this expansion in the literature then we would be done. But if not (which is the case here), how to prove such an expansion? The key is *orthogonal polynomials*.

A sequence $(p_n(x))_{n\ge 0}$ of polynomials is called (formally) *orthogonal* if $p_n(x)$ has degree n, $n = 0, 1, \dots$, and if there exists a linear functional L such that $L(p_n(x)p_m(x)) = \delta_{mn}c_n$ for some sequence $(c_n)_{n\ge 0}$ of nonzero numbers, with $\delta_{m,n}$ denoting the Kronecker delta (i.e., $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise).

The first important theorem in the theory of orthogonal polynomials is Favard's Theorem, which gives an unexpected characterization for sequences of orthogonal polynomials, in that it completely avoids the mention of the functional L.

Theorem 12. (Cf. [186, Théorème 9 on p. I-4] or [188, Theorem 50.1]). *Let $(p_n(x))_{n\ge 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree n, $n = 0, 1, \dots$. Then the sequence $(p_n(x))$ is (formally) orthogonal if and only if there exist sequences $(a_n)_{n\ge 1}$ and $(b_n)_{n\ge 1}$, with $b_n \ne 0$ for all $n \ge 1$, such that the three-term recurrence*

$$p_{n+1}(x) = (a_n + x)p_n(x) - b_n p_{n-1}(x), \qquad \text{for } n \ge 1, \tag{2.32}$$

holds, with initial conditions $p_0(x) = 1$ and $p_1(x) = x + a_0$. □

What is the connection between orthogonal polynomials and continued fractions? This question is answered by the next theorem, the link being the *generating function of the moments*.

Theorem 13. (Cf. [188, Theorem 51.1] or [186, Proposition 1, (7), on p. V-5]). *Let $(p_n(x))_{n\ge 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree n, which is orthogonal with respect to some functional L. Let*

$$p_{n+1}(x) = (a_n + x)p_n(x) - b_n p_{n-1}(x) \tag{2.33}$$

be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the generating function $\sum_{k=0}^{\infty} \mu_k x^k$ for the moments $\mu_k = L(x^k)$ satisfies (2.30) *with the a_i's and b_i's being the coefficients in the three-term recurrence* (2.33). □

Thus, what we have to do is to find orthogonal polynomials $(p_n(x))_{n\ge 0}$, the three-term recurrence of which is explicitly known, and which are orthogonal with respect to some linear functional L whose moments $L(x^k)$ are exactly equal to B_{k+2}. So, what would be very helpful at this point is some sort of table of orthogonal polynomials. Indeed, there is such a *table for hypergeometric and basic hypergeometric orthogonal polynomials*, proposed by Richard Askey (therefore called the *"Askey table"*), and compiled by Koekoek and Swarttouw [81].

Indeed, in Section 1.4 of [81], we find the family of orthogonal polynomials that is of relevance here, the *continuous Hahn polynomials*, first studied by Atakishiyev and Suslov [13] and Askey [12]. These polynomials depend on four parameters, a, b, c, d. It is just the special choice $a=b=c=d=1$ which is of interest to us. The theorem below lists the relevant facts about these special polynomials.

Theorem 14. *The continuous Hahn polynomials with parameters $a=b=c=d=1$, $(p_n(x))_{n\ge0}$, are the monic polynomials defined by*

$$p_n(x)=(\sqrt{-1})^n\frac{(n+1)!^2\,(n+2)!}{(2n+2)!}\sum_{k=0}^{\infty}\frac{(-n)_k\,(n+3)_k\,(1+x\sqrt{-1})_k}{k!\,(k+1)!^2},\tag{2.34}$$

with the shifted factorial $(a)_k$ defined as previously (see (2.19)). These polynomials satisfy the three-term recurrence

$$p_{n+1}(x)=xp_n(x)+\frac{n(n+1)^2(n+2)}{4(2n+1)(2n+3)}p_{n-1}(x).\tag{2.35}$$

They are orthogonal with respect to the functional L which is given by

$$L(p(x))=\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{x^2}{\sinh^2(\pi x)}p(x)\,dx\,.\tag{2.36}$$

Explicitly, the orthogonality relation is

$$L(p_m(x)p_n(x))=\frac{n!\,(n+1)!^4\,(n+2)!}{(2n+2)!\,(2n+3)!}\delta_{m,n}.\tag{2.37}$$

In particular, $L(1)=1/6$. □

Now, by combining Theorems 11, 13, and 14, and by using an integral representation of Bernoulli numbers (see [122, p. 75]),

$$B_\nu=\frac{1}{2\pi\sqrt{-1}}\int_{-\infty\sqrt{-1}}^{\infty\sqrt{-1}}z^\nu\left(\frac{\pi}{\sin\pi z}\right)^2dz$$

(if $\nu=0$ or $\nu=1$ then the path of integration is indented so that it avoids the singularity $z=0$, passing it on the negative side) we obtain without difficulty the desired determinant evaluation,

$$\begin{aligned}\det_{0\le i,j,\le n-1}(B_{i+j+2})&=(-1)^{\binom n2}\left(\frac16\right)^n\prod_{i=1}^{n-1}\left(\frac{i(i+1)^2(i+2)}{4(2i+1)(2i+3)}\right)^{n-i}\\&=(-1)^{\binom n2}\frac16\prod_{i=1}^{n-1}\frac{i!\,(i+1)!^4\,(i+2)!}{(2i+2)!\,(2i+3)!}.\end{aligned}\tag{2.38}$$

The general determinant evaluation which results from using continuous Hahn polynomials with generic nonnegative integers a,b,c,d is worked out in [51, Sec. 5].

Let me mention that, given a Hankel determinant evaluation such as (2.38), one has automatically proved a more general one, by means of the following simple fact (see for example [121, p. 419]):

Lemma 15. *Let x be an indeterminate. For any nonnegative integer n there holds*

$$\det_{0\le i,j\le n-1}(A_{i+j})=\det_{0\le i,j\le n-1}\left(\sum_{k=0}^{i+j}\binom{i+j}{k}A_kx^{i+j-k}\right).\tag{2.39}$$

□

The idea of using continued fractions and/or orthogonal polynomials for the evaluation of Hankel determinants has been also exploited in [1, 35, 113, 114, 115, 116]. Some of these results are exhibited in Theorem 52. See the remarks after Theorem 52 for pointers to further Hankel determinant evaluations.

2.8. MISCELLANEOUS. This section is a collection of various further results on determinant evaluation of the general sort, which I personally like, regardless whether they may be more or less useful.

Let me begin with a result by Strehl and Wilf [173, Sec. II], a special case of which was already in the seventies advertised by van der Poorten [131, Sec. 4] as 'a determinant evaluation that should be better known'. (For a generalization see [78].)

Lemma 16. *Let $f(x)$ be a formal power series. Then for any positive integer n there holds*

$$\det_{1\le i,j\le n}\left(\left(\frac{d}{dx}\right)^{i-1} f(x)^{a_j}\right)=\left(\frac{f'(x)}{f(x)}\right)^{\binom n2} f(x)^{a_1+\dots+a_n}\prod_{1\le i<j\le n}(a_j-a_i). \tag{2.40}$$

□

By specializing, this result allows for the quick proof of various, sometimes surprising, determinant evaluations, see Theorems 53 and 54.

An extremely beautiful determinant evaluation is the evaluation of the determinant of the *circulant matrix*.

Theorem 17. *Let n by a fixed positive integer, and let $a_0, a_1, \dots, a_{n-1}$ be indeterminates. Then*

$$\det\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1}\\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2}\\ a_{n-2} & a_{n-1} & a_0 & \dots & a_{n-3}\\ \dots & \dots & \dots & \dots & \dots\\ a_1 & a_2 & a_3 & \dots & a_0\end{pmatrix}=\prod_{i=0}^{n-1}(a_0+\omega^i a_1+\omega^{2i}a_2+\dots+\omega^{(n-1)i}a_{n-1}), \tag{2.41}$$

where ω is a primitive n-th root of unity. □

Actually, the *circulant determinant* is just a very special case in a whole family of determinants, called *group determinants*. This would bring us into the vast territory of group representation theory, and is therefore beyond the scope of this article. It must suffice to mention that the group determinants were in fact the *cause of birth of group representation theory* (see [99] for a beautiful introduction into these matters).

The next theorem does actually not give the evaluation of a determinant, but of a *Pfaffian*. The Pfaffian $\operatorname{Pf}(A)$ of a skew-symmetric $(2n)\times(2n)$ matrix A is defined by

$$\operatorname{Pf}(A)=\sum_\pi(-1)^{c(\pi)}\prod_{(ij)\in\pi}A_{ij},$$

where the sum is over all perfect matchings π of the complete graph on $2n$ vertices, where $c(\pi)$ is the *crossing number* of π, and where the product is over all edges (ij), $i<j$, in the matching π (see e.g. [169, Sec. 2]). What links Pfaffians so closely to

determinants is (aside from similarity of definitions) the fact that the Pfaffian of a skew-symmetric matrix is, up to sign, the square root of its determinant. That is, $\det(A) = \operatorname{Pf}(A)^2$ for any skew-symmetric $(2n) \times (2n)$ matrix A (cf. [169, Prop. 2.2]).[9] Pfaffians play an important role, for example, in the enumeration of plane partitions, due to the results by Laksov, Thorup and Lascoux [98, Appendix, Lemma (A.11)] and Okada [123, Theorems 3 and 4] on sums of minors of a given matrix (a combinatorial view as enumerating nonintersecting lattice paths with varying starting and/or ending points has been given by Stembridge [169, Theorems 3.1, 3.2, and 4.1]), and their generalization in form of the powerful *minor summation formulas* due to Ishikawa and Wakayama [69, Theorems 2 and 3].

Exactly in this context, the context of enumeration of plane partitions, Gordon [58, implicitly in Sec. 4, 5] (see also [169, proof of Theorem 7.1]) proved two extremely useful reductions of Pfaffians to determinants.

Lemma 18. *Let (g_i) be a sequence with the property $g_{-i} = g_i$, and let N be a positive integer. Then*

$$\operatorname{Pf}_{1\le i<j\le 2N}\left(\sum_{-(j-i)<\alpha\le j-i} g_\alpha\right) = \det_{1\le i,j\le N}\left(g_{i-j} + g_{i+j-1}\right), \tag{2.42}$$

and

$$\operatorname{Pf}_{1\le i<j\le 2N+2}\left(\begin{Bmatrix} \sum_{-(j-i)<\alpha\le j-i} g_\alpha & j\le 2N+1 \\ X & j=2N+2 \end{Bmatrix}\right) = X\cdot \det_{1\le i,j\le N}\left(g_{i-j} - g_{i+j}\right). \tag{2.43}$$

(*In these statements only one half of the entries of the Pfaffian is given, the other half being uniquely determined by skew-symmetry*). □

This result looks somehow technical, but its usefulness was sufficiently proved by its applications in the enumeration of plane partitions and tableaux in [58] and [169, Sec. 7].

Another technical, but useful result is due to Goulden and Jackson [61, Theorem 2.1].

Lemma 19. *Let $F_m(t)$, $G_m(t)$ and $H_m(t)$ by formal power series, with $H_m(0) = 0$, $m = 0, 1, \dots, n-1$. Then for any positive integer n there holds*

$$\det_{0\le i,j,\le n-1}\left(\operatorname{CT}\left(\frac{F_j(t)}{H_j(t)^i} G_i(H_j(t))\right)\right) = \det_{0\le i,j\le n-1}\left(\operatorname{CT}\left(\frac{F_j(t)}{H_j(t)^i} G_i(0)\right)\right), \tag{2.44}$$

where $\operatorname{CT}(f(t))$ *stands for the constant term of the Laurent series* $f(t)$. □

What is the value of this theorem? In some cases, out of a given determinant evaluation, it immediately implies a more general one, containing (at least) one more parameter. For example, consider the determinant evaluation (3.30). Choose $F_j(t) = t^j(1+t)^{\mu+j}$, $H_j(t) = t^2/(1+t)$, and $G_i(t)$ such that $G_i(t^2/(1+t)) = (1+t)^k + (1+t)^{-k}$ for a fixed k (such a choice does indeed exist; see [61, proof of Cor. 2.2]) in Lemma 19. This yields

$$\det_{0\le i,j\le n-1}\left(\binom{\mu+k+i+j}{2i-j} + \binom{\mu-k+i+j}{2i-j}\right) = \det_{0\le i,j\le n-1}\left(2\binom{\mu+i+j}{2i-j}\right).$$

[9] Another point of view, beautifully set forth in [79], is that "Pfaffians are more fundamental than determinants, in the sense that determinants are merely the bipartite special case of a general sum over matchings."

Thus, out of the validity of (3.30), this enables to establish the validity of (3.32), and even of (3.33), by choosing $F_j(t)$ and $H_j(t)$ as above, but $G_i(t)$ such that $G_i(t^2/(1+t)) = (1+t)^{x_i} + (1+t)^{-x_i}$, $i = 0, 1, \dots, n-1$.

3. A list of determinant evaluations

In this section I provide a list of determinant evaluations, some of which are very frequently met, others maybe not so often. In any case, I believe that all of them are useful or attractive, or even both. However, this is not intended to be, and cannot possibly be, an exhaustive list of known determinant evaluations. The selection depends totally on my taste. This may explain that many of these determinants arose in the enumeration of plane partitions and rhombus tilings. On the other hand, it is exactly this field (see [138, 148, 163, 165] for more information on these topics) which is a particular rich source of nontrivial determinant evaluations. If you do not find "your" determinant here, then, at least, the many references given in this section or the general results and methods from Section 2 may turn out to be helpful.

Throughout this section we use the standard hypergeometric and basic hypergeometric notations. To wit, for nonnegative integers k the *shifted factorial* $(a)_k$ is defined (as already before) by

$$(a)_k := a(a+1)\cdots(a+k-1),$$

so that in particular $(a)_0 := 1$. Similarly, for nonnegative integers k the *shifted q-factorial* $(a;q)_k$ is given by

$$(a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1}),$$

so that $(a;q)_0 := 1$. Sometimes we make use of the notations $[\alpha]_q := (1-q^\alpha)/(1-q)$, $[n]_q! := [n]_q[n-1]_q\cdots[1]_q$, $[0]_q! := 1$. The *q-binomial coefficient* is defined by

$$\begin{bmatrix}\alpha\\ k\end{bmatrix}_q := \frac{[\alpha]_q[\alpha-1]_q\cdots[\alpha-k+1]_q}{[k]_q!} = \frac{(1-q^\alpha)(1-q^{\alpha-1})\cdots(1-q^{\alpha-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}.$$

Clearly we have $\lim_{q\to1}\left[\begin{smallmatrix}\alpha\\ k\end{smallmatrix}\right]_q = \binom{\alpha}{k}$.

Occasionally shifted (q-)factorials will appear which contain a subscript which is a negative integer. By convention, a shifted factorial $(a)_k$, where k is a negative integer, is interpreted as $(a)_k := 1/(a-1)(a-2)\cdots(a+k)$, whereas a shifted q-factorial $(a;q)_k$, where k is a negative integer, is interpreted as $(a;q)_k := 1/(1-q^{a-1})(1-q^{a-2})\cdots(1-q^{a+k})$. (A uniform way to define the shifted factorial, for positive *and* negative k, is by $(a)_k := \Gamma(a+k)/\Gamma(a)$, respectively by an appropriate limit in case that a or $a+k$ is a nonpositive integer, see [62, Sec. 5.5, p. 211f]. A uniform way to define the shifted q-factorial is by means of $(a;q)_k := (a;q)_\infty/(aq^k;q)_\infty$, see [55, (1.2.30)].)

We begin our list with two determinant evaluations which generalize the Vandermonde determinant evaluation (2.1) in a nonstandard way. The determinants appearing in these evaluations can be considered as "augmentations" of the Vandermonde determinant by columns which are formed by differentiating "Vandermonde-type" columns. (Thus, these determinants can also be considered as certain *generalized Wronskians*.) Occurences of the first determinant can be found e.g. in [45], [107, App. A.16], [108, (7.1.3)], [154], [187]. (It is called "confluent alternant" in [107, 108].) The motivation in [45] to study these determinants came from *Hermite interpolation* and the *analysis of linear recursion relations*. In [107, App. A.16], special cases of these determinants are used in the context of *random matrices.* Special cases arose also in the context of *transcendental number theory* (see [131, Sec. 4]).

Theorem 20. *Let n be a nonnegative integer, and let $A_m(X)$ denote the $n\times m$ matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ X & 1 & 0 & 0 & \dots & 0 \\ X^2 & 2X & 2 & 0 & \dots & 0 \\ X^3 & 3X^2 & 6X & 6 & \dots & 0 \\ \hdotsfor{6} \\ X^{n-1} & (n-1)X^{n-2} & (n-1)(n-2)X^{n-3} & \dots & \dots & (n-1)\cdots(n-m+1)X^{n-m} \end{pmatrix},$$

i.e., any next column is formed by differentiating the previous column with respect to X. Given a composition of n, $n=m_1+\dots+m_\ell$, there holds

$$\det_{1\le i,j,\le n}\big(A_{m_1}(X_1)\,A_{m_2}(X_2)\dots A_{m_\ell}(X_\ell)\big)=\left(\prod_{i=1}^{\ell}\prod_{j=1}^{m_i-1}j!\right)\prod_{1\le i<j\le\ell}(X_j-X_i)^{m_im_j}. \tag{3.1}$$

□

The paper [45] has as well an "Abel-type" variation of this result.

Theorem 21. *Let n be a nonnegative integer, and let $B_m(X)$ denote the $n\times m$ matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ X & X & X & X & \dots & X \\ X^2 & 2X^2 & 4X^2 & 8X^2 & \dots & 2^{m-1}X^2 \\ X^3 & 3X^3 & 9X^3 & 27X^3 & \dots & 3^{m-1}X^3 \\ \hdotsfor{6} \\ X^{n-1} & (n-1)X^{n-1} & (n-1)^2X^{n-1} & \dots & \dots & (n-1)^{m-1}X^{n-1} \end{pmatrix},$$

i.e., any next column is formed by applying the operator $X(d/dX)$. Given a composition of n, $n=m_1+\dots+m_\ell$, there holds

$$\det_{1\le i,j,\le n}\big(B_{m_1}(X_1)\,B_{m_2}(X_2)\dots B_{m_\ell}(X_\ell)\big)=\left(\prod_{i=1}^{\ell}X_i^{\binom{m_i}{2}}\prod_{j=1}^{m_i-1}j!\right)\prod_{1\le i<j\le\ell}(X_j-X_i)^{m_im_j}. \tag{3.2}$$

□

As Alain Lascoux taught me, the natural environment for this type of determinants is *divided differences* and (generalized) *discrete Wronskians.* The divided difference $\partial_{x,y}$ is a linear operator which maps polynomials in x and y to polynomials symmetric in x and y, and is defined by

$$\partial_{x,y}f(x,y)=\frac{f(x,y)-f(y,x)}{x-y}.$$

Divided differences have been introduced by Newton to solve the interpolation problem in one variable. (See [100] for an excellent introduction to interpolation, divided differences, and related matters, such as *Schur functions* and *Schubert polynomials.*) In fact, given a polynomial $g(x)$ in x, whose coefficients do not depend on $a_1, a_2, \dots, a_m$, *Newton's interpolation formula* reads as follows (cf. e.g. [100, (Ni2)]),

$$g(x) = g(a_1) + (x - a_1)\partial_{a_1,a_2} g(a_1) + (x - a_1)(x - a_2)\partial_{a_2,a_3}\partial_{a_1,a_2} g(a_1) \\ + (x - a_1)(x - a_2)(x - a_3)\partial_{a_3,a_4}\partial_{a_2,a_3}\partial_{a_1,a_2} g(a_1) + \cdots . \quad (3.3)$$

Now suppose that $f_1(x), f_2(x), \dots, f_n(x)$ are polynomials in one variable x, whose coefficients do not depend on $a_1, a_2, \dots, a_n$, and consider the determinant

$$\det_{1\le i,j,\le n} (f_i(a_j)). \quad (3.4)$$

Let us for the moment concentrate on the first m_1 columns of this determinant. We may apply (3.3), and write

$$f_i(a_j) = f_i(a_1) + (a_j - a_1)\partial_{a_1,a_2} f_i(a_1) + (a_j - a_1)(a_j - a_2)\partial_{a_2,a_3}\partial_{a_1,a_2} f_i(a_1) \\ + \cdots + (a_j - a_1)(a_j - a_2)\cdots(a_j - a_{j-1})\partial_{a_{j-1},a_j}\cdots\partial_{a_2,a_3}\partial_{a_1,a_2} f_i(a_1),$$

$j = 1, 2, \dots, m_1$. Following [100, Proof of Lemma (Ni5)], we may perform column reductions to the effect that the determinant (3.4), with column j replaced by

$$(a_j - a_1)(a_j - a_2)\cdots(a_j - a_{j-1})\partial_{a_{j-1},a_j}\cdots\partial_{a_2,a_3}\partial_{a_1,a_2} f_i(a_1),$$

$j = 1, 2, \dots, m_1$, has the same value as the original determinant. Clearly, the product $\prod_{k=1}^{j-1}(a_j - a_k)$ can be taken out of column j, $j = 1, 2, \dots, m_1$. Similar reductions can be applied to the next m_2 columns, then to the next m_3 columns, etc.

This proves the following fact about generalized discrete Wronskians:

Lemma 22. *Let n be a nonnegative integer, and let $W_m(x_1, x_2, \dots, x_m)$ denote the $n \times m$ matrix $\big(\partial_{x_{j-1},x_j}\cdots\partial_{x_2,x_3}\partial_{x_1,x_2} f_i(x_1)\big)_{1\le i\le n,\ 1\le j\le m}$. Given a composition of n, $n = m_1 + \cdots + m_\ell$, there holds*

$$\det_{1\le i,j,\le n}\big(W_{m_1}(a_1, \dots, a_{m_1})\, W_{m_2}(a_{m_1+1}, \dots, a_{m_1+m_2}) \dots W_{m_\ell}(a_{m_1+\cdots+m_{\ell-1}+1}, \dots, a_n)\big) \\ = \det_{1\le i,j,\le n}(f_i(a_j)) \Big/ \prod_{k=1}^{\ell}\Bigg(\prod_{m_1+\cdots+m_{k-1}+1\le i<j\le m_1+\cdots+m_k}(a_j - a_i)\Bigg). \quad (3.5)$$

□

If we now choose $f_i(x) := x^{i-1}$, so that $\det_{1\le i,j,\le n}(f_i(a_j))$ is a Vandermonde determinant, then the right-hand side of (3.5) factors completely by (2.1). The final step to obtain Theorem 20 is to let $a_1 \to X_1$, $a_2 \to X_1$, ... , $a_{m_1} \to X_1$, $a_{m_1+1} \to X_2$, ... , $a_{m_1+m_2} \to X_2$, etc., in (3.5). This does indeed yield (3.1), because

$$\lim_{x_j\to x} \dots \lim_{x_2\to x}\lim_{x_1\to x} \partial_{x_{j-1},x_j}\cdots\partial_{x_2,x_3}\partial_{x_1,x_2} g(x_1) = \frac{1}{(j-1)!}\left(\frac{d}{dx}\right)^{j-1} g(x),$$

as is easily verified.

The Abel-type variation in Theorem 21 follows from Theorem 20 by multiplying column j in (3.1) by X_1^{j-1} for $j = 1, 2, \dots, m_1$, by $X_2^{j-m_1-1}$ for $j = m_1+1, m_1+2, \dots, m_2$, etc., and by then using the relation

$$X\frac{d}{dX} g(X) = \frac{d}{dX} X g(X) - g(X)$$

many times, so that a typical entry $X_k^{j-1}(d/dX_k)^{j-1}X_k^{i-1}$ in row i and column j of the k-th submatrix is expressed as $(X_k(d/dX_k))^{j-1}X_k^{i-1}$ plus a linear combination of terms $(X_k(d/dX_k))^sX_k^{i-1}$ with $s<j-1$. Simple column reductions then yield (3.2).

It is now not very difficult to adapt this analysis to derive, for example, q-analogues of Theorems 20 and 21. The results below do actually contain q-analogues of extensions of Theorems 20 and 21.

Theorem 23. *Let n be a nonnegative integer, and let $A_m(X)$ denote the $n\times m$ matrix*

$$\begin{pmatrix} 1 & [C]_q\,X^{-1} & [C]_q\,[C-1]_q\,X^{-2} & \dots & [C]_q\cdots[C-m+2]_q\,X^{1-m}\\ X & [C+1]_q & [C+1]_q\,[C]_q\,X^{-1} & \dots & [C+1]_q\cdots[C-m+3]_q\,X^{2-m}\\ X^2 & [C+2]_q\,X & [C+2]_q\,[C+1]_q & \dots & [C+2]_q\cdots[C-m+4]_q\,X^{3-m}\\ \hdotsfor{5}\\ X^{n-1} & [C+n-1]_q\,X^{n-2} & [C+n-1]_q\,[C+n-2]_q\,X^{n-3} & \dots & [C+n-1]_q\cdots[C+n-m+1]_q\,X^{n-m} \end{pmatrix},$$

i.e., any next column is formed by applying the operator $X^{-C}D_q\,X^C$, with D_q denoting the usual q-derivative, $D_qf(X):=(f(qX)-f(X))/(q-1)X$. Given a composition of n, $n=m_1+\dots+m_\ell$, there holds

$$\det_{1\le i,j,\le n}\big(A_{m_1}(X_1)\,A_{m_2}(X_2)\dots A_{m_\ell}(X_\ell)\big)\\ =q^{N_1}\left(\prod_{i=1}^{\ell}\prod_{j=1}^{m_i-1}[j]_q!\right)\prod_{1\le i<j\le\ell}\prod_{s=0}^{m_i-1}\prod_{t=0}^{m_j-1}(q^{t-s}X_j-X_i), \tag{3.6}$$

where N_1 is the quantity

$$\sum_{i=1}^{\ell}\sum_{j=1}^{m_i}\left((C+j+m_1+\dots+m_{i-1}-1)(m_i-j)-\binom{m_i}{3}\right)-\sum_{1\le i<j\le\ell}\left(m_i\binom{m_j}{2}-m_j\binom{m_i}{2}\right).$$

□

To derive (3.6) one would choose strings of geometric sequences for the variables a_j in Lemma 22, i.e., $a_1=X_1$, $a_2=qX_1$, $a_3=q^2X_1$, ... , $a_{m_1+1}=X_2$, $a_{m_1+2}=qX_2$, etc., and, in addition, use the relation

$$y^C\partial_{x,y}f(x,y)=\partial_{x,y}(x^Cf(x,y))-(\partial_{x,y}x^C)f(x,y) \tag{3.7}$$

repeatedly.

A "q-Abel-type" variation of this result reads as follows.

Theorem 24. *Let n be a nonnegative integer, and let $B_m(X)$ denote the $n\times m$ matrix*

$$\begin{pmatrix} 1 & [C]_q & [C]_q^2 & \dots & [C]_q^{m-1} \\ X & [C+1]_q\,X & [C+1]_q^2\,X & \dots & [C+1]_q^{m-1}\,X \\ X^2 & [C+2]_q\,X^2 & [C+2]_q^2\,X^2 & \dots & [C+2]_q^{m-1}\,X^2 \\ \hdotsfor{5} \\ X^{n-1} & [C+n-1]_q\,X^{n-1} & [C+n-1]_q^2\,X^{n-1} & \dots & [C+n-1]_q^{m-1}\,X^{n-1} \end{pmatrix},$$

i.e., any next column is formed by applying the operator $X^{1-C}D_qX^C$, with D_q denoting the q-derivative as in Theorem 23. Given a composition of n, $n=m_1+\dots+m_\ell$, there holds

$$\det_{1\le i,j,\le n}\big(B_{m_1}(X_1)\,B_{m_2}(X_2)\dots B_{m_\ell}(X_\ell)\big)\\ = q^{N_2}\left(\prod_{i=1}^{\ell}X_i^{\binom{m_i}{2}}\prod_{j=1}^{m_i-1}[j]_q!\right)\prod_{1\le i<j\le\ell}\prod_{s=0}^{m_i-1}\prod_{t=0}^{m_j-1}(q^{t-s}X_j-X_i),\qquad(3.8)$$

where N_2 is the quantity

$$\sum_{i=1}^{\ell}\sum_{j=1}^{m_i}((C+j+m_1+\dots+m_{i-1}-1)(m_i-j))-\sum_{1\le i<j\le\ell}\left(m_i\binom{m_j}{2}-m_j\binom{m_i}{2}\right).$$

□

Yet another generalization of the Vandermonde determinant evaluation is found in [21]. Multidimensional analogues are contained in [176, Theorem A.7, Eq. (A.14), Theorem B.8, Eq. (B.11)] and [182, Part I, p. 547].

Extensions of Cauchy's double alternant (2.7) can also be found in the literature (see e.g. [117, 149]). I want to mention here particularly Borchardt's variation [17] in which the (i,j)-entry in Cauchy's double alternant is replaced by its square,

$$\det_{1\le i,j\le n}\left(\frac{1}{(X_i-Y_j)^2}\right)=\frac{\prod_{1\le i<j\le n}(X_i-X_j)(Y_i-Y_j)}{\prod_{1\le i,j\le n}(X_i-Y_j)}\operatorname*{Per}_{1\le i,j\le n}\left(\frac{1}{X_i-Y_j}\right),\qquad(3.9)$$

where $\operatorname{Per}M$ denotes the *permanent* of the matrix M. Thus, there is no closed form expression such as in (2.7). This may not look that useful. However, most remarkably, there is a (q-)deformation of this identity which did indeed lead to a "closed form evaluation," thus solving a famous enumeration problem in an unexpected way, the problem of enumerating *alternating sign matrices.*[10] This q-deformation is equivalent to Izergin's evaluation [74, Eq. (5)] (building on results by Korepin [82]) of the partition function of the *six-vertex model* under certain boundary conditions (see also [97, Theorem 8] and [83, Ch. VII, (10.1)/(10.2)]).

[10] An alternating sign matrix is a square matrix with entries $0,1,-1$, with all row and column sums equal to 1, and such that, on disregarding the 0s, in each row and column the 1s and (-1)s alternate. Alternating sign matrix are currently the most fascinating, and most mysterious, objects in enumerative combinatorics. The reader is referred to [18, 19, 111, 148, 97, 198, 199] for more detailed material. Incidentally, the "birth" of alternating sign matrices came through — determinants, see [150].

Theorem 25. *For any nonnegative integer n there holds*

$$\det_{1\le i,j\le n}\left(\frac{1}{(X_i-Y_j)(qX_i-Y_j)}\right)=\frac{\prod_{1\le i<j\le n}(X_i-X_j)(Y_i-Y_j)}{\prod_{1\le i,j\le n}(X_i-Y_j)(qX_i-Y_j)}$$
$$\times\sum_A(1-q)^{2N(A)}\prod_{i=1}^n X_i^{N_i(A)}Y_i^{N^i(A)}\prod_{i,j\text{ such that }A_{ij}=0}(\alpha_{i,j}X_i-Y_j),\quad(3.10)$$

where the sum is over all $n\times n$ alternating sign matrices $A=(A_{ij})_{1\le i,j\le n}$, $N(A)$ is the number of (-1)s in A, $N_i(A)$ (respectively $N^i(A)$) is the number of (-1)s in the i-th row (respectively column) of A, and $\alpha_{ij}=q$ if $\sum_{k=1}^j A_{ik}=\sum_{k=1}^i A_{kj}$, and $\alpha_{ij}=1$ otherwise.

Clearly, equation (3.9) results immediately from (3.10) by setting $q=1$. Roughly, Kuperberg's solution [97] of the enumeration of alternating sign matrices consisted of suitably specializing the x_i's, y_i's and q in (3.10), so that each summand on the right-hand side would reduce to the same quantity, and, thus, the sum would basically count $n\times n$ alternating sign matrices, and in evaluating the left-hand side determinant for that special choice of the x_i's, y_i's and q. The resulting number of $n\times n$ alternating sign matrices is given in (A.1) in the Appendix. (The first, very different, solution is due to Zeilberger [198].) Subsequently, Zeilberger [199] improved on Kuperberg's approach and succeeded in proving the refined alternating sign matrix conjecture from [111, Conj. 2]. For a different expansion of the determinant of Izergin, in terms of *Schur functions*, and a variation, see [101, Theorem q, Theorem γ].

Next we turn to typical applications of Lemma 3. They are listed in the following theorem.

Theorem 26. *Let n be a nonnegative integer, and let $L_1,L_2,\dots,L_n$ and A,B be indeterminates. Then there hold*

$$\det_{1\le i,j\le n}\left(\begin{bmatrix}L_i+A+j\\L_i+j\end{bmatrix}_q\right)=q^{\sum_{i=1}^n(i-1)(L_i+i)}\frac{\prod_{1\le i<j\le n}[L_i-L_j]_q}{\prod_{i=1}^n[L_i+n]_q!}\frac{\prod_{i=1}^n[L_i+A+1]_q!}{\prod_{i=1}^n[A+1-i]_q!},\quad(3.11)$$

and

$$\det_{1\le i,j\le n}\left(q^{jL_i}\begin{bmatrix}A\\L_i+j\end{bmatrix}_q\right)=q^{\sum_{i=1}^n iL_i}\frac{\prod_{1\le i<j\le n}[L_i-L_j]_q}{\prod_{i=1}^n[L_i+n]_q!}\frac{\prod_{i=1}^n[A+i-1]_q!}{\prod_{i=1}^n[A-L_i-1]_q!},\quad(3.12)$$

and

$$\det_{1\le i,j\le n}\left(\binom{BL_i+A}{L_i+j}\right)$$
$$=\frac{\prod_{1\le i<j\le n}(L_i-L_j)}{\prod_{i=1}^n(L_i+n)!}\prod_{i=1}^n\frac{(BL_i+A)!}{((B-1)L_i+A-1)!}\prod_{i=1}^n(A-Bi+1)_{i-1},\quad(3.13)$$

and

$$\det_{1\le i,j\le n}\left(\frac{(A+BL_i)^{j-1}}{(j-L_i)!}\right)=\prod_{i=1}^n\frac{(A+Bi)^{i-1}}{(n-L_i)!}\prod_{1\le i<j\le n}(L_j-L_i).\quad(3.14)$$

□

(For derivations of (3.11) and (3.12) using Lemma 3 see the proofs of Theorems 6.5 and 6.6 in [85]. For a derivation of (3.13) using Lemma 3 see the proof of Theorem 5 in [86].)

Actually, the evaluations (3.11) and (3.12) are equivalent. This is seen by observing that

$$\begin{bmatrix} L_i+A+j \\ L_i+j \end{bmatrix}_q = (-1)^{L_i+j} q^{\binom{L_i}{2}+\binom{j}{2}+jL_i+(A+1)(L_i+j)} \begin{bmatrix} -A-1 \\ L_i+j \end{bmatrix}_q .$$

Hence, replacement of A by $-A-1$ in (3.11) leads to (3.12) after little manipulation.

The determinant evaluations (3.11) and (3.12), and special cases thereof, are rediscovered and reproved in the literature over and over. (This phenomenon will probably persist.) To the best of my knowledge, the evaluation (3.11) appeared in print explicitly for the first time in [22], although it was (implicitly) known earlier to people in group representation theory, as it also results from the principal specialization (i.e., set $x_i = q^i$, $i = 1, 2, \dots, N$) of a Schur function of arbitrary shape, by comparing the Jacobi–Trudi identity with the bideterminantal form (Weyl character formula) of the Schur function (cf. [105, Ch. I, (3.4), Ex. 3 in Sec. 2, Ex. 1 in Sec. 3]; the determinants arising in the bideterminantal form are Vandermonde determinants and therefore easily evaluated).

The main applications of (3.11)–(3.13) are in the enumeration of *tableaux, plane partitions and rhombus tilings.* For example, the *hook-content formula* [163, Theorem 15.3] for tableaux of a given shape with bounded entries follows immediately from the theory of nonintersecting lattice paths (cf. [57, Cor. 2] and [169, Theorem 1.2]) and the determinant evaluation (3.11) (see [57, Theorem 14] and [85, proof of Theorem 6.5]). MacMahon's *"box formula"* [106, Sec. 429; proof in Sec. 494] for the generating function of plane partitions which are contained inside a given box follows from nonintersecting lattice paths and the determinant evaluation (3.12) (see [57, Theorem 15] and [85, proof of Theorem 6.6]). The $q = 1$ special case of the determinant which is relevant here is the one in (1.2) (which is the one which was evaluated as an illustration in Section 2.2). To the best of my knowledge, the evaluation (3.13) is due to Proctor [133] who used it for enumerating *plane partitions of staircase shape* (see also [86]). The determinant evaluation (3.14) can be used to give closed form expressions in the enumeration of λ-*parking functions* (an extension of the notion of k-parking functions such as in [167]), if one starts with determinantal expressions due to Gessel (private communication). Further applications of (3.11), in the domain of *multiple (basic) hypergeometric series*, are found in [63]. Applications of these determinant evaluations in statistics are contained in [66] and [168].

It was pointed out in [34] that plane partitions in a given box are in bijection with *rhombus tilings of a "semiregular" hexagon.* Therefore, the determinant (1.2) counts as well rhombus tilings in a hexagon with side lengths a, b, n, a, b, n. In this regard, generalizations of the evaluation of this determinant, and of a special case of (3.13), appear in [25] and [27]. The theme of these papers is to enumerate *rhombus tilings of a hexagon with triangular holes.*

The next theorem provides a typical application of Lemma 4. For a derivation of this determinant evaluation using this lemma see [87, proofs of Theorems 8 and 9].

Theorem 27. *Let n be a nonnegative integer, and let $L_1, L_2, \dots, L_n$ and A be indeterminates. Then there holds*

$$\det_{1\le i,j\le n}\left(q^{jL_i}\begin{bmatrix}L_i+A-j\\ L_i+j\end{bmatrix}_q\right)\\
=q^{\sum_{i=1}^n iL_i}\prod_{i=1}^n\frac{[L_i+A-n]_q!}{[L_i+n]_q!\,[A-2i]_q!}\prod_{1\le i<j\le n}\big([L_i-L_j]_q\,[L_i+L_j+A+1]_q\big). \tag{3.15}$$

□

This result was used to compute generating functions for *shifted plane partitions of trapezoidal shape* (see [87, Theorems 8 and 9], [134, Prop. 4.1] and [135, Theorem 1]).

Now we turn to typical applications of Lemma 5, given in Theorems 28–31 below. All of them can be derived in just the same way as we evaluated the determinant (1.2) in Section 2.2 (the only difference being that Lemma 5 is invoked instead of Lemma 3).

The first application is the evaluation of a determinant whose entries are a product of two q-binomial coefficients.

Theorem 28. *Let n be a nonnegative integer, and let $L_1, L_2, \dots, L_n$ and A, B be indeterminates. Then there holds*

$$\det_{1\le i,j\le n}\left(\begin{bmatrix}L_i+j\\ B\end{bmatrix}_q\cdot\begin{bmatrix}L_i+A-j\\ B\end{bmatrix}_q\right)\\
=q^{\sum_{i=1}^n(i-1)L_i-B\binom n2+2\binom{n+1}3}\prod_{1\le i<j\le n}\big([L_i-L_j]_q\,[L_i+L_j+A-B+1]_q\big)\\
\times\prod_{i=1}^n\frac{[L_i+1]_q!\,[L_i+A-n]_q!}{[L_i-B+n]_q!\,[L_i+A-B-1]_q!}\frac{[A-2i-1]_q!}{[A-i-n-1]_q!\,[B+i-n]_q!\,[B]_q!}. \tag{3.16}$$

□

As is not difficult to verify, this determinant evaluation contains (3.11), (3.12), as well as (3.15) as special, respectively limiting cases.

This determinant evaluation found applications in basic hypergeometric functions theory. In [191, Sec. 3], Wilson used a special case to construct *biorthogonal rational functions*. On the other hand, Schlosser applied it in [157] to find several new summation theorems for *multidimensional basic hypergeometric series*.

In fact, as Joris Van der Jeugt pointed out to me, there is a generalization of Theorem 28 of the following form (which can be also proved by means of Lemma 5).

Theorem 29. *Let n be a nonnegative integer, and let $X_0, X_1, \dots, X_{n-1}$, $Y_0, Y_1, \dots, Y_{n-1}$, A and B be indeterminates. Then there holds*

$$\det_{0\le i,j\le n-1}\left(\frac{\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}_q\begin{bmatrix}Y_j+A-X_i\\ j\end{bmatrix}_q}{\begin{bmatrix}X_i+B\\ j\end{bmatrix}_q\begin{bmatrix}A+B-X_i\\ j\end{bmatrix}_q}\right)$$

$$= q^{2\binom{n}{3}+\sum_{i=0}^{n-1} i(X_i+Y_i-A-2B)}\prod_{0\le i<j\le n-1}[X_i-X_j]_q\,[X_i+X_j-A]_q$$

$$\times\prod_{i=0}^{n-1}\frac{(q^{B-Y_i-i+1})_i\,(q^{Y_i+A+B+2-2i})_i}{(q^{X_i-A-B})_{n-1}\,(q^{X_i+B-n+2})_{n-1}}. \quad (3.17)$$

□

As another application of Lemma 5 we list two evaluations of determinants (see below) where the entries are, up to some powers of q, a difference of two q-binomial coefficients. A proof of the first evaluation which uses Lemma 5 can be found in [88, proof of Theorem 7], a proof of the second evaluation using Lemma 5 can be found in [155, Ch. VI, §3]. Once more, the second evaluation was always (implicitly) known to people in group representation theory, as it also results from a principal specialization (set $x_i = q^{i-1/2}$, $i = 1, 2, \dots$) of a symplectic character of arbitrary shape, by comparing the symplectic dual Jacobi–Trudi identity with the bideterminantal form (Weyl character formula) of the symplectic character (cf. [52, Cor. 24.24 and (24.18)]; the determinants arising in the bideterminantal form are easily evaluated by means of (2.4)).

Theorem 30. *Let n be a nonnegative integer, and let $L_1, L_2, \dots, L_n$ and A be indeterminates. Then there hold*

$$\det_{1\le i,j\le n}\left(q^{j(L_j-L_i)}\left(\begin{bmatrix}A\\ j-L_i\end{bmatrix}_q - q^{j(2L_i+A-1)}\begin{bmatrix}A\\ -j-L_i+1\end{bmatrix}_q\right)\right)$$

$$=\prod_{i=1}^{n}\frac{[A+2i-2]_q!}{[n-L_i]_q!\,[A+n-1+L_i]_q!}\prod_{1\le i<j\le n}[L_j-L_i]_q\prod_{1\le i\le j\le n}[L_i+L_j+A-1]_q \quad (3.18)$$

and

$$\det_{1\le i,j\le n}\left(q^{j(L_j-L_i)}\left(\begin{bmatrix}A\\ j-L_i\end{bmatrix}_q - q^{j(2L_i+A)}\begin{bmatrix}A\\ -j-L_i\end{bmatrix}_q\right)\right)$$

$$=\prod_{i=1}^{n}\frac{[A+2i-1]_q!}{[n-L_i]_q!\,[A+n+L_i]_q!}\prod_{1\le i<j\le n}[L_j-L_i]_q\prod_{1\le i\le j\le n}[L_i+L_j+A]_q\,. \quad (3.19)$$

□

A special case of (3.19) was the second determinant evaluation which Andrews needed in [4, (1.4)] in order to prove the *MacMahon Conjecture* (since then, ex-Conjecture) about the q-enumeration of *symmetric plane partitions*. Of course, Andrews' evaluation proceeded by LU-factorization, while Schlosser [155, Ch. VI, §3] simplified Andrews' proof significantly by making use of Lemma 5. The determinant evaluation (3.18) was used in [88] in the proof of *refinements* of the *MacMahon (ex-)Conjecture* and the *Bender–Knuth (ex-)Conjecture.* (The latter makes an assertion about the generating function for tableaux with bounded entries and a bounded number of columns. The first proof is due to Gordon [59], the first published proof [3] is due to Andrews.)

Next, in the theorem below, we list two very similar determinant evaluations. This time, the entries of the determinants are, up to some powers of q, a sum of two q-binomial coefficients. A proof of the first evaluation which uses Lemma 5 can be found in [155, Ch. VI, §3]. A proof of the second evaluation can be established analogously. Again, the second evaluation was always (implicitly) known to people in group representation theory, as it also results from a principal specialization (set $x_i = q^i$, $i = 1, 2, \dots$) of an odd orthogonal character of arbitrary shape, by comparing the orthogonal dual Jacobi–Trudi identity with the bideterminantal form (Weyl character formula) of the orthogonal character (cf. [52, Cor. 24.35 and (24.28)]; the determinants arising in the bideterminantal form are easily evaluated by means of (2.3)).

Theorem 31. *Let n be a nonnegative integer, and let $L_1, L_2, \dots, L_n$ and A be indeterminates. Then there hold*

$$\det_{1\le i,j\le n}\left(q^{(j-1/2)(L_j-L_i)}\left(\begin{bmatrix}A\\ j-L_i\end{bmatrix}_q + q^{(j-1/2)(2L_i+A-1)}\begin{bmatrix}A\\ -j-L_i+1\end{bmatrix}_q\right)\right)$$
$$= \prod_{i=1}^{n}\frac{(1+q^{L_i+A/2-1/2})}{(1+q^{i+A/2-1/2})}\frac{[A+2i-1]_q!}{[n-L_i]_q!\,[A+n+L_i-1]_q!}$$
$$\times \prod_{1\le i<j\le n}[L_j-L_i]_q\,[L_i+L_j+A-1]_q \qquad (3.20)$$

and

$$\det_{1\le i,j\le n}\left(q^{(j-1/2)(L_j-L_i)}\left(\begin{bmatrix}A\\ j-L_i\end{bmatrix}_q + q^{(j-1/2)(2L_i+A-2)}\begin{bmatrix}A\\ -j-L_i+2\end{bmatrix}_q\right)\right)$$
$$= \frac{\prod_{i=1}^{n}(1+q^{L_i+A/2-1})}{\prod_{i=2}^{n}(1+q^{i+A/2-1})}\prod_{i=1}^{n}\frac{[A+2i-2]_q!}{[n-L_i]_q!\,[A+n+L_i-2]_q!}$$
$$\times \prod_{1\le i<j\le n}[L_j-L_i]_q\,[L_i+L_j+A-2]_q\,. \qquad (3.21)$$

□

A special case of (3.20) was the first determinant evaluation which Andrews needed in [4, (1.3)] in order to prove the *MacMahon Conjecture* on *symmetric plane partitions.* Again, Andrews' evaluation proceeded by LU-factorization, while Schlosser [155, Ch. VI, §3] simplified Andrews' proof significantly by making use of Lemma 5.

Now we come to determinants which belong to a different category what regards difficulty of evaluation, as it is not possible to introduce more parameters in a substantial way.

The first determinant evaluation in this category that we list here is a determinant evaluation due to Andrews [5, 6]. It solved, at the same time, Macdonald's problem of enumerating *cyclically symmetric plane partitions* and Andrews' own conjecture about the enumeration of *descending plane partitions*.

Theorem 32. *Let μ be an indeterminate. For nonnegative integers n there holds*

$$\det_{0\le i,j\le n-1}\left(\delta_{ij}+\binom{2\mu+i+j}{j}\right)$$
$$=\begin{cases} 2^{\lceil n/2\rceil}\prod_{i=1}^{n-2}(\mu+\lceil i/2\rceil+1)_{\lfloor (i+3)/4\rfloor} \\ \times\dfrac{\prod_{i=1}^{n/2}\left(\mu+\frac{3n}{2}-\left\lceil\frac{3i}{2}\right\rceil+\frac{3}{2}\right)_{\lceil i/2\rceil-1}\left(\mu+\frac{3n}{2}-\left\lceil\frac{3i}{2}\right\rceil+\frac{3}{2}\right)_{\lceil i/2\rceil}}{\prod_{i=1}^{n/2-1}(2i-1)!!\,(2i+1)!!} & \text{if } n \text{ is even,}\\ 2^{\lceil n/2\rceil}\prod_{i=1}^{n-2}(\mu+\lceil i/2\rceil+1)_{\lfloor (i+3)/4\rfloor} \\ \times\dfrac{\prod_{i=1}^{(n-1)/2}\left(\mu+\frac{3n}{2}-\left\lceil\frac{3i-1}{2}\right\rceil+1\right)_{\lceil (i-1)/2\rceil}\left(\mu+\frac{3n}{2}-\left\lceil\frac{3i}{2}\right\rceil\right)_{\lceil i/2\rceil}}{\prod_{i=1}^{(n-1)/2}(2i-1)!!^2} & \text{if } n \text{ is odd.}\end{cases} \tag{3.22}$$

□

The specializations of this determinant evaluation which are of relevance for the enumeration of cyclically symmetric plane partitions and descending plane partitions are the cases $\mu=0$ and $\mu=1$, respectively. In these cases, Macdonald, respectively Andrews, actually had conjectures about q-enumeration. These were proved by Mills, Robbins and Rumsey [110]. Their theorem which solves the q-enumeration of *cyclically symmetric plane partitions* is the following.

Theorem 33. *For nonnegative integers n there holds*

$$\det_{0\le i,j\le n-1}\left(\delta_{ij}+q^{3i+1}\begin{bmatrix} i+j\\ j\end{bmatrix}_{q^3}\right)=\prod_{i=1}^{n}\frac{1-q^{3i-1}}{1-q^{3i-2}}\prod_{1\le i\le j\le n}\frac{1-q^{3(n+i+j-1)}}{1-q^{3(2i+j-1)}}. \tag{3.23}$$

□

The theorem by Mills, Robbins and Rumsey in [110] which concerns the enumeration of *descending plane partitions* is the subject of the next theorem.

Theorem 34. *For nonnegative integers n there holds*

$$\det_{0\le i,j\le n-1}\left(\delta_{ij}+q^{i+2}\begin{bmatrix} i+j+2\\ j\end{bmatrix}_{q}\right)=\prod_{1\le i\le j\le n+1}\frac{1-q^{n+i+j}}{1-q^{2i+j-1}}. \tag{3.24}$$

□

It is somehow annoying that so far nobody was able to come up with a full q-analogue of the Andrews determinant (3.22) (i.e., not just in the cases $\mu=0$ and $\mu=1$). This issue is already addressed in [6, Sec. 3]. In particular, it is shown there that the result for a natural q-enumeration of a parametric family of descending plane partitions does not factor nicely in general, and thus does not lead to a q-analogue of (3.22). Yet, such a q-analogue should exist. Probably the binomial coefficient in (3.22) has to be replaced by something more complicated than just a q-binomial times some power of q.

On the other hand, there are surprising variations of the Andrews determinant (3.22), discovered by Douglas Zare. These can be interpreted as certain *weighted enumerations* of *cyclically symmetric plane partitions* and of *rhombus tilings of a hexagon with a triangular hole* (see [27]).

Theorem 35. *Let μ be an indeterminate. For nonnegative integers n there holds*

$$\det_{0\le i,j\le n-1}\left(-\delta_{ij}+\binom{2\mu+i+j}{j}\right)\\
=\begin{cases}0, & \text{if } n \text{ is odd,}\\ (-1)^{n/2}\prod_{i=0}^{n/2-1}\frac{i!^2\,(\mu+i)!^2\,(\mu+3i+1)!^2\,(2\mu+3i+1)!^2}{(2i)!\,(2i+1)!\,(\mu+2i)!^2\,(\mu+2i+1)!^2\,(2\mu+2i)!\,(2\mu+2i+1)!}, & \text{if } n \text{ is even.}\end{cases} \tag{3.25}$$

If ω is a primitive 3rd root of unity, then for nonnegative integers n there holds

$$\det_{0\le i,j\le n-1}\left(\omega\delta_{ij}+\binom{2\mu+i+j}{j}\right)=\frac{(1+\omega)^n 2^{\lfloor n/2\rfloor}}{\prod_{i=1}^{\lfloor n/2\rfloor}(2i-1)!!\prod_{i=1}^{\lfloor (n-1)/2\rfloor}(2i-1)!!}\\
\times\prod_{i\ge0}(\mu+3i+1)_{\lfloor (n-4i)/2\rfloor}\,(\mu+3i+3)_{\lfloor (n-4i-3)/2\rfloor}\\
\cdot\left(\mu+n-i+\tfrac12\right)_{\lfloor (n-4i-1)/2\rfloor}\left(\mu+n-i-\tfrac12\right)_{\lfloor (n-4i-2)/2\rfloor}, \tag{3.26}$$

where, in abuse of notation, by $\lfloor\alpha\rfloor$ we mean the usual floor function if $\alpha\ge0$, however, if $\alpha<0$ then $\lfloor\alpha\rfloor$ must be read as 0, so that the product over i in (3.26) is indeed a finite product.

If ω is a primitive 6th root of unity, then for nonnegative integers n there holds

$$\det_{0\le i,j\le n-1}\left(\omega\delta_{ij}+\binom{2\mu+i+j}{j}\right)=\frac{(1+\omega)^n\left(\frac23\right)^{\lfloor n/2\rfloor}}{\prod_{i=1}^{\lfloor n/2\rfloor}(2i-1)!!\prod_{i=1}^{\lfloor (n-1)/2\rfloor}(2i-1)!!}\\
\times\prod_{i\ge0}\left(\mu+3i+\tfrac32\right)_{\lfloor (n-4i-1)/2\rfloor}\left(\mu+3i+\tfrac52\right)_{\lfloor (n-4i-2)/2\rfloor}\\
\cdot(\mu+n-i)_{\lfloor (n-4i)/2\rfloor}\,(\mu+n-i)_{\lfloor (n-4i-3)/2\rfloor}, \tag{3.27}$$

where again, in abuse of notation, by $\lfloor\alpha\rfloor$ we mean the usual floor function if $\alpha\ge0$, however, if $\alpha<0$ then $\lfloor\alpha\rfloor$ must be read as 0, so that the product over i in (3.27) is indeed a finite product. □

There are no really simple proofs of Theorems 32–35. Let me just address the issue of proofs of the evaluation of the Andrews determinant, Theorem 32. The only *direct* proof of Theorem 32 is the original proof of Andrews [5], who worked out the LU-factorization of the determinant. Today one agrees that the "easiest" way of evaluating the determinant (3.22) is by first employing a magnificent factorization theorem [112, Theorem 5] due to Mills, Robbins and Rumsey, and then evaluating each of the two resulting determinants. For these, for some reason, more elementary evaluations exist (see in particular [10] for such a derivation). What I state below is a (straightforward) generalization of this factorization theorem from [92, Lemma 2].

Theorem 36. *Let $Z_n(x;\mu,\nu)$ be defined by*

$$Z_n(x;\mu,\nu) := \det_{0\le i,j\le n-1}\left(\delta_{ij}+\sum_{t=0}^{n-1}\sum_{k=0}^{n-1}\binom{i+\mu}{t}\binom{k+\nu}{k-t}\binom{j-k+\mu-1}{j-k}x^{k-t}\right),$$

let $T_n(x;\mu,\nu)$ be defined by

$$T_n(x;\mu,\nu) := \det_{0\le i,j\le n-1}\left(\sum_{t=i}^{2j}\binom{i+\mu}{t-i}\binom{j+\nu}{2j-t}x^{2j-t}\right),$$

and let $R_n(x;\mu,\nu)$ be defined by

$$R_n(x;\mu,\nu) := \det_{0\le i,j\le n-1}\left(\sum_{t=i}^{2j+1}\left(\binom{i+\mu}{t-i-1}+\binom{i+\mu+1}{t-i}\right)\right.$$
$$\left.\cdot\left(\binom{j+\nu}{2j+1-t}+\binom{j+\nu+1}{2j+1-t}\right)x^{2j+1-t}\right).$$

Then for all positive integers n there hold

$$Z_{2n}(x;\mu,\nu) = T_n(x;\mu,\nu/2)\,R_n(x;\mu,\nu/2) \tag{3.28}$$

and

$$Z_{2n-1}(x;\mu,\nu) = 2\,T_n(x;\mu,\nu/2)\,R_{n-1}(x;\mu,\nu/2). \tag{3.29}$$

□

The reader should observe that $Z_n(1;\mu,0)$ is identical with the determinant in (3.22), as the sums in the entries simplify by means of Chu–Vandermonde summation (see e.g. [62, Sec. 5.1, (5.27)]). However, also the entries in the determinants $T_n(1;\mu,0)$ and $R_n(1;\mu,0)$ simplify. The respective evaluations read as follows (see [112, Theorem 7] and [9, (5.2)/(5.3)]).

Theorem 37. *Let μ be an indeterminate. For nonnegative integers n there holds*

$$\det_{0\le i,j\le n-1}\left(\binom{\mu+i+j}{2i-j}\right)$$
$$= (-1)^{\chi(n\equiv 3 \mod 4)}2^{\binom{n-1}{2}}\prod_{i=1}^{n-1}\frac{(\mu+i+1)_{\lfloor (i+1)/2\rfloor}\,\left(-\mu-3n+i+\frac{3}{2}\right)_{\lfloor i/2\rfloor}}{(i)_i}, \tag{3.30}$$

where $\chi(\mathcal{A})=1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A})=0$ otherwise, and

$$\det_{0\le i,j\le n-1}\left(\binom{\mu+i+j}{2i-j}+2\binom{\mu+i+j+2}{2i-j+1}\right)$$
$$= 2^n\prod_{i=1}^{n}\frac{(\mu+i)_{\lfloor i/2\rfloor}\left(\mu+3n-\left\lfloor\frac{3i-1}{2}\right\rfloor+\frac{1}{2}\right)_{\lfloor (i+1)/2\rfloor}}{(2i-1)!!}. \tag{3.31}$$

□

The reader should notice that the determinant in (3.30) is the third determinant from the Introduction, (1.3). Originally, in [112, Theorem 7], Mills, Robbins and Rumsey proved (3.30) by applying their factorization theorem (Theorem 36) the other way round, relying on Andrews' Theorem 32. However, in the meantime there exist short *direct* proofs of (3.30), see [10, 91, 129], either by LU-factorization, or by "identification of factors". A proof based on the determinant evaluation (3.35) and some combinatorial considerations is given in [29, Remark 4.4], see the remarks after Theorem 40. As shown in [9, 10], the determinant (3.31) can easily be transformed into a special case of the determinant in (3.35) (whose evaluation is easily proved using condensation, see the corresponding remarks there). Altogether, this gives an alternative, and simpler, proof of Theorem 32.

Mills, Robbins and Rumsey needed the evaluation of (3.30) because it allowed them to prove the (at that time) conjectured enumeration of *cyclically symmetric transpose-complementary plane partitions* (see [112]). The unspecialized determinants $Z_n(x;\mu,\nu)$ and $T_n(x;\mu,\nu)$ have combinatorial meanings as well (see [110, Sec. 4], respectively [92, Sec. 3]), as the *weighted enumeration of certain descending plane partitions and triangularly shaped plane partitions.*

It must be mentioned that the determinants $Z_n(x;\mu,\nu)$, $T_n(x;\mu,\nu)$, $R_n(x;\mu,\nu)$ do also factor nicely for $x=2$. This was proved by Andrews [7] using LU-factorization, thus confirming a conjecture by Mills, Robbins and Rumsey (see [92] for an alternative proof by "identification of factors").

It was already mentioned in Section 2.8 that there is a general theorem by Goulden and Jackson [61, Theorem 2.1] (see Lemma 19 and the remarks thereafter) which, given the evaluation (3.30), immediately implies a generalization containing one more parameter. (This property of the determinant (3.30) is called by Goulden and Jackson the *averaging property.*) The resulting determinant evaluation had been earlier found by Andrews and Burge [9, Theorem 1]. They derived it by showing that it can be obtained by multiplying the matrix underlying the determinant (3.30) by a suitable triangular matrix.

Theorem 38. *Let x and y be indeterminates. For nonnegative integers n there holds*

$$\det_{0\le i,j\le n-1}\left(\binom{x+i+j}{2i-j}+\binom{y+i+j}{2i-j}\right)$$
$$=(-1)^{\chi(n\equiv 3 \mod 4)}2^{\binom{n}{2}+1}\prod_{i=1}^{n-1}\frac{\left(\frac{x+y}{2}+i+1\right)_{\lfloor (i+1)/2\rfloor}\left(-\frac{x+y}{2}-3n+i+\frac{3}{2}\right)_{\lfloor i/2\rfloor}}{(i)_i},\qquad (3.32)$$

where $\chi(\mathcal{A})=1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A})=0$ otherwise. □

(The evaluation (3.32) does indeed reduce to (3.30) by setting $x=y$.)

The above described procedure of Andrews and Burge to multiply a matrix, whose determinant is known, by an appropriate triangular matrix, and thus obtain a new determinant evaluation, was systematically exploited by Chu [23]. He derives numerous variations of (3.32), (3.31), and special cases of (3.13). We content ourselves with displaying two typical identities from [23, (3.1a), (3.5a)], just enough to get an idea of the character of these.

Theorem 39. *Let $x_0, x_1, \dots, x_{n-1}$ and c be indeterminates. For nonnegative integers n there hold*

$$\det_{0\le i,j\le n-1}\left(\binom{c+x_i+i+j}{2i-j}+\binom{c-x_i+i+j}{2i-j}\right)$$
$$=(-1)^{\chi(n\equiv 3 \mod 4)}2^{\binom{n}{2}+1}\prod_{i=1}^{n-1}\frac{(c+i+1)_{\lfloor (i+1)/2\rfloor}\left(-c-3n+i+\frac{3}{2}\right)_{\lfloor i/2\rfloor}}{(i)_i} \tag{3.33}$$

and

$$\det_{0\le i,j\le n-1}\left(\frac{(2i-j)+(2c+3j+1)(2c+3j-1)}{(c+i+j+\frac{1}{2})(c+i+j-\frac{1}{2})}\binom{c+i+j+\frac{1}{2}}{2i-j}\right)$$
$$=(-1)^{\chi(n\equiv 3 \mod 4)}2^{\binom{n+1}{2}+1}\prod_{i=1}^{n-1}\frac{\left(c+i+\frac{1}{2}\right)_{\lfloor (i+1)/2\rfloor}(-c-3n+i+2)_{\lfloor i/2\rfloor}}{(i)_i}, \tag{3.34}$$

where $\chi(\mathcal{A})=1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A})=0$ otherwise. □

The next determinant (to be precise, the special case $y=0$), whose evaluation is stated in the theorem below, seems to be closely related to the Mills–Robbins–Rumsey determinant (3.30), although it is in fact a lot easier to evaluate. Indications that the evaluation (3.30) is much deeper than the following evaluation are, first, that it does not seem to be possible to introduce a second parameter into the Mills–Robbins–Rumsey determinant (3.30) in a similar way, and, second, the much more irregular form of the right-hand side of (3.30) (it contains many floor functions!), as opposed to the right-hand side of (3.35).

Theorem 40. *Let x, y, n be nonnegative integers. Then there holds*

$$\det_{0\le i,j\le n-1}\left(\frac{(x+y+i+j-1)!}{(x+2i-j)!\,(y+2j-i)!}\right)$$
$$=\prod_{i=0}^{n-1}\frac{i!\,(x+y+i-1)!\,(2x+y+2i)_i\,(x+2y+2i)_i}{(x+2i)!\,(y+2i)!}. \tag{3.35}$$

□

This determinant evaluation is due to the author, who proved it in [90, (5.3)] as an aside to the (much more difficult) determinant evaluations which were needed there to settle a conjecture by Robbins and Zeilberger about a generalization of the enumeration of *totally symmetric self-complementary plane partitions*. (These are the determinant evaluations of Theorems 43 and 45 below.) It was proved there by "identification of factors". However, Amdeberhan [2] observed that it can be easily proved by "condensation".

Originally there was no application for (3.35). However, not much later, Ciucu [29] found not just one application. He observed that if the determinant evaluation (3.35) is suitably combined with his beautiful *Matchings Factorization Theorem* [26, Theorem 1.2] (and some combinatorial considerations), then not only does one obtain simple proofs for the evaluation of the Andrews determinant (3.22) and the Mills–Robbins–Rumsey determinant (3.30), but also simple proofs for the enumeration of four different symmetry classes of plane partitions, *cyclically symmetric plane partitions, cyclically symmetric self-complementary plane partitions* (first proved by Kuperberg [96]), *cyclically symmetric transpose-complementary plane partitions* (first proved by Mills, Robbins and Rumsey [112]), and *totally symmetric self-complementary plane partitions* (first proved by Andrews [8]).

A q-analogue of the previous determinant evaluation is contained in [89, Theorem 1]. Again, Amdeberhan [2] observed that it can be easily proved by means of "condensation".

Theorem 41. *Let x, y, n be nonnegative integers. Then there holds*

$$\det_{0\le i,j\le n-1}\left(\frac{(q;q)_{x+y+i+j-1}}{(q;q)_{x+2i-j}\,(q;q)_{y+2j-i}}\frac{q^{-2ij}}{(-q^{x+y+1};q)_{i+j}}\right)\\ =\prod_{i=0}^{n-1} q^{-2i^2}\frac{(q^2;q^2)_i\,(q;q)_{x+y+i-1}\,(q^{2x+y+2i};q)_i\,(q^{x+2y+2i};q)_i}{(q;q)_{x+2i}\,(q;q)_{y+2i}\,(-q^{x+y+1};q)_{n-1+i}}. \tag{3.36}$$

□

The reader should observe that this is not a straightforward q-analogue of (3.35) as it does contain the terms $(-q^{x+y+1};q)_{i+j}$ in the determinant, respectively $(-q^{x+y+1};q)_{n-1+i}$ in the denominator of the right-hand side product, which can be cleared only if $q=1$.

A similar determinant evaluation, with some overlap with (3.36), was found by Andrews and Stanton [10, Theorem 8] by making use of LU-factorization, in their "étude" on the Andrews and the Mills–Robbins–Rumsey determinant.

Theorem 42. *Let x and E be indeterminates and n be a nonnegative integer. Then there holds*

$$\det_{0\le i,j\le n-1}\left(\frac{(E/xq^i;q^2)_{i-j}\,(q/Exq^i;q^2)_{i-j}\,(1/x^2q^{2+4i};q^2)_{i-j}}{(q;q)_{2i+1-j}\,(1/Exq^{2i};q)_{i-j}\,(E/xq^{1+2i};q)_{i-j}}\right)\\ =\prod_{i=0}^{n-1}\frac{(x^2q^{2i+1};q)_i\,(xq^{3+i}/E;q^2)_i\,(Exq^{2+i};q^2)_i}{(x^2q^{2i+2};q^2)_i\,(q;q^2)_{i+1}\,(Exq^{1+i};q)_i\,(xq^{2+i}/E;q)_i}. \tag{3.37}$$

□

The next group of determinants is (with one exception) from [90]. These determinants were needed in the proof of a conjecture by Robbins and Zeilberger about a generalization of the enumeration of *totally symmetric self-complementary plane partitions.*

Theorem 43. *Let x, y, n be nonnegative integers. Then*

$$\det_{0\le i,j\le n-1}\left(\frac{(x+y+i+j-1)!\,(y-x+3j-3i)}{(x+2i-j+1)!\,(y+2j-i+1)!}\right)\\ =\prod_{i=0}^{n-1}\left(\frac{i!\,(x+y+i-1)!\,(2x+y+2i+1)_i\,(x+2y+2i+1)_i}{(x+2i+1)!\,(y+2i+1)!}\right)\\ \cdot\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x)_k\,(y)_{n-k}. \tag{3.38}$$

□

This is Theorem 8 from [90]. A q-analogue, provided in [89, Theorem 2], is the following theorem.

Theorem 44. *Let x, y, n be nonnegative integers. Then there holds*

$$\det_{0\le i,j\le n-1}\left(\frac{(q;q)_{x+y+i+j-1}\,(1-q^{y+2j-i}-q^{y+2j-i+1}+q^{x+y+i+j+1})}{(q;q)_{x+2i-j+1}\,(q;q)_{y+2j-i+1}}\cdot\frac{q^{-2ij}}{(-q^{x+y+2};q)_{i+j}}\right)$$
$$=\prod_{i=0}^{n-1}\left(q^{-2i^2}\frac{(q^2;q^2)_i\,(q;q)_{x+y+i-1}\,(q^{2x+y+2i+1};q)_i\,(q^{x+2y+2i+1};q)_i}{(q;q)_{x+2i+1}\,(q;q)_{y+2i+1}\,(-q^{x+y+2};q)_{n-1+i}}\right)$$
$$\times\sum_{k=0}^{n}(-1)^k q^{nk}\begin{bmatrix}n\\k\end{bmatrix}_q q^{yk}\,(q^x;q)_k\,(q^y;q)_{n-k}. \tag{3.39}$$

□

Once more, Amdeberhan observed that, in principle, Theorem 43 as well as Theorem 44 could be proved by means of "condensation". However, as of now, nobody provided a proof of the double sum identities which would establish (2.16) in these cases.

We continue with Theorems 2 and Corollary 3 from [90].

Theorem 45. *Let x, m, n be nonnegative integers with $m \le n$. Under the convention that sums are interpreted by*

$$\sum_{r=A+1}^{B}\operatorname{Expr}(r)=\begin{cases}\sum_{r=A+1}^{B}\operatorname{Expr}(r) & A<B\\ 0 & A=B\\ -\sum_{r=B+1}^{A}\operatorname{Expr}(r) & A>B,\end{cases}$$

there holds

$$\det_{0\le i,j\le n-1}\left(\sum_{x+2i-j<r\le x+m+2j-i}\binom{2x+m+i+j}{r}\right)$$
$$=\prod_{i=1}^{n-1}\left(\frac{(2x+m+i)!\,(3x+m+2i+2)_i\,(3x+2m+2i+2)_i}{(x+2i)!\,(x+m+2i)!}\right)$$
$$\times\frac{(2x+m)!}{(x+\lfloor m/2\rfloor)!\,(x+m)!}\cdot\prod_{i=0}^{\lfloor n/2\rfloor-1}(2x+2\lceil m/2\rceil+2i+1)\cdot P_1(x;m,n), \tag{3.40}$$

where $P_1(x;m,n)$ is a polynomial in x of degree $\le\lfloor m/2\rfloor$.

In particular, for $m=0$ the determinant equals

$$\begin{cases}\prod_{i=0}^{n-1}\left(\frac{i!\,(2x+i)!\,(3x+2i+2)_i^2}{(x+2i)!^2}\right)\frac{\prod_{i=0}^{n/2-1}(2x+2i+1)}{(n-1)!!} & n \text{ even}\\ 0 & n \text{ odd},\end{cases} \tag{3.41}$$

for $m=1$, $n\ge 1$, it equals

$$\prod_{i=0}^{n-1}\left(\frac{i!\,(2x+i+1)!\,(3x+2i+3)_i\,(3x+2i+4)_i}{(x+2i)!\,(x+2i+1)!}\right)\frac{\prod_{i=0}^{\lfloor n/2\rfloor-1}(2x+2i+3)}{(2\lfloor n/2\rfloor-1)!!},\tag{3.42}$$

for $m=2$, $n\ge 2$, it equals

$$\prod_{i=0}^{n-1}\left(\frac{i!\,(2x+i+2)!\,(3x+2i+4)_i\,(3x+2i+6)_i}{(x+2i)!\,(x+2i+2)!}\right)\frac{\prod_{i=0}^{\lfloor n/2\rfloor-1}(2x+2i+3)}{(2\lfloor n/2\rfloor-1)!!}\\
\times\frac{1}{(x+1)}\cdot\begin{cases}(x+n+1) & n\text{ even}\\(2x+n+2) & n\text{ odd,}\end{cases}\tag{3.43}$$

for $m=3$, $n\ge 3$, it equals

$$\prod_{i=0}^{n-1}\left(\frac{i!\,(2x+i+3)!\,(3x+2i+5)_i\,(3x+2i+8)_i}{(x+2i)!\,(x+2i+3)!}\right)\frac{\prod_{i=0}^{\lfloor n/2\rfloor-1}(2x+2i+5)}{(2\lfloor n/2\rfloor-1)!!}\\
\times\frac{1}{(x+1)}\cdot\begin{cases}(x+2n+1) & n\text{ even}\\(3x+2n+5) & n\text{ odd,}\end{cases}\tag{3.44}$$

and for $m=4$, $n\ge 4$, it equals

$$\prod_{i=0}^{n-1}\left(\frac{i!\,(2x+i+4)!\,(3x+2i+6)_i\,(3x+2i+10)_i}{(x+2i)!\,(x+2i+4)!}\right)\frac{\prod_{i=0}^{\lfloor n/2\rfloor-1}(2x+2i+5)}{(2\lfloor n/2\rfloor-1)!!}\\
\times\frac{1}{(x+1)(x+2)}\cdot\begin{cases}(x^2+(4n+3)x+2(n^2+4n+1)) & n\text{ even}\\(2x+n+4)(2x+2n+4) & n\text{ odd.}\end{cases}\tag{3.45}$$

□

One of the most embarrassing failures of "identification of factors," respectively of LU-factorization, is the problem of q-enumeration of *totally symmetric plane partitions*, as stated for example in [164, p. 289] or [165, p. 106]. It is now known for quite a while that also this problem can be reduced to the evaluation of a certain determinant, by means of Okada's result [123, Theorem 4] about the sum of all minors of a given matrix, that was already mentioned in Section 2.8. In fact, in [123, Theorem 5], Okada succeeded to transform the resulting determinant into a reasonably simple one, so that the problem of q-enumerating totally symmetric plane partitions reduces to resolving the following conjecture.

Conjecture 46. *For any nonnegative integer n there holds*

$$\det_{1\le i,j\le n}\left(T_n^{(1)}+T_n^{(2)}\right)=\prod_{1\le i\le j\le k\le n}\left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2,\tag{3.46}$$

where

$$T_n^{(1)} = \left(q^{i+j-1} \left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_q + q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q \right) \right)_{1\le i,j\le n}$$

and

$$T_n^{(2)} = \begin{pmatrix} 1+q & & & & & \\ -1 & 1+q^2 & & & 0 & \\ & -1 & 1+q^3 & & & \\ & & -1 & 1+q^4 & & \\ & 0 & & \ddots & \ddots & \\ & & & & -1 & 1+q^n \end{pmatrix}.$$

While the problem of (plain) enumeration of totally symmetric plane partitions was solved a few years ago by Stembridge [170] (by some ingenious transformations of the determinant which results directly from Okada's result on the sum of all minors of a matrix), the problem of q-enumeration is still wide open. "Identification of factors" cannot even get started because so far nobody came up with a way of introducing a parameter in (3.46) or any equivalent determinant (as it turns out, the parameter q cannot serve as a parameter in the sense of Section 2.4), and, apparently, guessing the LU-factorization is too difficult.

Let us proceed by giving a few more determinants which arise in the enumeration of rhombus tilings.

Our next determinant evaluation is the evaluation of a determinant which, on disregarding the second binomial coefficient, would be just a special case of (3.13), and which, on the other hand, resembles very much the $q=1$ case of (3.18). (It is the determinant that was shown as (1.4) in the Introduction.) However, neither Lemma 3 nor Lemma 5 suffice to give a proof. The proof in [30] by means of "identification of factors" is unexpectedly difficult.

Theorem 47. *Let n be a positive integer, and let x and y be nonnegative integers. Then the following determinant evaluation holds:*

$$\det_{1\le i,j\le n}\left(\binom{x+y+j}{x-i+2j} - \binom{x+y+j}{x+i+2j} \right) \\ = \prod_{j=1}^{n} \frac{(j-1)!\,(x+y+2j)!\,(x-y+2j+1)_j\,(x+2y+3j+1)_{n-j}}{(x+n+2j)!\,(y+n-j)!}. \tag{3.47}$$

□

This determinant evaluation is used in [30] to enumerate rhombus tilings of a certain pentagonal subregion of a hexagon.

To see an example of different nature, I present a determinant evaluation from [50, Lemma 2.2], which can be considered as a determinant of a mixture of two matrices, out of one we take all rows except the l-th, while out of the other we take just the l-th row. The determinants of both of these matrices could be straightforwardly evaluated by means of Lemma 3. (They are in fact equivalent to special cases of (3.13).) However, to evaluate this mixture is much more challenging. In particular, the mixture does not anymore factor completely into "nice" factors, i.e., the result is of the form (2.21), so that for a proof one has to resort to the extension of "identification of factors" described there.

Theorem 48. *Let n, m, l be positive integers such that $1 \le l \le n$. Then there holds*

$$\det_{1\le i,j\le n}\left(\begin{cases}\binom{n+m-i}{m+i-j}\frac{(m+\frac{n-j+1}{2})}{(n+j-2i+1)} & \text{if } i\ne l\\ \binom{n+m-i}{m+i-j} & \text{if } i=l\end{cases}\right)$$

$$=\prod_{i=1}^{n}\frac{(n+m-i)!}{(m+i-1)!\,(2n-2i+1)!}\prod_{i=1}^{\lfloor n/2\rfloor}\left((m+i)_{n-2i+1}\,(m+i+\frac{1}{2})_{n-2i}\right)$$

$$\times\, 2^{\frac{(n-1)(n-2)}{2}}\frac{(m)_{n+1}\prod_{j=1}^{n}(2j-1)!}{n!\prod_{i=1}^{\lfloor n/2\rfloor}(2i)_{2n-4i+1}}\sum_{e=0}^{l-1}(-1)^e\binom{n}{e}\frac{(n-2e)\,(\frac{1}{2})_e}{(m+e)\,(m+n-e)\,(\frac{1}{2}-n)_e}. \tag{3.48}$$

□

In [50], this result was applied to enumerate all rhombus tilings of a symmetric hexagon that contain a fixed rhombus. In Section 4 of [50] there can be found several conjectures about the enumeration of rhombus tilings with more than just one fixed rhombus, all of which amount to evaluating other mixtures of the above-mentioned two determinants.

As last binomial determinants, I cannot resist to show the, so far, weirdest determinant evaluations that I am aware of. They arose in an attempt [16] by Bombieri, Hunt and van der Poorten to improve on Thue's method of approximating an algebraic number. In their paper, they conjectured the following determinant evaluation, which, thanks to van der Poorten [132], has recently become a theorem (see the subsequent paragraphs and, in particular, Theorem 51 and the remark following it).

Theorem 49. *Let N and l be positive integers. Let M be the matrix with rows labelled by pairs (i_1, i_2) with $0 \le i_1 \le 2l(N-i_2)-1$ (the intuition is that the points (i_1, i_2) are the lattice points in a right-angled triangle), with columns labelled by pairs (j_1, j_2) with $0 \le j_2 \le N$ and $2l(N-j_2) \le j_1 \le l(3N-2j_2)-1$ (the intuition is that the points (j_1, j_2) are the lattice points in a lozenge), and entry in row (i_1, i_2) and column (j_1, j_2) equal to*

$$\binom{j_1}{i_1}\binom{j_2}{i_2}.$$

Then the determinant of M is given by

$$\pm\left(\frac{\prod_{k=0}^{l-1}k!\prod_{k=2l}^{3l-1}k!}{\prod_{k=l}^{2l-1}k!}\right)^{\binom{N+2}{3}}.$$

This determinant evaluation is just one in a whole series of conjectured determinant evaluations and greatest common divisors of minors of a certain matrix, many of them reported in [16]. These conjectures being settled, the authors of [16] expect important implications in the approximation of algebraic numbers.

The case $N = 1$ of Theorem 49 is a special case of (3.11), and, thus, on a shallow level. On the other hand, the next case, $N = 2$, is already on a considerably deeper level. It was first proved in [94], by establishing, in fact, a much more general result, given in the next theorem. It reduces to the $N=2$ case of Theorem 49 for $x=0$, $b=4l$, and $c=2l$. (In fact, the $x=0$ case of Theorem 50 had already been conjectured in [16].)

Theorem 50. *Let b,c be nonnegative integers, $c\le b$, and let $\Delta(x;b,c)$ be the determinant of the $(b+c)\times(b+c)$ matrix*

$$\begin{array}{cc} & \begin{array}{ccc} 0\le j<c & c\le j<b & b\le j<b+c \end{array} \\ \begin{array}{c} 0\le i<c \\[2ex] c\le i<b \\[2ex] b\le i<b+c \end{array} & \left(\begin{array}{c:c:c} \binom{x+j}{i} & \binom{x+j}{i} & \binom{2x+j}{i} \\ \hdashline 0 & \binom{x+j}{i} & \binom{2x+j}{i} \\ \hdashline 2\binom{x+j}{i-b} & \binom{x+j}{i-b} & 0 \end{array}\right) \end{array}. \tag{3.49}$$

Then

(i) $\Delta(x;b,c)=0$ *if b is even and c is odd;*
(ii) *if any of these conditions does not hold, then*

$$\Delta(x;b,c)=(-1)^c2^c\prod_{i=1}^{b-c}\frac{\left(i+\frac{1}{2}-\lceil\frac{b}{2}\rceil\right)_c}{(i)_c}$$
$$\times\prod_{i=1}^{c}\frac{\left(x+\lceil\frac{c+i}{2}\rceil\right)_{b-c+\lceil i/2\rceil-\lceil (c+i)/2\rceil}\ \left(x+\lceil\frac{b-c+i}{2}\rceil\right)_{\lceil (b+i)/2\rceil-\lceil (b-c+i)/2\rceil}}{\left(\frac{1}{2}-\lceil\frac{b}{2}\rceil+\lceil\frac{c+i}{2}\rceil\right)_{b-c+\lceil i/2\rceil-\lceil (c+i)/2\rceil}\ \left(\frac{1}{2}-\lceil\frac{b}{2}\rceil+\lceil\frac{b-c+i}{2}\rceil\right)_{\lceil (b+i)/2\rceil-\lceil (b-c+i)/2\rceil}}. \tag{3.50}$$

□

The proof of this result in [94] could be called "heavy". It proceeded by "identification of factors". Thus, it was only the introduction of the parameter x in the determinant in (3.49) that allowed the attack on this special case of the conjecture of Bombieri, Hunt and van der Poorten. However, the authors of [94] (this includes myself) failed to find a way to introduce a parameter into the determinant in Theorem 49 for generic N (that is, in a way such the determinant would still factor nicely). This was accomplished by van der Poorten [132]. He first changed the entries in the determinant slightly, without changing the value of the determinant, and then was able to introduce a parameter. I state his result, [132, Sec. 5, Main Theorem], in the theorem below. For the proof of his result he used "identification of factors" as well, thereby considerably simplifying and illuminating arguments from [94].

Theorem 51. *Let N and l be positive integers. Let M be the matrix with rows labelled by pairs (i_1,i_2) with $0\le i_1\le 2l(N-i_2)-1$, with columns labelled by pairs (j_1,j_2) with $0\le j_2\le N$ and $0\le j_1\le lN-1$, and entry in row (i_1,i_2) and column (j_1,j_2) equal to*

$$(-1)^{i_1-j_1}\binom{-x(N-j_2)}{i_1-j_1}\binom{j_2}{i_2}. \tag{3.51}$$

Then the determinant of M is given by

$$\pm\left(\prod_{i=1}^{l}\binom{x+i-1}{2i-1}\Big/\binom{l+i-1}{2i-1}\right)^{\binom{N+2}{3}}.$$

Although not completely obvious, the special case $x=-2l$ establishes Theorem 49, see [132]. Van der Poorten proves as well an evaluation that overlaps with the $x=0$ case of Theorem 50, see [132, Sec. 6, Example Application].

Let us now turn to a few remarkable Hankel determinant evaluations.

Theorem 52. *Let n be a positive integer. Then there hold*

$$\det_{0\le i,j\le n-1}\left(E_{2i+2j}\right)=\prod_{i=0}^{n-1}(2i)!^2, \tag{3.52}$$

where E_{2k} is the $(2k)$-th (signless) Euler number*, defined through the generating function $1/\cos z=\sum_{k=0}^{\infty}E_{2k}z^{2k}/(2k)!$, and*

$$\det_{0\le i,j\le n-1}\left(E_{2i+2j+2}\right)=\prod_{i=0}^{n-1}(2i+1)!^2. \tag{3.53}$$

Furthermore, define the Bell polynomials *$\mathcal{B}_m(x)$ by $\mathcal{B}_m(x)=\sum_{k=1}^{m}S(m,k)x^k$, where $S(m,k)$ is a* Stirling number of the second kind *(the number of partitions of an m-element set into k blocks; cf.* [166, p. 33]*). Then*

$$\det_{0\le i,j\le n-1}\left(\mathcal{B}_{i+j}(x)\right)=x^{n(n-1)/2}\prod_{i=0}^{n-1}i!. \tag{3.54}$$

Next, there holds

$$\det_{0\le i,j\le n-1}\left(H_{i+j}(x)\right)=(-1)^{n(n-1)/2}\prod_{i=0}^{n-1}i!, \tag{3.55}$$

where $H_m(x)=\sum_{k\ge0}\frac{m!}{k!\,(m-2k)!}\left(-\frac{1}{2}\right)^k x^{m-2k}$ is the m-th Hermite polynomial*.*

Finally, the following Hankel determinant evaluations featuring Bernoulli numbers *hold,*

$$\det_{0\le i,j,\le n-1}\left(B_{i+j}\right)=(-1)^{\binom{n}{2}}\prod_{i=1}^{n-1}\frac{i!^6}{(2i)!\,(2i+1)!}, \tag{3.56}$$

and

$$\det_{0\le i,j,\le n-1}\left(B_{i+j+1}\right)=(-1)^{\binom{n+1}{2}}\frac{1}{2}\prod_{i=1}^{n-1}\frac{i!^3\,(i+1)!^3}{(2i+1)!\,(2i+2)!}, \tag{3.57}$$

and

$$\det_{0\le i,j,\le n-1}\left(B_{i+j+2}\right)=(-1)^{\binom{n}{2}}\frac{1}{6}\prod_{i=1}^{n-1}\frac{i!\,(i+1)!^4\,(i+2)!}{(2i+2)!\,(2i+3)!}, \tag{3.58}$$

and

$$\det_{0\le i,j,\le n-1}(B_{2i+2j+2}) = \prod_{i=0}^{n-1} \frac{(2i)!\,(2i+1)!^4\,(2i+2)!}{(4i+2)!\,(4i+3)!}, \tag{3.59}$$

and

$$\det_{0\le i,j,\le n-1}(B_{2i+2j+4}) = (-1)^n \prod_{i=1}^{n} \frac{(2i-1)!\,(2i)!^4\,(2i+1)!}{(4i)!\,(4i+1)!}. \tag{3.60}$$

□

All these evaluations can be deduced from continued fractions and orthogonal polynomials, in the way that was described in Section 2.7. To prove (3.52) and (3.53) one would resort to suitable special cases of *Meixner–Pollaczek polynomials* (see [81, Sec. 1.7]), and use an integral representation for Euler numbers, given for example in [122, p. 75],

$$E_{2\nu} = (-1)^{\nu+1}\sqrt{-1}\int_{-\infty\sqrt{-1}}^{\infty\sqrt{-1}} \frac{(2z)^{2\nu}}{\cos \pi z}dz.$$

Slightly different proofs of (3.52) can be found in [1] and [108, App. A.5], together with more Hankel determinant evaluations (among which are also (3.56) and (3.58), respectively). The evaluation (3.54) can be derived by considering *Charlier polynomials* (see [35] for such a derivation in a special case). The evaluation (3.55) follows from the fact that Hermite polynomials are moments of slightly shifted Hermite polynomials, as explained in [71]. In fact, the papers [71] and [72] contain more examples of orthogonal polynomials which are moments, thus in particular implying Hankel determinant evaluations whose entries are *Laguerre polynomials*, *Meixner polynomials*, and *Al-Salam–Chihara polynomials*. Hankel determinants where the entries are (classical) orthogonal polynomials are also considered in [77], where they are related to *Wronskians* of orthogonal polynomials. In particular, there result Hankel determinant evaluations with entries being *Legendre*, *ultraspherical*, and *Laguerre polynomials* [77, (12.3), (14.3), (16.5), § 28], respectively. The reader is also referred to [103], where illuminating proofs of these identities between Hankel determinants and Wronskians are given, by using the fact that Hankel determinants can be seen as certain *Schur functions* of rectangular shape, and by applying a 'master identity' of Turnbull [178, p. 48] on minors of a matrix. (The evaluations (3.52), (3.55) and (3.56) can be found in [103] as well, as corollaries to more general results.) Alternative proofs of (3.52), (3.54) and (3.55) can be found in [141], see also [139] and [140].

Clearly, to prove (3.56)–(3.58) one would proceed in the same way as in Section 2.7. (Identity (3.58) is in fact the evaluation (2.38) that we derived in Section 2.7.) The evaluations (3.59) and (3.60) are equivalent to (3.58), because the matrix underlying the determinant in (3.58) has a checkerboard pattern (recall that Bernoulli numbers with odd indices are zero, except for B_1), and therefore decomposes into the product of a determinant of the form (3.59) and a determinant of the form (3.60). Very interestingly, variations of (3.56)–(3.60) arise as normalization constants in *statistical mechanics models*, see e.g. [14, (4.36)], [32, (4.19)], and [108, App. A.5]. A common generalization of (3.56)–(3.58) can be found in [51, Sec. 5]. Strangely enough, it was needed there in the enumeration of rhombus tilings.

In view of Section 2.7, any continued fraction expansion of the form (2.30) gives rise to a Hankel determinant evaluation. Thus, many more Hankel determinant evaluations follow e.g. from work by Rogers [151], Stieltjes [171, 172], Flajolet [44], Han, Randrianarivony and Zeng [65, 64, 142, 143, 144, 145, 146, 201], Ismail, Masson and Valent [70, 73] or Milne [113, 114, 115, 116], in particular, evaluations of Hankel determinant featuring *Euler numbers* with odd indices (these are given through the generating function $\tan z = \sum_{k=0}^{\infty} E_{2k+1} z^{2k+1}/(2k+1)!$), *Genocchi numbers*, *q*- and other extensions of *Catalan, Euler* and *Genocchi numbers*, and coefficients in the power series expansion of *Jacobi elliptic functions*. Evaluations of the latter type played an important role in Milne's recent beautiful results [113, 114] on the number of representations of integers as sums of m-th powers (see also [108, App. A.5]).

For further evaluations of Hankel determinants, which apparently do not follow from known results about continued fractions or orthogonal polynomials, see [68, Prop. 14] and [51, Sec. 4].

Next we state two charming applications of Lemma 16 (see [189]).

Theorem 53. *Let x be a nonnegative integer. For any nonnegative integer n there hold*

$$\det_{0\le i,j\le n}\left(\frac{(xi)!}{(xi+j)!}S(xi+j,xi)\right)=\left(\frac{x}{2}\right)^{\binom{n+1}{2}} \tag{3.61}$$

where $S(m,k)$ is a Stirling number of the second kind (*the number of partitions of an m-element set into k blocks; cf.* [166, p. 33]), *and*

$$\det_{0\le i,j\le n}\left(\frac{(xi)!}{(xi+j)!}s(xi+j,xi)\right)=\left(-\frac{x}{2}\right)^{\binom{n+1}{2}}, \tag{3.62}$$

where $s(m,k)$ is a Stirling number of the first kind (*up to sign, the number of permutations of m elements with exactly k cycles; cf.* [166, p. 18]). □

Theorem 54. *Let A_{ij} denote the number of representations of j as a sum of i squares of nonnegative integers. Then $\det_{0\le i,j\le n}(A_{ij})=1$ for any nonnegative integer n. The same is true if "squares" is replaced by "cubes," etc.* □

After having seen so many determinants where rows and columns are indexed by integers, it is time for a change. There are quite a few interesting determinants whose rows and columns are indexed by (other) combinatorial objects. (Actually, we already encountered one in Conjecture 49.)

We start by a determinant where rows and columns are indexed by permutations. Its beautiful evaluation was obtained at roughly the same time by Varchenko [184] and Zagier [193].

Theorem 55. *For any positive integer n there holds*

$$\det_{\sigma,\pi\in\mathfrak{S}_n}\left(q^{\operatorname{inv}(\sigma\pi^{-1})}\right)=\prod_{i=2}^{n}(1-q^{i(i-1)})^{\binom{n}{i}(i-2)!\,(n-i+1)!}, \tag{3.63}$$

where $\mathfrak{S}_n$ denotes the symmetric group on n elements. □

This determinant evaluation appears in [193] in the study of certain models in *infinite statistics*. However, as Varchenko et al. [20, 153, 184] show, this determinant evaluation is in fact just a special instance in a whole series of determinant evaluations. The latter papers give evaluations of determinants corresponding to certain bilinear forms associated to *hyperplane arrangements* and *matroids*. Some of these bilinear forms are relevant to the study of *hypergeometric functions* and the *representation theory of quantum groups* (see also [185]). In particular, these results contain analogues of (3.63) for all *finite Coxeter groups* as special cases. For other developments related to Theorem 55 (and different proofs) see [36, 37, 40, 67], tying the subject also to the *representation theory of the symmetric group*, to *noncommutative symmetric functions*, and to *free Lie algebras*, and [109]. For more remarkable determinant evaluations related to hyperplane arrangements see [39, 182, 183]. For more determinant evaluations related to hypergeometric functions and quantum groups and algebras, see [175, 176], where determinants arising in the context of *Knizhnik-Zamolodchikov equations* are computed.

The results in [20, 153] may be considered as a generalization of the *Shapovalov determinant* evaluation [159], associated to the *Shapovalov form* in *Lie theory*. The latter has since been extended to *Kac–Moody algebras* (although not yet in full generality), see [31].

There is a result similar to Theorem 55 for another prominent permutation statistics, MacMahon's *major index*. (The major index $\operatorname{maj}(\pi)$ is defined as the sum of all positions of descents in the permutation π, see e.g. [46].)

Theorem 56. *For any positive integer n there holds*

$$\det_{\sigma,\pi\in\mathfrak{S}_n}\left(q^{\operatorname{maj}(\sigma\pi^{-1})}\right)=\prod_{i=2}^{n}(1-q^i)^{n!\,(i-1)/i}. \tag{3.64}$$

As Jean–Yves Thibon explained to me, this determinant evaluation follows from results about the *descent algebra* of the symmetric group given in [95], presented there in an equivalent form, in terms of *noncommutative symmetric functions*. For the details of Thibon's argument see Appendix C. Also the bivariate determinant $\det_{\sigma,\pi\in\mathfrak{S}_n}\left(x^{\operatorname{des}(\sigma\pi^{-1})}q^{\operatorname{maj}(\sigma\pi^{-1})}\right)$ seems to possess an interesting factorization.

The next set of determinant evaluations shows determinants where the rows and columns are indexed by *set partitions*. In what follows, the set of all partitions of $\{1,2,\dots,n\}$ is denoted by Π_n. The number of blocks of a partition π is denoted by $\operatorname{bk}(\pi)$. A partition π is called *noncrossing*, if there do not exist $i<j<k<l$ such that both i and k belong to one block, B_1 say, while both j and l belong to another block which is different from B_1. The set of all noncrossing partitions of $\{1,2,\dots,n\}$ is denoted by NC_n. (For more information about noncrossing partitions see [160].)

Further, poset-theoretic, notations which are needed in the following theorem are: Given a poset P, the *join* of two elements x and y in P is denoted by $x\vee_P y$, while the *meet* of x and y is denoted by $x\wedge_P y$. The *characteristic polynomial* of a poset P is written as $\chi_P(q)$ (that is, if the maximum element of P has rank h and μ is the *Möbius function* of P, then $\chi_P(q):=\sum_{p\in P}\mu(\hat 0,p)q^{h-\operatorname{rank}(p)}$, where $\hat 0$ stands for the minimal element of P). The symbol $\tilde\chi_P(q)$ denotes the *reciprocal polynomial* $q^h\chi_P(1/q)$ of $\chi_P(q)$. Finally, P^* is the *order-dual* of P.

Theorem 57. *Let n be a positive integer. Then*

$$\det_{\pi,\gamma\in\Pi_n}\left(q^{\operatorname{bk}(\pi\wedge_{\Pi_n}\gamma)}\right)=\prod_{i=1}^{n}\left(q\,\tilde\chi_{\Pi_i^*}(q)\right)^{\binom{n}{i}B(n-i)},\tag{3.65}$$

where $B(k)$ denotes the k-th Bell number (*the total number of partitions of a k-element set; cf.* [166, p. 33]). *Furthermore,*

$$\det_{\pi,\gamma\in\Pi_n}\left(q^{\operatorname{bk}(\pi\vee_{\Pi_n}\gamma)}\right)=\prod_{i=1}^{n}\left(q\,\chi_{\Pi_i}(q)\right)^{S(n,i)},\tag{3.66}$$

where $S(m,k)$ is a Stirling number of the second kind (*the number of partitions of an m-element set into k blocks; cf.* [166, p. 33]). *Next,*

$$\det_{\pi,\gamma\in\mathrm{NC}_n}\left(q^{\operatorname{bk}(\pi\wedge_{\mathrm{NC}_n}\gamma)}\right)=q^{\binom{2n-1}{n}}\prod_{i=1}^{n}\left(\tilde\chi_{\mathrm{NC}_i}(q)\right)^{\binom{2n-1-i}{n-1}},\tag{3.67}$$

and

$$\det_{\pi,\gamma\in\mathrm{NC}_n}\left(q^{\operatorname{bk}(\pi\vee_{\mathrm{NC}_n}\gamma)}\right)=q^{\frac{1}{n+1}\binom{2n}{n}}\prod_{i=1}^{n}\left(\chi_{\mathrm{NC}_i}(q)\right)^{\binom{2n-1-i}{n-1}},\tag{3.68}$$

Finally,

$$\det_{\pi,\gamma\in\mathrm{NC}_n}\left(q^{\operatorname{bk}(\pi\vee_{\Pi_n}\gamma)}\right)=q^{\binom{2n-1}{n}}\prod_{i=1}^{n-1}\left(\frac{U_{i+1}(\sqrt{q}/2)}{qU_{i-1}(\sqrt{q}/2)}\right)^{\frac{i+1}{n}\binom{2n}{n-1-i}},\tag{3.69}$$

where $U_m(x):=\sum_{j\ge0}(-1)^j\binom{m-j}{j}(2x)^{m-2j}$ is the m-th Chebyshev polynomials of the second kind. □

The evaluations (3.65)–(3.68) are due to Jackson [75]. The last determinant evaluation, (3.69), is the hardest among those. It was proved independently by Dahab [33] and Tutte [181]. All these determinants are related to the so-called *Birkhoff–Lewis equation* from chromatic graph theory (see [33, 180] for more information).

A determinant of somewhat similar type appeared in work by Lickorish [104] on 3-*manifold invariants*. Let $\mathrm{NCmatch}(2n)$ denote the set of all noncrossing perfect matchings of $2n$ elements. Equivalently, $\mathrm{NCmatch}(2n)$ can be considered as the set of all noncrossing partitions of $2n$ elements with all blocks containing exactly 2 elements. Lickorish considered a bilinear form on the linear space spanned by $\mathrm{NCmatch}(2n)$. The corresponding determinant was evaluated by Ko and Smolinsky [80] using an elegant recursive approach, and independently by Di Francesco [47], whose calculations are done within the framework of the *Temperley–Lieb algebra* (see also [49]).

Theorem 58. *For $\alpha,\beta\in\mathrm{NCmatch}(2n)$, let $c(\alpha,\beta)$ denote the number of connected components when the matchings α and β are superimposed. Equivalently, $c(\alpha,\beta)=\operatorname{bk}(\alpha\vee_{\Pi_{2n}}\beta)$. For any positive integer n there holds*

$$\det_{\alpha,\beta\in\mathrm{NCmatch}(2n)}\left(q^{c(\alpha,\beta)}\right)=\prod_{i=1}^{n}U_i(q/2)^{a_{2n,2i}},\tag{3.70}$$

where $U_m(q)$ is the Chebyshev polynomial of the second kind *as given in Theorem 57, and where $a_{2n,2i}=c_{2n,2i}-c_{2n,2i+2}$ with $c_{n,h}=\binom{n}{(n-h)/2}-\binom{n}{(n-h)/2-1}$.* □

Di Francesco [47, Theorem 2] does also provide a generalization to partial matchings, and in [48] a generalization in an $SU(n)$ setting, the previously mentioned results being situated in the $SU(2)$ setting. While the derivations in [47] are mostly combinatorial, the derivations in [48] are based on computations in quotients of type A Hecke algebras.

There is also an interesting determinant evaluation, which comes to my mind, where rows and columns of the determinant are indexed by *integer partitions*. It is a result due to Reinhart [147]. Interestingly, it arose in the analysis of *algebraic differential equations*.

In concluding, let me attract your attention to other determinant evaluations which I like, but which would take too much space to state and introduce properly.

For example, there is a determinant evaluation, conjectured by Good, and proved by Goulden and Jackson [60], which arose in the calculation of cumulants of a statistic analogous to *Pearson's chi-squared for a multinomial sample*. Their method of derivation is very combinatorial, in particular making use of generalized *ballot sequences*.

Determinants arising from certain *raising operators of* $\mathfrak{sl}(2)$*-representations* are presented in [136]. As special cases, there result beautiful determinant evaluations where rows and columns are indexed by integer partitions and the entries are numbers of *standard Young tableaux* of skew shapes.

In [84, p. 4] (see also [192]), an interesting mixture of linear algebra and combinatorial matrix theory yields, as a by-product, the evaluation of the determinant of certain *incidence mappings*. There, rows and columns of the relevant matrix are indexed by all subsets of an n-element set of a fixed size.

As a by-product of the analysis of an interesting matrix in *quantum information theory* [93, Theorem 6], the evaluation of a determinant of a matrix whose rows and columns are indexed by all subsets of an n-element set is obtained.

Determinant evaluations of *q-hypergeometric functions* are used in [177] to compute *q-Selberg integrals*.

And last, but not least, let me once more mention the remarkable determinant evaluation, arising in connection with *holonomic q-difference equations*, due to Aomoto and Kato [11, Theorem 3], who thus proved a conjecture by Mimachi [118].

Appendix A: A word about guessing

The problem of guessing a formula for the generic element a_n of a sequence $(a_n)_{n\ge0}$ out of the first few elements was present at many places, in particular this is crucial for a successful application of the "identification of factors" method (see Section 2.4) or of LU-factorization (see Section 2.6). Therefore some elaboration on guessing is in order.

First of all, as I already claimed, *guessing can be largely automatized*. This is due to the following tools[11]:

1. `Superseeker`, the electronic version of the "Encyclopedia of Integer Sequences" [162, 161] by Neil Sloane and Simon Plouffe (see Footnote 2 in the Introduction),

[11]In addition, one has to mention Frank Garvan's `qseries` [54], which is designed for guessing and computing within the territory of q-series, q-products, eta and theta functions, and the like. Procedures like `prodmake` or `qfactor`, however, might also be helpful for the evaluation of "q-determinants". The package is available from `http://www.math.ufl.edu/~frank/qmaple.html`.

2. `gfun` by Bruno Salvy and Paul Zimmermann and `Mgfun` by Frederic Chyzak (see Footnote 3 in the Introduction),
3. `Rate` by the author (see Footnote 4 in the Introduction).

If you send the first few elements of your sequence to `Superseeker` then, if it overlaps with a sequence that is stored there, you will receive information about your sequence such as where your sequence already appeared in the literature, a formula, generating function, or a recurrence for your sequence.

The *Maple* package `gfun` provides tools for finding a generating function and/or a recurrence for your sequence. (In fact, `Superseeker` does also automatically invoke features from `gfun`.) `Mgfun` does the same in a multidimensional setting.

Within the "hypergeometric paradigm," the most useful is the *Mathematica* program `Rate` ("Rate!" is German for "Guess!"), respectively its *Maple* equivalent `GUESS`. Roughly speaking, it allows to automatically guess "closed forms".[12] The program is based on the observation that any "closed form" sequence $(a_n)_{n\ge0}$ that appears within the "hypergeometric paradigm" is either given by a rational expression, like $a_n = n/(n+1)$, or the sequence of successive quotients $(a_{n+1}/a_n)_{n\ge0}$ is given by a rational expression, like in the case of central binomial coefficients $a_n = \binom{2n}{n}$, or the sequence of successive quotients of successive quotients $((a_{n+2}/a_{n+1})/(a_{n+1}/a_n))_{n\ge0}$ is given by a rational expression, like in the case of the famous sequence of numbers of *alternating sign matrices* (cf. the paragraphs following (3.9), and [18, 19, 111, 148, 97, 198, 199] for information on alternating sign matrices),

$$a_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}, \tag{A.1}$$

etc. Given enough special values, a rational expression is easily found by rational interpolation.

This is implemented in `Rate`. Given the first m terms of a sequence, it takes the first $m-1$ terms and applies rational interpolation to these, then it applies rational interpolation to the successive quotients of these $m-1$ terms, etc. For each of the obtained results it is tested if it does also give the m-th term correctly. If it does, then the corresponding result is added to the output, if it does not, then nothing is added to the output.

Here is a short demonstration of the *Mathematica* program `Rate`. The output shows guesses for the $i0$-th element of the sequence.

```
In[1]:= ⟨⟨rate.m

In[2]:= Rate[1,2,3]

Out[2]= {i0}

In[3]:= Rate[2/3,3/4,4/5,5/6]
```

[12]Commonly, by "closed form" ("NICE" in Zeilberger's "terminology") one means an expression which is built by forming products and quotients of factorials. A strong indication that you encounter a sequence $(a_n)_{n\ge0}$ for which a "closed form" exists is that the prime factors in the prime factorization of a_n do not grow rapidly as n becomes larger. (In fact, they should grow linearly.)

```
          1 + i0
Out[3]= {------}
          2 + i0
```

Now we try the central binomial coefficients:

```
In[4]:= Rate[1,2,6,20,70]

          -1 + i0
           ----- 2 (-1 + 2 i1)
Out[4]= {   | |  -------------}
            | |       i1
            | |
           i1=1
```

It needs the first 8 values to guess the formula (A.1) for the numbers of alternating sign matrices:

```
In[5]:= Rate[1,2,7,42,429,7436,218348,10850216]

          -1 + i0   -1 + i1
           -----     ----- 3 (2 + 3 i2) (4 + 3 i2)
Out[5]= {   | |   2 ( | |  -----------------------)}
            | |       | |  4 (1 + 2 i2) (3 + 2 i2)
            | |       | |
           i1=1      i2=1
```

However, what if we encounter a sequence where all these nice automatic tools fail? Here are a few hints. First of all, it is not uncommon to encounter a sequence $(a_n)_{n\ge0}$ which has actually a split definition. For example, it may be the case that the subsequence $(a_{2n})_{n\ge0}$ of even-numbered terms follows a "nice" formula, and that the subsequence $(a_{2n+1})_{n\ge0}$ of odd-numbered terms follows as well a "nice," but different, formula. Then `Rate` will fail on any number of first terms of $(a_n)_{n\ge0}$, while it will give you something for sufficiently many first terms of $(a_{2n})_{n\ge0}$, and it will give you something else for sufficiently many first terms of $(a_{2n+1})_{n\ge0}$.

Most of the subsequent hints apply to a situation where you encounter a sequence $p_0(x), p_1(x), p_2(x), \dots$ of polynomials $p_n(x)$ in x for which you want to find (i.e., guess) a formula. This is indeed the situation in which you are generally during the guessing for "identification of factors," and also usually when you perform a guessing where a parameter is involved.

To make things concrete, let us suppose that the first 10 elements of your sequence of polynomials are

$$
\begin{gathered}
1,\ 1+2x,\ 1+x+3x^2,\ \frac{1}{6}(6+31x-15x^2+20x^3),\ \frac{1}{12}(12-58x+217x^2-98x^3+35x^4),\\
\frac{1}{20}(20+508x-925x^2+820x^3-245x^4+42x^5),\ \frac{1}{120}(120-8042x+20581x^2-17380x^3+7645x^4-1518x^5+154x^6),\\
\frac{1}{1680}(1680+386012x-958048x^2+943761x^3-455455x^4+123123x^5-17017x^6+1144x^7),\\
\frac{1}{20160}(20160-15076944x+40499716x^2-42247940x^3+23174515x^4-7234136x^5+1335334x^6-134420x^7+6435x^8),\\
\frac{1}{181440}(181440+462101904x-1283316876x^2+1433031524x^3-853620201x^4+303063726x^5\\
-66245634x^6+8905416x^7-678249x^8+24310x^9),\ \dots
\end{gathered}
\tag{A.2}
$$

You may of course try to guess the coefficients of powers of x in these polynomials. But within the "hypergeometric paradigm" this does usually not work. In particular, that does not work with the above sequence of polynomials.

A first very useful idea is *to guess through interpolation.* (For example, this is what helped to guess coefficients in [43].) What this means is that, for each $p_n(x)$ you try to find enough values of x for which $p_n(x)$ appears to be "nice" (the prime factorization of $p_n(x)$ has small prime factors, see Footnote 12). Then you guess these special evaluations of $p_n(x)$ (by, possibly, using `Rate` or `GUESS`), and, by interpolation, are able to write down a guess for $p_n(x)$ itself.

Let us see how this works for our sequence (A.2). A few experiments will convince you that $p_n(x)$, $0 \le n \le 9$ (this is all we have), appears to be "nice" for $x = 0, 1, \dots, n$. Furthermore, using `Rate` or `GUESS`, you will quickly find that, apparently, $p_n(e) = \binom{2n+e}{e}$ for $e = 0, 1, \dots, n$. Therefore, as it also appears to be the case that $p_n(x)$ is of degree n, our sequence of polynomials should be given (using Lagrange interpolation) by

$$p_n(x) = \sum_{e=0}^{n} \binom{2n+e}{e} \frac{x(x-1)\cdots(x-e+1)(x-e-1)\cdots(x-n)}{e(e-1)\cdots 1\cdot(-1)\cdots(e-n)}. \tag{A.3}$$

Another useful idea is to try to *expand your polynomials with respect to a "suitable" basis.* (For example, this is what helped to guess coefficients in [30] or [94, e.g., (3.15), (3.33)].) Now, of course, you do not know beforehand what "suitable" could be in your situation. Within the "hypergeometric paradigm" candidates for a suitable basis are always more or less sophisticated shifted factorials. So, let us suppose that we know that we were working within the "hypergeometric paradigm" when we came across the example (A.2). Then the simplest possible bases are $(x)_k$, $k = 0, 1, \dots$, or $(-x)_k$, $k = 0, 1, \dots$. It is just a matter of taste, which of these to try first. Let us try the latter. Here are the expansions of $p_3(x)$ and $p_4(x)$ in terms of this basis (actually, in terms of the equivalent basis $\binom{x}{k}$, $k = 0, 1, \dots$):

$$
\begin{aligned}
\tfrac{1}{6}\big(6+31x-15x^2+20x^3\big) &= 1 + 6\tbinom{x}{1} + 15\tbinom{x}{2} + 20\tbinom{x}{3},\\
\tfrac{1}{12}\big(12-58x+217x^2-98x^3+35x^4\big) &= 1 + 8\tbinom{x}{1} + 28\tbinom{x}{2} + 56\tbinom{x}{3} + 70\tbinom{x}{4}.
\end{aligned}
$$

I do not know how you feel. For me this is enough to guess that, apparently,

$$p_n(x) = \sum_{k=0}^{n} \binom{2n}{k}\binom{x}{k}.$$

(Although this is not the same expression as in (A.3), it is identical by means of a ${}_3F_2$-transformation due to Thomae, see [55, (3.1.1)]).

As was said before, we do not know beforehand what a "suitable" basis is. Therefore you are advised to get as much a priori information about your polynomials as possible. For example, in [28] it was "known" to the authors that the result which they wanted to guess (before being able to think about a proof) is of the form (NICE PRODUCT) $\times$ (IRREDUCIBLE POLYNOMIAL). (I.e., experiments indicated that.) Moreover, they knew that their (IRREDUCIBLE POLYNOMIAL), a polynomial in m, $p_n(m)$ say, would have the property $p_n(-m-2n+1) = p_n(m)$. Now, if we are asking ourselves what a "suitable" basis could be that has this property as well, and which is built in the way of shifted factorials, then the most obvious candidate is $(m+n-k)_{2k} = (m+n-k)(m+n-k+1)\cdots(m+n+k-1)$, $k = 0, 1, \dots$. Indeed, it was very easy to guess the expansion coefficients with respect to this basis. (See Theorems 1 and 2 in [28]. The polynomials that I was talking about are represented by the expression in big parentheses in [28, (1.2)].)

If the above ideas do not help, then I have nothing else to offer than to *try some*, more or less arbitrary, *manipulations*. To illustrate what I could possibly mean, let us again consider an example. In the course of working on [90], I had to guess the result of a determinant evaluation (which became Theorem 8 in [90]; it is reproduced here as Theorem 43). Again, the difficult part of guessing was to guess the "ugly" part of the result. As the dimension of the determinant varied, this gave a certain sequence $p_n(x, y)$ of polynomials in two variables, x and y, of which I display $p_4(x, y)$:

```
In[1]:= VPol[4]

                     2       3     4                         2         3          2
Out[1]= 6 x + 11 x  + 6 x  + x  + 6 y - 10 x y - 6 x  y - 4 x  y + 11 y  -

              2       2  2        3         3     4
>    6 x y  + 6 x  y  + 6 y  - 4 x y  + y
```

(What I subsequently describe is the actual way in which the expression for $p_n(x, y)$ in terms of the sum on the right-hand side of (3.38) was found.) What caught my eyes was the part of the polynomial independent of y, $x^4 + 6x^3 + 11x^2 + 6x$, which I recognized as $(x)_4 = x(x+1)(x+2)(x+3)$. For the fun of it, I subtracted that, just to see what would happen:

```
In[2]:= Factor[%-x(x+1)(x+2)(x+3)]

                          2       3                         2         2           2
Out[2]= y (6 - 10 x - 6 x  - 4 x  + 11 y - 6 x y + 6 x  y + 6 y  - 4 x y  +

          3
>        y )
```

Of course, a y factors. Okay, let us cancel that:

```
In[3]:= %/y

                      2       3                         2         2         2    3
Out[3]= 6 - 10 x - 6 x  - 4 x  + 11 y - 6 x y + 6 x  y + 6 y  - 4 x y  + y
```

One day I had the idea to continue in a "systematic" manner: Let us subtract/add an appropriate multiple of $(x)_3$! Perhaps, "appropriate" in this context is to add $4(x)_3$, because that does at least cancel the third powers of x:

```
In[4]:= Factor[%+4x(x+1)(x+2)]

                                2                     2
Out[4]= (1 + y) (6 - 2 x + 6 x  + 5 y - 4 x y + y )
```

I assume that I do not have to comment the rest:

```
In[5]:= %/(y+1)

                      2
Out[5]= 6 - 2 x + 6 x  + 5 y - 4 x y + y

In[6]:= Factor[%-6x(x+1)]

Out[6]= (2 + y) (3 - 4 x + y)

In[7]:= %/(y+2)

Out[7]= 3 - 4 x + y

In[8]:= Factor[%+4x]

Out[8]= 3 + y
```

What this shows is that

$$p_4(x,y) = (x)_4 - 4(x)_3\,(y)_1 + 6(x)_2\,(y)_2 - 4(x)_1\,(y)_3 + (y)_4.$$

No doubt that, at this point, you would have immediately guessed (as I did) that, in general, we "must" have (compare (3.38))

$$p_n(x,y) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x)_k\,(y)_{n-k}.$$

Appendix B: Turnbull's polarization of Bazin's theorem implies most of the identities in Section 2.2

In this appendix we show that all the determinant lemmas from Section 2.2, with the exception of Lemmas 8 and 9, follow from the evaluation of a certain determinant of minors of a given matrix, an observation which I owe to Alain Lascoux. This evaluation, due to Turnbull [179, p. 505], is a polarized version of a theorem of Bazin [119, II, pp. 206–208] (see also [102, Sec. 3.1 and 3.4]).

For the statement of Turnbull's theorem we have to fix an n-rowed matrix A, in which we label the columns, slightly unconventionally, by $a_2,\dots,a_m, b_{21}, b_{31}, b_{32}, b_{41},\dots,b_{n,n-1}$, $x_1,x_2,\dots,x_n$, for some $m\ge n$, i.e., A is an $n\times(n+m-1+\binom{n}{2})$ matrix. Finally, let $[a,b,c,\dots]$ denote the minor formed by concatenating columns $a,b,c,\dots$ of A, in that order.

Proposition 59. (Cf. [179, p. 505], [102, Sec. 3.4]). *With the notation as explained above, there holds*

$$\det_{1\le i,j\le n}\big([b_{j,1},b_{j,2},\dots,b_{j,j-1},x_i,a_{j+1},\dots,a_m]\big)\\ =[x_1,x_2,\dots,x_n,a_{n+1},\dots,a_m]\prod_{j=2}^{n}[b_{j,1},b_{j,2},\dots,b_{j,j-1},a_j,\dots,a_m]. \tag{B.1}$$

□

Now, in order to derive Lemma 3 from (B.1), we choose $m=n$ and for A the matrix

$$\begin{array}{c}
\begin{array}{ccccccccccccc} a_2 & \dots & a_n & b_{21} & b_{31} & b_{32} & \dots & b_{n,n-1} & x_1 & x_2 & \dots & x_n \end{array}\\
\begin{pmatrix}
1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1\\
-A_2 & \dots & -A_n & -B_2 & -B_2 & -B_3 & \dots & -B_n & X_1 & X_2 & \dots & X_n\\
(-A_2)^2 & \dots & (-A_n)^2 & (-B_2)^2 & (-B_2)^2 & (-B_3)^2 & \dots & (-B_n)^2 & X_1^2 & X_2^2 & \dots & X_n^2\\
\hdotsfor{12}\\
(-A_2)^{n-1} & \dots & (-A_n)^{n-1} & (-B_2)^{n-1} & (-B_2)^{n-1} & (-B_3)^{n-1} & \dots & (-B_n)^{n-1} & X_1^{n-1} & X_2^{n-1} & \dots & X_n^{n-1}
\end{pmatrix}
\end{array},$$

with the unconventional labelling of the columns indicated on top. I.e., column b_{st} is filled with powers of $-B_{t+1}$, $1\le t<s\le n$. With this choice of A, all the minors in (B.1) are Vandermonde determinants. In particular, due to the Vandermonde determinant evaluation (2.1), we then have for the (i,j)-entry of the determinant in (B.1)

$$[b_{j,1},b_{j,2},\dots,b_{j,j-1},x_i,a_{j+1},\dots,a_m]\\ =\prod_{2\le s<t\le j}(B_s-B_t)\prod_{j+1\le s<t\le n}(A_s-A_t)\prod_{s=2}^{j}\prod_{t=j+1}^{n}(A_t-B_s)\\ \times\prod_{s=j+1}^{n}(X_i+A_s)\prod_{s=2}^{j}(X_i+B_s),$$

which is, up to factors that only depend on the column index j, exactly the (i,j)-entry of the determinant in (2.8). Thus, Turnbull's identity (B.1) gives the evaluation (2.8) immediately, after some obvious simplification.

In order to derive Lemma 5 from (B.1), we choose $m=n$ and for A the matrix

$$\begin{array}{c}
\begin{array}{ccccc} a_2 & \dots & a_n & b_{21} & b_{31} \end{array}\\
\left(\begin{matrix}
1 & \dots & 1 & 1 & 1\\
-A_2-C/A_2 & \dots & -A_n-C/A_n & -B_{2,1}-C/B_{2,1} & -B_{3,1}-C/B_{3,1}\\
(-A_2-C/A_2)^2 & \dots & (-A_n-C/A_n)^2 & (-B_{2,1}-C/B_{2,1})^2 & (-B_{3,1}-C/B_{3,1})^2\\
\hdotsfor{5}\\
(-A_2-C/A_2)^{n-1} & \dots & (-A_n-C/A_n)^{n-1} & (-B_{2,1}-C/B_{2,1})^{n-1} & (-B_{3,1}-C/B_{3,1})^{n-1}
\end{matrix}\right.
\end{array}$$

$$\begin{array}{c}
\begin{array}{cccccc} b_{32} & \dots & b_{n,n-1} & x_1 & \dots & x_n \end{array}\\
\left.\begin{matrix}
1 & \dots & 1 & 1 & \dots & 1\\
-B_{3,2}-C/B_{3,2} & \dots & -B_{n,n-1}-C/B_{n,n-1} & X_1+C/X_1 & \dots & X_n+C/X_n\\
(-B_{3,2}-C/B_{3,2})^2 & \dots & (-B_{n,n-1}-C/B_{n,n-1})^2 & (X_1+C/X_1)^2 & \dots & (X_n+C/X_n)^2\\
\hdotsfor{6}\\
(-B_{3,2}-C/B_{3,2})^{n-1} & \dots & (-B_{n,n-1}-C/B_{n,n-1})^{n-1} & (X_1+C/X_1)^{n-1} & \dots & (X_n+C/X_n)^{n-1}
\end{matrix}\right)
\end{array}.$$

(In this display, the first line contains columns $a_2, \dots, b_{31}$ of A, while the second line contains the remaining columns.) Again, with this choice of A, all the minors in (B.1) are Vandermonde determinants. Therefore, by noting that $(S + C/S) - (T + C/T) = (S - T)(C/S - T)/(-T)$, and by writing $p_{j-1}(X)$ for

$$\prod_{s=1}^{j-1} (X + B_{j,s})(C/X + B_{j,s}), \tag{B.2}$$

we have for the (i, j)-entry of the determinant in (B.1)

$$\begin{aligned}
[b_{j,1}, b_{j,2}, \dots, b_{j,j-1}, x_i, a_{j+1}, \dots, a_m] &= \prod_{1\le s<t\le j-1} (B_{j,s} + C/B_{j,s} - B_{j,t} - C/B_{j,t}) \\
\times \prod_{j+1\le s<t\le n} (A_s + C/A_s - A_t - C/A_t) &\prod_{s=1}^{j-1} \prod_{t=j+1}^{n} (A_t + C/A_t - B_{j,s} - C/B_{j,s}) \\
&\times \left(\prod_{s=j+1}^{n} (X_i + A_s)(C/X_i + A_s)\, A_s^{-1} \right) p_{j-1}(X_i) \prod_{s=1}^{j-1} B_{j,s}^{-1}
\end{aligned}$$

for the (i, j)-entry of the determinant in (B.1). This is, up to factors which depend only on the column index j, exactly the (i, j)-entry of the determinant in (2.11). The polynomials $p_{j-1}(X)$, $j = 1, 2, \dots, n$, can indeed be regarded as arbitrary Laurent polynomials satisfying the conditions of Lemma 5, because any Laurent polynomial $q_{j-1}(X)$ over the complex numbers of degree at most $j - 1$ and with $q_{j-1}(X) = q_{j-1}(C/X)$ can be written in the form (B.2). Thus, Turnbull's identity (B.1) implies the evaluation (2.11) as well.

Similar choices for A are possible in order to derive Lemmas 4, 6 and 7 (which are in fact just limiting cases of Lemma 5) from Proposition 59.

Appendix C: Jean-Yves Thibon's proof of Theorem 56

Obviously, the determinant in (3.64) is the determinant of the linear operator $K_n(q) := \sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{maj}\sigma}\sigma$ acting on the group algebra $\mathbb{C}[\mathfrak{S}_n]$ of the symmetric group. Thus, if we are able to determine all the eigenvalues of this operator, together with their multiplicities, we will be done. The determinant is then just the product of all the eigenvalues (with multiplicities).

The operator $K_n(q)$ is also an element of Solomon's descent algebra (because permutations with the same descent set must necessarily have the same major index). The descent algebra is canonically isomorphic to the algebra of noncommutative symmetric functions (see [56, Sec. 5]). It is shown in [95, Prop. 6.3] that, as a noncommutative symmetric function, $K_n(q)$ is equal to $(q; q)_n\, S_n(A/(1 - q))$, where $S_n(B)$ denotes the complete (noncommutative) symmetric function of degree n of some alphabet B.

The inverse element of $S_n(A/(1 - q))$ happens to be $S_n((1 - q)A)$, i.e., $S_n((1 - q)A) * S_n(A/(1-q)) = S_n(A)$,[13] with $*$ denoting the internal multiplication of noncommutative symmetric functions (corresponding to the multiplication in the descent algebra). This is seen as follows. As in [95, Sec. 2.1] let us write $\sigma(B; t) = \sum_{n\ge0} S_n(B)t^n$ for the

[13]By definition of the isomorphism between noncommutative symmetric functions and elements in the descent algebra, $S_n(A)$ corresponds to the identity element in the descent algebra of $\mathfrak{S}_n$.

generating function for complete symmetric functions of some alphabet B, and $\lambda(B;t) = \sum_{n\ge0} \Lambda_n(B)t^n$ for the generating function for elementary symmetric functions, which are related by $\lambda(B;-t)\sigma(B;t) = 1$. Then, by [95, Def. 4.7 and Prop. 4.15], we have $\sigma((1-q)B;1) = \lambda(B;-q)\sigma(B;1)$. Let X be the ordered alphabet $\cdots < q^2 < q < 1$, so that $XA = A/(1-q)$. According to [95, Theorem 4.17], it then follows that

$$\begin{aligned}\sigma((1-q)A;1) * \sigma(XA;1) = \sigma((1-q)XA;1) &= \lambda(XA;-q)\sigma(XA;1)\\ &= \lambda(XA;-q)\sigma(XA;q)\sigma(A;1) = \sigma(A;1),\end{aligned}$$

since by definition of X, $\sigma(XA;1)$ is equal to $\sigma(XA;q)\sigma(A;1)$ (see [95, Def. 6.1]). Therefore, $S_n((1-q)A) * S_n(XA) = S_n(A)$, as required.

Hence, we infer that $K_n(q)$ is the inverse of $S_n((1-q)A)/(q;q)_n$.

The eigenvalues of $S_n((1-q)A)$ are given in [95, Lemma 5.13]. Their multiplicities follow from a combination of Theorem 5.14 and Theorem 3.24 in [95], since the construction in Sec. 3.4 of [95] yields idempotents e_μ such that the commutative immage of $\alpha(e_\mu)$ is equal to p_μ/z_μ. Explicitly, the eigenvalues of $S_n((1-q)A)$ are $\prod_{i\ge1}(1-q^{\mu_i})$, where $\mu = (\mu_1, \mu_2, \dots)$ varies through all partitions of n, with corresponding multiplicities $n!/z_\mu$, the number of permutations of cycle type μ, i.e., $z_\mu = 1^{m_1}m_1!\,2^{m_2}m_2!\cdots$, where m_i is the number of occurences of i in the partition μ, $i = 1, 2, \dots$. Hence, the eigenvalues of $K_n(q)$ are $(q;q)_n/\prod_{i\ge1}(1-q^{\mu_i})$, with the same multiplicities.

Knowing all the eigenvalues of $K_n(q)$ and their multiplicities explicitly, it is now not extremely difficult to form the product of all these and, after a short calculation, recover the right-hand side of (3.64).

Acknowledgements

I wish to thank an anonymous referee, Joris Van der Jeugt, Bernard Leclerc, Madan Lal Mehta, Alf van der Poorten, Volker Strehl, Jean-Yves Thibon, Alexander Varchenko, and especially Alain Lascoux, for the many useful comments and discussions which helped to improve the contents of this paper considerably.

References

(At the end of each reference, it is indicated on which page(s) of this article the reference is cited, including multiple occurences.)

[1] W. A. Al-Salam and L. Carlitz, *Some determinants of Bernoulli, Euler and related numbers*, Portugaliae Math. **18** (1959), 91–99. (p. 23, 47)

[2] T. Amdeberhan, *Lewis strikes again, and again!*, unpublished manuscript, available at `http://www.math.temple.edu/~tewodros/programs/kratdet.html` and `http://www.math.temple.edu/~tewodros/programs/qkratdet.html`. (p. 12, 39, 40)

[3] G. E. Andrews, *Plane partitions (II): The equivalence of the Bender–Knuth and the MacMahon conjectures*, Pacific J. Math. **72** (1977), 283–291. (p. 34)

[4] G. E. Andrews, *Plane partitions (I): The MacMahon conjecture*, in: Studies in Foundations and Combinatorics, G.-C. Rota, ed., Adv. in Math. Suppl. Studies, vol. 1, 1978, 131–150. (p. 19, 33, 34)

[5] G. E. Andrews, *Plane partitions (III): The weak Macdonald conjecture*, Invent. Math. **53** (1979), 193–225. (p. 19, 34, 36)

[6] G. E. Andrews, *Macdonald's conjecture and descending plane partitions*, in: Combinatorics, representation theory and statistical methods in groups, Young Day Proceedings, T. V. Narayana, R. M. Mathsen, J. G. Williams, eds., Lecture Notes in Pure Math., vol. 57, Marcel Dekker, New York, Basel, 1980, 91–106. (p. 19, 34, 35)

[7] G. E. Andrews, *Plane partitions (IV): A conjecture of Mills–Robbins–Rumsey*, Aequationes Math. **33** (1987), 230–250. (p. 19, 38)

[8] G. E. Andrews, *Plane partitions (V): The t.s.s.c.p.p. conjecture*, J. Combin. Theory Ser. A **66** (1994), 28–39. (p. 19, 40)

[9] G. E. Andrews and W. H. Burge, *Determinant identities*, Pacific J. Math. **158** (1993), 1–14. (p. 37, 38, 38)

[10] G. E. Andrews and D. W. Stanton, *Determinants in plane partition enumeration*, Europ. J. Combin. **19** (1998), 273–282. (p. 18, 19, 36, 38, 38, 40)

[11] K. Aomoto and Y. Kato, *Derivation of q-difference equation from connection matrix for Selberg type Jackson integrals*, J. Difference Equ. Appl. **4** (1998), 247–278. (p. 19, 51)

[12] R. Askey, *Continuous Hahn polynomials*, J. Phys. A – Math. Gen. **18** (1985), L1017–L1019. (p. 21)

[13] N. M. Atakishiyev and S. K. Suslov, *The Hahn and Meixner polynomials of an imaginary argument and some of their applications*, J. Phys. A – Math. Gen. **18** (1985), 1583–1596. (p. 21)

[14] H. Au-Yang and J. H. H. Perk, *Critical correlations in a Z-invariant inhomogeneous Ising model*, Physica **144A** (1987), 44–104. (p. 47)

[15] E. L. Basor and P. J. Forrester, *Formulas for the evaluation of Toeplitz determinants with rational generating functions*, Mathematische Nachrichten **170** (1994), 5–18. (p. 6)

[16] E. Bombieri, D. C. Hunt and A. J. van der Poorten, *Determinants in the study of Thue's method and curves with prescribed singularities*, Experimental Math. **4** (1995), 87–96. (p. 44, 44, 44, 45)

[17] C. W. Borchardt, *Bestimmung der symmetrischen Verbindungen ihrer erzeugenden Funktion*, Crelle J. **53** (1855), 193–198. (p. 29)

[18] D. M. Bressoud, *Proofs and confirmations — The story of the alternating sign matrix conjecture*, Cambridge University Press, Cambridge, 1999. (p. 11, 29, 52)

[19] D. M. Bressoud and J. Propp, *The proofs of the alternating sign matrix conjecture*, Notices Amer. Math. Soc. (to appear). (p. 52)

[20] T. Brylawski and A. Varchenko, *The determinant formula for a matroid bilinear form*, Adv. in Math. **129** (1997), 1–24. (p. 49, 49)

[21] M. W. Buck, R. A. Coley and D. P. Robbins, *A generalized Vandermonde determinant*, J. Alg. Combin. **1** (1992), 105–109. (p. 29)

[22] L. Carlitz, *Some determinants of q-binomial coefficients*, J. Reine Angew. Math. **226** (1967), 216–220. (p. 31)

[23] W. C. Chu, *Binomial convolutions and determinant identities*, Discrete Math. (Gould Anniversary Volume) (1999), (to appear). (p. 38, 38)

[24] F. Chyzak, *Holonomic systems and automatic proofs of identities*, INRIA Research Report no. 2371, 61 pp, 1994. (p. 3)

[25] M. Ciucu, *Plane partitions I: A generalization of MacMahon's formula*, preprint, `math/9808017`. (p. 31)

[26] M. Ciucu, *Enumeration of perfect matchings in graphs with reflective symmetry*, J. Combin. Theory Ser. A **77** (1997), 67-97. (p. 39)

[27] M. Ciucu, T. Eisenkölbl, C. Krattenthaler and D. Zare, *Enumeration of lozenge tilings of hexagons with a central triangular hole*, in preparation. (p. 15, 31, 36)

[28] M. Ciucu and C. Krattenthaler, *The number of centered lozenge tilings of a symmetric hexagon*, J. Combin. Theory Ser. A **86** (1999), 103–126. (p. 16, 55, 55, 55)

[29] M. Ciucu and C. Krattenthaler, *Plane partitions II: $5\frac{1}{2}$ symmetry classes*, preprint, `math/9808018`. (p. 38, 39)

[30] M. Ciucu and C. Krattenthaler, *Enumeration of lozenge tilings of hexagons with cut off corners*, in preparation. (p. 15, 43, 43, 54)

[31] A. J. Coleman and L. B. Cao, *Factoring the Shapovalov determinant, II*, J. Algebra **187** (1997), 373–387. (p. 49)

[32] T. L. Curtright and C. B. Thorn, *Symmetry patterns in the mass spectra of dual string models*, Nuclear Phys. **B274** (1986), 520–558. (p. 47)

[33] R. Dahab, *The Birkhoff–Lewis equations*, Ph.D. Dissertation, University of Waterloo, 1993. (p. 50, 50)

[34] G. David and C. Tomei, *The problem of the calissons*, Amer. Math. Monthly **96** (1989), 429–431. (p. 31)

[35] Ph. Delsarte, *Nombres de Bell et polynômes de Charlier*, Comptes Rendus Acad. Sciences Paris Sér. A **287** (1978), 271–273. (p. 23, 47)

[36] G. Denham and Ph. Hanlon, *On the Smith normal form of the Varchenko bilinear form of a hyperplane arrangement*, Pacific J. Math. (Olga Taussky–Todd memorial issue) (1997), 123–146. (p. 49)

[37] G. Denham and Ph. Hanlon, *Some algebraic properties of the Schechtman–Varchenko bilinear form*, MSRI preprint No. 1998–068. (p. 49)

[38] C. L. Dodgson, *Condensation of determinants*, Proc. Royal Soc. London **15** (1866), 150–155. (p. 11)

[39] A. Douai and H. Terao, *The determinant of a hypergeometric period matrix*, Invent. Math. **128** (1997), 417–436. (p. 49)

[40] G. Duchamp, A. Klyachko, D. Krob and J.-Y. Thibon, *Noncommutative symmetric functions III: Deformations of Cauchy and convolution algebras*, Discrete Math. Theoret. Computer Science **1** (1997), 159–216. (p. 49)

[41] T. Eisenkölbl, *Rhombus tilings of a hexagon with three missing border tiles*, J. Combin. Theory Ser. A (to appear). (p. 12)

[42] T. Eisenkölbl, *Rhombus tilings of a hexagon with two triangles missing on the symmetry axis*, preprint, `math/9810019`. (p. 15)

[43] I. Fischer, *Enumeration of rhombus tilings of a hexagon which contain a fixed rhombus in the centre*, preprint. (p. 16, 54)

[44] P. Flajolet, *Combinatorial aspects of continued fractions*, Discrete Math. **32** (1980), 125–161. (p. 48)

[45] R. P. Flowe and G. A. Harris, *A note on generalized Vandermonde determinants*, SIAM J. Matrix Anal. Appl. **14** (1993), 1146–1151. (p. 25, 25, 26)

[46] D. Foata, *On the Netto inversion number of a sequence*, Proc. Amer. Math. Soc. **19** (1968), 236–240. (p. 49)

[47] P. Di Francesco, *Meander determinants*, Comm. Math. Phys. **191** (1998), 543–583. (p. 50, 51, 51)

[48] P. Di Francesco, *$SU(n)$ meander determinants*, J. Math. Phys. **38**, (1997), 5904–5943. (p. 51, 51)

[49] P. Di Francesco, O. Golinelli and E. Guitter, *Meanders and the Temperley–Lieb algebra*, Comm. Math. Phys. **186** (1997), 1–59. (p. 50)

[50] M. Fulmek and C. Krattenthaler, *The number of rhombus tilings of a symmetric hexagon which contain a fixed rhombus on the symmetry axis, I*, Ann. Combin. 2 (1998), 19–40. (p. 16, 43, 44, 44)

[51] M. Fulmek and C. Krattenthaler, *The number of rhombus tilings of a symmetric hexagon which contain a fixed rhombus on the symmetry axis, II*, preprint. (p. 16, 22, 47, 48)

[52] W. Fulton and J. Harris, *Representation Theory*, Springer–Verlag, New York, 1991. (p. 6, 6, 6, 6, 6, 6, 33, 34)

[53] F. R. Gantmacher, *The theory of matrices*, translated from the Russian, reprinted by Chelsea, New York, 1959. (p. 18)

[54] F. Garvan, *A q-product tutorial for a Maple q-series package*, Séminaire Lotharingien Combin. **42** ("The Andrews Festschrift") (1999), paper B42d, 27 pp. (p. 51)

[55] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics And Its Applications 35, Cambridge University Press, Cambridge, 1990. (p. 25, 55)

[56] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. in Math. **112** (1995), 218–348. (p. 58)

[57] I. M. Gessel and X. Viennot, *Determinants, paths, and plane partitions*, preprint, 1989. (p. 31, 31, 31)

[58] B. Gordon, *Notes on plane partitions V*, J. Combin. Theory **11** (1971), 157–168. (p. 24, 24)

[59] B. Gordon, *A proof of the Bender–Knuth conjecture*, Pacific J. Math. **108** (1983), 99–113. (p. 34)

[60] I. P. Goulden and D. M. Jackson, *Ballot sequences and a determinant of Good's*, J. Combin. Theory Ser. A **40** (1985), 55–61. (p. 51)

[61] I. P. Goulden and D. M. Jackson, *Further determinants with the averaging property of Andrews-Burge*, J. Combin. Theory Ser. A **73** (1996), 368–375. (p. 24, 24, 38)

[62] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison Wesley, Reading, Massachusetts, 1988. (p. 25, 37)

[63] R. A. Gustafson and C. Krattenthaler, *Determinant evaluations and $U(n)$ extensions of Heine's ${}_2\phi_1$-transformations*, Special Functions, q-Series and Related Topics, M. E. H. Ismail, D. R. Masson and M. Rahman, eds., Fields Institute Communications, vol. 14, Amer. Math. Soc., Providence, R. I., 83–90, 1997. (p. 31)

[64] G.-N. Han, A. Randrianarivony and J. Zeng, *Un autre q-analogue des nombres d'Euler*, Séminaire Lotharingien Combin. **42** ("The Andrews Festschrift") (1999), paper B42e, 22 pp. (p. 48)

[65] G.-N. Han and J. Zeng, *q-Polynômes de Ghandi et statistique de Denert*, Discrete Math. (to appear). (p. 48)

[66] B. R. Handa and S. G. Mohanty, *Enumeration of higher-dimensional paths under restrictions*, Discrete Math. **26** (1979), 119–128. (p. 31)

[67] Ph. Hanlon and R. P. Stanley, *A q-deformation of a trivial symmetric group action*, Trans. Amer. Math. Soc. **350** (1999), 4445–4459. (p. 49)

[68] H. Helfgott and I. M. Gessel, *Exact enumeration of tilings of diamonds and hexagons with defects*, Electron. J. Combin. **6** (1) (1999), #R16, 26 pp. (p. 48)

[69] M. Ishikawa and M. Wakayama, *Minor summation formula for pfaffians*, Linear and Multilinear Algebra **39** (1995), 285–305. (p. 24)

[70] M. E. H. Ismail and D. R. Masson, *Some continued fractions related to elliptic functions*, Contemporary Math. (to appear). (p. 48)

[71] M. E. H. Ismail and D. Stanton, *Classical orthogonal polynomials as moments*, Can. J. Math. **49** (1997), 520–542. (p. 47, 47)

[72] M. E. H. Ismail and D. Stanton, *More orthogonal polynomials as moments*, in: Mathematical Essays in Honor of Gian-Carlo Rota, B. E. Sagan, R. P. Stanley, eds., Progress in Math., vol. 161, Birkhäuser, Boston, 1998, pp. 377–396. (p. 47)

[73] M. E. H. Ismail and G. Valent, *On a family of orthogonal polynomials related to elliptic functions*, Illinois J. Math. **42** (1998), 294–312. (p. 48)

[74] A. G. Izergin, *Partition function of the six-vertex model in a finite volume*, Soviet Phys. Dokl. **32** (1987), 878–879. (p. 29)

[75] D. M. Jackson, *The lattice of noncrossing partitions and the Birkhoff–Lewis equations*, Europ. J. Combin. **15** (1994), 245–250. (p. 50)

[76] W. B. Jones and W. J. Thron, *Continued Fractions*, Addison–Wesley, Reading, Massachusetts, 1980. (p. 20)

[77] S. Karlin and G. Szegő, *On certain determinants whose elements are orthogonal polynomials*, Journal d'Analyse Math. **8** (1961), 1–157. (p. 47, 47)

[78] K. S. Kedlaya, *Another combinatorial determinant*, preprint, `math/9810127`. (p. 23)

[79] D. E. Knuth, *Overlapping pfaffians*, Electron. J. Combin. **3** (no. 2, "The Foata Festschrift") (1996), paper #R5, 13 pp. (p. 11, 24)

[80] K. H. Ko and L. Smolinsky, *A combinatorial matrix in 3-manifold theory*, Pacific J. Math. **149** (1991), 319–336. (p. 50)

[81] R. Koekoek and R. F. Swarttouw, *The Askey–scheme of hypergeometric orthogonal polynomials and its q-analogue*, TU Delft, The Netherlands, 1998; on the WWW: `ftp://ftp.twi.tudelft.nl/TWI/publications/tech-reports/1998/DUT-TWI-98-17.ps.gz`. (p. 20, 21, 21, 47)

[82] V. Korepin, *Calculation of norms of Bethe wave functions*, Commun. Math. Phys. **86** (1982), 391–418. (p. 29)

[83] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, Cambridge, 1993. (p. 29)

[84] H. Krämer, *Eigenspace decompositions with respect to symmetrized incidence mappings*, Séminaire Lotharingien Combin. **41** (1998), paper B41c, 20 pp. (p. 51)

[85] C. Krattenthaler, *Generating functions for plane partitions of a given shape*, Manuscripta Math. **69**, (1990), 173–202. (p. 7, 8, 31, 31, 31)

[86] C. Krattenthaler, *A determinant evaluation and some enumeration results for plane partitions*, in: Number-Theoretic Analysis, E. Hlawka, R. F. Tichy, eds., Lect. Notes in Math. 1452, Springer-Verlag, Berlin, 1990. (p. 31, 31)

[87] C. Krattenthaler, *Generating functions for shifted plane partitions*, J. Statist. Plann. Inference **34** (1993), 197–208. (p. 8, 31, 32)

[88] C. Krattenthaler, *The major counting of nonintersecting lattice paths and generating functions for tableaux*, Mem. Amer. Math. Soc. 115, no. 552, Providence, R. I., 1995. (p. 9, 33, 34)

[89] C. Krattenthaler, *Some q-analogues of determinant identities which arose in plane partition enumeration*, Séminaire Lotharingien Combin. **36**, (1996), paper B36e, 23 pp. (p. 15, 16, 40, 41)

[90] C. Krattenthaler, *Determinant identities and a generalization of the number of totally symmetric self-complementary plane partitions*, Electron. J. Combin. **4**(1) (1997), #R27, 62 pp. (p. 15, 16, 39, 40, 41, 41, 55, 55)

[91] C. Krattenthaler, *An alternative evaluation of the Andrews–Burge determinant*, in: Mathematical Essays in Honor of Gian-Carlo Rota, B. E. Sagan, R. P. Stanley, eds., Progress in Math., vol. 161, Birkhäuser, Boston, 1998, pp. 263–270. (p. 13, 14, 15, 38)

[92] C. Krattenthaler, *A new proof of the M–R–R conjecture — including a generalization*, J. Difference Equ. Appl. (to appear). (p. 15, 36, 38, 38)

[93] C. Krattenthaler and P. Slater, *Asymptotic redundancies for universal quantum coding*, preprint, `quant-ph/9612043`. (p. 51)

[94] C. Krattenthaler and D. Zeilberger, *Proof of a determinant evaluation conjectured by Bombieri, Hunt and van der Poorten*, New York J. Math. **3** (1997), 54–102. (p. 15, 45, 45, 45, 45, 54)

[95] D. Krob, B. Leclerc and J.-Y. Thibon, *Noncommutative symmetric functions II: Transformations of alphabets*, Int. J. Algebra Comput. **7** (1997), 181–264. (p. 49, 58, 58, 59, 59, 59, 59, 59, 59)

[96] G. Kuperberg, *Symmetries of plane partitions and the permanent determinant method*, J. Combin. Theory Ser. A **68** (1994), 115–151. (p. 40)

[97] G. Kuperberg, *Another proof of the alternating sign matrix conjecture*, Math. Research Lett. **3** (1996), 139–150. (p. 15, 29, 29, 30, 52)

[98] D. Laksov, A. Thorup and A. Lascoux, *On Giambelli's theorem on complete correlations*, Acta Math. **162** (1989), 163–199. (p. 24)

[99] T. Y. Lam, *Representations of finite groups: A hundred years, Part I*, Notices Amer. Math. Soc. **45** (1998), 361–372. (p. 23)

[100] A. Lascoux, *Interpolation*, preprint. (p. 26, 27, 27)
[101] A. Lascoux, *Square ice enumeration*, Séminaire Lotharingien Combin. **42** ("The Andrews Festschrift") (1999) (to appear). (p. 30)
[102] B. Leclerc, *On identities satisfied by minors of a matrix*, Adv. in Math. **100** (1993), 101–132. (p. 56, 57)
[103] B. Leclerc, *On certain formulas of Karlin and Szegő*, Séminaire Lotharingien Combin. **41** (1998), paper B41d, 21 pp. (p. 47, 47)
[104] W. B. R. Lickorish, *Invariants for 3-manifold from the combinatorics of the Jones polynomial*, Pacific J. Math. **149** (1991), 337–347. (p. 50)
[105] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Oxford University Press, New York/London, 1995. (p. 6, 6, 31)
[106] P. A. MacMahon, *Combinatory Analysis*, vol. 2, Cambridge University Press, 1916; reprinted by Chelsea, New York, 1960. (p. 31)
[107] M. L. Mehta, *Random Matrices and the statistical theory of energy levels*, Academic Press, New York, 1967. (p. 25, 25, 25)
[108] M. L. Mehta, *Matrix theory*, Editions de Physique, Orsay, France, 1989. (p. 20, 20, 25, 25, 47, 47, 48)
[109] St. Meljanac and D. Svrtan, *Study of Gram matrices in Fock representation of multiparametric canonical commutation relations, extended Zagier's conjecture, hyperplane arrangements and quantum groups*, Math. Commun. **1** (1996), 1–24. (p. 49)
[110] W. H. Mills, D. H. Robbins and H. Rumsey, *Proof of the Macdonald conjecture*, Invent. Math. **66** (1982), 73–87. (p. 35, 35, 38)
[111] W. H. Mills, D. H. Robbins and H. Rumsey, *Alternating sign matrices and descending plane partitions*, J. Combin. Theory Ser. A **34** (1983), 340–359. (p. 30, 52)
[112] W. H. Mills, D. H. Robbins and H. Rumsey, *Enumeration of a symmetry class of plane partitions*, Discrete Math. **67** (1987), 43–55. (p. 2, 36, 37, 38, 38, 40)
[113] S. C. Milne, *New infinite families of exact sums of squares formulas, Jacobi elliptic functions, and Ramanujan's tau function*, Proc. Natl. Acad. Sci. USA **93** (1996), 15004–15008. (p. 23, 48, 48)
[114] S. C. Milne, *Infinite families of exact sums of squares formulas, Jacobi elliptic functions, continued fractions, and Schur functions*, Adv. in Math. (to appear). (p. 23, 48, 48)
[115] S. C. Milne, *Continued fractions, Hankel determinants, and further identities for powers of classical theta functions*, preprint. (p. 23, 48)
[116] S. C. Milne, *Hankel determinants of Eisenstein series*, preprint. (p. 23, 48)
[117] S. C. Milne, *Some extensions of Cauchy's determinant*, preprint. (p. 29)
[118] K. Mimachi, *Holonomic q-difference system of the 1st order associated with a Jackson integral of Selberg type*, Duke Math. J. **73** (1994), 453–468. (p. 19, 51)
[119] T. Muir, *The theory of determinants in the historical order of development*, 4 vols., Macmillan, London, 1906–1923. (p. 2, 2, 6, 11, 56)
[120] T. Muir, *Contributions to the history of determinants, 1900–1920*, Blackie & Son, London and Glasgow, 1930. (p. 2)
[121] T. Muir, *A treatise on the theory of determinants*, Rev. and enlarged by W. H. Metzler, Dover, New York, 1960. (p. 2, 22)
[122] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer, Berlin, 1924. (p. 22, 47)
[123] S. Okada, *On the generating functions for certain classes of plane partitions*, J. Combin. Theory A **51** (1989), 1–23. (p. 24, 42, 42)
[124] S. Okada, *Applications of minor summation formulas to rectangular-shaped representations of classical groups*, J. Algebra **205** (1998), 337–368. (p. 6)
[125] S. Okada and C. Krattenthaler, *The number of rhombus tilings of a "punctured" hexagon and the minor summation formula*, Adv. Appl. Math. **21** (1998), 381–404. (p. 6)
[126] A. M. Ostrowski, *On some determinants with combinatorial numbers*, J. Reine Angew. Math. **216** (1964), 25–30. (p. 2)
[127] E. Pascal, *Die Determinanten*, Teubner, Leipzig, 1900. (p. 2)
[128] O. Perron, *Die Lehre von den Kettenbrüchen*, B. G. Teubner, Stuttgart, 1977. (p. 20)

[129] M. Petkovšek and H. Wilf, *A high-tech proof of the Mills-Robbins-Rumsey determinant formula*, Electron. J. Combin. **3** (no. 2, "The Foata Festschrift") (1996), #R19, 3 pp. (p. 38)
[130] M. Petkovšek, H. Wilf and D. Zeilberger, *A=B*, A. K. Peters, Wellesley, 1996. (p. 3, 15)
[131] A. J. van der Poorten, *Some determinants that should be better known*, J. Austral. Math. Soc. **21** (Ser. A) (1976), 278–288. (p. 23, 26)
[132] A. J. van der Poorten, *A powerful determinant*, preprint. (p. 15, 44, 45, 45, 46, 46)
[133] R. A. Proctor, unpublished research announcement, 1984. (p. 31)
[134] R. A. Proctor, *Odd symplectic groups*, Invent. Math. **92** (1988), 307–332. (p. 32)
[135] R. A. Proctor, *New symmetric plane partition identities from invariant theory work of DeConcini and Procesi*, Europ. J. Combin. **11** (1990), 289–300. (p. 32)
[136] R. A. Proctor, *Product evaluations of Lefschetz determinants for Grassmannians and of determinants of multinomial coefficients*, J. Combin. Theory Ser. A **54** (1990), 235–247. (p. 51)
[137] R. A. Proctor, *Young tableaux, Gelfand patterns, and branching rules for classical groups*, J. Algebra **164** (1994), 299–360. (p. 6, 6)
[138] J. Propp, *Twenty open problems on enumeration of matchings*, manuscript, 1996, `math/9801060`. (p. 12, 25)
[139] C. Radoux, *Déterminant de Hankel construit sur les polynômes de Hérmite*, Ann. Soc. Sci. Brux., Ser. I **104** (1990), 59–61. (p. 47)
[140] C. Radoux, *Déterminant de Hankel construit sur des polynômes liés aux nombres de dérangements*, Europ. J. Combin. **12** (1991), 327–329. (p. 47)
[141] C. Radoux, *Déterminants de Hankel et théorème de Sylvester*, in: Actes de la 28e session du Séminaire Lotharingien de Combinatoire, publication de l'I.R.M.A. no 498/S–28, Strasbourg, 1992, pp 115–122. (p. 47)
[142] A. Randrianarivony, *Fractions continues, combinatoire et extensions de nombres classiques*, Ph.D. thesis, Univ. Louis Pasteur, Strasbourg, France, 1994. (p. 48)
[143] A. Randrianarivony, *Fractions continues, q-nombres de Catalan et q-polynômes de Genocchi*, Europ. J. Combin. **18** (1997), 75–92. (p. 48)
[144] A. Randrianarivony, *q,p-analogue des nombres de Catalan*, Discrete Math. **178** (1998), 199–211. (p. 48)
[145] A. Randrianarivony and J. A. Zeng, *Extension of Euler numbers and records of up-down permutations*, J. Combin. Theory Ser. A **68** (1994), 86–99. (p. 48)
[146] A. Randrianarivony and J. Zeng, *A family of polynomials interpolating several classical series of numbers*, Adv. in Appl. Math. **17** (1996), 1–26. (p. 48)
[147] G. M. Reinhart, *Determinants of partition matrices*, J. Number Theory **56** (1996), 283–297. (p. 51)
[148] D. P. Robbins. *The story of* $1, 2, 7, 42, 429, 7436, \dots$, The Math. Intelligencer **13** (1991), 12–19. (p. 25, 29, 52)
[149] D. P. Robbins, *Solution to problem 10387**, Amer. Math. Monthly **104** (1997), 366–367. (p. 29)
[150] D. P. Robbins and H. Rumsey, *Determinants and alternating sign matrices*, Adv. in Math. **62** (1986), 169–184. (p. 29)
[151] L. J. Rogers, *On the representation of certain asymptotic series as convergent continued fractions*, Proc. London Math. Soc. (2) **4** (1907), 72–89. (p. 48)
[152] B. Salvy and P. Zimmermann, *Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable*, ACM Trans. Math. Software **20** (1994). (p. 3)
[153] V. Schechtman and A. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991), 139–194. (p. 49, 49)
[154] L. Schendel, *Das alternirende Exponentialdifferenzenproduct*, Zeitschrift Math. Phys. (1891) 84–87; see [119], vol. IV, p. 178. (p. 25)
[155] M. Schlosser, *Symmetrieklassen von Plane Partitions*, diploma thesis, University of Vienna, 1993. (p. 33, 33, 34, 34)
[156] M. Schlosser, *Multidimensional matrix inversion and A_r and D_r basic hypergeometric series*, The Ramanujan J. **1** (1997), 243–276. (p. 11)
[157] M. Schlosser, *Summation theorems for multidimensional basic hypergeometric series by determinant evaluations*, Discrete Math. (to appear). (p. 32)

[158] M. Schlosser, *A new multidimensional matrix inversion in* A_r, preprint. (p. 11)

[159] N. N. Shapovalov, *On a bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra*, Funct. Anal. Appl. **6** (1972), 65–70. (p. 49)

[160] R. Simion, *Noncrossing partitions*, Discrete Math. (to appear). (p. 49)

[161] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, Elect. J. Combin. **1** (1994), feature #F1, 5 pp. (p. 51)

[162] N. J. A. Sloane and S. Plouffe, *The encyclopedia of integer sequences*, Academic Press, San Diego, 1995. (p. 51)

[163] R. P. Stanley, *Theory and applications of plane partitions: Parts 1,2*, Stud. Appl. Math **50** (1971), 167–188, 259–279. (p. 25, 31)

[164] R. P. Stanley, *A baker's dozen of conjectures concerning plane partitions*, in: Combinatoire énumérative, G. Labelle, P. Leroux, eds., Lecture Notes in Math. 1234, Springer, Berlin, New York, 1986, pp. 285–293. (p. 42)

[165] R. P. Stanley, *Symmetries of plane partitions*, J. Combin. Theory A **43** (1986), 103–113; Erratum: ibd. **44** (1987), 310. (p. 25, 42)

[166] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, Wadsworth & Brooks/Cole, Pacific Grove, California, 1986; reprinted by Cambridge University Press, Cambridge, 1998. (p. 50, 50)

[167] R. P. Stanley, *Hyperplane arrangements, parking functions and tree inversions*, in: Mathematical Essays in Honor of Gian-Carlo Rota, B. E. Sagan, R. P. Stanley, eds., Progress in Math., vol. 161, Birkhäuser, Boston, 1998, 359–376. (p. 31)

[168] G. P. Steck, *Evaluation of some Steck determinants with applications*, Comm. Statist. **3** (1974), 121–138. (p. 31)

[169] J. R. Stembridge, *Nonintersecting paths, pfaffians and plane partitions*, Adv. in Math. **83** (1990), 96–131. (p. 23, 24, 24, 24, 24, 31)

[170] J. R. Stembridge, *The enumeration of totally symmetric plane partitions*, Adv. in Math. **111** (1995), 227–245. (p. 43)

[171] T. J. Stieltjes, *Sur la réduction en fraction continue d'une série précédent suivant les puissances descendants d'une variable*, Ann. Fac. Sci. Toulouse **3** (1889), **H**, 1–17; Œuvres, Vol. 2, 184–200. (p. 48)

[172] T. J. Stieltjes, *Sur quelques intégrales définies et leur développement en fractions continues*, Quart. J. Math. **24** (1890), 370–382. (p. 48)

[173] V. Strehl and H. S. Wilf, *Five surprisingly simple complexities*, J. Symbolic Comput. **20** (1995), 725–729. (p. 23)

[174] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications Vol. XXIII, New York, 1959. (p. 20)

[175] V. Tarasov and A. Varchenko, *Geometry of q-hypergeometric functions as a bridge between Yangians and quantum affine algebras*, Invent. Math. **128** (1997), 501–588. (p. 18, 49)

[176] V. Tarasov and A. Varchenko, *Geometry of q-hypergeometric functions, quantum affine algebras and elliptic quantum groups*, Astérisque **246** (1997), 1–135. (p. 18, 29, 49)

[177] T. Terasoma, *Determinants of q-hypergeometric functions and another proof of the Askey conjecture*, Math. Zeitschrift **226** (1997), 499–512. (p. 51)

[178] H. W. Turnbull, *The theory of determinants, matrices, and invariants*, Blackie & Son, London and Glasgow, 1929. (p. 2, 47)

[179] H. W. Turnbull, *The general symbolic notation for the principal of duality, and its application to determinants*, Proc. London Math. Soc. (2) **22** (1923), 495–507. (p. 7, 56, 57)

[180] W. T. Tutte, *On the Birkhoff–Lewis equation*, Discrete Math. **92** (1991), 417–425. (p. 50)

[181] W. T. Tutte, *The matrix of chromatic joins*, J. Combin. Theory Ser. B **57** (1993), 269–288. (p. 50)

[182] A. Varchenko, *The Euler beta function, the Vandermonde determinant, Legendre's equation, and critical points of linear function on a configuration of hyperplanes*, I, II, Math. USSR Izv. **35** (1990), 542–571; **36** (1991), 155–168. (p. 29, 49)

[183] A. Varchenko, *Determinant formula for Selberg type integrals*, Funct. Anal. Appl. **4** (1991), 65–66. (p. 49)

[184] A. Varchenko, *Bilinear form of real configuration of hyperplanes*, Adv. in Math. **97** (1993), 110–144. (p. 48, 49)

[185] A. Varchenko, *Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups*, Advanced Series in Math. Phys., vol. 21, World Scientific, 1995. (p. 49)

[186] X. Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux*, UQAM, Montreal, Quebec, 1983. (p. 20, 20, 21, 21)

[187] H. Voigt, *Sur l'apolarité des formes binaires*, Nouv. Annales de Math. **4** (1901), 337–365; see [120], p. 183. (p. 25)

[188] H. S. Wall, *Analytic Theory of Continued Fractions*, Van Nostrand, New York, 1948. (p. 20, 20, 21, 21)

[189] H. S. Wilf, *A combinatorial determinant*, preprint, `math/9809120`. (p. 48)

[190] H. S. Wilf and D. Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Invent. Math. **108** (1992), 575–633. (p. 3, 15)

[191] J. A. Wilson, *Orthogonal functions from Gram determinants*, SIAM J. Math. Anal. **22** (1991), 1147–1155. (p. 32)

[192] R. A. Wilson, *A diagonal form for the incidence matrices of t-subsets versus k-subsets*, Europ. J. Combin. **11** (1990), 609–615. (p. 51)

[193] D. Zagier, *Realizability of a model in infinite statistics*, Comm. Math. Phys. **147** (1992), 199–210. (p. 48, 49)

[194] D. Zeilberger, *A fast algorithm for proving terminating hypergeometric identities*, Discrete Math. **80** (1990), 207–211. (p. 3, 14, 15)

[195] D. Zeilberger, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. **32** (1990), 321–368. (p. 3, 15)

[196] D. Zeilberger, *The method of creative telescoping*, J. Symbolic Comput. **11** (1991), 195–204. (p. 3, 14, 15)

[197] D. Zeilberger, *Reverend Charles to the aid of Major Percy and Fields medalist Enrico*, Amer. Math. Monthly, **103** (1996), 501–502. (p. 12)

[198] D. Zeilberger, *Proof of the alternating sign matrix conjecture*, Electron. J. Combin. **3** (no. 2, "The Foata Festschrift") (1996), #R13, 84 pp. (p. 30, 52)

[199] D. Zeilberger, *Proof of the refined alternating sign matrix conjecture*, New York J. Math. **2** (1996), 59–68. (p. 30, 52)

[200] D. Zeilberger, *Dodgson's determinant-evaluation rule proved by TWO-TIMING MEN and WOMEN*, Electron. J. Combin. **4** (no. 2, "The Wilf Festschrift") (1997), #R22, 2 pp. (p. 12)

[201] J. Zeng, *Sur quelques propriétes de symétrie des nombres de Genocchi*, Discrete Math. **153** (1996), 319–333. (p. 48)

Institut für Mathematik der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria.

e-mail: KRATT@Pap.Univie.Ac.At, WWW: `http://radon.mat.univie.ac.at/People/kratt`