EVALUATION OF BINOMIAL DOUBLE SUMS INVOLVING ABSOLUTE VALUES

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Abstract. We show that double sums of the form

$$\sum_{i,j=-n}^{n} |i^s j^t (i^k - j^k)^\beta| \binom{2n}{n+i} \binom{2n}{n+j}$$

can always be expressed in terms of a linear combination of just four functions, namely $\binom{4n}{2n}$, $\binom{2n}{n}^2$, $4^n\binom{2n}{n}$, and 16^n, with coefficients that are rational in n. We provide two different proofs: one is algorithmic and uses the second author's computer algebra package `Sigma`; the second is based on complex contour integrals. In many instances, these results are extended to double sums of the above form where $\binom{2n}{n+j}$ is replaced by $\binom{2m}{m+j}$ with independent parameter m.

1. Introduction

Motivated by work in [6] concerning the Hadamard maximal determinant problem [10], Brent and Osborn [5] proved the double sum evaluation

$$\sum_{i,j=-n}^{n} |i^2 - j^2| \binom{2n}{n+i} \binom{2n}{n+j} = 2n^2 \binom{2n}{n}^2. \tag{1.1}$$

It should be noted that the difficulty in evaluating this sum lies in the appearance of the absolute value. Without the absolute value, the summand would become antisymmetric in i and j so that the sum would trivially vanish. Together with Ohtsuka and Prodinger, they went on in [3] (see [4] for the published version) to consider more general double sums of the form

$$\sum_{i,j=-n}^{n} |i^s j^t (i^k - j^k)^\beta| \binom{2n}{n+i} \binom{2n}{n+j}, \tag{1.2}$$

mostly for small positive integers s, t, k, β. Again, without the absolute value, the summation would not pose any particular problem since it could be carried out separately in i and j by means of a relatively straightforward application of the binomial theorem.

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In several cases, they found explicit evaluations of such sums — sometimes with proof, sometimes conjecturally.

The purpose of the current paper is to provide a complete treatment of double sums of the form (1.2) and of the more general form

$$\sum_{i,j} |i^s j^t (i^k - j^k)^\beta| \binom{2n}{n+i} \binom{2m}{m+j}, \tag{1.3}$$

with an independent parameter m. More precisely, using the computer algebra package `Sigma` [15], we were led to the conjecture that these double sums of the form (1.2) can always be expressed in terms of a linear combination of just four functions, namely $\binom{4n}{2n}$, $\binom{2n}{n}^2$, $4^n\binom{2n}{n}$, and 16^n, with coefficients that are rational in n, while in many instances double sums of the form (1.3) can be expressed in terms of a linear combination of the four functions $\binom{2n+2m}{n+m}$, $\binom{2n}{n}\binom{2m}{m}$, $4^n\binom{2m}{m}$, and $4^m\binom{2n}{n}$, with coefficients that are rational in n and m. We demonstrate this observation in Theorems 15–18, in a much more precise form.

It is not difficult to see that the problem of evaluation of double sums of the form (1.2) and (1.3) can be reduced to the evaluation of sums of the form

$$\sum_{0 \le i \le j} i^s j^t \binom{2n}{n+i} \binom{2m}{m+j} \tag{1.4}$$

(and a few simpler *single* sums). See the proofs of Theorems 15–18 in Section 7 and Remark 19(1). We furthermore show (see the proofs of Propositions 9 and 11 in Section 5, which may be considered as the actual main result of the present paper) that for the evaluation of double sums of the form (1.4) it suffices to evaluate four *fundamental* double sums, given in Lemmas 1–4 in Section 2. While Lemmas 2–4 are relatively easy to prove by telescoping arguments (see the proofs in Section 2), the proof of Lemma 1 is more challenging. We provide two different proofs, one using computer algebra, and one using complex contour integrals. We believe that both proofs are of intrinsic interest. The algorithmic proof is described in Section 3. There, we explain that the computer algebra package `Sigma` can be used in a completely automatic fashion to evaluate double sums of the form (1.4). In particular, the reader can see how we empirically discovered our main results in Sections 5 and 7. The second proof, based on the power of complex integration, is explained in Section 4.

We close our paper by proving another conjecture from [3, Conj. 3.1], namely the inequality (see Theorem 20 in Section 8)

$$\sum_{i,j} \left| j^2 - i^2 \right| \binom{2n}{n+i} \binom{2m}{m+j} \ge 2nm \binom{2n}{n} \binom{2m}{m}.$$

We show moreover that equality holds if and only if $m = n$, in which case the evaluation (1.1) applies. Although Lemmas 1–4 would provide a good starting point for a proof of the inequality, we prefer to use a more direct approach, involving an application of Gosper's algorithm [7] at a crucial point.

We wish to point out that Bostan, Lairez and Salvy [1] have developed an algorithmic approach — based on contour integrals — that is capable of automatically finding a recurrence for the double sum (1.2) for any particular choice of s, t, k, β, and, thus, is able to establish an evaluation of such a sum (such as (1.1), for example) once the right-hand side is found.

Our final remark is that some of the double sums (1.2) and (1.3) can be embedded into infinite families of multidimensional sums that still allow for closed form evaluations, see [2].

2. The fundamental lemmas

In this section, we state the summation identities which form the basis of the evaluation of double sums of the form (1.4) (and, thus, of double sums of the form (1.2) and (1.3)). As it turns out, Lemmas 2–4 are very easy to prove since at least one summation of the double sum can be put in telescoping form, see the proofs below. Lemma 1 is much more subtle. We provide two different proofs, the first being algorithmic — see Section 3, the second making use of complex integration — see Section 4.

Lemma 1. *For all non-negative integers n and m, we have*

$$\sum_{0\le i\le j}\binom{2n}{n+i}\binom{2m}{m+j}=2^{2n+2m-3}+\frac{1}{4}\binom{2n+2m}{n+m}+\frac{1}{2}\binom{2n}{n}\binom{2m}{m}\\ +2^{2m-2}\binom{2n}{n}-\frac{1}{8}\sum_{\ell=0}^{n-m}\binom{2n-2\ell}{n-\ell}\binom{2m+2\ell}{m+\ell}, \tag{2.1}$$

where the sum on the right-hand side has to be interpreted as explained in Lemma 8.

Lemma 2. *For all non-negative integers n and m, we have*

$$\sum_{0\le i\le j} i\binom{2n}{n+i}\binom{2m}{m+j}=-\frac{n}{4}\binom{2n+2m}{n+m}+n\,2^{2m-2}\binom{2n}{n}+\frac{nm}{4(n+m)}\binom{2n}{n}\binom{2m}{m}. \tag{2.2}$$

Lemma 3. *For all non-negative integers n and m, we have*

$$\sum_{0\le i\le j} j\binom{2n}{n+i}\binom{2m}{m+j}=\frac{m}{4}\binom{2n+2m}{n+m}+\frac{m(m+2n)}{4(n+m)}\binom{2n}{n}\binom{2m}{m}. \tag{2.3}$$

Lemma 4. *For all non-negative integers n and m, we have*

$$\sum_{0\le i\le j} i\,j\binom{2n}{n+i}\binom{2m}{m+j}=\frac{mn}{2(n+m)}\binom{2n+2m-2}{n+m-1}+\frac{nm^2}{4(n+m)}\binom{2n}{n}\binom{2m}{m}. \tag{2.4}$$

Proof of Lemma 2. We have[1]

$$i\binom{2n}{n+i}=\frac{n+i}{2}\binom{2n}{n+i}-\frac{n+i+1}{2}\binom{2n}{n+i+1}.$$

Thus, we obtain

$$\begin{aligned}\sum_{0\le i\le j} i\binom{2n}{n+i}\binom{2m}{m+j}&=\frac{1}{2}\sum_{j\ge0}\left(n\binom{2n}{n}-(n+j+1)\binom{2n}{n+j+1}\right)\binom{2m}{m+j}\\&=\frac{n}{2}\binom{2n}{n}\sum_{j\ge0}\binom{2m}{m+j}-\frac{1}{2}\sum_{j\ge0}(n-j)\binom{2n}{n+j}\binom{2m}{m+j}.\end{aligned}$$

The first sum is, essentially, one half of a binomial theorem,

$$\sum_{j\ge0}\binom{2m}{m+j}=\frac{1}{2}\binom{2m}{m}+2^{2m-1}.$$

In order to evaluate the second sum, we observe that[2]

$$\begin{aligned}\sum_{j\ge0}(n-j)\binom{2n}{n+j}\binom{2m}{m+j}&=n\sum_{j\ge0}\binom{2n}{n+j}\binom{2m}{m+j}-\sum_{j\ge0}j\binom{2n}{n+j}\binom{2m}{m+j}\\&=\frac{n}{2}\sum_{j=-\infty}^{\infty}\binom{2n}{n+j}\binom{2m}{m+j}+\frac{n}{2}\binom{2n}{n}\binom{2m}{m}\\&\quad-\sum_{j\ge0}\left(\frac{(n+j)(m+j)}{2(m+n)}\binom{2n}{n+j}\binom{2m}{m+j}\right.\\&\qquad\left.-\frac{(n+j+1)(m+j+1)}{2(m+n)}\binom{2n}{n+j+1}\binom{2m}{m+j+1}\right)\\&=\frac{n}{2}\sum_{j=-\infty}^{\infty}\binom{2n}{n+j}\binom{2m}{m-j}+\frac{n}{2}\binom{2n}{n}\binom{2m}{m}-\frac{nm}{2(m+n)}\binom{2n}{n}\binom{2m}{m}.\end{aligned}$$

The sum in the last line can be evaluated by means of the Chu–Vandermonde summation formula (cf. [9, Sec. 5.1, (5.27)]). Substitution of these findings and little simplification then leads to the right-hand side of (2.2). □

Proof of Lemma 3. We have

$$j\binom{2m}{m+j}=\frac{m+j}{2}\binom{2m}{m+j}-\frac{m+j+1}{2}\binom{2m}{m+j+1}. \tag{2.5}$$

[1] The informed reader will have guessed that the telescoping form of the summand was discovered by using Gosper's algorithm [7] (see also [14]). The particular implementation that we applied is the one due to Paule and Schorn [13].

[2] For the finding of the telescoping form of the sum over $j\ge0$ below see Footnote 1.

Thus, we obtain

$$\begin{aligned}\sum_{0\le i\le j} j\binom{2n}{n+i}\binom{2m}{m+j} &= \frac{1}{2}\sum_{i\ge 0}\binom{2n}{n+i}(m+i)\binom{2m}{m+i}\\ &= \frac{m}{2}\sum_{i\ge 0}\binom{2n}{n+i}\binom{2m}{m+i}+\frac{1}{2}\sum_{i\ge 0} i\binom{2n}{n+i}\binom{2m}{m+i}.\end{aligned}$$

We have evaluated the same sums in the previous proof. We leave it to the reader to fill in the details in order to arrive at the right-hand side of (2.3). □

Proof of Lemma 4. Using (2.5), we have

$$\begin{aligned}\sum_{0\le i\le j} i\,j\binom{2n}{n+i}\binom{2m}{m+j} &= \frac{1}{2}\sum_{i\ge 0} i\binom{2n}{n+i}(m+i)\binom{2m}{m+i}\\ &= \frac{1}{2}\sum_{i\ge 0}(n+i)\binom{2n}{n+i}(m+i)\binom{2m}{m+i}\\ &\qquad -\frac{n}{2}\sum_{i\ge 0}\binom{2n}{n+i}(m+i)\binom{2m}{m+i}\\ &= 2nm\sum_{i\ge 0}\binom{2n-1}{n+i-1}\binom{2m-1}{m+i-1}-\frac{n}{2}\sum_{i\ge 0}\binom{2n}{n+i}(m+i)\binom{2m}{m+i}.\end{aligned}$$

We have evaluated the second sum in the previous proof. In order to evaluate the first sum, we do the substitution $i\to -i+1$ and obtain

$$\begin{aligned}\sum_{i\ge 0}\binom{2n-1}{n+i-1}\binom{2m-1}{m+i-1} &= \frac{1}{2}\sum_{i\ge 0}\binom{2n-1}{n+i-1}\binom{2m-1}{m+i-1}\\ &\qquad +\frac{1}{2}\sum_{i\le 1}\binom{2n-1}{n-i}\binom{2m-1}{m-i}\\ &= \frac{1}{2}\sum_{i=-\infty}^{\infty}\binom{2n-1}{n+i-1}\binom{2m-1}{m+i-1}\\ &\qquad +\frac{1}{2}\binom{2n-1}{n-1}\binom{2m-1}{m-1}+\frac{1}{2}\binom{2n-1}{n}\binom{2m-1}{m}\\ &= \frac{1}{2}\sum_{i=-\infty}^{\infty}\binom{2n-1}{n+i-1}\binom{2m-1}{m-i}+\binom{2n-1}{n}\binom{2m-1}{m}.\end{aligned}$$

Again, the sum can be evaluated by means of the Chu–Vandermonde summation formula, and then substitution of these findings and little simplification leads to the right-hand side of (2.4). □

3. Proof of Lemma 1 using the computer algebra package Sigma

Here we show how Lemma 1 can be established by using the algorithmic tools provided by the summation package `Sigma` [15] of the second author. Algorithmic proofs of Lemmas 2–4 are much simpler and could be obtained completely analogously.

We seek an alternative representation of the double sum

$$S(n,m)=\sum_{0\le i\le j}\binom{2n}{n+i}\binom{2m}{m+j} \tag{3.1}$$

for all non-negative integers m,n with the following property: if one specialises m (respectively n) to a non-negative integer or if one knows the distance between n and m, then the evaluation of the double sum should be performed in a direct and simple fashion. In order to accomplish this task, we utilise the summation package `Sigma` [15].

The sum (3.1) can be rewritten in the form

$$S(n,m)=\sum_{j=0}^{m} f(n,m,j) \tag{3.2}$$

with

$$f(n,m,j)=\binom{2m}{j+m}\sum_{i=0}^{j}\binom{2n}{i+n}. \tag{3.3}$$

Given this sum representation we will exploit the following summation spiral that is built into `Sigma`:

(1) Calculate a linear recurrence in m of order d (for an appropriate positive integer d) for the sum $S(n,m)$ by the creative telescoping paradigm;
(2) solve the recurrence in terms of (indefinite) nested sums over hypergeometric products with respect to m (the corresponding sequences are also called *d'Alembertian solutions*, see [14]);
(3) combine the solutions into an expression $\mathrm{RHS}(n,m)$ such that $S(n,l)=\mathrm{RHS}(n,l)$ holds for all n and $l=0,1,\dots,d-1$.

Then this implies that $S(n,m)=\mathrm{RHS}(n,m)$ holds for all non-negative integers m,n.

Remark 5. This summation engine can be considered as a generalisation of [14] that works not only for hypergeometric products but for expressions in terms of nested sums over such hypergeometric products. It is based on a constructive summation theory of difference rings and fields [17, 18] that enhances Karr's summation approach [11] in various directions.

In the following paragraphs, we assume that $m\le n$. We activate `Sigma`'s summation spiral.

Step 1. Observe that our sum (3.2) with summand given in (3.3) is already in the right input form for `Sigma`: the summation objects of (3.3) are given in terms of nested sums over hypergeometric products. More precisely, let $\mathcal{S}_j$ denote the shift operator with respect to j, that is, $\mathcal{S}_j F(j) := F(j+1)$. Then, if one applies this shift operator to the arising objects of $f(n,m,j)$, one can rewrite them again in their non-shifted versions:

$$\begin{aligned} \mathcal{S}_j \binom{2m}{j+m} &= \frac{m-j}{1+j+m}\binom{2m}{j+m}, \\ \mathcal{S}_j \sum_{i=0}^{j}\binom{2n}{i+n} &= \sum_{i=0}^{j}\binom{2n}{i+n} + \frac{n-j}{1+j+n}\binom{2n}{j+n}. \end{aligned} \tag{3.4}$$

With the help of these identities, we can look straightforwardly for a linear recurrence in the free integer parameter m as follows. First, we load `Sigma` into the computer algebra system *Mathematica*,

```
In[1]:= << Sigma.m
Sigma - A summation package by Carsten Schneider © RISC-Linz
```

and enter our definite sum $S(n,m)$:

```
In[2]:= mySum = SigmaSum[Binomial[2m, j + m]SigmaSum[Binomial[2n, i + n], {i, 0, j}], {j, 0, m}]
```

Out[2]= $\sum_{j=0}^{m}\binom{2m}{j+m}\sum_{i=0}^{j}\binom{2n}{i+n}$

Then we compute a recurrence in m by executing the function call

```
In[3]:= rec = GenerateRecurrence[mySum, m][[1]]
```

Out[3]= $\mathtt{SUM}[m+1] - 4\mathtt{SUM}[m] == -\frac{1}{1+m+n}\sum_{i=0}^{m}\binom{2m}{i+m}\binom{2n}{i+n} + \frac{mn}{(m+1)(1+m+n)}\binom{2m}{m}\binom{2n}{n}$

This means that $\mathtt{SUM}[m] = S(n,m)(= \mathtt{mySum})$ is a solution of the output recurrence. But what is going on behind the scenes? Roughly speaking, Zeilberger's creative telescoping paradigm [14] is carried out in the setting of difference rings. More precisely, one tries to compute a recurrence for the summand $f(n,m,j)$ of the form

$$\begin{aligned} c_0(n,m)f(n,m,j) + c_1(n,m)f(n,m+1,j) + \cdots + c_d(n,m)f(n,m+d,j) \\ = g(n,m,j+1) - g(n,m,j), \end{aligned} \tag{3.5}$$

for $d = 0,1,2,\ldots$. In our particular instance, `Sigma` is successful for $d=1$ and delivers the solution $c_0(n,m) = -4$, $c_1(n,m) = 1$, and

$$\begin{aligned} g(n,m,j) = {}& \frac{(2j-1)}{-1+j-m}\binom{2m}{j+m}\sum_{i=0}^{j}\binom{2n}{i+n} \\ &+ \frac{j-n}{1+m+n}\binom{2m}{j+m}\binom{2n}{j+n} + \frac{1}{-1-m-n}\sum_{i=0}^{j}\binom{2m}{i+m}\binom{2n}{i+n}, \end{aligned} \tag{3.6}$$

which holds for all non-negative integers j,m,n with $0 \le j \le m \le n$. The correctness can be verified by substituting the right-hand side of (3.3) into (3.5), rewriting the summation objects in terms of $\binom{2m}{j+m}$ and $\sum_{i=0}^{j}\binom{2n}{i+n}$ using the relations given in (3.4) and $\mathcal{S}_m\binom{2m}{j+m} = \frac{2(m+1)(2m+1)}{(m-j+1)(1+j+m)}\binom{2m}{j+m}$, and applying simple rational function arithmetic. We recall that we assumed $m \le n$, and this restriction is indeed essential for being allowed to use `Sigma` in the described setup. However, the above check reveals that the result is in fact correct without any restriction on the relative sizes of m and n.

Finally, by summing (3.5) over j from 0 to m, we obtain the linear recurrence

$$\sum_{j=0}^{m} f(n,m+1,j) - 4\sum_{j=0}^{m} f(n,m,j) = -\sum_{j=0}^{m+1}\binom{2n}{i+n} + \frac{1}{-1-m-n}\sum_{i=0}^{m}\binom{2m}{i+m}\binom{2n}{i+n} + \frac{mn}{(m+1)(1+m+n)}\binom{2m}{m}\binom{2n}{n}.$$

which, by the above remark, holds for all non-negative integers m, n. As is straightforward to see, this is indeed equivalent to `Out[3]`.

STEP 2. We now apply our summation toolbox to the definite sum $\sum_{i=0}^{m}\binom{2m}{i+m}\binom{2n}{i+n}$ and obtain

$$\sum_{i=0}^{m}\binom{2m}{m+i}\binom{2n}{n+i} = \frac{1}{2}\binom{2m}{m}\binom{2n}{n} + \frac{1}{2}\binom{2m+2n}{m+n}. \tag{3.7}$$

Note that the calculations can be verified rigorously and as a consequence we obtain a proof that the identity holds for all non-negative integers m, n. Since we remain in this particular case purely in the hypergeometric world, one could also use the classical toolbox described in [14]. Yet another (classical) proof consists in observing that the sum on the left-hand side of (3.7) can be rewritten as

$$\frac{1}{2}\left(\sum_{i=0}^{m}\binom{2m}{m+i}\binom{2n}{n-i} + \sum_{i=0}^{m}\binom{2m}{m-i}\binom{2n}{n+i}\right) = \frac{1}{2}\left(\sum_{i=0}^{2m}\binom{2m}{i}\binom{2n}{n+m-i} + \binom{2m}{m}\binom{2n}{n}\right),$$

and then evaluating the sum on the right-hand side by means of the Chu–Vandermonde summation formula.

As a consequence, we arrive at the linear recurrence

In[4]:= $\mathbf{rec = rec /. \sum_{i=0}^{m}\binom{2m}{i+m}\binom{2n}{i+n} \to \frac{1}{2}\binom{2m}{m}\binom{2n}{n} + \frac{1}{2}\binom{2m+2n}{m+n}}$

Out[4]= $\mathtt{SUM[m+1] - 4SUM[m] == -\frac{\binom{2m+2n}{m+n}}{1+m+n}\frac{1}{2} + \frac{(-1-m+2mn)\binom{2m}{m}\binom{2n}{n}}{2(m+1)(1+m+n)}}$

Now we can activate `Sigma`'s recurrence solver with the function call

In[5]:= **recSol = SolveRecurrence[rec, SUM[m]]**

Out[5]= $\{\{0, 2^{2m}\}, \{1, \frac{1}{4}\binom{2m}{m}\binom{2n}{n} + \frac{1}{4}\binom{2m+2n}{m+n} + 2^{2m}\binom{2n}{n}\left(-\frac{1}{4} + \frac{1}{4}n\sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n}\right)\}\}$

This means that the first entry of the output is the solution of the homogeneous version of the recurrence, and the second entry is a solution of the recurrence itself. Hence, the general solution is

$$c\,2^{2m} + \frac{1}{4}\binom{2m}{m}\binom{2n}{n} + \frac{1}{4}\binom{2m+2n}{m+n} + 2^{2m}\binom{2n}{n}\left(-\frac{1}{4} + \frac{1}{4}n\sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n}\right), \tag{3.8}$$

where the constant c (free of m) can be freely chosen. We note that this solution can be easily verified by substituting it into `rec` computed in Out[4] and using the relations

$$\begin{aligned}
\mathcal{S}_m\binom{2m}{m} &= \frac{2(2m+1)}{m+1}\binom{2m}{m},\\
\mathcal{S}_m\binom{2m+2n}{m+n} &= \frac{2(2m+2n+1)}{m+n+1}\binom{2m+2n}{m+n},\\
\mathcal{S}_m\sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n} &= \sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n} + \frac{2^{-2m}(2m+1)}{2(m+1)(1+m+n)}\binom{2m}{m}.
\end{aligned}$$

Step 3. Looking at the initial value $S(n,0) = \binom{2n}{n}$, we conclude that the specialisation $c = \frac{1}{2}\binom{2n}{n}$ in (3.8) equals $S(n,m)$ for all $n \geq 0$ and $m = 0$.

Summarising, we have found (together with a proof) the representation

$$S(n,m) = 2^{2m-2}\binom{2n}{n}n\sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n} + 2^{2m-2}\binom{2n}{n} + \frac{1}{4}\binom{2m}{m}\binom{2n}{n} + \frac{1}{4}\binom{2m+2n}{m+n}, \tag{3.9}$$

which holds for all non-negative integers m, n. This last calculation step can be also carried out within `Sigma`, by making use of the function call

In[6]:= **FindLinearCombination[recSol, {0, {$\binom{2n}{n}$}}, m, 1]**

Out[6]= $2^{2m-2}\binom{2n}{n}n\sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n} + 2^{2m-2}\binom{2n}{n} + \frac{1}{4}\binom{2m}{m}\binom{2n}{n} + \frac{1}{4}\binom{2m+2n}{m+n}$

Strictly speaking, the above derivations contained one "human" (= non-automatic) step, namely at the point where we checked (3.6) and observed that this relation actually holds without the restriction $m \leq n$. For the algorithmic "purist" we point out that it is also possible to set up the problem appropriately under the restriction $m > n$ (by splitting the double sum $S(n,m)$ into two parts) so that `Sigma` is applicable. Not surprisingly, `Sigma` finds (3.9) again.

In this article, we are particularly interested in the evaluation of $S(n,m)$ if one fixes the distance $r = n - m \geq 0$ (or $r = m - n \geq 0$). In order to find such a representation for the case $m \leq n$, we manipulate the obtained sum

$$\sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n} = \sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+r+m} := T(m,r) \tag{3.10}$$

in (3.9) further by applying once more `Sigma`'s summation spiral (where r takes over the role of m).

Step 1. Using Sigma (alternatively one could use the Paule and Schorn implementation [13] of Zeilberger's algorithm), we obtain the recurrence

$$2(m+r)T(m,r)+(-1-2m-2r)T(m,r+1)=\frac{2^{-2m}(2m+1)\binom{2m}{m}}{2m+r+1}.$$

Step 2. Using Sigma's recurrence solver we obtain the general solution

$$d\,\frac{2^{2r}m\binom{2m}{m}}{\binom{2m+2r}{m+r}(m+r)}+\frac{2^{-2m}\binom{2m}{m}}{m+r}-\frac{2^{-2m+2r}(4m+1)\binom{2m}{m}^2}{2\binom{2m+2r}{m+r}(m+r)}-\frac{2^{2r-2m}m\binom{2m}{m}}{\binom{2m+2r}{m+r}(m+r)}\sum_{i=0}^{r}\frac{2^{-2i}\binom{2m+2i}{m+i}}{2m+i},$$

where the constant d (free of r) can be freely chosen.

Step 3. Looking at the initial value

$$T(m,0)=\sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+m}=\frac{2^{2m-1}}{m\binom{2m}{m}}+\frac{2^{-2m-1}\binom{2m}{m}}{m},$$

which we simplified by another round of Sigma's summation spiral, we conclude that we have to specialise d to

$$d=\frac{2^{2m-1}}{m\binom{2m}{m}}+\frac{2^{-2m-1}(4m+1)\binom{2m}{m}}{m}.$$

With this choice, we end up at the identity

$$T(m,r)=-\frac{2^{2r-2m}m\binom{2m}{m}}{\binom{2m+2r}{m+r}(m+r)}\sum_{i=0}^{r}\frac{2^{-2i}\binom{2i+2m}{i+m}}{i+2m}+\frac{2^{-2m}\binom{2m}{m}}{m+r}+\frac{2^{2m+2r-1}}{\binom{2m+2r}{m+r}(m+r)},$$

being valid for all non-negative integers r,m. Finally, performing the substitution $r\to n-m$, we find the identity

$$T(m,n-m)=-\frac{2^{2n-4m}\binom{2m}{m}}{n\binom{2n}{n}}m\sum_{i=0}^{n-m}\frac{2^{-2i}\binom{2i+2m}{i+m}}{i+2m}+\frac{2^{2n-1}}{n\binom{2n}{n}}+\frac{2^{-2m}\binom{2m}{m}}{n},\tag{3.11}$$

which holds for all non-negative integers n,m with $n\ge m$. By substituting this result into (3.9), we see that we have discovered *and* proven that

$$\begin{aligned}S(n,m)=-2^{-2m+2n-2}\binom{2m}{m}m\sum_{i=0}^{n-m}\frac{2^{-2i}\binom{2i+2m}{i+m}}{i+2m}\\+2^{2m-2}\binom{2n}{n}+\frac{1}{2}\binom{2m}{m}\binom{2n}{n}+\frac{1}{4}\binom{2m+2n}{m+n}+2^{2m+2n-3},\end{aligned}\tag{3.12}$$

which is valid for all non-negative integers n, m with $n \geq m$. In a similar fashion, if $m \geq n$, we obtain

$$S(n,m) = 2^{2m-2n-2}\binom{2n}{n} n \sum_{i=0}^{m-n} \frac{2^{-2i}\binom{2i+2n}{i+n}}{i+2n} + 2^{2m-2}\binom{2n}{n} + \frac{1}{4}\binom{2m}{m}\binom{2n}{n} + \frac{1}{4}\binom{2m+2n}{m+n} + 2^{2m+2n-3}. \quad (3.13)$$

We note that the interaction of the summation steps 1–3 is carried out at various places in a recursive manner. In order to free the user from all these mechanical but rather subtle calculation steps, the additional package `EvaluateMultiSums` [16] has been developed recently. It coordinates all these calculation steps cleverly and discovers identities as above completely automatically whenever such a simplification in terms of nested sums over hypergeometric products is possible. For instance, after loading the package

In[7]:= $<<$ **EvaluateMultiSum.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

we can transform the sum (3.1) into the desired form by executing the function call

In[8]:= $\mathbf{res = EvaluateMultiSum[\binom{2n}{n+i}\binom{2m}{m+j}, \{\{i,0,j\},\{j,0,m\}\}, \{m,n\}, \{0,0\}, \{n,\infty\}]}$

Out[8]= $$\frac{(2n+1)2^{2m-3}(2n)!}{n^2((n-1)!)^2}\sum_{i=1}^{m}\frac{2^{-2i}\binom{2i}{i}}{1+i+n} + \frac{(4n+3)2^{2m-3}(2n)!}{n^2(n+1)((n-1)!)^2} + \frac{(3+4m+2n)\binom{2m}{m}(2n)!}{8n^2(1+m+n)((n-1)!)^2} + \frac{(2m+2n)!}{4n^2((n-1)!)^2\left((n+1)_m\right)^2}$$

Here, `Sigma` uses the *Pochhammer symbol* $(\alpha)_m$ defined by

$$(\alpha)_m = \begin{cases} \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1), & \text{for } m>0, \\ 1, & \text{for } m=0, \\ 1/(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha+m), & \text{for } m<0, \end{cases} \quad (3.14)$$

which we shall also use later. The parameters m, n in the calculation above are bounded from below by $0, 0$ and from above by n, ∞, respectively. If one prefers a representation purely in terms of binomial coefficients, one may execute the following function calls:

In[9]:= $\mathbf{res = SigmaReduce[res, m, Tower \to \{\binom{2m}{m}, \binom{2n+2m}{n+m}\}];}$

In[10]:= $\mathbf{res = SigmaReduce[res, n, Tower \to \{\binom{2n}{n}\}];}$

Out[10]= $$2^{2m-3}(2n+1)\binom{2n}{n}\sum_{i=1}^{m}\frac{2^{-2i}\binom{2i}{i}}{1+i+n} + \frac{(4n+3)2^{2m-3}\binom{2n}{n}}{n+1} + \frac{(3+4m+2n)\binom{2m}{m}\binom{2n}{n}}{8(1+m+n)} + \frac{1}{4}\binom{2m+2n}{m+n}$$

If one rewrites the arising sum manually by means of the function call below, one finally ends up exactly at the result given in (3.9):

In[11]:= $\mathbf{res = SigmaReduce[res, m, Tower \to \{\sum_{i=1}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n}\}]}$

Out[11]= $2^{2m-2}\binom{2n}{n}n\sum_{i=1}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n}+2^{2m-1}\binom{2n}{n}+\frac{1}{4}\binom{2m}{m}\binom{2n}{n}+\frac{1}{4}\binom{2m+2n}{m+n}$

Analogously one can carry out these calculation steps to calculate the simplification given in (3.11) automatically.

Comparison with Lemma 1 reveals that (3.12) or (3.13) do not quite agree with the right-hand side of (2.1). For example, in order to prove that (3.12) is equivalent with (2.1), we would have to establish the identity

$$\frac{1}{8}\sum_{l=0}^{n-m}\binom{2m+2l}{m+l}\binom{2n-2l}{n-l}=2^{-2m+2n-2}\binom{2m}{m}m\sum_{i=0}^{n-m}\frac{2^{-2i}\binom{2i+2m}{i+m}}{i+2m}.$$

This can, of course, be routinely achieved by using the Paule and Schorn implementation [13] of Zeilberger's algorithm. Alternatively, we may use our `Sigma` summation technology again. Let

$$T'(n,m):=\sum_{l=0}^{n-m}\binom{2m+2l}{m+l}\binom{2n-2l}{n-l}.$$

The above described summation spiral leads to

$$T'(n,m)=-2^{2m+1}n\binom{2n}{n}\sum_{i=0}^{m}\frac{2^{-2i}\binom{2i}{i}}{i+n}+2\binom{2m}{m}\binom{2n}{n}+2^{2m+2n}.$$

If this relation is substituted in (3.9), then we arrive exactly at the assertion of Lemma 1.

Clearly, the case where $m\geq n$ can be treated in a similar fashion. This finishes the algorithmic proof of Lemma 1. □

4. Proof of Lemma 1 using complex contour integrals

In this section, we show how to prove Lemma 1 by making use of complex contour integrals. Before we can embark on the proof of the lemma, we need to establish several auxiliary evaluations of specific contour integrals.

Remark. In order to avoid a confusion of the summation index i with the usual short notation for $\sqrt{-1}$, throughout this section we write $\mathbf{i}$ for $\sqrt{-1}$.

Lemma 6. *For all non-negative integers n, we have*

$$\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}}\frac{dz}{z^{n+1}(1-z)^{n+1}}\frac{1}{(1-2z)}=2^{2n}, \tag{4.1}$$

where $\mathcal{C}$ is a contour close to 0, which encircles 0 once in the positive direction.

Proof. Let I_1 denote the expression on the left-hand side of (4.1). We blow up the contour $\mathcal{C}$ so that it is sent to infinity. While doing this, we must pass over the poles $z=1/2$ and $z=1$ of the integrand. This must be compensated by taking the residues at these points into account. Since the integrand is of the order $O(z^{-2})$ as $|z|\to\infty$, the integral along the contour near infinity vanishes. Thus, we obtain

$$\begin{aligned} I_1 &= -\operatorname{Res}_{z=1/2} \frac{1}{z^{n+1}(1-z)^{n+1}} \frac{1}{(1-2z)} - \operatorname{Res}_{z=1} \frac{1}{z^{n+1}(1-z)^{n+1}} \frac{1}{(1-2z)} \\ &= 2^{2n+1} - \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{1}{(1+z)^{n+1}(1-(1+z))^{n+1}} \frac{1}{(1-2(1+z))}\,dz. \end{aligned}$$

As the substitution $z\to -z$ shows, the last integral is identical with I_1. Thus, we have obtained an equation for I_1, from which we easily get the claimed result. ☐

Lemma 7. *For all non-negative integers n and m, we have*

$$\frac{1}{(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{1}{(u-t)} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} = -\frac{1}{2}\binom{2n+2m}{n+m}, \tag{4.2}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are contours close to 0, which encircle 0 once in the positive direction, and C_2 is entirely in the interior of C_1.

Proof. We treat here the case where $n\ge m$. The other case can be disposed of completely analogously.

Let I_2 denote the expression on the left-hand side of (4.2). Clearly, interchange of u and t in the integrand does not change I_2. In that case however, we must also interchange the corresponding contours. Hence, I_2 is also equal to one half of the sum of the original expression and the one where u and t are exchanged, that is,

$$\begin{aligned} I_2 = \frac{1}{2\,(2\pi\mathbf{i})^2} &\int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{1}{(u-t)} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \\ &- \frac{1}{2\,(2\pi\mathbf{i})^2} \int_{\mathcal{C}_2} \int_{\mathcal{C}_1} \frac{1}{(u-t)} \frac{dt}{t^{n+1}(1-t)^{n+1}} \frac{du}{u^m(1-u)^m}. \end{aligned}$$

We would like to put both expressions under one integral. In order to do so, we must blow up the contour C_2 in the second integral (the contour for t) so that it passes across C_1. When doing so, the term $u-t$ in the denominator will vanish, and so we shall collect a residue at $t=u$. This yields

$$\begin{aligned} I_2 &= \frac{1}{2\,(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{du\,dt}{(u-t)\big(u(1-u)\,t(1-t)\big)^{n+1}} \Big(\big(t(1-t)\big)^{n-m+1} - \big(u(1-u)\big)^{n-m+1}\Big) \\ &\quad + \frac{1}{2\,(2\pi\mathbf{i})} \int_{\mathcal{C}_1} \operatorname{Res}_{t=u} \frac{1}{(u-t)} \frac{dt}{t^{n+1}(1-t)^{n+1}} \frac{du}{u^m(1-u)^m} \\ &= \frac{1}{2\,(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{du\,dt\,(u+t-1)}{\big(u(1-u)\,t(1-t)\big)^{n+1}} \sum_{\ell=0}^{n-m} \big(t(1-t)\big)^{\ell} \big(u(1-u)\big)^{n-m-\ell} \\ &\quad - \frac{1}{2\,(2\pi\mathbf{i})} \int_{\mathcal{C}_1} \frac{du}{u^{n+m+1}(1-u)^{n+m+1}} \end{aligned}$$

$$= \sum_{\ell=0}^{n-m} \frac{1}{2\,(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{du\,dt}{u^{m+\ell}(1-u)^{m+\ell+1}\big(t(1-t)\big)^{n-\ell+1}}$$
$$- \sum_{\ell=0}^{n-m} \frac{1}{2\,(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{du\,dt}{\big(u(1-u)\big)^{m+\ell+1} t^{n-\ell+1}(1-t)^{n-\ell}} - \frac{1}{2}\binom{2n+2m}{n+m}$$
$$= \frac{1}{2} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell}\binom{2m+2\ell-1}{m+\ell} - \frac{1}{2} \sum_{\ell=0}^{n-m} \binom{2n-2\ell-1}{n-\ell-1}\binom{2m+2\ell}{m+\ell}$$
$$- \frac{1}{2}\binom{2n+2m}{n+m} = -\frac{1}{2}\binom{2n+2m}{n+m},$$

the last equality following from $\binom{2k}{k} = 2\binom{2k-1}{k}$. □

Lemma 8. *For all non-negative integers n and m with $n \ge m$, we have*

$$\frac{1}{(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{1}{(u-t)(1-2t)} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m}$$
$$= -\frac{1}{4} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell}\binom{2m+2\ell}{m+\ell} - 3 \cdot 2^{2n+2m-2}, \quad (4.3)$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are contours close to 0, which encircle 0 once in the positive direction, and C_2 is entirely in the interior of C_1. The sum on the right-hand side must be interpreted according to

$$\sum_{k=M}^{N-1} \operatorname{Expr}(k) = \begin{cases} \sum_{k=M}^{N-1} \operatorname{Expr}(k), & N > M, \\ 0, & N = M, \\ -\sum_{k=N}^{M-1} \operatorname{Expr}(k), & N < M. \end{cases} \quad (4.4)$$

Proof. Again, here we treat the case where $n \ge m$. The other case can be disposed of completely analogously.

Let I_3 denote the expression on the left-hand side of (4.3). We apply the same trick as in the proof of Lemma 7 and observe that I_3 is equal to one half of the sum of the original expression and the one where u and t are exchanged, plus the residue of the latter at $t = u$. To be precise,

$$I_3 = \frac{1}{2\,(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{du\,dt}{(u-t)\,(1-2u)\,(1-2t)\,\big(u(1-u)\,t(1-t)\big)^{n+1}}$$
$$\cdot \Big((1-2u)\,\big(t(1-t)\big)^{n-m+1} - (1-2t)\,\big(u(1-u)\big)^{n-m+1}\Big)$$
$$+ \frac{1}{2\,(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1} \operatorname{Res}_{t=u} \frac{1}{(u-t)(1-2u)} \frac{1}{t^{n+1}(1-t)^{n+1}} \frac{du}{u^m(1-u)^m}$$
$$= \frac{1}{2\,(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{du\,dt}{(u-t)\,(1-2t)\,\big(u(1-u)\,t(1-t)\big)^{n+1}}$$

$$
\begin{aligned}
&\qquad\qquad \cdot \Big(\big(t(1-t)\big)^{n-m+1} - \big(u(1-u)\big)^{n-m+1}\Big) \\
&\quad - \frac{1}{(2\pi \mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{du\,dt}{(1-2u)\,(1-2t)\,\big(u(1-u)\big)^m \big(t(1-t)\big)^{n+1}} \\
&\quad - \frac{1}{2\,(2\pi \mathbf{i})^2} \int_{\mathcal{C}_1} \frac{1}{(1-2u)} \frac{du}{u^{n+m+1}(1-u)^{n+m+1}} \\
&= \frac{1}{2\,(2\pi \mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{du\,dt\,(u+t-1)}{(1-2t)\,\big(u(1-u)\,t(1-t)\big)^{n+1}} \sum_{\ell=0}^{n-m} \big(t(1-t)\big)^{\ell} \big(u(1-u)\big)^{n-m-\ell} \\
&\qquad\qquad - 2^{2m-2+2n} - 2^{2n+2m-1} \\
&= \sum_{\ell=0}^{n-m} \frac{1}{2\,(2\pi \mathbf{i})^2} \int_{\mathcal{C}_1'} \int_{\mathcal{C}_2'} \frac{du\,dt}{(1-2t)\,u^{m+\ell}(1-u)^{m+\ell+1}\big(t(1-t)\big)^{n-\ell+1}} \\
&\quad - \sum_{\ell=0}^{n-m} \frac{1}{2\,(2\pi \mathbf{i})^2} \int_{\mathcal{C}_1'} \int_{\mathcal{C}_2'} \frac{du\,dt}{(1-2t)\,\big(u(1-u)\big)^{m+\ell+1} t^{n-\ell+1}(1-t)^{n-\ell}} - 3\cdot 2^{2m+2n-2} \\
&= \frac{1}{2} \sum_{\ell=0}^{n-m} \binom{2m+2\ell-1}{m+\ell} 2^{2n-2\ell} \\
&\quad - \frac{1}{2} \sum_{\ell=0}^{n-m} \binom{2m+2\ell}{m+\ell} \left(2^{2n-2\ell-1} + \frac{1}{2}\binom{2n-2\ell}{n-\ell} \right) - 3\cdot 2^{2n+2m-2} \\
&= -\frac{1}{4} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell}{m+\ell} - 3\cdot 2^{2n+2m-2},
\end{aligned}
$$

which is again seen by observing $\binom{2k}{k} = 2\binom{2k-1}{k}$. $\square$

We are now in the position to prove Lemma 1 from Section 2.

Proof of Lemma 1. Using complex contour integrals, we may write

$$
\begin{aligned}
\sum_{0\le i\le j} \binom{2n}{n+i}\binom{2m}{m+j} &= \sum_{0\le i\le j} \binom{2n}{n-i}\binom{2m}{m-j} \\
&= \sum_{0\le i\le j} \frac{1}{(2\pi \mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{(1+x)^{2n}}{x^{n-i+1}} \frac{(1+y)^{2m}}{y^{m-j+1}}\,dx\,dy \\
&= \frac{1}{(2\pi \mathbf{i})^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{(1+x)^{2n}}{x^{n+1}} \frac{(1+y)^{2m}}{y^{m+1}} \frac{dx\,dy}{(1-xy)(1-y)},
\end{aligned}
$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are contours close to 0, which encircle 0 once in the positive direction.

Now we do the substitutions $x = u/(1-u)$ and $y = t/(1-t)$, implying $dx = du/(1-u)^2$ and $dy = dt/(1-t)^2$. This leads to

$$\sum_{0\le i\le j} \binom{2n}{n+i}\binom{2m}{m+j}$$
$$= \frac{1}{(2\pi\mathbf{i})^2}\int_{\mathcal{C}_1'}\int_{\mathcal{C}_2'} \frac{du}{u^{n+1}(1-u)^{n+1}}\frac{dt}{t^{m+1}(1-t)^{m+1}}\frac{(1-u)(1-t)^2}{(1-u-t)(1-2t)}$$
$$= \frac{1}{2\,(2\pi\mathbf{i})^2}\int_{\mathcal{C}_1'}\int_{\mathcal{C}_2'} \frac{du}{u^{n+1}(1-u)^{n+1}}\frac{dt}{t^{m+1}(1-t)^{m+1}}$$
$$- \frac{1}{(2\pi\mathbf{i})^2}\int_{\mathcal{C}_1'}\int_{\mathcal{C}_2'} \frac{du}{u^{n+1}(1-u)^{n+1}}\frac{dt}{t^{m}(1-t)^{m}}\frac{1}{(1-2t)}$$
$$+ \frac{1}{2\,(2\pi\mathbf{i})^2}\int_{\mathcal{C}_1'}\int_{\mathcal{C}_2'} \frac{du}{u^{n+1}(1-u)^{n+1}}\frac{dt}{t^{m+1}(1-t)^{m+1}}\frac{1}{(1-2t)}$$
$$+ \frac{1}{2\,(2\pi\mathbf{i})^2}\int_{\mathcal{C}_1'}\int_{\mathcal{C}_2'} \frac{du}{u^{n+1}(1-u)^{n+1}}\frac{dt}{t^{m}(1-t)^{m}}\frac{1}{(1-u-t)}$$
$$+ \frac{1}{2\,(2\pi\mathbf{i})^2}\int_{\mathcal{C}_1'}\int_{\mathcal{C}_2'} \frac{du}{u^{n+1}(1-u)^{n+1}}\frac{dt}{t^{m}(1-t)^{m}}\frac{1}{(1-u-t)(1-2t)}. \tag{4.5}$$

We now discuss the evaluation of the five integrals on the right-hand side one by one. First of all, we have

$$\frac{1}{2\,(2\pi\mathbf{i})^2}\int_{\mathcal{C}_1'}\int_{\mathcal{C}_2'} \frac{du}{u^{n+1}(1-u)^{n+1}}\frac{dt}{t^{m+1}(1-t)^{m+1}} = \frac{1}{2}\left\langle u^n\right\rangle(1-u)^{-n-1}\ \left\langle t^m\right\rangle(1-t)^{-m-1}$$
$$= \frac{1}{2}\binom{2n}{n}\binom{2m}{m}. \tag{4.6}$$

Next, by Lemma 6, we have

$$\frac{1}{(2\pi\mathbf{i})^2}\int_{\mathcal{C}_1'}\int_{\mathcal{C}_2'} \frac{du}{u^{n+1}(1-u)^{n+1}}\frac{dt}{t^{m}(1-t)^{m}}\frac{1}{(1-2t)} = 2^{2m-2}\binom{2n}{n} \tag{4.7}$$

and

$$\frac{1}{2\,(2\pi\mathbf{i})^2}\int_{\mathcal{C}_1'}\int_{\mathcal{C}_2'} \frac{du}{u^{n+1}(1-u)^{n+1}}\frac{dt}{t^{m+1}(1-t)^{m+1}}\frac{1}{(1-2t)} = 2^{2m-1}\binom{2n}{n}. \tag{4.8}$$

In order to evaluate

$$I_4 := \frac{1}{2\,(2\pi\mathbf{i})^2}\int_{\mathcal{C}_1'}\int_{\mathcal{C}_2'} \frac{du}{u^{n+1}(1-u)^{n+1}}\frac{dt}{t^{m}(1-t)^{m}}\frac{1}{(1-u-t)},$$

we blow up the contour $\mathcal{C}_1'$ (the contour for u) so that it is sent to infinity. While doing this, we pass over the poles $u = 1-t$ and $u = 1$ of the integrand. This must be compensated by taking the residues at these points into account. Since the integrand is of the order $O(u^{-2})$ as $|u| \to \infty$, the integral along the contour near infinity vanishes. Thus, we obtain

$$\begin{aligned} I_4 &= -\frac{1}{2\,(2\pi\mathbf{i})} \int_{\mathcal{C}_2'} \operatorname{Res}_{u=1-t} \frac{1}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)} \\ &\qquad - \frac{1}{2\,(2\pi\mathbf{i})} \int_{\mathcal{C}_2'} \operatorname{Res}_{u=1} \frac{1}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)} \\ &= \frac{1}{2\,(2\pi\mathbf{i})} \int_{\mathcal{C}_2'} \frac{dt}{t^{n+m+1}(1-t)^{n+m+1}} \\ &\qquad - \frac{1}{2\,(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1'} \int_{\mathcal{C}_2'} \frac{du}{(1+u)^{n+1}(1-(1+u))^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-(1+u)-t)} \\ &= \frac{1}{2}\binom{2n+2m}{n+m} - \frac{1}{4}\binom{2n+2m}{n+m} = \frac{1}{4}\binom{2n+2m}{n+m}, \end{aligned} \tag{4.9}$$

which is seen by performing the substitution $u \to -u$ in the second expression in the next-to-last line and applying Lemma 7.

Finally, in order to evaluate

$$I_5 := \frac{1}{2\,(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1'} \int_{\mathcal{C}_2'} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)(1-2t)} \tag{4.10}$$

we again blow up the contour $\mathcal{C}_1$ so that it is sent to infinity. While doing this, we pass over the poles $u = 1-t$ and $u = 1$ of the integrand. This must be compensated by taking the residues at these points into account. Since the integrand is of the order $O(u^{-2})$ as $|u| \to \infty$, the integral along the contour near infinity vanishes. Thus, we obtain

$$\begin{aligned} I_5 &= -\frac{1}{2\,(2\pi\mathbf{i})} \int_{\mathcal{C}_2'} \operatorname{Res}_{u=1-t} \frac{1}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)(1-2t)} \\ &\qquad - \frac{1}{2\,(2\pi\mathbf{i})} \int_{\mathcal{C}_2'} \operatorname{Res}_{u=1} \frac{1}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)(1-2t)} \\ &= \frac{1}{2\,(2\pi\mathbf{i})} \int_{\mathcal{C}_2'} \frac{dt}{t^{n+m+1}(1-t)^{n+m+1}} \frac{1}{(1-2t)} \\ &\qquad - \frac{1}{2\,(2\pi\mathbf{i})^2} \int_{\mathcal{C}_1'} \int_{\mathcal{C}_2'} \frac{du}{(1+u)^{n+1}(1-(1+u))^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-(1+u)-t)(1-2t)} \\ &= 2^{2n+2m-1} - \frac{1}{8} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell}{m+\ell} - 3 \cdot 2^{2n+2m-3}, \end{aligned} \tag{4.11}$$

which is seen by applying Lemma 6 to the first expression in the next-to-last line, performing the substitution $u \to -u$ in the second expression, and applying Lemma 8. By combining (4.5)–(4.11) and simplifying, we obtain the right-hand side of (2.1). ☐

5. Main results

This section contains our main results concerning double sums of the form

$$\sum_{0\le i\le j} i^s j^t \binom{2n}{n+i}\binom{2m}{m+j}.$$

If both s and t are even, then we are only able to provide a result in the special case where $m=n$. (It would also be possible to provide a similar result for the case where the difference $n-m$ is some fixed integer.) The reason is that the identity in Lemma 1, on which an evaluation of the above sum will have to be based, contains the sum over ℓ that cannot be simplified if n and m are generic. Proposition 9 restricts attention to this special case. On the other hand, if s and t are not both even, then it is possible to provide a general result for the above double sum without any restriction on n and m. The evaluations are then based on Lemmas 2–4, and the corresponding results are presented in Proposition 11. It should be noted that, for the three cases of parity of s and t that are treated in both propositions, it is not true that Proposition 9 is a direct consequence of Proposition 11 as the assertions in Proposition 9 are more refined.

Proposition 9. *For all non-negative integers s,t,k and n, we have*

$$\sum_{0\le i\le j\le n} i^s j^t \binom{2n}{n+i}\binom{2n}{n+j} = \frac{P^{(1)}_{s,t}(n)}{(4n-1)(4n-3)\cdots(4n-2S-2T+1)}\binom{4n}{2n}$$
$$+\frac{P^{(2)}_{s,t}(n)}{(2n-1)(2n-3)\cdots(2n-2\lfloor (S+T)/2\rfloor+1)}\binom{2n}{n}^2 + P^{(3)}_{s,t}(n)\cdot 4^n\binom{2n}{n} + P^{(4)}_{s,t}(n)\cdot 16^n, \tag{5.1}$$

where the $P^{(i)}_{s,t}(n)$, $i=1,2,3,4$, are polynomials in n, $S=\lfloor s/2\rfloor$ and $T=\lfloor t/2\rfloor$. More specifically,

1. *if s and t are even, then, as polynomials in n, $P^{(1)}_{s,t}(n)$ is of degree at most $3S+3T$, $P^{(2)}_{s,t}(n)$ is of degree at most $2S+2T+\lfloor (S+T)/2\rfloor$, $P^{(3)}_{s,t}(n)$ is identically zero if $s\ne 0$, $P^{(3)}_{0,t}(n)$ is of degree at most $2T$, and $P^{(4)}_{s,t}(n)$ is of degree at most $2S+2T$;*
2. *if s is odd and t is even, then, as polynomials in n, $P^{(1)}_{s,t}(n)$ is of degree at most $3S+3T+1$, $P^{(2)}_{s,t}(n)$ is of degree at most $2S+2T+1+\lfloor (S+T)/2\rfloor$, $P^{(3)}_{s,t}(n)$ is of degree at most $2S+2T+1$, and $P^{(4)}_{s,t}(n)$ is identically zero;*
3. *if s is even and t is odd, then, as polynomials in n, $P^{(1)}_{s,t}(n)$ is of degree at most $3S+3T+1$, $P^{(2)}_{s,t}(n)$ is of degree at most $2S+2T+1+\lfloor (S+T)/2\rfloor$, and $P^{(3)}_{s,t}(n)$ and $P^{(4)}_{s,t}(n)$ are identically zero;*
4. *if s and t are odd, then, as polynomials in n, $P^{(1)}_{s,t}(n)$ is of degree at most $3S+3T+2$, $P^{(2)}_{s,t}(n)$ is of degree at most $2S+2T+2+\lfloor (S+T)/2\rfloor$, and $P^{(3)}_{s,t}(n)$ and $P^{(4)}_{s,t}(n)$ are identically zero.*

Remark 10. As the proof below shows, explicit formulae for the polynomials $P_{s,t}^{(i)}(n)$, $i = 1,2,3,4$, can be given that involve the coefficients $c_{a,S}(n)$ and $c_{b,T}(n)$ in (5.2) and (5.3), for which an explicit formula exists as well, see Lemma 14. Admittedly, these explicit formulae are somewhat cumbersome, and therefore we refrain from presenting them in full here.

Proof of Proposition 9. We start with the case in which both s and t are even. With the notation of the proposition, we have $s = 2S$ and $t = 2T$. We write

$$i^{2S} = \sum_{a=0}^{S} c_{a,S}(n)\,\left(n^2 - i^2\right)\left((n-1)^2 - i^2\right)\cdots\left((n-a+1)^2 - i^2\right), \tag{5.2}$$

where $c_{a,S}(n)$ is a polynomial in n of degree $2S - 2a$, $a = 0, 1, \dots, S$, and

$$j^{2T} = \sum_{b=0}^{T} c_{b,T}(n)\,\left(n^2 - j^2\right)\left((n-1)^2 - j^2\right)\cdots\left((n-b+1)^2 - j^2\right), \tag{5.3}$$

where $c_{b,T}(n)$ is a polynomial in n of degree $2T - 2b$, $b = 0, 1, \dots, T$. It should be noted that $c_{S,S}(n) = (-1)^S$ and $c_{T,T}(m) = (-1)^T$. For an explicit formula for the coefficients $c_{a,S}(n)$ see Lemma 14.

If we use the expansions (5.2) and (5.3) on the left-hand side of (5.1), then we obtain the expression

$$\begin{aligned}
\sum_{a=0}^{S}\sum_{b=0}^{T} & c_{a,S}(n)\, c_{b,T}(n)\left((2n-2a+1)_{2a}\,(2n-2b+1)_{2b} \sum_{0\le i\le j}\binom{2n-2a}{n+i-a}\binom{2n-2b}{n+j-b}\right)\\
&= \sum_{a=0}^{S}\sum_{b=0}^{T} c_{a,S}(n)\, c_{b,T}(n)\Bigg((2n-2a+1)_{2a}\,(2n-2b+1)_{2b}\\
&\qquad\cdot\Bigg(2^{4n-2a-2b-3} + \frac{1}{4}\binom{4n-2a-2b}{2n-a-b} + \frac{1}{2}\binom{2n-2a}{n-a}\binom{2n-2b}{n-b}\\
&\qquad\quad + 2^{2n-2b-2}\binom{2n-2a}{n-a} - \frac{1}{8}\sum_{\ell=0}^{b-a}\binom{2n-2a-2\ell}{n-a-\ell}\binom{2n-2b+2\ell}{n-b+\ell}\Bigg)\Bigg),
\end{aligned}$$

due to Lemma 1 with n replaced by $n - a$ and $m = n - b$. This expression can be further simplified by noting that

$$\sum_{a=0}^{S} c_{a,S}(n)\,(2n-2a+1)_{2a}\binom{2n-2a}{n-a} = 0^{2S}\binom{2n}{n}, \tag{5.4}$$

which is equivalent to the expansion (5.2) for $i=0$. Thus, we obtain

$$\frac{1}{2}0^{2S+2T}\binom{2n}{n}^2+0^{2S}\binom{2n}{n}\sum_{b=0}^{T}c_{b,T}(n)\,2^{2n-2b-2}\,(2n-2b+1)_{2b}$$
$$+\sum_{a=0}^{S}\sum_{b=0}^{T}c_{a,S}(n)\,c_{b,T}(n)\Bigg((2n-2a+1)_{2a}\,(2n-2b+1)_{2b}$$
$$\cdot\left(2^{4n-2a-2b-3}+\frac{1}{4}\binom{4n-2a-2b}{2n-a-b}-\frac{1}{8}\sum_{\ell=0}^{b-a}\binom{2n-2a-2\ell}{n-a-\ell}\binom{2n-2b+2\ell}{n-b+\ell}\right)\Bigg).$$

Taking into account the properties of $c_{a,S}(n)$ and $c_{b,T}(n)$, from this expression it is clear that $P^{(4)}_{s,t}(n)$, the coefficient of $2^{4n}=16^n$, has degree at most $2S+2T$ as a polynomial in n. It is furthermore obvious that, due to the term $0^{2S}=0^s$, the polynomial $P^{(3)}_{s,t}(n)$, the coefficient of $2^{2n}\binom{2n}{n}=4^n\binom{2n}{n}$, vanishes for $s\neq 0$, while its degree is at most $2T$ if $s=0$.

In order to verify the claim about $P^{(1)}_{s,t}(n)$, the coefficient of $\binom{4n}{2n}$, we write

$$c_{a,S}(n)\,c_{b,T}(n)\,(2n-2a+1)_{2a}\,(2n-2b+1)_{2b}\binom{4n-2a-2b}{2n-a-b}$$
$$=c_{a,S}(n)\,c_{b,T}(n)\,\frac{(2n-2a+1)_{2a}\,(2n-2b+1)_{2b}\,(2n-a-b+1)^2_{a+b}}{(4n-2a-2b+1)_{2a+2b}}\binom{4n}{2n}.$$

It is easy to see that $(2n-a-b+1)_{a+b}$ divides numerator and denominator. After this division, the denominator becomes

$$2^{a+b}(4n-1)(4n-3)\cdots(4n-2a-2b+1),$$

that is, part of the denominator below $P^{(1)}(n)$ in (5.1). The terms which are missing are

$$(4n-2a-2b-1)(4n-2a-2b-3)\cdots(4n-2S-2T+1).$$

Thus, if we put everything on the denominator

$$(4n-1)(4n-3)\cdots(4n-2S-2T+1),$$

then we see that the numerator of the coefficient of $\binom{4n}{2n}$ has degree at most

$$(2S-2a)+(2T-2b)+2a+2b+2(a+b)+(S+T-a-b)-(a+b)=3S+3T,$$

as desired.

Finally, we turn our attention to $P^{(2)}_{s,t}(n)$, the coefficient of $\binom{2n}{n}^2$. We have

$$c_{a,S}(n)\,c_{b,T}(n)\,(2n-2a+1)_{2a}\,(2n-2b+1)_{2b}\binom{2n-2a-2\ell}{n-a-\ell}\binom{2n-2b+2\ell}{n-b+\ell}$$
$$=c_{a,S}(n)\,c_{b,T}(n)\,\frac{(n-a-\ell+1)^2_{a+\ell}\,(n-b+\ell+1)^2_{b-\ell}\,(2n-2b+1)_{2\ell}}{(2n-2a-2\ell+1)_{2\ell}}\binom{2n}{n}^2 \tag{5.5a}$$
$$=c_{a,S}(n)\,c_{b,T}(n)\,\frac{(n-a-\ell+1)^2_{a+\ell}\,(n-b+\ell+1)^2_{b-\ell}\,(2n-2b+1)_{2b-2a-2\ell}}{(2n-2b+2\ell+1)_{2b-2a-2\ell}}\binom{2n}{n}^2. \tag{5.5b}$$

Let us assume $a\le b$, in which case we need to consider non-negative indices ℓ. (If $a>b$, then, according to the convention (4.4), we have to consider negative ℓ. Using the definition (3.14) of the Pochhammer symbol for negative indices, the arguments would be completely analogous.) We make the further assumption that $\ell\le\frac{1}{2}(b-a)$ and use expression (5.5a). (If $\ell>\frac{1}{2}(b-a)$, then analogous arguments work starting from expression (5.5b).)

It is easy to see that $(n-a-\ell+1)_\ell$ divides numerator and denominator (as polynomials in n) of the prefactor in (5.5a). Second, the (remaining) factor $2^{2\ell}(n-a-\ell+\frac{1}{2})_\ell$ in the denominator and the factor $(2n-2b+1)_{2\ell}$ in the numerator do not have common factors for $\ell\le\frac{1}{2}(b-a)$. The denominator is a factor of the denominator below $P^{(2)}_{s,t}(n)$ in (5.1). If in (5.5a) we extend denominator and numerator by the "missing" factor

$$(n-\lfloor (S+T)/2\rfloor+\tfrac{1}{2})_{\lfloor (T+S)/2\rfloor-\lfloor b+a\rfloor/2}\,(n-a+\tfrac{1}{2})_a,$$

then, due to the properties of $c_{a,S}(n)$ and $c_{b,T}(n)$, the numerator polynomial is of degree at most

$$\begin{aligned}(2S-2a)+(2T-2b)+2(a+\ell)+2(b-\ell)+2\ell-\ell+\lfloor (T+S)/2\rfloor-\lfloor (b+a)/2\rfloor+a\\ =2S+2T+\ell+\lfloor (T+S)/2\rfloor-\lfloor (b+a)/2\rfloor+a\\ \le 2S+2T+\lfloor (b-a)/2\rfloor+\lfloor (T+S)/2\rfloor-\lfloor (b+a)/2\rfloor+a\\ \le 2S+2T+\lfloor (S+T)/2\rfloor,\end{aligned}$$

as desired.

For the other cases, namely (s,t) being (odd, even), (even, odd), respectively (odd, odd), we proceed in the same way. That is, we apply the expansions (5.2) and (5.3) on the left-hand side of (5.1). Then, however, instead of Lemma 1, we apply Lemma 2, Lemma 3, and Lemma 4, respectively. The remaining arguments are completely analogous to those from the case of (s,t) being (even,even) (and, in fact, much simpler since the right-hand sides of the identities in Lemmas 2–4 are simpler than the one in Lemma 1). $\square$

Proposition 11. *Let s,t and n,m be non-negative integers.*

If s and t are not both even or both odd, then

$$\sum_{0\le i\le j} i^s j^t \binom{2n}{n+i}\binom{2m}{m+j}$$
$$= \frac{Q^{(1)}_{s,t}(n,m)}{(2n+2m-1)(2n+2m-3)\cdots(2n+2m-2S-2T+1)}\binom{2n+2m}{n+m}$$
$$+ \frac{Q^{(2)}_{s,t}(n,m)}{(n+m)(n+m-1)(n+m-2)\cdots(n+m-S-T)}\binom{2n}{n}\binom{2m}{m}$$
$$+ Q^{(3)}_{s,t}(n,m)\cdot 4^m\binom{2n}{n}, \quad (5.6)$$

where the $Q^{(i)}_{s,t}(n,m)$, $i=1,2,3$, are polynomials in n and m, $S=\lfloor s/2\rfloor$ and $T=\lfloor t/2\rfloor$. More specifically,

(1) *if s is odd and t is even, then, as polynomials in n and m, $Q^{(1)}_{s,t}(n,m)$ is of degree at most $3S+3T+1$, $Q^{(2)}_{s,t}(n,m)$ is of degree at most $3S+3T+2$, and $Q^{(3)}_{s,t}(n,m)$ is of degree at most $2S+2T+1$;*

(2) *if s is even and t is odd, then, as polynomials in n and m, $Q^{(1)}_{s,t}(n,m)$ is of degree at most $3S+3T+1$, $Q^{(2)}_{s,t}(n,m)$ is of degree at most $3S+3T+2$, and $Q^{(3)}_{s,t}(n,m)$ is identically zero.*

If s and t are odd, then

$$\sum_{0\le i\le j} i^s j^t \binom{2n}{n+i}\binom{2m}{m+j}$$
$$= \frac{Q^{(1)}_{s,t}(n,m)}{(2n+2m-1)(2n+2m-3)\cdots(2n+2m-2S-2T-1)}\binom{2n+2m}{n+m}$$
$$+ \frac{Q^{(2)}_{s,t}(n,m)}{(n+m)(n+m-1)(n+m-2)\cdots(n+m-S-T)}\binom{2n}{n}\binom{2m}{m}, \quad (5.7)$$

where $S=\lfloor s/2\rfloor$ and $T=\lfloor t/2\rfloor$, and, as polynomials in n and m, $Q^{(1)}_{s,t}(n,m)$ and $Q^{(2)}_{s,t}(n,m)$ are of degree at most $3S+3T+3$.

The proof of this proposition is completely analogous to the proof of Proposition 9 and is therefore left to the reader. Also here (cf. Remark 10), explicit formulae for the polynomials $Q^{(i)}_{s,t}(n,m)$, $i=1,2,3$, can be given that involve coefficients $c_{a,S}(n)$ and $c_{b,T}(m)$ for which an explicit formula exists (see Lemma 14).

6. Some more auxiliary results

In this section we derive some single sum evaluations that we shall need in the proofs in Section 7.

Lemma 12. *For all non-negative integers n and k, we have*

$$\sum_{j=1}^{n} j^{2k}\binom{2n}{n+j} = -\frac{0^{2k}}{2}\binom{2n}{n} + 4^n \sum_{b=0}^{k} c_{b,k}(n)\,(2n-2b+1)_{2b}\,2^{-2b-1}, \tag{6.1}$$

and

$$\sum_{j=1}^{n} j^{2k+1}\binom{2n}{n+j} = \frac{1}{2}\binom{2n}{n}\sum_{b=0}^{k} c_{b,k}(n)\,(n-b)_{b+1}\,(n-b+1)_b\,, \tag{6.2}$$

where the coefficients $c_{b,k}(n)$ are defined in (5.3) (*with explicit formula provided in Lemma 14*).

Proof. We use the expansion (5.3) with $T = k$ on the left-hand side of (6.1). This gives

$$\begin{aligned}
\sum_{j=1}^{n} j^{2k}\binom{2n}{n+j} &= \sum_{j=1}^{n}\sum_{b=0}^{k} c_{b,k}(n)\,(2n-2b+1)_{2b}\binom{2n-2b}{n+j-b} \\
&= \sum_{b=0}^{k} c_{b,k}(n)\,(2n-2b+1)_{2b}\left(2^{2n-2b-1} - \frac{1}{2}\binom{2n-2b}{n-b}\right) \\
&= -\frac{0^{2k}}{2}\binom{2n}{n} + \sum_{b=0}^{k} c_{b,k}(n)\,(2n-2b+1)_{2b}\,2^{2n-2b-1},
\end{aligned}$$

where we used (5.3) with $T = k$ and $j = 0$ in the last line. This is exactly the right-hand side of (6.1).

Now we do the same on the left-hand side of (6.2). This leads to

$$\begin{aligned}
\sum_{j=1}^{n} j^{2k+1}\binom{2n}{n+j} &= \sum_{j=1}^{n} j\cdot\sum_{b=0}^{k} c_{b,k}(n)\,(2n-2b+1)_{2b}\binom{2n-2b}{n+j-b} \\
&= \sum_{b=0}^{k} c_{b,k}(n)\,(2n-2b)_{2b+1}\sum_{j=1}^{n}\left(\binom{2n-2b-1}{n+j-b-1} - \frac{1}{2}\binom{2n-2b}{n+j-b}\right) \\
&= \sum_{b=0}^{k} c_{b,k}(n)\,(2n-2b)_{2b+1}\left(2^{2n-2b-2} - \frac{1}{2}2^{2n-2b-1} + \frac{1}{4}\binom{2n-2b}{n-b}\right) \\
&= \frac{1}{2}\sum_{b=0}^{k} c_{b,k}(n)\,(n-b)_{b+1}\,(n-b+1)_b\binom{2n}{n}.
\end{aligned}$$

This is exactly the right-hand side of (6.2). ☐

Lemma 13. *For all non-negative integers n and h, k, we have*

$$\sum_{j\ge 1} j^{2k}\binom{2n}{n+j}\binom{2m}{m+j} = -\frac{0^{2k}}{2}\binom{2n}{n}\binom{2m}{m} + \frac{1}{2}\sum_{b=0}^{k} c_{b,k}(n)\,(2n-2b+1)_{2b}\binom{2n+2m-2b}{n+m-b} \tag{6.3}$$

and

$$\sum_{j\ge1} j^{2h+2k+1}\binom{2n}{n+j}\binom{2m}{m+j}$$
$$=\sum_{a=0}^{h}\sum_{b=0}^{k} c_{a,h}(n)\,c_{b,k}(m)\,(n-a+1)_a^2\,(m-b+1)_b^2\,\frac{(n-a)(m-b)}{2(n+m-a-b)}\binom{2n}{n}\binom{2m}{m}, \tag{6.4}$$

where the coefficients $c_{a,h}(n)$ *and* $c_{b,k}(m)$ *are defined in* (5.2) (*with explicit formula provided in Lemma 14*).

Proof. We start by using the expansion (5.3) with $T=k$ on the left-hand side of (6.3). This gives

$$\sum_{j\ge1} j^{2k}\binom{2n}{n+j}\binom{2m}{m+j}=\sum_{j\ge1}\sum_{b=0}^{k} c_{b,k}(n)\,(2n-2b+1)_{2b}\binom{2n-2b}{n+j-b}\binom{2m}{m+j}. \tag{6.5}$$

We have

$$\sum_{j\ge1}\binom{2n-2b}{n+j-b}\binom{2m}{m+j}=\sum_{j\le-1}\binom{2n-2b}{n+j-b}\binom{2m}{m+j}$$

and hence

$$\begin{aligned}\sum_{j\ge1}\binom{2n-2b}{n+j-b}\binom{2m}{m+j}&=-\frac{1}{2}\binom{2n-2b}{n-b}\binom{2m}{m}+\frac{1}{2}\sum_{j}\binom{2n-2b}{n+j-b}\binom{2m}{m+j}\\&=-\frac{1}{2}\binom{2n-2b}{n-b}\binom{2m}{m}+\frac{1}{2}\binom{2n+2m-2b}{n+m-b},\end{aligned}$$

due to the Chu–Vandermonde summation. We substitute this back into (6.5) and obtain

$$\begin{aligned}\sum_{j\ge1} j^{2k}\binom{2n}{n+j}\binom{2m}{m+j}&=\sum_{b=0}^{k} c_{b,k}(n)\,(2n-2b+1)_{2b}\\&\qquad\cdot\left(-\frac{1}{2}\binom{2n-2b}{n-b}\binom{2m}{m}+\frac{1}{2}\binom{2n+2m-2b}{n+m-b}\right)\\&=-\frac{0^{2k}}{2}\binom{2n}{n}\binom{2m}{m}+\frac{1}{2}\sum_{b=0}^{k} c_{b,k}(n)\,(2n-2b+1)_{2b}\binom{2n+2m-2b}{n+m-b},\end{aligned}$$

where we used (5.3) with $T=k$ and $j=0$ in the last line.

In order to establish (6.4), we write $j^{2h+2k+1} = j \cdot j^{2h} \cdot j^{2k}$ and use (5.3) with $T = h$ and with $T = k$. This leads to

$$\sum_{j=1}^{n} j^{2h+2k+1} \binom{2n}{n+j} \binom{2m}{m+j}$$
$$= \sum_{j\ge 1} j \cdot \sum_{a=0}^{h} \sum_{b=0}^{k} c_{a,h}(n)\, c_{b,k}(m)\, (2n-2a+1)_{2a}\, (2m-2b+1)_{2b} \binom{2n-2a}{n+j-a} \binom{2m-2b}{m+j-b}. \tag{6.6}$$

Using the standard hypergeometric notation

$${}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right] = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{m!\,(b_1)_m \cdots (b_q)_m} z^m \ ,$$

where the Pochhammer symbol $(\alpha)_m$ is defined in (3.14), we have

$$\sum_{j\ge 1} j \binom{2n-2a}{n+j-a} \binom{2m-2b}{m+j-b}$$
$$= \binom{2n-2a}{n-a+1} \binom{2m-2b}{m-b+1}\, {}_3F_2\left[\begin{matrix} 2, -n+a+1, -m+b+1 \\ n-a+2, m-b+2 \end{matrix}; 1\right].$$

This ${}_3F_2$-series can be evaluated by means of (the terminating version of) Dixon's summation (see [19, Appendix (III.9)])

$${}_3F_2\left[\begin{matrix} A, B, -N \\ 1+A-B, 1+A+N \end{matrix}; 1\right] = \frac{(1+A)_N\, (1+\frac{A}{2}-B)_N}{(1+\frac{A}{2})_N\, (1+A-B)_N},$$

where N is a non-negative integer. Indeed, if we choose $A = 2$, $B = -n+a+1$, and $N = m-b-1$ in this summation formula, then we obtain

$$\sum_{j=1}^{n} j \binom{2n-2a}{n+j-a} \binom{2m-2b}{m+j-b} = \frac{(n-a+1)(m-b+1)}{2(n+m-a-b)} \binom{2n-2a}{n-a+1} \binom{2m-2b}{m-b+1}.$$

If this is substituted back in (6.6), then we obtain the right-hand side of (6.4) after little manipulation. □

For the proof of our theorems it is not necessary to have an explicit formula for the coefficients $c_{a,S}(n)$ in the expansion (5.2) — the coefficients that appeared in the proof of Proposition 9, and in Lemmas 12 and 13 — at our disposal. However, it is still of intrinsic interest to provide such an explicit formula.

Lemma 14. *The coefficient $c_{a,S}(n)$ in the expansion* (5.2) *is given by*

$$c_{a,S}(n) = \sum_{r=0}^{a} \frac{2(-1)^{a+r}(n-r)^{2S+1}}{r!\,(a-r)!\,(2n-a-r)_{a+1}}. \tag{6.7}$$

Proof. Substituting $n-b$ for i in (5.2), $b=0,1,\dots,S$, we obtain the triangular system of linear equations for the $c_{a,S}$'s

$$(n-b)^{2S}=\sum_{a=0}^{b} c_{a,S}(n)\,(2n-a-b+1)_a\,(b-a+1)_a,\quad b=0,1,\dots,S. \tag{6.8}$$

Now, using the inversion formula of Gould and Hsu [8] (in the statement [12, Eq. (1.1)] of the formula put $n=b$, $k=a$, $l=r$, $a_j=2n-j$, $b_j=-1$, in this order), we see that the matrix

$$\big((2n-a-b+1)_a\,(b-a+1)_a\big)_{b,a\ge 0}$$

is inverse to the matrix

$$\left(\frac{(-1)^{a+r}(2n-2r)}{r!\,(a-r)!\,(2n-a-r)_{a+1}}\right)_{a,r\ge 0}.$$

Hence, if the system (6.8) is inverted, that is, the coefficients $c_{a,S}(n)$, $a=0,1,\dots,S$, are expressed in terms of the $(n-b)^{2S}$, $b=0,1,\dots,S$, then the connection coefficients are given by the latter matrix. This proves (6.7). $\square$

7. Summation formulae for binomial double sums involving absolute values

In this section we present the implications of Propositions 9 and 11 on sums of the form (1.2) and (1.3) with $\beta=1$. As we point out in Remark 19(1) below, it would also be possible to derive similar theorems for arbitrary β. (An example of an evaluation with $\beta=3$ is given in (7.17).)

We start with results for double sums of the form (1.3) with even k (and $\beta=1$). First, we also let $m=n$. The corresponding evaluations are given in Theorem 15 below. In Theorem 16 we address these same double sums for generic n and m. Similarly to Proposition 11, for that case we have results only if s and t are not both even.

Theorem 15. *Let s,t,k and n be non-negative integers.*

If s and t are even, then

$$\sum_{-n\le i,j\le n}\left|i^s j^t(j^{2k}-i^{2k})\right|\binom{2n}{n+i}\binom{2n}{n+j}\\ =\frac{U^{(2)}_{s,t,k}(n)}{(2n-1)(2n-3)\cdots(2n-2\lfloor (S+T+k)/2\rfloor+1)}\binom{2n}{n}^2, \tag{7.1}$$

where $U^{(2)}_{s,t,k}(n)$ is of degree at most $2S+2T+2k+\lfloor (S+T+k)/2\rfloor$.

If s and t are both odd, then

$$\sum_{-n\le i,j\le n} \left|i^s j^t (j^{2k}-i^{2k})\right| \binom{2n}{n+i}\binom{2n}{n+j}$$
$$= \frac{U^{(2)}_{s,t,k}(n)}{(2n-1)(2n-3)\cdots(2n-2\lceil (S+T+k)/2\rceil+1)}\binom{2n}{n}^2, \quad (7.2)$$

where $U^{(2)}_{s,t,k}(n)$ is of degree at most $2S+2T+2k+\lceil (S+T+k)/2\rceil$.

If s and t have different parity, then

$$\sum_{-n\le i,j\le n} \left|i^s j^t (j^{2k}-i^{2k})\right| \binom{2n}{n+i}\binom{2n}{n+j}$$
$$= \frac{U^{(1)}_{s,t,k}(n)}{(4n-1)(4n-3)\cdots(4n-2S-2T-2k+1)}\binom{4n}{2n} + U^{(3)}_{s,t,k}(n)\cdot 4^n\binom{2n}{n}, \quad (7.3)$$

where $U^{(1)}_{s,t,k}(n)$ and $U^{(3)}_{s,t,k}(n)$ are polynomials in n, $S=\lfloor s/2\rfloor$ and $T=\lfloor t/2\rfloor$.

More specifically,

(1) *if s is odd and t is even, then, as polynomials in n, $U^{(1)}_{s,t,k}(n)$ is of degree at most $3S+3T+3k+1$, and $U^{(3)}_{s,t,k}(n)$ is of degree at most $2S+2T+2k+1$;*

(2) *if s is even and t is odd, then, as polynomials in n, $U^{(1)}_{s,t,k}(n)$ is of degree at most $3S+3T+3k+1$, and $U^{(3)}_{s,t,k}(n)$ is of degree at most $2S+2T+2k+1$.*

Remark. As the proof below shows, also here (cf. Remark 10) explicit formulae for the polynomials $U^{(i)}_{s,t,k}(n)$, $i=1,2,3$, can be given that involve coefficients $c_{a,A}(n)$, for various specific choices of A. As we pointed out at several places already, Lemma 14 provides an explicit formula for these coefficients.

Proof of Theorem 15. The claim is trivially true for $k=0$. Therefore we may assume from now on that $k>0$.

Using the operations $(i,j)\to(-i,j)$, $(i,j)\to(i,-j)$, and $(i,j)\to(j,i)$, which do not change the summand, we see that

$$\sum_{-n\le i,j\le n} \left|i^s j^t (j^{2k}-i^{2k})\right| \binom{2n}{n+i}\binom{2n}{n+j}$$
$$= 4\sum_{0\le i\le j\le n} \alpha(i=0)\,\alpha(j=0)\,\left(i^s j^t+i^t j^s\right)\left(j^{2k}-i^{2k}\right)\binom{2n}{n+i}\binom{2n}{n+j}$$
$$= 4\sum_{0\le i\le j\le n} \left(i^s j^t+i^t j^s\right)\left(j^{2k}-i^{2k}\right)\binom{2n}{n+i}\binom{2n}{n+j}$$
$$-2\binom{2n}{n}\sum_{j=1}^{n}\left(0^s j^t+0^t j^s\right) j^{2k}\binom{2n}{n+j}, \quad (7.4)$$

where $\alpha(\mathcal{A}) = \frac{1}{2}$ if $\mathcal{A}$ is true and $\alpha(\mathcal{A}) = 1$ otherwise. Now one splits the sums into several sums of the form

$$\sum_{0\le i\le j\le n} i^A j^B \binom{2n}{n+i}\binom{2n}{n+j}, \quad \text{respectively} \quad \sum_{j=1}^{n} j^B \binom{2n}{n+j}.$$

To sums of the second form, we apply Lemma 12. In order to evaluate the sums of the first form, we proceed as in the proof of Proposition 9. That is, we apply the expansions (5.2) and (5.3), and subsequently we use Lemmas 1–4 to evaluate the sums over i and j. Inspection of the result makes all assertions of the theorem obvious, except for the implicit claims in (7.1) and (7.2) that the term $4^n\binom{2n}{n}$ does not appear.

In order to verify these claims, we have to figure out what the coefficients of $4^n\binom{2n}{n}$ of the various sums in (7.4) are precisely. For the case of even s and t, from Lemma 1 we obtain that the coefficient of $4^n\binom{2n}{n}$ in the expression (7.4) equals

$$\begin{aligned}
&4\sum_{a=0}^{S}\sum_{b=0}^{T+k} c_{a,S}(n)\, c_{b,T+k}(n)\, (2n-2a+1)_{2a}\, (2n-2b+1)_{2b}\, 2^{-2b-2}\binom{2n-2a}{n-a}\binom{2n}{n}^{-1}\\
&\quad+4\sum_{a=0}^{T}\sum_{b=0}^{S+k} c_{a,T}(n)\, c_{b,S+k}(n)\, (2n-2a+1)_{2a}\, (2n-2b+1)_{2b}\, 2^{-2b-2}\binom{2n-2a}{n-a}\binom{2n}{n}^{-1}\\
&\quad-4\sum_{a=0}^{S+k}\sum_{b=0}^{T} c_{a,S+k}(n)\, c_{b,T}(n)\, (2n-2a+1)_{2a}\, (2n-2b+1)_{2b}\, 2^{-2b-2}\binom{2n-2a}{n-a}\binom{2n}{n}^{-1}\\
&\quad-4\sum_{a=0}^{T+k}\sum_{b=0}^{S} c_{a,T+k}(n)\, c_{b,S}(n)\, (2n-2a+1)_{2a}\, (2n-2b+1)_{2b}\, 2^{-2b-2}\binom{2n-2a}{n-a}\binom{2n}{n}^{-1}\\
&\qquad\qquad-2\cdot 0^{2S}\sum_{b=0}^{T+k} c_{b,T+k}(n)\, (2n-2b+1)_{2b}\, 2^{-2b-1}\\
&\qquad\qquad\qquad-2\cdot 0^{2T}\sum_{b=0}^{S+k} c_{b,S+k}(n)\, (2n-2b+1)_{2b}\, 2^{-2b-1}.
\end{aligned}$$

We may use (5.4) to simplify the double sums. In this manner, we arrive at the expression

$$\begin{aligned}
&0^{2S}\sum_{b=0}^{T+k} c_{b,T+k}(n)\, (2n-2b+1)_{2b}\, 2^{-2b} + 0^{2T}\sum_{b=0}^{S+k} c_{b,S+k}(n)\, (2n-2b+1)_{2b}\, 2^{-2b}\\
&\quad-0^{2S+2k}\sum_{b=0}^{T} c_{b,T}(n)\, (2n-2b+1)_{2b}\, 2^{-2b} - 0^{2T+2k}\sum_{b=0}^{S} c_{b,S}(n)\, (2n-2b+1)_{2b}\, 2^{-2b}\\
&\quad-0^{2S}\sum_{b=0}^{T+k} c_{b,T+k}(n)\, (2n-2b+1)_{2b}\, 2^{-2b} - 0^{2T}\sum_{b=0}^{S+k} c_{b,S+k}(n)\, (2n-2b+1)_{2b}\, 2^{-2b},
\end{aligned}$$

which visibly vanishes due to our assumption that $k > 0$.

The proof for the analogous claim in the case of odd s and t proceeds along the same lines. The only difference is that, instead of Lemma 1, here we need Lemma 4, and instead of (6.1) we need (6.2). $\square$

Theorem 16. *Let s, t, k and n, m be non-negative integers. If s and t are not both even, then*

$$\sum_{i,j} \left| i^s j^t (j^{2k} - i^{2k}) \right| \binom{2n}{n+i} \binom{2m}{m+j}$$
$$= \frac{V^{(1)}_{s,t,k}(n,m)}{(2n+2m-1)(2n+2m-3)\cdots(2n+2m-2S-2T-2k+1)} \binom{2n+2m}{n+m}$$
$$+ \frac{V^{(2)}_{s,t,k}(n,m)}{(n+m-1)(n+m-2)\cdots(n+m-S-T-k)} \binom{2n}{n}\binom{2m}{m}$$
$$+ V^{(3)}_{s,t,k}(n,m)\cdot 4^m \binom{2n}{n} + V^{(4)}_{s,t,k}(n,m)\cdot 4^n \binom{2m}{m}, \quad (7.5)$$

where the $V^{(i)}_{s,t,k}(n,m)$, $i = 1,2,3,4$, are polynomials in n and m, $S = \lfloor s/2 \rfloor$ and $T = \lfloor t/2 \rfloor$. More specifically,

1. *if s is odd and t is even, then, as polynomials in n and m, $V^{(1)}_{s,t,k}(n,m)$ is of degree at most $3S+3T+3k+1$, $V^{(3)}_{s,t,k}(n,m)$ is of degree at most $2S+2T+2k+1$, and $V^{(2)}_{s,t,k}(n,m)$ and $V^{(4)}_{s,t,k}(n,m)$ are identically zero,*
2. *if s is even and t is odd, then, as polynomials in n and m, $V^{(1)}_{s,t,k}(n,m)$ is of degree at most $3S+3T+3k+1$, $V^{(4)}_{s,t,k}(n,m)$ is of degree at most $2S+2T+2k+1$, and $V^{(2)}_{s,t,k}(n,m)$ and $V^{(3)}_{s,t,k}(n,m)$ are identically zero,*
3. *if s and t are odd, then, as polynomials in n and m, $V^{(2)}_{s,t,k}(n,m)$ is of degree at most $3S+3T+3k+2$, and $V^{(1)}_{s,t,k}(n,m)$, $V^{(3)}_{s,t,k}(n,m)$, and $V^{(4)}_{s,t,k}(n,m)$ are identically zero.*

Remark. Again (cf. Remark 10), from the proof below it is obvious that explicit formulae for the polynomials $V^{(i)}_{s,t,k}(n,m)$, $i = 1,2,3,4$, are available in terms of coefficients $c_{a,A}(n)$ and $c_{b,B}(m)$, for various specific choices of A and B, with Lemma 14 providing an explicit formula for these coefficients.

Proof of Theorem 16. Again, the claim is trivially true for $k = 0$. Therefore we may assume from now on that $k > 0$.

We follow the same idea as in the proof of Theorem 15, that is, we observe that the operations $(i,j) \to (-i,j)$ and $(i,j) \to (i,-j)$ leave the summand invariant. However, a notable difference here is that the interchange of summation indices $(i,j) \to (j,i)$ does not leave the summand invariant. Consequently, here we see that

$$\begin{aligned}
\sum_{i,j} \left|i^s j^t (j^{2k}-i^{2k})\right| &\binom{2n}{n+i}\binom{2m}{m+j}\\
&= 4 \sum_{0\le i\le j} \alpha(i=0)\,\alpha(j=0)\, i^s j^t \left(j^{2k}-i^{2k}\right)\binom{2n}{n+i}\binom{2m}{m+j}\\
&\qquad + 4 \sum_{0\le i\le j} \alpha(i=0)\,\alpha(j=0)\, i^t j^s \left(j^{2k}-i^{2k}\right)\binom{2n}{n+j}\binom{2m}{m+i}\\
&= 4 \sum_{0\le i\le j} i^s j^t \left(j^{2k}-i^{2k}\right)\binom{2n}{n+i}\binom{2m}{m+j}\\
&\qquad + 4 \sum_{0\le i\le j} i^t j^s \left(j^{2k}-i^{2k}\right)\binom{2n}{n+j}\binom{2m}{m+i}\\
&\qquad - 2\binom{2n}{n}\sum_{j=1}^{m} 0^s j^{t+2k}\binom{2m}{m+j} - 2\binom{2m}{m}\sum_{j=1}^{n} 0^t j^{s+2k}\binom{2n}{n+j},
\end{aligned} \tag{7.6}$$

where $\alpha(\mathcal{A})$ has the same meaning as in the proof of Theorem 15. Now one splits the sums into several sums of the form

$$\sum_{0\le i\le j} i^A j^B \binom{2n}{n+i}\binom{2m}{m+j} \quad \text{and} \quad \sum_{0\le i\le j} i^A j^B \binom{2m}{m+i}\binom{2n}{n+j},$$

$$\text{respectively} \quad \sum_{j=1}^{n} j^B \binom{2n}{n+j} \quad \text{and} \quad \sum_{j=1}^{m} j^B \binom{2m}{m+j}.$$

To sums of the second form, we apply Lemma 12. In order to evaluate the sums of the first form, we proceed as in the proof of Proposition 9. That is, we apply the expansions (5.2) and (5.3) (with n replaced by m if appropriate), and subsequently we use Lemmas 2–4 to evaluate the sums over i and j. Inspection of the result makes all assertions of the theorem obvious, except for the claims in Items (1) and (2) that the polynomial $V^{(2)}_{s,t,k}(n,m)$, the coefficient of $\binom{2n}{n}\binom{2m}{m}$ in (7.5), vanishes.

Below we treat Item (1), that is, the case where s is odd and t is even. Item (2) can be handled completely analogously.

After having done the above described manipulations, we see that, for odd s and even t, the coefficient of $\binom{2n}{n}\binom{2m}{m}$ in the expression (7.6) equals

$$\sum_{a=0}^{S}\sum_{b=0}^{T+k} c_{a,S}(n)\, c_{b,T+k}(m)\, (2n-2a+1)_{2a}\, (2m-2b+1)_{2b} \frac{(n-a)(m-b)}{n+m-a-b} \\
\times \binom{2n-2a}{n-a}\binom{2m-2b}{m-b}\binom{2n}{n}^{-1}\binom{2m}{m}^{-1}$$

$$
\begin{aligned}
&-\sum_{a=0}^{S+k}\sum_{b=0}^{T} c_{a,S+k}(n)\, c_{b,T}(m)\, (2n-2a+1)_{2a}\, (2m-2b+1)_{2b} \frac{(n-a)(m-b)}{n+m-a-b}\\
&\qquad\qquad\qquad\qquad\times \binom{2n-2a}{n-a}\binom{2m-2b}{m-b}\binom{2n}{n}^{-1}\binom{2m}{m}^{-1}\\
&+\sum_{a=0}^{T}\sum_{b=0}^{S+k} c_{a,T}(m)\, c_{b,S+k}(n)\, (2m-2a+1)_{2a}\, (2n-2b+1)_{2b} \frac{(n-b)(n-b+2(m-a))}{n+m-a-b}\\
&\qquad\qquad\qquad\qquad\times \binom{2n-2b}{n-b}\binom{2m-2a}{m-a}\binom{2n}{n}^{-1}\binom{2m}{m}^{-1}\\
&-\sum_{a=0}^{T+k}\sum_{b=0}^{S} c_{a,T+k}(m)\, c_{b,S}(n)\, (2m-2a+1)_{2a}\, (2n-2b+1)_{2b} \frac{(n-b)(n-b+2(m-a))}{n+m-a-b}\\
&\qquad\qquad\qquad\qquad\times \binom{2n-2b}{n-b}\binom{2m-2a}{m-a}\binom{2n}{n}^{-1}\binom{2m}{m}^{-1}\\
&-0^t\sum_{b=0}^{S+k} c_{b,S+k}(n)\, (n-b)_{b+1}\, (n-b+1)_b.
\end{aligned}
$$

In the last two double sums above, we interchange the summation indices a and b. Then the first and fourth double sum can be combined into one double sum, as well as the second and third double sum. Thus, the above expression simplifies to

$$
\begin{aligned}
&-\sum_{a=0}^{S}\sum_{b=0}^{T+k} c_{a,S}(n)\, c_{b,T+k}(m)\, (2n-2a+1)_{2a}\, (2m-2b+1)_{2b}(n-a)\\
&\qquad\qquad\qquad\qquad\times \binom{2n-2a}{n-a}\binom{2m-2b}{m-b}\binom{2n}{n}^{-1}\binom{2m}{m}^{-1}\\
&+\sum_{a=0}^{S+k}\sum_{b=0}^{T} c_{a,S+k}(n)\, c_{b,T}(m)\, (2n-2a+1)_{2a}\, (2m-2b+1)_{2b}(n-a)\\
&\qquad\qquad\qquad\qquad\times \binom{2n-2a}{n-a}\binom{2m-2b}{m-b}\binom{2n}{n}^{-1}\binom{2m}{m}^{-1}\\
&-0^t\sum_{b=0}^{S+k} c_{b,S+k}(n)\, (n-b)_{b+1}\, (n-b+1)_b.
\end{aligned}
$$

In both double sums, the sum over b can be evaluated by means of (5.4). This leads us to the expression

$$-\,0^{2T+2k}\sum_{a=0}^{S} c_{a,S}(n)\,(n-a)_{a+1}\,(n-a+1)_a + 0^{2T}\sum_{a=0}^{S+k} c_{a,S+k}(n)\,(n-a)_{a+1}\,(n-a+1)_a \\ -\,0^{t}\sum_{b=0}^{S+k} c_{b,S+k}(n)\,(n-b)_{b+1}\,(n-b+1)_b,$$

which visibly vanishes due to our assumptions that $k>0$ and that t is even. □

We now turn to our results for double sums of the form (1.3) with odd k (and $\beta=1$). We first state our results for $m=n$ and immediately thereafter the one we obtain for generic n and m in the case where s and t are both odd. We then indicate the proofs of both theorems.

Theorem 17. *Let s,t,k and n be non-negative integers.*
If s and t are not both odd, then

$$\sum_{-n\le i,j\le n}\left|i^sj^t(j^{2k+1}-i^{2k+1})\right|\binom{2n}{n+i}\binom{2n}{n+j} \\ =\frac{X^{(1)}_{s,t,k}(n)}{(4n-1)(4n-3)\cdots(4n-2S-2T-2k+1)}\binom{4n}{2n} \\ +\frac{X^{(2)}_{s,t,k}(n)}{(2n-1)(2n-3)\cdots(2n-2\left\lceil (S+T+k)/2\right\rceil+1)}\binom{2n}{n}^2 \\ +X^{(3)}_{s,t,k}(n)\cdot 4^n\binom{2n}{n}+X^{(4)}_{s,t,k}(n)\cdot 16^n, \quad (7.7)$$

where the $X^{(i)}_{s,t,k}(n)$, $i=1,2,3,4$, are polynomials in n, $S=\lfloor s/2\rfloor$ and $T=\lfloor t/2\rfloor$.
More specifically,

1. *if s and t are even, then, as polynomials in n, $X^{(1)}_{s,t,k}(n)$ is of degree at most $3S+3T+3k$, and $X^{(2)}_{s,t,k}(n)$, $X^{(3)}_{s,t,k}(n)$, and $X^{(4)}_{s,t,k}(n)$ are identically zero;*
2. *if s is odd and t is even, then, as polynomials in n, $X^{(2)}_{s,t,k}(n)$ is of degree at most $2S+2T+2k+1+\lceil (S+T+k)/2\rceil$, $X^{(4)}_{s,t,k}(n)$ is of degree at most $2S+2T+2k+1$, and $X^{(1)}_{s,t,k}(n)$ and $X^{(3)}_{s,t,k}(n)$ are identically zero;*
3. *if s is even and t is odd, then, as polynomials in n, $X^{(2)}_{s,t,k}(n)$ is of degree at most $2S+2T+2k+1+\lceil (S+T+k)/2\rceil$, $X^{(4)}_{s,t,k}(n)$ is of degree at most $2S+2T+2k+1$, and $X^{(1)}_{s,t,k}(n)$ and $X^{(3)}_{s,t,k}(n)$ are identically zero.*

If s and t are odd, then

$$\sum_{-n\le i,j\le n} \left|i^s j^t (j^{2k+1}-i^{2k+1})\right| \binom{2n}{n+i}\binom{2n}{n+j} \\ = \frac{X^{(1)}_{s,t,k}(n)}{(4n-1)(4n-3)\cdots(4n-2S-2T-2k-1)}\binom{4n}{2n} + X^{(3)}_{s,t,k}(n)\cdot 4^n\binom{2n}{n}, \quad (7.8)$$

where $S=\lfloor s/2\rfloor$ and $T=\lfloor t/2\rfloor$, and, as polynomials in n, $X^{(1)}_{s,t,k}(n)$ is of degree at most $3S+3T+3k+2$, and $X^{(3)}_{s,t,k}(n)$ is of degree at most $2S+2T+2k+2$.

Theorem 18. *Let s,t,k and n,m be non-negative integers. If s and t are both odd, then*

$$\sum_{i,j} \left|i^s j^t (j^{2k+1}-i^{2k+1})\right| \binom{2n}{n+i}\binom{2m}{m+j} \\ = \frac{Y^{(1)}_{s,t,k}(n,m)}{(2n+2m-1)(2n+2m-3)\cdots(2n+2m-2S-2T-2k-1)}\binom{2n+2m}{n+m} \\ + Y^{(3)}_{s,t,k}(n,m)\cdot 4^m\binom{2n}{n} + Y^{(4)}_{s,t,k}(n,m)\cdot 4^n\binom{2m}{m}, \quad (7.9)$$

where $S=\lfloor s/2\rfloor$ and $T=\lfloor t/2\rfloor$, and, as polynomials in n and m, $Y^{(1)}_{s,t,k}(n,m)$ is of degree at most $3S+3T+3k+3$, and $Y^{(3)}_{s,t,k}(n,m)$ and $Y^{(4)}_{s,t,k}(n,m)$ are of degree at most $2S+2T+2k+1$.

Remark. From the proof below it is obvious that also here (cf. Remark 10) explicit formulae for the polynomials $X^{(i)}_{s,t,k}(n)$ and $Y^{(i)}_{s,t,k}(n,m)$, $i=1,2,3,4$, exist in terms of coefficients $c_{a,A}(n)$ and $c_{b,B}(m)$, for various specific choices of A and B, with Lemma 14 providing an explicit formula for these coefficients.

Proof of Theorems 17 and 18. We use the operations $(i,j)\to(-i,j)$ and $(i,j)\to(i,-j)$ (but not $(i,j)\to(j,i)$). What we get is (for the proof of Theorem 17 we have to assume that $m=n$)

$$\sum_{i,j} \left|i^s j^t (j^{2k+1}-i^{2k+1})\right| \binom{2n}{n+i}\binom{2m}{m+j} \\ = \frac{1}{2}\sum_{i,j}\Big(\left|i^s j^t (j^{2k+1}-i^{2k+1})\right| + \left|i^s j^t (j^{2k+1}+i^{2k+1})\right|\Big)\binom{2n}{n+i}\binom{2m}{m+j} \\ = 2\sum_{0\le i,j}\alpha(i=0)\,\alpha(j=0)\Big(\left|i^s j^t (j^{2k+1}-i^{2k+1})\right| \\ + \left|i^s j^t (j^{2k+1}+i^{2k+1})\right|\Big)\binom{2n}{n+i}\binom{2m}{m+j} \\ = 2\sum_{0\le i\le j}\Big(\left|i^s j^t (j^{2k+1}-i^{2k+1})\right| + \left|i^s j^t (j^{2k+1}+i^{2k+1})\right|\Big)\binom{2n}{n+i}\binom{2m}{m+j}$$

$$
\begin{aligned}
&\qquad + 2 \sum_{0\le i<j} \left(\left| i^t j^s (j^{2k+1} - i^{2k+1}) \right| + \left| i^t j^s (j^{2k+1} + i^{2k+1}) \right| \right) \binom{2n}{n+j}\binom{2m}{m+i}\\
&\qquad - 2\binom{2n}{n} 0^s \sum_{0\le j} j^{t+2k+1}\binom{2m}{m+j} - 2\binom{2m}{m} 0^t \sum_{0\le i} i^{s+2k+1}\binom{2n}{n+i}\\
&= 4\sum_{0\le i\le j} i^s j^{t+2k+1}\binom{2n}{n+i}\binom{2m}{m+j} + 4\sum_{0\le i\le j} i^t j^{s+2k+1}\binom{2n}{n+j}\binom{2m}{m+i}\\
&\qquad - 2\binom{2n}{n} 0^s \sum_{0\le j} j^{t+2k+1}\binom{2m}{m+j} - 2\binom{2m}{m} 0^t \sum_{0\le i} i^{s+2k+1}\binom{2n}{n+i}\\
&\qquad - 4\sum_{j\ge1} j^{s+t+2k+1}\binom{2n}{n+j}\binom{2m}{m+j},
\end{aligned}
\tag{7.10}
$$

where $\alpha(\mathcal{A})$ has the same meaning as in the proof of Theorem 15. To the single sums over i and over j, we apply Lemmas 12 and 13. In order to evaluate the sums over $0 \le i \le j$, we proceed as in the proof of Proposition 9. That is, we apply the expansions (5.2) and (5.3) (with n replaced by m if appropriate), and subsequently we use Lemmas 1–4 to evaluate the sums over $0 \le i \le j$. Inspection of the result makes all assertions of the theorem obvious, except for the claims of the vanishing of the polynomial $X^{(2)}_{s,t,k}(n)$ in Theorem 17, Item (1), of the vanishing of the polynomial $X^{(1)}_{s,t,k}(n)$ in Theorem 17, Items (2) and (3), and of the claim that the coefficient of $\binom{2n}{n}^2$ in Theorem 17, right-hand side of (7.8), vanishes, as well as the coefficient of $\binom{2n}{n}\binom{2m}{m}$ in Theorem 18, right-hand side of (7.9).

Below we treat the last case, that is, the case of generic n and m where s and t are both odd. The other claims can be handled completely analogously.

Following the above described procedure, using (6.4) with $h = S + T + k + 1$ and $k = 0$ for the evaluation of the sum over j in the last line of (7.10), we obtain from Lemma 2 that the coefficient of $\binom{2n}{n}\binom{2m}{m}$ in the expression (7.10) equals

$$
\begin{aligned}
4\sum_{a=0}^{S}\sum_{b=0}^{T+k+1} c_{a,S}(n)\, c_{b,T+k+1}(m)\, (n-a+1)_a^2\, (m-b+1)_b^2\, \frac{(n-a)(m-b)}{4(n+m-a-b)}&\\
+ 4\sum_{b=0}^{T}\sum_{a=0}^{S+k+1} c_{b,T}(m)\, c_{a,S+k+1}(n)\, (n-a+1)_a^2\, (m-b+1)_b^2\, \frac{(n-a)(m-b)}{4(n+m-a-b)}&\\
- 4\sum_{b=0}^{S+T+k+1} c_{b,S+T+k+1}(n)\, (n-b)_{b+1}\, (n-b+1)_b \frac{m}{2(n+m-b)}&.
\end{aligned}
$$

If we now use (6.4) with $(S, T + k + 1)$, $(S + k + 1, T)$, and $(S + T + k + 1, 0)$ in place of (h, k), we see that the above expression vanishes. This establishes the assertion about the "non-appearance" of the term $\binom{2n}{n}\binom{2m}{m}$ in Theorem 18, and thus also the assertion about the "non-appearance" of $\binom{2n}{n}^2$ in Eq. (7.8) of Theorem 17. □

Remark 19. (1) It is obvious from the proofs of Theorems 15–18 that we could deduce analogous theorems for the more general sums (1.2) and (1.3). We omit this here for the sake of brevity, but provide an example of such an evaluation in (7.17) below.

(2) Theorems 15–18 imply an obvious algorithm to evaluate a sum of the form (1.2) or (1.3) for any given s, t, k and $\beta = 1$. (Again, an extension to arbitrary β would be possible.) Namely, addressing the case of odd k and $m = n$, one makes an indeterminate Ansatz for the polynomials $X^{(1)}_{s,t}(n), X^{(2)}_{s,t}(n), X^{(3)}_{s,t}(n), X^{(4)}_{s,t}(n)$ in Theorem 17, one evaluates the sum on the left-hand side of (7.7) for $n = S+T+k, \dots, N+S+T+k$, where N is the number of indeterminates involved in the Ansatz, giving rise to a system of $N+1$ linear equations for the N indeterminates. One solves the system and substitutes the solutions on the right-hand side of (7.7).

In this manner, we can establish any of the proved or conjectured double sum evaluations in [3]. For example, we obtain

$$\sum_{-n\le i,j\le n} \left|j^3-i^3\right| \binom{2n}{n+i}\binom{2n}{n+j} = \frac{4n^2(5n-2)}{4n-1}\binom{4n-1}{2n-1}, \tag{7.11}$$

$$\sum_{-n\le i,j\le n} \left|j^5-i^5\right| \binom{2n}{n+i}\binom{2n}{n+j} = \frac{8n^2(43n^3-70n^2+36n-6)}{(4n-2)(4n-3)}\binom{4n-2}{2n-2}, \tag{7.12}$$

$$\sum_{i,j} \left|ij(j^2-i^2)\right| \binom{2n}{n+i}\binom{2m}{m+j} = \frac{mn(n^2-n+m^2-m)}{n+m-1}\binom{2n}{n}\binom{2m}{m}, \tag{7.13}$$

$$\sum_{i,j} \left|i^3j^3(j^2-i^2)\right| \binom{2n}{n+i}\binom{2m}{m+j} = \frac{2n^2m^2P_1(n,m)}{(n+m-1)(n+m-2)(n+m-3)} \times \binom{2n}{n}\binom{2m}{m}, \tag{7.14}$$

$$\sum_{-n\le i,j\le n} \left|j^7-i^7\right| \binom{2n}{n+i}\binom{2n}{n+j} = \frac{16n^2P_2(n)}{(4n-3)(4n-4)(4n-5)}\binom{4n-3}{2n-3}, \tag{7.15}$$

where

$$P_1(n,m) = n^4+2n^3m-6n^3-6n^2m+11n^2+2nm^3-6nm^2+12nm \\ -10n+m^4-6m^3+11m^2-10m+4$$

and

$$P_2(n) = 531n^5-1960n^4+2800n^3-1952n^2+668n-90.$$

These identities (with $m = n$ for (7.13) and (7.14)) establish the conjectured identities (5.7)–(5.9), (5.12), (5.14) from [3]. However, our machinery also yields

$$\sum_{-n\le i,j\le n} \left|i^4 j^3 (j^5 - i^5)\right| \binom{2n}{n+i}\binom{2n}{n+j} \\ = \frac{n^4\left(414n^6 - 2968n^5 + 8332n^4 - 11853n^3 + 9105n^2 - 3592n + 565\right)}{2(2n-5)(2n-3)(2n-1)}\binom{2n}{n}^2 \\ + \frac{1}{128}n^2(3n-1)\left(105n^3 - 210n^2 + 147n - 34\right)16^n \quad (7.16)$$

or

$$\sum_{-n\le i,j\le n} \left|ij(j^3 - i^3)^3\right| \binom{2n}{n+i}\binom{2n}{n+j} \\ = \frac{1}{16}n^2\left(1377n^4 - 3870n^3 + 4503n^2 - 2442n + 496\right)4^n\binom{2n}{n} \\ - \frac{4n^3 P_3(n)}{(4n-7)(4n-5)(4n-3)(4n-1)}\binom{4n}{2n}, \quad (7.17)$$

where

$$P_3(n) = 1917n^7 - 11160n^6 + 26439n^5 - 33189n^4 + 23945n^3 - 9951n^2 + 2206n - 201,$$

for example. Obviously, one could also use the summation tools ddescribed in Section 3 to simplify the left-hand sides to their right-hand sides.

(3) In case the reader wonders what would happen if, instead of double sums of the form (1.3), we would consider double sums of the form

$$\sum_{i,j} |i^s j^t (i^k - j^k)^\beta| \binom{2n+1}{n+i}\binom{2m+1}{m+j} \quad (7.18)$$

or mixed sums

$$\sum_{i,j} |i^s j^t (i^k - j^k)^\beta| \binom{2n+1}{n+i}\binom{2m}{m+j}, \quad (7.19)$$

we point out that

$$\binom{2n+1}{n+i} = \frac{n+i+1}{2n+2}\binom{2(n+1)}{n+1+i} = \frac{1}{2}\binom{2(n+1)}{n+1+i} + \frac{i}{2n+2}\binom{2(n+1)}{n+1+i},$$

and thus double sums of the form (7.18) or (7.19) can be written as a linear combination of our familiar double sums (1.3).

8. An inequality for a binomial double sum

In this final section, we establish Conjecture 3.1 from [3], which provides a lower bound on sums of the form (1.3) with $s = t = 0$, $k = 2$, $\beta = 1$.

Theorem 20. *For all non-negative integers m and n, we have*

$$\sum_{i,j} \left|j^2 - i^2\right| \binom{2n}{n+i}\binom{2m}{m+j} \ge 2nm\binom{2n}{n}\binom{2m}{m}, \tag{8.1}$$

and equality holds if and only if $m = n$.

Proof. Without loss of generality, we assume $m \ge n$.

Using the operations $(i,j) \to (-i,j)$ and $(i,j) \to (i,-j)$, which do not change the summand, we see that (8.1) is equivalent to

$$\sum_{0\le i,j} \alpha(i=0)\,\alpha(j=0)\,\left|j^2 - i^2\right| \binom{2n}{n+i}\binom{2m}{m+j} \ge \frac{nm}{2}\binom{2n}{n}\binom{2m}{m}, \tag{8.2}$$

where $\alpha(i=0)$ has the same meaning as in the proof of Proposition 15. By Lemma 21, we see that the claim would be established if we were able to show that

$$\sum_{0\le i<j} \alpha(i=0)\left(\binom{2n}{n+i}\binom{2m-2}{m+j-1} - \binom{2n-2}{n+j-1}\binom{2m}{m+i}\right) \tag{8.3}$$

is non-negative, with equality holding only if $m = n$. Indeed, Lemma 23 says that these two last assertions hold even for each summand in (8.3) individually. (It is at this point that our assumption $m \ge n$ comes into play.) This completes the proof of the theorem. $\square$

Lemma 21. *For all non-negative integers m and n, we have*

$$\begin{aligned}\sum_{0\le i,j} &\alpha(i=0)\,\alpha(j=0)\,\left|j^2 - i^2\right| \binom{2n}{n+i}\binom{2m}{m+j}\\ &= \frac{nm}{2}\binom{2n}{n}\binom{2m}{m}\\ &\quad + 2(m-n)\sum_{0\le i<j} \alpha(i=0)\left(\binom{2n}{n+i}\binom{2m-2}{m+j-1} - \binom{2n-2}{n+j-1}\binom{2m}{m+i}\right).\end{aligned} \tag{8.4}$$

Proof. We write

$$j^2 - i^2 = (n^2 - i^2) - (m^2 - j^2) + (m^2 - n^2)$$

and decompose the sum on the left-hand side of (8.4) into two parts according to whether $i<j$ or $i>j$. Thereby, the sum on the left-hand side of (8.4) becomes

$$\begin{aligned}(2n-1)_2 \sum_{0\le i<j} \alpha(i=0) &\binom{2n-2}{n+i-1}\binom{2m}{m+j}\\ &- (2m-1)_2 \sum_{0\le i<j} \alpha(i=0) \binom{2n}{n+i}\binom{2m-2}{m+j-1}\\ &- (2n-1)_2 \sum_{0\le j<i} \alpha(j=0) \binom{2n-2}{n+i-1}\binom{2m}{m+j}\\ &+ (2m-1)_2 \sum_{0\le j<i} \alpha(j=0) \binom{2n}{n+i}\binom{2m-2}{m+j-1}\\ &+ (m^2-n^2) \sum_{0\le i<j} \alpha(i=0) \left(\binom{2n}{n+i}\binom{2m}{m+j} - \binom{2n}{n+j}\binom{2m}{m+i}\right).\end{aligned} \tag{8.5}$$

We next show how to evaluate the first two (double) sums in (8.5). In the first line of (8.5), we use the decomposition

$$\binom{2m}{m+j} = \binom{2m-2}{m+j} + 2\binom{2m-2}{m+j-1} + \binom{2m-2}{m+j-2}, \tag{8.6}$$

while in the second line we use the same decomposition with m replaced by n and j by i. This leads to

$$\begin{aligned}(2n-1)_2 \sum_{0\le i<j} &\alpha(i=0) \binom{2n-2}{n+i-1}\binom{2m}{m+j}\\ &- (2m-1)_2 \sum_{0\le i<j} \alpha(i=0) \binom{2n}{n+i}\binom{2m-2}{m+j-1}\\ =& (2n-1)_2 \sum_{0\le i<j} \alpha(i=0) \binom{2n-2}{n+i-1}\binom{2m-2}{m+j}\\ &+ (2n-1)_2 \sum_{0\le i<j} \alpha(i=0) \binom{2n-2}{n+i-1}\binom{2m-2}{m+j-2}\\ &- (2n-1)_2 \sum_{0\le i<j} \alpha(i=0) \binom{2n-2}{n+i}\binom{2m-2}{m+j-1}\\ &- (2n-1)_2 \sum_{0\le i<j} \alpha(i=0) \binom{2n-2}{n+i-2}\binom{2m-2}{m+j-1}\\ &+ \big((2n-1)_2-(2m-1)_2\big) \sum_{0\le i<j} \alpha(i=0) \binom{2n}{n+i}\binom{2m-2}{m+j-1}.\end{aligned}$$

By a simultaneous shift of i and j by one, one sees that the first and fourth sum on the right-hand side cancel each other largely, and the same is true for the second and the third sum. Thus, we have

$$\begin{aligned}
&(2n-1)_2 \sum_{0\le i<j} \alpha(i=0)\binom{2n-2}{n+i-1}\binom{2m}{m+j}\\
&\qquad - (2m-1)_2 \sum_{0\le i<j} \alpha(i=0)\binom{2n}{n+i}\binom{2m-2}{m+j-1}\\
&\quad = -\frac{1}{2}(2n-1)_2 \sum_{0<j}\binom{2n-2}{n-1}\binom{2m-2}{m+j}\\
&\qquad - \frac{1}{2}(2n-1)_2 \sum_{0<j}\binom{2n-2}{n-2}\binom{2m-2}{m+j-1}\\
&\qquad + \frac{1}{2}(2n-1)_2 \sum_{0<j}\binom{2n-2}{n-1}\binom{2m-2}{m+j-2}\\
&\qquad + \frac{1}{2}(2n-1)_2 \sum_{0<j}\binom{2n-2}{n}\binom{2m-2}{m+j-1}\\
&\qquad + \big((2n-1)_2-(2m-1)_2\big) \sum_{0\le i<j} \alpha(i=0)\binom{2n}{n+i}\binom{2m-2}{m+j-1}.
\end{aligned}$$

Here, there is more cancellation: the second and fourth sum on the right-hand side cancel each other, while the first and third cancel each other in large parts, with only two terms remaining. As a result, we obtain

$$\begin{aligned}
&(2n-1)_2 \sum_{0\le i<j} \alpha(i=0)\binom{2n-2}{n+i-1}\binom{2m}{m+j}\\
&\qquad - (2m-1)_2 \sum_{0\le i<j} \alpha(i=0)\binom{2n}{n+i}\binom{2m-2}{m+j-1}\\
&\quad = \frac{1}{2}(2n-1)_2 \binom{2n-2}{n-1}\binom{2m-1}{m}\\
&\qquad + \big((2n-1)_2-(2m-1)_2\big) \sum_{0\le i<j} \alpha(i=0)\binom{2n}{n+i}\binom{2m-2}{m+j-1}\\
&\quad = \frac{n^2}{4}\binom{2n}{n}\binom{2m}{m}\\
&\qquad + \big((2n-1)_2-(2m-1)_2\big) \sum_{0\le i<j} \alpha(i=0)\binom{2n}{n+i}\binom{2m-2}{m+j-1}.
\end{aligned}$$

The same calculation, with n and m interchanged, yields

$$\begin{aligned}
&-(2n-1)_2\sum_{0\le j<i}\alpha(j=0)\binom{2n-2}{n+i-1}\binom{2m}{m+j}\\
&\qquad+(2m-1)_2\sum_{0\le j<i}\alpha(j=0)\binom{2n}{n+i}\binom{2m-2}{m+j-1}\\
&=\frac{m^2}{4}\binom{2n}{n}\binom{2m}{m}\\
&\qquad+\big((2m-1)_2-(2n-1)_2\big)\sum_{0\le i<j}\alpha(i=0)\binom{2m}{m+i}\binom{2n-2}{n+j-1}.
\end{aligned}$$

If we put everything together, then we have shown that the sum on the left-hand side of (8.4) equals

$$\begin{aligned}
&\frac{n^2+m^2}{4}\binom{2n}{n}\binom{2m}{m}\\
&\quad+\big(4(m^2-n^2)-2(m-n)\big)\\
&\qquad\times\sum_{0\le i<j}\alpha(i=0)\left(\binom{2n-2}{n+j-1}\binom{2m}{m+i}-\binom{2n}{n+i}\binom{2m-2}{m+j-1}\right)\\
&\quad+(m^2-n^2)\sum_{0\le i<j}\alpha(i=0)\left(\binom{2n}{n+i}\binom{2m}{m+j}-\binom{2n}{n+j}\binom{2m}{m+i}\right).
\end{aligned}$$

If we finally use Lemma 22 in this expression, then the result is the right-hand side of (8.4). □

Lemma 22. *For all non-negative integers m and n, we have*

$$\begin{aligned}
&4\sum_{0\le i<j}\alpha(i=0)\left(\binom{2n-2}{n+j-1}\binom{2m}{m+i}-\binom{2n}{n+i}\binom{2m-2}{m+j-1}\right)\\
&\qquad+\sum_{0\le i<j}\alpha(i=0)\left(\binom{2n}{n+i}\binom{2m}{m+j}-\binom{2n}{n+j}\binom{2m}{m+i}\right)\\
&\qquad\qquad=-\frac{m-n}{4(m+n)}\binom{2n}{n}\binom{2m}{m}. \qquad (8.7)
\end{aligned}$$

Proof. Using the decomposition (8.6) in the second line of (8.7), we compute

$$\begin{aligned}
&4\sum_{0\le i<j}\alpha(i=0)\left(\binom{2n-2}{n+j-1}\binom{2m}{m+i}-\binom{2n}{n+i}\binom{2m-2}{m+j-1}\right)\\
&\qquad+\sum_{0\le i<j}\alpha(i=0)\left(\binom{2n}{n+i}\binom{2m}{m+j}-\binom{2n}{n+j}\binom{2m}{m+i}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{0\le i<j} \alpha(i=0)\Bigg(2\binom{2n-2}{n+j-1}\binom{2m}{m+i} \\
&\qquad\qquad - \binom{2n-2}{n+j}\binom{2m}{m+i} - \binom{2n-2}{n+j-2}\binom{2m}{m+i} \\
&\qquad + \binom{2n}{n+i}\binom{2m-2}{m+j} + \binom{2n}{n+i}\binom{2m-2}{m+j-2} - 2\binom{2n}{n+i}\binom{2m-2}{m+j-1}\Bigg) \\
&= \sum_{0\le i} \alpha(i=0)\Bigg(\binom{2n-2}{n+i}\binom{2m}{m+i} - \binom{2n-2}{n+i-1}\binom{2m}{m+i} \\
&\qquad\qquad + \binom{2n}{n+i}\binom{2m-2}{m+i-1} - \binom{2n}{n+i}\binom{2m-2}{m+i}\Bigg) \\
&= \frac{m-n}{m+n}\sum_{0\le i}\alpha(i=0)\Bigg(\frac{(2n-2)!\,(2m-2)!\,(4nm-4(i+1)n-4(i+1)m+1)}{(n+i)!\,(n-i-1)!\,(m+i)!\,(m-i-1)!} \\
&\qquad\qquad - \frac{(2n-2)!\,(2m-2)!\,(4nm-4in-4im+1)}{(n+i-1)!\,(n-i)!\,(m+i-1)!\,(m-i)!}\Bigg) \\
&= \frac{m-n}{m+n}\Bigg(-\frac{1}{2}\frac{(2n-2)!\,(2m-2)!\,(4nm-4n-4m+1)}{n!\,(n-1)!\,m!\,(m-1)!} \\
&\qquad\qquad - \frac{1}{2}\frac{(2n-2)!\,(2m-2)!\,(4nm+1)}{(n-1)!\,n!\,(m-1)!\,m!}\Bigg) \\
&= -\frac{m-n}{4(m+n)}\binom{2n}{n}\binom{2m}{m},
\end{aligned}
$$

which is the desired result.[3] $\square$

Lemma 23. *For all non-negative integers m, n, i, j with $m \ge n$ and $i < j$, we have*

$$\binom{2n}{n+i}\binom{2m-2}{m+j-1} \ge \binom{2n-2}{n+j-1}\binom{2m}{m+i},$$

with equality if and only if $m = n$.

Proof. We have

$$
\begin{aligned}
\frac{\binom{2n}{n+i}\binom{2m-2}{m+j-1}}{\binom{2n-2}{n+j-1}\binom{2m}{m+i}} &= \frac{2n(2n-1)}{2m(2m-1)}\frac{(m-j+1)(m-j)}{(n-j+1)(n-j)}\prod_{k=i+1}^{j-1}\frac{(n+k)(m-k+1)}{(n-k+1)(m+k)} \\
&= \frac{\left(2+\frac{2j-2}{n-j+1}\right)\left(2+\frac{2j-1}{n-j}\right)}{\left(2+\frac{2j-2}{m-j+1}\right)\left(2+\frac{2j-1}{m-j}\right)}\prod_{k=i+1}^{j-1}\frac{nm+km-(k-1)n-k(k-1)}{nm-(k-1)m+kn-k(k-1)} \ge 1,
\end{aligned}
$$

and visibly equality holds if and only if $m = n$. $\square$

[3] For the finding of the telescoping form of the sum over i see Footnote 1.

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