2-adic valuations of certain ratios of products of factorials and applications

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Abstract

We prove the conjecture of Falikman–Friedland–Loewy on the parity of the degrees of projective varieties of $n \times n$ complex symmetric matrices of rank at most k. We also characterize the parity of the degrees of projective varieties of $n \times n$ complex skew symmetric matrices of rank at most $2p$. We give recursive relations which determine the parity of the degrees of projective varieties of $m \times n$ complex matrices of rank at most k. In the case the degrees of these varieties are odd, we characterize the minimal dimensions of subspaces of $n \times n$ skew symmetric real matrices and of $m \times n$ real matrices containing a nonzero matrix of rank at most k. The parity questions studied here are also of combinatorial interest since they concern the parity of the number of plane partitions contained in a given box, on the one hand, and the parity of the number of symplectic tableaux of rectangular shape, on the other hand.

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1 Introduction

Consider the polynomial equation $z^d + a_1 z^{d-1} + \cdots + a_d = 0$ over the field of complex numbers $\mathbb{C}$. The fundamental theorem of algebra says that this polynomial system has always a nontrivial complex solution ζ. Assume that $a_1, \ldots, a_d$ are real valued. Clearly, this does not imply that the polynomial equation is solvable over the field of real numbers $\mathbb{R}$. Since the complex solutions come in complex pairs, it is well-known that a sufficient condition for an existence of a real solution ζ is that the degree d is odd. A similar situation holds in a more general setting.

Let $\mathbb{PR}^n$ and $\mathbb{P}^n$ be the real and the complex projective space of dimension n, respectively. Let $V \subset \mathbb{PR}^n$ be an algebraic variety, such that its complexification $V_{\mathbb{C}} \subset \mathbb{P}^n$ is irreducible and has codimension $m \geq 1$. Hence $B(V_{\mathbb{C}}) = V_{\mathbb{C}}$, where $B : \mathbb{P}^n \to \mathbb{P}^n$ is the involution $z \mapsto \bar{z}$. Recall that any linear space $M \subset \mathbb{P}^n$ of dimension m intersects $V_{\mathbb{C}}$. Furthermore, for a generic M, the set $V_{\mathbb{C}} \cap M$ consists of a finite number, d say, of points. The positive integer d is called the degree of $V_{\mathbb{C}}$ and is denoted by $\deg V_{\mathbb{C}}$. As in the previous simple case, it is well-known that if $\deg V_{\mathbb{C}}$ is odd then any linear space $L \subset \mathbb{PR}^n$ of dimension m intersects V. Indeed, for a generic $L \subset \mathbb{PR}^n$, the set $V_{\mathbb{C}} \cap L_{\mathbb{C}}$ consists of $\deg V_{\mathbb{C}}$ points. As this set is invariant under the involution B, we deduce that there exists $\zeta \in V_{\mathbb{C}} \cap L_{\mathbb{C}}$ such that $B(\zeta) = \zeta$, that is, $\zeta \in \mathbb{PR}^n$. A continuity argument yields that $V \cap L \neq \emptyset$ for any L in the Graßmannian $\mathrm{Gr}\,(m+1, n+1, \mathbb{R})$. We refer to this result as the *odd degree theorem*.

Assume now that $\deg V_{\mathbb{C}}$ is even. Then it is not difficult to find nontrivial examples where $V \cap L' = \emptyset$ for some $L' \in \mathrm{Gr}\,(m+1, n+1, \mathbb{R})$. Moreover examples can be found among determinantal varieties, such that for any integer $k \in [0, p)$ there exists $L' \in \mathrm{Gr}\,(m+k+1, n+1, \mathbb{R})$ satisfying $V \cap L' = \emptyset$, while $V \cap L \neq \emptyset$ for any $L \in \mathrm{Gr}\,(m+p+1, n+1, \mathbb{R})$ (see [7, §2]).

The following generalization of the odd degree theorem is proved in [7]: Assume that $\deg V_{\mathbb{C}}$ is even and let r be a positive integer. Suppose that the codimension of the variety of the singular points of $V_{\mathbb{C}}$ in $V_{\mathbb{C}}$ is at least $2r+1$. Suppose furthermore that for a generic $M \in \mathrm{Gr}\,(m+2r+1, n+1, \mathbb{C})$ the Euler characteristic of $V_{\mathbb{C}} \cap M$ is odd. Then $V \cap L \neq \emptyset$ for any $L \in \mathrm{Gr}\,(m+2r+1, n+1, \mathbb{R})$.

The degree of $V_{\mathbb{C}}$ and the Euler characteristic of $V_{\mathbb{C}} \cap M$ can be often computed using the projectivized complex bundles and the corresponding Chern classes of their tangent bundles, see for example [13] and [7]. It turns out that the degrees of determinantal varieties (see the special cases discussed below) are quotients of products of certain binomial coefficients (see [12] and [2, Ch. II]). Binomial coefficients appear also in Stiefel–Whitney classes [19].

Let $\mathrm{M}_{m,n}(\mathbb{F})$, $\mathrm{M}_n(\mathbb{F})$, $\mathrm{S}_n(\mathbb{F})$, $\mathrm{A}_n(\mathbb{F})$ be the spaces of $m \times n$ matrices, $n \times n$ matrices, $n \times n$ symmetric matrices, and $n \times n$ skew (antisymmetric) matrices with entries in $\mathbb{F} = \mathbb{R}, \mathbb{C}$, respectively. Let $\mathrm{U}_{k,m,n}(\mathbb{F})$, $\mathrm{V}_{k,n}(\mathbb{F})$, $\mathrm{W}_{k,n}(\mathbb{F})$ be the varieties of all matrices in $\mathrm{M}_{m,n}(\mathbb{F})$, $\mathrm{S}_n(\mathbb{F})$, $\mathrm{A}_n(\mathbb{F})$ of rank $k \leq \min(m,n)$ or less, respectively. It is known that any $A \in \mathrm{A}_n(\mathbb{F})$ has an even rank, see e.g. [9, §11.4] or §5. Hence it is enough to consider $\mathrm{W}_{2p,n}(\mathbb{F})$, where $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$. Then the projectivizations $\mathbb{P}\mathrm{U}_{k,m,n}$, $\mathbb{P}\mathrm{V}_{k,n}(\mathbb{F})$, $\mathbb{P}\mathrm{W}_{2p,n}(\mathbb{F})$ are irreducible varieties of codimension $(m-k)(n-k)$, $\binom{n-k+1}{2}$, $\binom{n-2p}{2}$ in the projective spaces $\mathbb{P}\mathrm{M}_{m,n}(\mathbb{F})$, $\mathbb{P}\mathrm{S}_n(\mathbb{F})$, $\mathbb{P}\mathrm{A}_n(\mathbb{F})$, respectively. Note that $\mathbb{P}\mathrm{U}_{k-1,m,n}(\mathbb{F})$, $\mathbb{P}\mathrm{V}_{k-1,n}(\mathbb{F})$, $\mathbb{P}\mathrm{W}_{2(p-1),n}(\mathbb{F})$ are the varieties of the singular points of $\mathbb{P}\mathrm{U}_{k,m,n}(\mathbb{F})$, $\mathbb{P}\mathrm{V}_{k,n}(\mathbb{F})$, $\mathbb{P}\mathrm{W}_{2p,n}(\mathbb{F})$, respectively. For $\mathbb{P}\mathrm{U}_{k,m,n}(\mathbb{F})$, $\mathbb{P}\mathrm{V}_{k,n}(\mathbb{F})$

see for example [2, II], and for $\mathbb{P}\mathrm{W}_{2p,n}(\mathbb{F})$ see §5. Let $d(m,n,k,\mathbb{F})$, $d_s(n,k,\mathbb{F})$, $d_a(n,2p,\mathbb{F})$ be the smallest integer ℓ such that every ℓ-dimensional subspace of $\mathrm{M}_{m,n}(\mathbb{F})$, $\mathrm{S}_n(\mathbb{F})$, $\mathrm{A}_n(\mathbb{F})$ contains a nonzero matrix whose rank is at most $k, 2p$, respectively. Then

$$d(m,n,k,\mathbb{C}) = (m-k)(n-k)+1,\ d_s(n,k,\mathbb{C}) = \binom{n-k+1}{2}+1,$$
$$d_a(n,2p,\mathbb{C}) = \binom{n-2p}{2}+1, \quad (1.1)$$

and the problem is to determine $d(m,n,k,\mathbb{R})$, $d_s(n,k,\mathbb{R})$, $d_a(n,2p,\mathbb{R})$. The degrees of $\mathbb{P}\mathrm{U}_{k,m,n}(\mathbb{C})$, $\mathbb{P}\mathrm{V}_{k,n}(\mathbb{C})$, $\mathbb{P}\mathrm{W}_{2p,n}(\mathbb{C})$ were computed by Harris and Tu in [12],

$$\gamma_{k,m,n} := \deg \mathbb{P}\mathrm{U}_{k,m,n}(\mathbb{C}) = \prod_{j=0}^{n-k-1} \frac{\binom{m+j}{m-k}}{\binom{m-k+j}{m-k}} = \prod_{j=0}^{n-k-1} \frac{(m+j)!\,j!}{(k+j)!\,(m-k+j)!} \quad (1.2)$$

$$\delta_{k,n} := \deg \mathbb{P}\mathrm{V}_{k,n}(\mathbb{C}) = \prod_{j=0}^{n-k-1} \frac{\binom{n+j}{n-k-j}}{\binom{2j+1}{j}}, \quad \varepsilon_{2p,n} := \deg \mathbb{P}\mathrm{W}_{2p,n} = \frac{\delta_{2p+1,n}}{2^{n-2p-1}} \quad (1.3)$$

For the curiosity of the reader we remark that these quantities have also combinatorial interpretations. The quantity $\gamma_{k,m,n}$ counts plane partitions which are contained in an $(n-k)\times(m-k)\times k$ box (see [4] and §6). On the other hand, the quantity $\varepsilon_{2p,n}$ counts symplectic tableaux (see [15]) of a rectangular shape of size $n\times p$, and thus several other sets of combinatorial objects (see [21] and [10] for more information on these topics).

It was shown in [6] that $\delta_{n-q,n}$ is odd if

$$n \equiv \pm q \pmod{2^{\lceil \log_2 2q \rceil}}. \quad (1.4)$$

For values of q and n which satisfy this condition,

$$d_s(n,n-q,\mathbb{C}) = d_s(n,n-q,\mathbb{R}) = \binom{q+1}{2}+1. \quad (1.5)$$

It was furthermore shown in [6] that this equality implies the following interesting result. Assume that $n \ge q$ and that n satisfies (1.4), then any $\binom{q+1}{2}$-dimensional subspace of $\mathrm{S}_n(\mathbb{R})$ contains a nonzero matrix with an eigenvalue of multiplicity at least q. This statement for $q=2$ yields Lax's result [16] that any 3-dimensional subspace of $\mathrm{S}_n(\mathbb{R})$ contains a nonzero matrix with a multiple eigenvalue for $n \equiv 2 \pmod 4$. (This result and its generalization in [8] is of importance in the study of singularities of hyperbolic systems.)

On the other hand, it was conjectured in [6] that also the converse holds, that is, that if $\delta_{n-q,n}$ is odd then (1.4) holds. In particular, for n and q which do not satisfy (1.4), we do not have a *simple* way to compute $d_s(n,n-q,\mathbb{R})$. One of the main purposes of this paper is to prove this conjecture, see §§3,4. Our results yield in addition that $\varepsilon_{2p,n}$ is odd if and only if (1.4) holds with $q=n-2p$, see §5. Hence for these values of p,n we have

$$d_a(n,2p,\mathbb{C}) = d_a(n,2p,\mathbb{R}) = \binom{n-2p}{2}+1. \quad (1.6)$$

In particular, for $n \equiv 2 \pmod 4$ any two-dimensional subspace of real $n \times n$ skew symmetric matrices contains a nonzero singular matrix. For n and $q = n - 2p$ which do not satisfy the condition (1.4), we do not have a *simple* way to compute $d_a(n, 2p, \mathbb{R})$.

We also consider the problem of characterizing the values of k, m, n for which $\gamma_{k,m,n}$ is odd. This problem seems to have a rather intricate solution. We give some partial results on this problem in §6. In particular, we provide an algorithm for computing the parity of $\gamma_{k,m,n}$ from the binary expansions of k, m, n directly, without having to actually compute $\gamma_{k,m,n}$ (see Remark 6.2 and Proposition 6.5). As above, if $\gamma_{k,m,n}$ is odd then

$$d(m, n, k, \mathbb{C}) = d(m, n, k, \mathbb{R}) = (m - k)(n - k) + 1. \tag{1.7}$$

See Corollary 6.8 for the corresponding geometric results which we are able to prove.

Another purpose of this paper is to estimate the 2-adic valuation of $\delta_{k,n}$, i.e., the largest power of 2 that divides $\delta_{k,n}$. There are two reasons for these estimations. First, we show that our methods can give good estimates for the complex behavior of the 2-adic valuation of $\delta_{k,n}$. Second, we recall the classical results of Radon [18] and Hurwitz [14] on the maximal dimension of the spaces of $n \times n$ scaled orthogonal matrices, and the Adams result [1] on maximal number of vectors fields on the tangent bundle of $n-1$-dimensional sphere, which are functions of the 2-adic valuation of n. We believe that the 2-adic valuation of $\deg \mathbb{P}V_{k,n}(\mathbb{C})$ is related to a lower bound for the problem raised in Friedland–Libgober [7].

Consider the variety of $n \times n$ singular matrices in $\mathrm{M}_n(\mathbb{F})$. Clearly the degree of this variety is n and its codimension is 1. Hence any two-dimensional complex subspace $L \subset \mathrm{M}_n(\mathbb{C})$ contains a nonzero singular matrix. For n odd, any two-dimensional real subspace of $\mathrm{M}_n(\mathbb{R})$ contains a nonzero singular matrix. For n even, the situation is much more complicated. For $n \in \mathbb{N}$, let $c + 4d$ be the 2-adic valuation of n. Thus $n = (2a+1)2^{c+4d}$, where a and d are nonnegative integers and $c \in \{0, 1, 2, 3\}$. Then the Radon–Hurwitz number $\rho(n)$ is defined by $\rho(n) = 2^c + 8d$. The classical results of Radon [18] and Hurwitz [14] state that $\rho(n)$ is the maximal dimension of an n-dimensional subspace U of $\mathrm{M}_n(\mathbb{R})$ such that each nonzero $A \in U$ is an orthogonal matrix times $r \in \mathbb{R}^*$. In his famous work [1], Adams gave a nonlinear version of the Radon–Hurwitz result by showing that $\rho(n) - 1$ is the maximal number of linearly independent vector fields on the $(n-1)$-dimensional sphere in $\mathbb{R}^n$. In particular, Adams's result implies that any $(\rho(n)+1)$-dimensional subspace of $\mathrm{M}_n(\mathbb{R})$ contains a nonzero singular matrix.

The parity of binomial coefficients plays a role in generalized Radon–Hurwitz numbers [3, Prop. I (f)]. Similarly, we believe that the answer to the following problem raised in [7] significantly depends on the 2-adic valuation of $\delta_{k,n}$:

Problem 1.1. *Assume that $\delta_{k,n}$ is even. Find an integer $r \geq 1$, preferably the smallest possible, such that $2r < \binom{n-k+2}{2} - \binom{n-k+1}{2}$, and such that the Euler characteristic of $\mathbb{P}\mathrm{V}_{n,k}(\mathbb{C}) \cap M$ is odd for a generic $M \in \mathrm{Gr}\left(\binom{n-k+1}{2} + 2r + 1, \binom{n+1}{2}, \mathbb{C}\right)$.*

For the above minimal value of r, we have $\mathbb{P}\mathrm{V}_{n,k}(\mathbb{R}) \cap L \neq \emptyset$ for any

$$L \in \mathrm{Gr}\left(\binom{n-k+1}{2} + 2r + 1, \binom{n+1}{2}, \mathbb{R}\right).$$

We now briefly survey the contents of this paper. In §2 we give some auxiliary results on the 2-adic valuation of $\delta_{n-q,n}$. In §3 we prove the conjecture from [6] characterizing the values of q and n for which $\delta_{n-q,n}$ is odd. In §4 we discuss properties of the 2-adic valuation of $\delta_{n-q,n}$ when the condition (1.4) is not satisfied. In particular, we characterize the values of q and n for which the 2-adic valuation of $\delta_{n-q,n}$ is 1, and we show that, for fixed q, the 2-adic valuations of $\delta_{n-q,n}$ have a wave-like behavior as n increases. In §5 we show that $\varepsilon_{2p,n}$ is odd if and only if (1.4) holds with $q = n - 2p$. Hence for these values of n and p the equality (1.6) holds. Finally, in §6 we study the parity of the number of plane partitions contained in an $a \times b \times c$ box, and thus the parity of $\gamma_{k,m,n}$. Some partial results are given, as well as the above-mentioned algorithm for computing this parity.

2 Preliminary results on the 2-adic valuation of $\delta_{n-q,n}$

For a nonzero integer i we write $\nu_2(i)$ for the 2-adic valuation of i. That is $i = (2j+1)2^{\nu_2(i)}$ for some $j \in \mathbb{Z}$. For positive integers q and n define

$$\theta_{q,n} := \prod_{j=0}^{q-1} \frac{\binom{n+j}{q-j}}{\binom{2j+1}{j}}. \tag{2.1}$$

The reader should note that $\theta_{q,n} = \delta_{n-q,n}$ (compare (1.3)). It will be convenient later to extend the definition of $\theta_{q,n}$ to all *nonnegative* integers q and n, that is, to allow $n = 0$, respectively $q = 0$, in (2.1) also. In particular, for $q = 0$ we set $\theta_{0,n} = 1$ by interpreting the empty product as 1.

Clearly, we have $\theta_{q,n} = 0$ for $q > n$. We want to study the behavior of the 2-adic valuation of $\theta_{q,n}$ for $n \geq q$. The following proposition simplifies this study as it exhibits a simple relationship between the 2-adic valuation of $\theta_{q,n}$ when $n - q$ is even and those when $n-q$ is odd. In particular, this result allows one to concentrate on the analysis of just one case, which will be the case where $n - q$ is even.

Proposition 2.1. *Let n and q be nonnegative integers, $n \geq q + 1$. Then*

$$\nu_2(\theta_{q,n}) = \nu_2(\theta_{q+1,n}) + q - \sum_{j=0}^{q} \nu_2(n - q + 2j). \tag{2.2}$$

In particular, if $n - q$ is odd then $\nu_2(\theta_{q,n}) = \nu_2(\theta_{q+1,n}) + q \geq q$. Hence, if n and q are both positive, and if $n - q$ is positive and odd, then $\theta_{q,n}$ is even.

Proof. The ratio of $\theta_{q+1,n}$ and $\theta_{q,n}$ is

$$\begin{aligned}
\frac{\theta_{q+1,n}}{\theta_{q,n}} &= \frac{1}{\binom{2q+1}{q}} \prod_{j=0}^{q} \frac{(n+j)!}{(q+1-j)!\,(n-q-1+2j)!} \prod_{j=0}^{q-1} \frac{(q-j)!\,(n-q+2j)!}{(n+j)!} \\
&= \frac{(q)!\,(q+1)!}{(2q+1)!} \frac{(n+q)!}{1!\,(n+q-1)!} \prod_{j=0}^{q-1} \frac{(q-j)!\,(n-q+2j)!}{(q+1-j)!\,(n-q-1+2j)!} \\
&= \frac{(q+1)!}{2^q(2q+1)!!}(n+q) \prod_{j=0}^{q-1} \frac{(n-q+2j)}{(q+1-j)} = \frac{\prod_{j=0}^{q}(n-q+2j)}{2^q\,(2q+1)!!}.
\end{aligned}$$

(Here $(2q+1)!! := (2q+1)\cdot(2q-1)\cdots 3\cdot 1$.) As $(2q+1)!!$ is odd, we deduce (2.2). Assume that $n-q$ is odd. Then $n-q+2j$ is odd for $j=0,\dots,q$, and the last part of the proposition follows. $\square$

We now concentrate on the case where $n-q$ is even.

Proposition 2.2. *Let n and q be nonnegative integers, $n \geq q$, such that the difference of n and q is even, say $n-q=2p$. Then*

$$\nu_2(\theta_{q,n}) = (n-1-p)p - \nu_2\left(\prod_{k=1}^{p} \frac{(n-k)!}{(k-1)!}\right). \tag{2.3}$$

Here again, in case that $p=0$, the empty product has to be interpreted as 1. In particular, $(n-1-p)p \geq \nu_2\left(\prod_{k=1}^{p} \frac{(n-k)!}{(k-1)!}\right)$.

Proof. We prove (2.3) by a reverse induction on q. By the definition (2.1) of $\theta_{q,n}$ we have $\theta_{n,n}=1$. Hence $\nu_2(\theta_{n,n})=0$, which confirms (2.3) in this case.

Proposition 2.1 implies that for any positive integer k we have

$$\nu_2(\theta_{n-2k+1,n}) = \nu_2(\theta_{n-2k+2,n}) + n - 2k + 1.$$

We now use (2.2) for $q = n-2k$ to obtain the recursive formula

$$\begin{aligned}\nu_2(\theta_{n-2k,n}) &= \nu_2(\theta_{n-2k+2,n}) + n - 2k + 1 + n - 2k - \sum_{j=0}^{n-2k} \nu_2(2k+2j)\\ &= \nu_2(\theta_{n-2k+2,n}) + n - 2k - \sum_{j=0}^{n-2k} \nu_2(k+j)\\ &= \nu_2(\theta_{n-2k+2,n}) + n - 2k - \nu_2\left(\frac{(n-k)!}{(k-1)!}\right).\end{aligned}$$

Use this recursive relation for $k=p,p-1,\dots,1$ to obtain (2.3). Since $\nu_2(\theta_{q,n}) \geq 0$ we obtain that the right-hand side of (2.3) is nonnegative. $\square$

Our next goal is to give an explicit expression of the 2-adic valuation of $\theta_{q,n}$ in terms of binary digit sums. More precisely, for a nonnegative integer m let $s(m)$ denote the sum of the digits of m when written in binary notation. Then $s(0)=0$, $s(2m)=s(m)$, $s(2m+1)=1+s(m)$, and

$$s(2^e-1-k) = e - s(k) \quad \text{for any integers } e \geq 0,\ k \in [0, 2^e-1]. \tag{2.4}$$

The basic result which ties together the 2-adic valuation of factorials and binary digit sums is the following one due to Legendre (cf. [11, Sec. 4.4] and [17]). We bring its proof for completeness.

Proposition 2.3. *Let n be a nonnegative integer. Then $\nu_2(n!) = n - s(n)$.*

Proof. We prove the proposition by induction. Clearly $\nu_2(1) = 0 = 0 - s(0) = 1 - s(1)$. Assume that the proposition holds for $n \leq m-1$. Let $n=m$. Suppose first that $m=2l$. Then $\nu_2((2l)!) = \nu_2((2l)!!) = \nu_2(2^l l!) = l + \nu_2(l!) = l + l - s(l) = m - s(m)$. Assume now that $m = 2l+1$. Then $\nu_2(m!) = \nu_2((2l)!) = 2l - s(2l) = m - s(m)$. $\square$

In what follows we are going to make extensive use of the following lemma and particularly of its corollary.

Lemma 2.4. *Let p and q be nonnegative integers, and assume that $n-q=2p$. Then*

$$\nu_2(\theta_{q,n}) = -p + \sum_{k=1}^{p} s(n-k) - \sum_{k=1}^{p} s(k-1). \tag{2.5}$$

If $p=0$, we have to interpret empty sums as 0, as before. The equation holds also for $q=0$ if we interpret, as earlier, $\theta_{0,2p}$ as 1 for any p.

Proof. Combine (2.3) and Proposition 2.3. □

Since the 2-adic valuation in (2.5) must always be nonnegative, we obtain the following corollary.

Corollary 2.5. *For all nonnegative integers l and p, we have*

$$\sum_{j=l}^{l+p-1} s(j) \geq p + \sum_{j=0}^{p-1} s(j) \quad \text{if } l \geq p, \tag{2.6}$$

and

$$\sum_{j=l}^{l+p-1} s(j) \geq l + \sum_{j=0}^{p-1} s(j) \quad \text{if } l \leq p. \tag{2.7}$$

For $l=p+1,p$, there holds equality,

$$0 = -\sum_{j=0}^{p-1} s(j) + \sum_{j=p+1}^{2p} s(j) - p = -\sum_{j=0}^{p-1} s(j) + \sum_{j=p}^{2p-1} s(j) - p. \tag{2.8}$$

In particular,

$$\sum_{j=l}^{l+p-1} s(j) \geq \sum_{j=0}^{p-1} s(j), \tag{2.9}$$

and equality holds if and only if either $p=0$ or $l=0$.

Proof. Use (2.5) with $l=n-p=p+q$ to deduce (2.6). Assume that $0 \leq l \leq p$. Then by cancelling out the common terms in (2.7) we deduce that (2.7) follows from (2.6) with the roles of l and p being interchanged.

Put $q=0$ in (2.5) and recall that $\theta_{0,n}=1$. This implies the second part of (2.8). Use the equality $s(2p)=s(p)$ to deduce the first part of (2.8).

The inequality (2.9) follows trivially from (2.6) and (2.7). □

3 Proof of the Falikman–Friedland–Loewy Conjecture

In this section, we use the results from the previous section to prove the conjecture from [6] characterizing the values of q and n for which $\delta_{n-q,n} = \theta_{q,n}$ is odd. For the sake of convenience, we state the result in form of the following theorem. The "if" direction was already shown in [6]. Our proof will not only show the "only if" direction, but, in passing, it will also provide an independent proof of the "if" direction.

Theorem 3.1. *For positive integers q and n, the quantity $\theta_{q,n}$ is odd if and only if $n \geq q \geq 1$ and*

$$n \equiv \pm q \pmod{2^{\lceil \log_2 2q \rceil}}.$$

Proof. For $n-q$ odd, the theorem follows from Proposition 2.1. Therefore, for the rest of the proof, let $n-q$ be even.

We repeatedly use subsequently the following observation. Let r, t, j be nonnegative integers such that $2^t > j$. Then $s(r2^t + j) = s(r) + s(j)$. We divide the proof into the two following cases.

CASE 1. $n = 2n_1$. It is enough to assume that $q = 2q_1$ and $n_1 \geq q_1$. Write $p = n_1 - q_1$ and substitute in (2.5) to obtain

$$\nu_2(\theta_{q,n}) = -(n_1 - q_1) - \sum_{j=0}^{n_1-q_1-1} s(j) + \sum_{j=n_1+q_1}^{2n_1-1} s(j). \tag{3.1}$$

We now show that the right-hand side of (3.1) is zero if and only if

$$n_1 \equiv \pm q_1 \pmod{2^{\lceil \log_2 (2q_1) \rceil}},$$

which is obviously equivalent to the theorem in this case.

In CASE 1 we always use the abbreviation $Q = 2^{\lceil \log_2 2q_1 \rceil}$. Write $n_1 = cQ + q_1 + d$, where $0 \leq d < Q$. We know that the quantity from (3.1),

$$-\sum_{j=0}^{n_1-q_1-1} s(j) + \sum_{j=n_1+q_1}^{2n_1-1} s(j) - (n_1 - q_1) = -\sum_{j=0}^{cQ+d-1} s(j) + \sum_{j=cQ+2q_1+d}^{2cQ+2q_1+2d-1} s(j) - (cQ+d), \tag{3.2}$$

is nonnegative. We have to show that it is zero only if $d = 0$ or $d = Q - 2q_1$. To do so, we distinguish various subcases, depending on the size of d.

CASE 1A: $2q_1 + 2d \leq Q$. In this case, the quantity (3.2) becomes

$$\begin{aligned}
-\sum_{r=0}^{c-1}\sum_{j=0}^{Q-1} s(rQ+j) &- \sum_{j=cQ}^{cQ+d-1} s(j) + \sum_{j=cQ+2q_1+d}^{(c+1)Q-1} s(j) \\
&+ \sum_{r=c+1}^{2c-1}\sum_{j=0}^{Q-1} s(rQ+j) + \sum_{j=2cQ}^{2cQ+2q_1+2d-1} s(j) - (cQ+d) \\
= -Q\sum_{r=0}^{c-1} s(r) &- c\sum_{j=0}^{Q-1} s(j) - ds(c) - \sum_{j=0}^{d-1} s(j) + (Q - 2q_1 - d)s(c) + \sum_{j=2q_1+d}^{Q-1} s(j) \\
+ Q\sum_{r=c+1}^{2c-1} s(r) &+ (c-1)\sum_{j=0}^{Q-1} s(j) + (2q_1+2d)s(2c) + \sum_{j=0}^{2q_1+2d-1} s(j) - (cQ+d) \\
= Q\left(-\sum_{r=0}^{c-1} s(r) + \sum_{r=c+1}^{2c} s(r) - c\right) &- \sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d}^{2q_1+2d-1} s(j) - d.
\end{aligned}$$

Using (2.8), we see that the above expression is equal to

$$-\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d}^{2q_1+2d-1} s(j) - d. \tag{3.3}$$

By the definitions of Q and d, we have $Q/2 < 2q_1 < 2q_1 + 2d - 1 < Q$. Thus, we have

$$-\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d}^{2q_1+2d-1} s(j) - d = -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d-\frac{Q}{2}}^{2q_1+2d-\frac{Q}{2}-1} s(j). \tag{3.4}$$

It should be noted that, by the definitions of Q and d, we have $Q/2 < 2q_1 + d$. By (2.9), the quantity on the right-hand side is zero only if the sums on the right-hand side are empty, i.e., if $d = 0$.

CASE 1B: $2q_1 + d \le Q < 2q_1 + 2d$. In this case, the quantity (3.2) becomes

$$\begin{aligned}
&-\sum_{r=0}^{c-1}\sum_{j=0}^{Q-1} s(rQ+j) - \sum_{j=cQ}^{cQ+d-1} s(j) + \sum_{j=cQ+2q_1+d}^{(c+1)Q-1} s(j)\\
&\qquad + \sum_{r=c+1}^{2c}\sum_{j=0}^{Q-1} s(rQ+j) + \sum_{j=(2c+1)Q}^{2cQ+2q_1+2d-1} s(j) - (cQ+d)\\
&= -Q\sum_{r=0}^{c-1} s(r) - c\sum_{j=0}^{Q-1} s(j) - ds(c) - \sum_{j=0}^{d-1} s(j) + (Q-2q_1-d)s(c) + \sum_{j=2q_1+d}^{Q-1} s(j)\\
&\quad + Q\sum_{r=c+1}^{2c} s(r) + c\sum_{j=0}^{Q-1} s(j) + (2q_1+2d-Q)s(2c) + \sum_{j=Q}^{2q_1+2d-1} s(j) - (cQ+d)\\
&= Q\left(-\sum_{r=0}^{c-1} s(r) + \sum_{r=c+1}^{2c} s(r) - c\right) - \sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d}^{2q_1+2d-1} s(j) - d.
\end{aligned}$$

Using (2.8) again, we deduce that the above expression is equal to

$$-\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d}^{2q_1+2d-1} s(j) - d. \tag{3.5}$$

By the definitions of Q and d, we have $Q/2 < 2q_1 < 2q_1+d \le Q \le 2q_1+2d-1 < 2Q$. Thus, the above quantity can be further modified to

$$\begin{aligned}
&-\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d}^{Q-1} s(j) + \sum_{j=Q}^{2q_1+2d-1} s(j) - d\\
&\qquad\qquad = -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d-\frac{Q}{2}}^{\frac{Q}{2}-1} s(j) + \sum_{j=0}^{2q_1+2d-Q-1} s(j)\\
&\qquad\qquad = -\sum_{j=2q_1+2d-Q}^{d-1} s(j) + \sum_{j=2q_1+d-\frac{Q}{2}}^{\frac{Q}{2}-1} s(j).
\end{aligned}$$

From $Q/2 < 2q_1$ and $2q_1 + d \le Q$, we infer that $d < Q/2$. Now we use identity (2.4) with $2^e = \frac{Q}{2}$ for all the digit sums in the last expression. This leads to the expression

$$\sum_{j=\frac{Q}{2}-d}^{\frac{3}{2}Q-2q_1-2d-1} s(j) - \sum_{j=0}^{Q-2q_1-d-1} s(j).$$

We recall that $Q-2q_1-d \geq 0$. Apply (2.9) again to conclude that the last expression, and hence, $\nu_2(\theta_{q,n})$, is zero only if the sums in the last line are empty, that is, if $d = Q - 2q_1$.

Case 1c: $Q < 2q_1 + d$ and $2q_1 + 2d \leq 2Q$. In this case, the quantity (3.2) becomes

$$\begin{aligned}
&-\sum_{r=0}^{c-1}\sum_{j=0}^{Q-1} s(rQ+j) - \sum_{j=cQ}^{cQ+d-1} s(j) + \sum_{j=cQ+2q_1+d}^{(c+2)Q-1} s(j) \\
&\qquad + \sum_{r=c+2}^{2c}\sum_{j=0}^{Q-1} s(rQ+j) + \sum_{j=(2c+1)Q}^{2cQ+2q_1+2d-1} s(j) - (cQ+d) \\
&= -Q\sum_{r=0}^{c-1} s(r) - c\sum_{j=0}^{Q-1} s(j) - ds(c) - \sum_{j=0}^{d-1} s(j) + (2Q-2q_1-d)s(c+1) \\
&\qquad + \sum_{j=2q_1+d-Q}^{Q-1} s(j) + Q\sum_{r=c+2}^{2c} s(r) + (c-1)\sum_{j=0}^{Q-1} s(j) \\
&\qquad + (2q_1+2d-Q)s(2c+1) + \sum_{j=0}^{2q_1+2d-Q-1} s(j) - (cQ+d) \\
&= Q\left(-\sum_{r=0}^{c-1} s(r) + \sum_{r=c+1}^{2c} s(r) - c\right) - \sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d-Q}^{2q_1+2d-Q-1} s(j) \\
&\qquad + (2q_1+d-Q)(s(c)-s(c+1)+1) \\
&= -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d-Q}^{2q_1+2d-Q-1} s(j) + (2q_1+d-Q)(s(c)-s(c+1)+1). \qquad (3.6)
\end{aligned}$$

In the second step we used the equality $s(2c+1) = s(c)+1$, and in the last step we used again (2.8). Since $Q < 2q_1+d$, we have $2q_1+d-Q > 0$. As $s(c)-s(c+1)+1 \geq 0$ for any $c \geq 0$, the third term in the last line is nonnegative. As $d > 0$, the inequality (2.9) implies that the sum of the first two terms is strictly positive.

Case 1d: $Q < 2q_1 + d$ and $2q_1 + 2d > 2Q$. In this case, the quantity (3.2) becomes

$$\begin{aligned}
&-\sum_{r=0}^{c-1}\sum_{j=0}^{Q-1} s(rQ+j) - \sum_{j=cQ}^{cQ+d-1} s(j) + \sum_{j=cQ+2q_1+d}^{(c+2)Q-1} s(j) \\
&\qquad + \sum_{r=c+2}^{2c+1}\sum_{j=0}^{Q-1} s(rQ+j) + \sum_{j=(2c+2)Q}^{2cQ+2q_1+2d-1} s(j) - (cQ+d) \\
&= -Q\sum_{r=0}^{c-1} s(r) - c\sum_{j=0}^{Q-1} s(j) - ds(c) - \sum_{j=0}^{d-1} s(j) + (2Q-2q_1-d)s(c+1)
\end{aligned}$$

$$+\sum_{j=2q_1+d-Q}^{Q-1} s(j) + Q\sum_{r=c+2}^{2c+1} s(r) + c\sum_{j=0}^{Q-1} s(j)$$
$$+ (2q_1+2d-2Q)s(2c+2) + \sum_{j=0}^{2q_1+2d-2Q-1} s(j) - (cQ+d).$$

We now do the following substitutions. First, $s(2(c+1)) = s(c+1)$. Second, in the sum over $j = 0, \ldots, 2q_1+2d-2Q-1$ (where $2q_1+2d-2Q-1 \le Q+2(Q-1)-2Q-1 = Q-1$), we let $s(j) = s(Q+j)-1$. Third,

$$\sum_{r=c+2}^{2c+1} s(r) = s(2c+1) - s(c+1) + \sum_{r=c+1}^{2c} s(r) = s(c)+1-s(c+1) + \sum_{r=c+1}^{2c} s(r).$$

Hence, the above expression is equal to

$$Q\left(-\sum_{r=0}^{c-1} s(r) + \sum_{r=c+1}^{2c} s(r) - c\right) - \sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d-Q}^{Q-1} s(j)$$
$$+ \sum_{j=Q}^{2q_1+2d-Q-1} s(j) - (2q_1+2d-2Q) + (Q-d)(s(c)-s(c+1)+1)$$
$$= -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d-Q}^{2q_1+2d-Q-1} s(j) - (2q_1+2d-2Q) + (Q-d)(s(c)-s(c+1)+1), \tag{3.7}$$

where we used again (2.8). Since $Q > d$ and $s(c)-s(c+1)+1 \ge 0$ for any $c \ge 0$, the fourth term in the last line is nonnegative. We have $\frac{Q}{2} < d < Q$ and $d \ge 2q_1+d-Q$. Thus, we may apply (2.7) to deduce that the sum of the first two terms is at least $2q_1+d-Q$. Hence the expression (3.7) is not less than $Q-d > 0$.

The proof of CASE 1 is completed.

CASE 2. Let $n = 2n_1+1$, $q = 2q_1+1$, where $n_1 \ge q_1 \ge 0$. Write $p = n_1 - q_1$ and substitute in (2.5) to obtain

$$\nu_2(\theta_{q,n}) = -(n_1-q_1) - \sum_{j=0}^{n_1-q_1-1} s(j) + \sum_{j=n_1+q_1+1}^{2n_1} s(j). \tag{3.8}$$

We now show that the right-hand side of (3.8) is zero if and only if

$$n_1 \equiv q_1 \pmod{2^{\lceil \log_2(2q_1+1)\rceil}}$$

or

$$n_1 \equiv -q_1 - 1 \pmod{2^{\lceil \log_2(2q_1+1)\rceil}},$$

which is obviously equivalent to the theorem in this case.

In CASE 2 we always use the abbreviation $Q = 2^{\lceil \log_2(2q_1+1)\rceil}$. Write $n_1 = cQ + q_1 + d$, where $0 \le d < Q$. We have to show that the expression (3.8), that is,

$$- \sum_{j=0}^{n_1-q_1-1} s(j) + \sum_{j=n_1+q_1+1}^{2n_1} s(j) - (n_1 - q_1)$$
$$= - \sum_{j=0}^{cQ+d-1} s(j) + \sum_{j=cQ+2q_1+d+1}^{2cQ+2q_1+2d} s(j) - (cQ+d), \quad (3.9)$$

is zero only if $d = 0$ or $d = Q - 2q_1 - 1$. To do so, we distinguish again various cases, depending on the size of d.

Case 2a: $2q_1 + 2d < Q$. In this case, the quantity (3.9) becomes

$$\begin{aligned}
&- \sum_{r=0}^{c-1} \sum_{j=0}^{Q-1} s(rQ+j) - \sum_{j=cQ}^{cQ+d-1} s(j) + \sum_{j=cQ+2q_1+d+1}^{(c+1)Q-1} s(j) \\
&\qquad\qquad + \sum_{r=c+1}^{2c-1} \sum_{j=0}^{Q-1} s(rQ+j) + \sum_{j=2cQ}^{2cQ+2q_1+2d} s(j) - (cQ+d) \\
&= -Q \sum_{r=0}^{c-1} s(r) - c \sum_{j=0}^{Q-1} s(j) - ds(c) - \sum_{j=0}^{d-1} s(j) \\
&\quad + (Q - 2q_1 - d - 1)s(c) + \sum_{j=2q_1+d+1}^{Q-1} s(j) \\
&\quad + Q \sum_{r=c+1}^{2c-1} s(r) + (c-1) \sum_{j=0}^{Q-1} s(j) + (2q_1 + 2d + 1)s(2c) + \sum_{j=0}^{2q_1+2d} s(j) - (cQ+d) \\
&= Q \left(- \sum_{r=0}^{c-1} s(r) + \sum_{r=c+1}^{2c} s(r) - c \right) - \sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d+1}^{2q_1+2d} s(j) - d \\
&= - \sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d+1}^{2q_1+2d} s(j) - d,
\end{aligned}$$

where in the last line we used again (2.8). The reader should compare this expression with the one in (3.3). Indeed, the remaining arguments are completely analogous to those after (3.3) in Case 1a of the current proof, which are therefore left to the reader.

Case 2b: $2q_1 + d < Q \le 2q_1 + 2d$. In this case, the quantity (3.9) becomes

$$\begin{aligned}
&- \sum_{r=0}^{c-1} \sum_{j=0}^{Q-1} s(rQ+j) - \sum_{j=cQ}^{cQ+d-1} s(j) + \sum_{j=cQ+2q_1+d+1}^{(c+1)Q-1} s(j) \\
&\qquad\qquad + \sum_{r=c+1}^{2c} \sum_{j=0}^{Q-1} s(rQ+j) + \sum_{j=(2c+1)Q}^{2cQ+2q_1+2d} s(j) - (cQ+d) \\
&= -Q \sum_{r=0}^{c-1} s(r) - c \sum_{j=0}^{Q-1} s(j) - ds(c) - \sum_{j=0}^{d-1} s(j)
\end{aligned}$$

$$+ (Q - 2q_1 - d - 1)s(c) + \sum_{j=2q_1+d+1}^{Q-1} s(j)$$

$$+ Q \sum_{r=c+1}^{2c} s(r) + c \sum_{j=0}^{Q-1} s(j) + (2q_1 + 2d + 1 - Q)s(2c) + \sum_{j=Q}^{2q_1+2d} s(j) - (cQ + d)$$

$$= Q \left(-\sum_{r=0}^{c-1} s(r) + \sum_{r=c+1}^{2c} s(r) - c \right) - \sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d+1}^{2q_1+2d} s(j) - d$$

$$= -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d+1}^{2q_1+2d} s(j) - d,$$

where in the last line we used again (2.8). The reader should compare this expression with the one in (3.5). Indeed, the remaining arguments are completely analogous to those after (3.5) in Case 1b of the current proof, which are therefore left to the reader.

Case 2c. $Q \le 2q_1 + d$ and $2q_1 + 2d < 2Q$. In this case, the quantity (3.9) becomes

$$-\sum_{r=0}^{c-1} \sum_{j=0}^{Q-1} s(rQ + j) - \sum_{j=cQ}^{cQ+d-1} s(j) + \sum_{j=cQ+2q_1+d+1}^{(c+2)Q-1} s(j)$$

$$+ \sum_{r=c+2}^{2c} \sum_{j=0}^{Q-1} s(rQ + j) + \sum_{j=(2c+1)Q}^{2cQ+2q_1+2d} s(j) - (cQ + d)$$

$$= -Q \sum_{r=0}^{c-1} s(r) - c \sum_{j=0}^{Q-1} s(j) - ds(c) - \sum_{j=0}^{d-1} s(j) + (2Q - 2q_1 - d - 1)s(c + 1)$$

$$+ \sum_{j=2q_1+d+1-Q}^{Q-1} s(j) + Q \sum_{r=c+2}^{2c} s(r) + (c - 1) \sum_{j=0}^{Q-1} s(j)$$

$$+ (2q_1 + 2d + 1 - Q)s(2c + 1) + \sum_{j=0}^{2q_1+2d-Q} s(j) - (cQ + d)$$

$$= Q \left(-\sum_{r=0}^{c-1} s(r) + \sum_{r=c+1}^{2c} s(r) - c \right) - \sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d+1-Q}^{2q_1+2d-Q} s(j)$$

$$+ (2q_1 + d + 1 - Q)(s(c) - s(c + 1) + 1)$$

$$= -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d+1-Q}^{2q_1+2d-Q} s(j) + (2q_1 + d + 1 - Q)(s(c) - s(c + 1) + 1),$$

where in the last step we used again (2.8). Since $Q < 2q_1 + d + 1$, we have $2q_1 + d + 1 - Q > 0$. In particular, since also $s(c) - s(c + 1) + 1 \ge 0$ for any c, the third term in the last line is nonnegative, and, because $d > 0$, the inequality (2.9) says that the sum of the first two terms is strictly positive.

CASE 2D. $Q \le 2q_1 + d$ and $2q_1 + 2d \ge 2Q$. In this case, the quantity (3.9) becomes

$$
\begin{aligned}
&-\sum_{r=0}^{c-1}\sum_{j=0}^{Q-1} s(rQ+j) - \sum_{j=cQ}^{cQ+d-1} s(j) + \sum_{j=cQ+2q_1+d+1}^{(c+2)Q-1} s(j)\\
&\qquad + \sum_{r=c+2}^{2c+1}\sum_{j=0}^{Q-1} s(rQ+j) + \sum_{j=(2c+2)Q}^{2cQ+2q_1+2d} s(j) - (cQ+d)\\
&= -Q\sum_{r=0}^{c-1} s(r) - c\sum_{j=0}^{Q-1} s(j) - ds(c) - \sum_{j=0}^{d-1} s(j) + (2Q-2q_1-d-1)s(c+1)\\
&\qquad + \sum_{j=2q_1+d+1-Q}^{Q-1} s(j) + Q\sum_{r=c+2}^{2c+1} s(r) + c\sum_{j=0}^{Q-1} s(j)\\
&\qquad + (2q_1+2d+1-2Q)s(2c+2) + \sum_{j=0}^{2q_1+2d-2Q} s(j) - (cQ+d)\\
&= Q\left(-\sum_{r=0}^{c-1} s(r) + \sum_{r=c+1}^{2c} s(r) - c\right) - \sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d+1-Q}^{Q-1} s(j)\\
&\qquad + \sum_{j=Q}^{2q_1+2d-Q} s(j) - (2q_1+2d-2Q+1) + (Q-d)(s(c)-s(c+1)+1)\\
&= -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d+1-Q}^{2q_1+2d-Q} s(j)\\
&\qquad - (2q_1+2d-2Q+1) + (Q-d)(s(c)-s(c+1)+1),
\end{aligned}
$$

where in the last step we used again (2.8). We have $d < Q$. In particular, since also $s(c) - s(c+1) + 1 \ge 0$ for any c, the fourth term in the last line is nonnegative. Moreover, because $2q_1 + d + 1 - Q \le d$, the inequality (2.7) says that the sum of the first two terms is at least $2q_1 + d + 1 - Q = (2q_1 + 2d - 2Q + 1) + (Q - d)$. Thus, the sum of the first three terms in the last line is strictly positive. Combining both findings, we infer that the whole last line is strictly positive (in fact, at least $Q - d$).

This finishes the proof of CASE 2 and, thus, of the theorem. □

4 Additional results on the 2-adic valuation of $\delta_{n-q,n} = \theta_{q,n}$

In this section, we examine the 2-adic behavior of $\theta_{q,n} = \delta_{n-q,n}$ in more detail. Keeping in mind Proposition 2.1, we concentrate throughout this section on the case that n and q have the same parity.

If we fix q and let $n = q + 2i$, $i = 0, 1, 2, \dots,$ vary, then we know that whenever (1.4) is satisfied, the 2-adic valuation of $\theta_{q,n} = \theta_{q,q+2i}$ will be zero. However, what happens in between? By looking at some random values of q, one might get the impression that, between two successive occurrences of zero, the 2-adic valuations $\nu_2(\theta_{q,q+2i})$ are unimodal, that is, they first grow (weakly) monotone until they reach their maximum value half-way, and then they drop (weakly) monotone until they come back to zero in the end. Moreover, one is led to guess that the 2-adic valuations are symmetric around the place where the maximum is attained. As it turns out (see Example 4.1 below), the unimodality conjecture is not true, while the symmetry conjecture is indeed true. However, in some sense, unimodality is "almost" true. As we demonstrate in Theorems 4.2, 4.4, 4.6, 4.8 below, between two successive values of zero, the 2-adic valuations $\nu_2(\theta_{q,q+2i})$ stay above a linear function of slope 1 which is tight at the opening zero, and at the same time they stay below another linear function of slope 1 which is tight at the maximum (see (4.1) and (4.2) in Theorem 4.2 and the analogous inequalities in the subsequent theorems), until they reach the maximum value, which is attained exactly half-way, and the 2-adic valuations beyond are the symmetric images of those before. In the theorems, we determine in addition the maximal value for each of these intervals.

The results which are found on the way to prove these theorems allow one also to address the following question: characterize all values of q and n for which the 2-adic valuation of $\theta_{q,n}$ has a certain fixed (small) value. Clearly, Theorem 3.1 does this if we fix this value to 0. In Corollaries 4.3, 4.5, 4.7, 4.9 we work out the analogous characterization if we fix this value to 1. We could move on to 2, 3, etc., but at the cost of a considerable increase of complication the further we go.

Example 4.1. The sequence $(\nu_2(\theta_{q,q+2i}))_{i\ge 0}$ is not unimodal in the intervals discussed in Theorems 4.2–4.8 (although the theorems show that it comes very close). Here we give two counter-examples.

Consider $q = 39$. In this case, the first 200 terms of the sequence $(\nu_2(\theta_{q,q+2i}))_{i\ge 0}$ are:

$$0,1,3,5,7,8,8,8,8,9,12,15,18,18,15,12,9,8,8,8,8,7,5,3,1,0,3,6,9,10,9,8,7,9,\\ 12,15,18,20,21,22,23,25,29,33,37,37,33,29,25,23,22,21,20,18,15,12,9,7,8,9,\\ 10,9,6,3,0,1,3,5,7,8,8,8,8,9,12,15,18,18,15,12,9,8,8,8,8,7,5,3,1,0,4,8,12,\\ 14,14,14,14,17,21,25,29,32,34,36,38,41,46,51,56,56,51,46,41,38,36,34,32,\\ 29,25,21,17,14,14,14,14,12,8,4,0,1,3,5,7,8,8,8,8,9,12,15,18,18,15,12,9,8,\\ 8,8,8,7,5,3,1,0,3,6,9,10,9,8,7,9,12,15,18,20,21,22,23,25,29,33,37,37,33,\\ 29,25,23,22,21,20,18,15,12,9,7,8,9,10,9,6,3,0,1,3,5,7,8,8,8,8.$$

As predicted by Theorem 3.1, zeros occur for all multiples of $\frac{1}{2}2^{\lceil \log_2 2\cdot 39\rceil} = 64$ and for multiples of 64 reduced by 39. However, while the first interval (the interval comprising the first 25 values) is unimodal, the second one (comprising the next 39 values) is not, as can be seen from the subsequence $0,3,6,9,10,9,8,7,9,12,15,\dots,37,\dots$, which rises first to 10, then drops to 7, to rise again beyond 10 until it reaches the maximum of 37 in this interval.

Consider now $q = 46$. In this case, the first 200 terms are:

$$0,4,2,5,6,10,10,13,14,19,14,13,10,10,6,5,2,4,0,3,4,8,8,11,12,17,14,15,14,16,\\ 14,15,14,19,18,22,24,29,30,34,36,42,36,34,30,29,24,22,18,19,14,15,14,16,\\ 14,15,14,17,12,11,8,8,4,3,0,4,2,5,6,10,10,13,14,19,14,13,10,10,6,5,2,4,0,\\ 4,6,11,12,16,18,24,22,24,24,27,26,28,28,34,34,39,42,48,50,55,58,65,58,55,\\ 50,48,42,39,34,34,28,28,26,27,24,24,22,24,18,16,12,11,6,4,0,4,2,5,6,10,10,\\ 13,14,19,14,13,10,10,6,5,2,4,0,3,4,8,8,11,12,17,14,15,14,16,14,15,14,19,$$

$18, 22, 24, 29, 30, 34, 36, 42, 36, 34, 30, 29, 24, 22, 18, 19, 14, 15, 14, 16, 14, 15, 14, 17,$
$12, 11, 8, 8, 4, 3, 0, 4, 2, 5, 6, 10, 10, 13, 14.$

This is an example where *both* types of intervals are not unimodular. The first interval (comprising the first $64 - 46 = 18$ values) begins $0, 4, 2, 5, 6, 10, 10, 13, 14, 19, \dots$, that is, rises to 4, drops to 2, before rising again up to the maximum of 19 in this interval. The second one (comprising the next 46 values) starts by $0, 3, 4, 8, 8, 11, 12, 17, 14, 15, \dots, 42, \dots$, that is, rises to 17, drops to 14, before rising again up to the maximum of 42 in this interval. Similar remarks apply to the subsequent intervals.

Nevertheless, between two successive zeros, although the 2-adic valuations $\nu_2(\theta_{q,q+2i})$ are not unimodal in general, the 2-adic valuations still seem to exhibit an overall increase until a maximum halfway and then a decrease which is the symmetric image of the increasing values. In the theorems below, we quantify this statement. We split our results into four separate theorems. On the one hand, we have to distinguish between the two possible parities of q, and, on the other hand, for each integer q, there are two types of intervals to be considered.

Theorem 4.2. *Let q be a fixed positive even integer, $Q = 2^{\lceil \log_2 q \rceil}$, and let c be any nonnegative integer. Then the values of the sequence $(\nu_2(\theta_{q,q+2i}))_{i \geq 0}$ for $i \in [cQ, (c+1)Q - q]$ are symmetric around its center $i_1 = cQ + (Q - q)/2$. The values at the extreme points of the interval $i = cQ$ and $i = (c+1)Q - q$ are 0, and the value at the center is*

$$\nu_2(\theta_{q,q+2i_1}) = \frac{Q-q}{2} \log_2 \frac{Q}{2} - 2 \sum_{j=0}^{\frac{Q-q-2}{2}} s(j),$$

independent of c. Furthermore, for $i \in [cQ, i_1]$ the inequalities

$$\nu_2(\theta_{q,q+2i}) \geq i - cQ \tag{4.1}$$

and

$$\nu_2(\theta_{q,q+2i}) \leq \nu_2(\theta_{q,q+2i_1}) - (i_1 - i) \tag{4.2}$$

hold.

Proof. In this proof, we use again the notation $q_1 = q/2$. To show the symmetry write $i = cQ + d$ with $0 \leq d \leq Q - 2q_1$. In particular, we have $2q_1 + d \leq Q$, and, hence (cf. (3.3) and (3.5)), the 2-adic valuation of $\theta_{q,q+2i}$ is given by

$$\nu_2(\theta_{q,q+2i}) = \nu_2(\theta_{q,q+2(cQ+d)}) = -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d}^{2q_1+2d-1} s(j) - d. \tag{4.3}$$

On the other hand, for the same reason, the 2-adic valuation of $\theta_{q,q+2(cQ+Q-q-d)}$ is given by

$$\nu_2(\theta_{q,q+2(cQ+Q-2q_1-d)}) = -\sum_{j=0}^{Q-2q_1-d-1} s(j) + \sum_{j=2q_1+(Q-2q_1-d)}^{2q_1+2(Q-2q_1-d)-1} s(j) - (Q - 2q_1 - d)$$

$$
\begin{aligned}
&= -\sum_{j=0}^{Q-2q_1-d-1} s(j) + \sum_{j=Q-d}^{2Q-2q_1-2d-1} s(j) - (Q-2q_1-d)\\
&= -\sum_{j=0}^{Q-1} s(j) + \sum_{j=Q-2q_1-d}^{Q-1} s(j)\\
&\qquad + \sum_{j=0}^{2Q-1} s(j) - \sum_{j=0}^{Q-d-1} s(j) - \sum_{j=2Q-2q_1-2d}^{2Q-1} s(j) - (Q-2q_1-d)\\
&= \sum_{j=Q-2q_1-d}^{Q-1} s(j) + \sum_{j=Q}^{2Q-1} s(j) - \sum_{j=0}^{Q-d-1} s(j) - \sum_{j=2Q-2q_1-2d}^{2Q-1} s(j) - (Q-2q_1-d)\\
&= \sum_{j=Q-2q_1-d}^{Q-1} s(j) + \sum_{j=Q-d}^{Q-1} s(j) - \sum_{j=2Q-2q_1-2d}^{2Q-1} s(j) + (2q_1+d).
\end{aligned}
$$

Now we apply the reflection identity (2.4) to all the sum of digit functions. Thus, we obtain

$$
\begin{aligned}
\nu_2(\theta_{q,q+2(cQ+Q-2q_1-d)}) &= (2q_1+2d)\log_2 Q - \sum_{j=0}^{2q_1+d-1} s(j) - \sum_{j=0}^{d-1} s(j)\\
&\qquad - (2q_1+2d)\log_2(2Q) + \sum_{j=0}^{2q_1+2d-1} s(j) + (2q_1+d)\\
&= -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d}^{2q_1+2d-1} s(j) - d\\
&= \nu_2(\theta_{q,q+2(cQ+d)}),
\end{aligned}
$$

proving the symmetry. That the values of the extreme points of the interval $i = cQ$ and $i = (c+1)Q - q$ are 0, was already shown in Theorem 3.1, Case 1.

Next we determine the 2-adic value of $\theta_{q,q+2i}$ at the center $i_1 = cQ + (Q-q)/2$ of the interval. By (4.3), we have

$$
\begin{aligned}
\nu_2(\theta_{q,q+2i_1}) &= \nu_2(\theta_{q,q+2(cQ+Q/2-q_1)})\\
&= -\sum_{j=0}^{\frac{Q}{2}-q_1-1} s(j) + \sum_{j=\frac{Q}{2}+q_1}^{Q-1} s(j) - \left(\frac{Q}{2}-q_1\right)\\
&= -\sum_{j=0}^{\frac{Q}{2}-q_1-1} s(j) + \sum_{j=q_1}^{\frac{Q}{2}-1} s(j)\\
&= \left(\frac{Q}{2}-q_1\right)\log_2\frac{Q}{2} - 2\sum_{j=0}^{\frac{Q}{2}-q_1-1} s(j).
\end{aligned}
$$

To prove the inequality (4.1), we write again $i = cQ + d$, with $d \le Q/2 - q_1$. Under this condition, we showed in the proof of Theorem 3.1, Case 1a (see (3.4)), that $\nu_2(\theta_{q,q+2i})$ can be written in the form

$$\nu_2(\theta_{q,q+2i}) = \nu_2(\theta_{q,q+2(cQ+d)}) = -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d-\frac{Q}{2}}^{2q_1+2d-\frac{Q}{2}-1} s(j). \tag{4.4}$$

Since $d \leq 2q_1 + d - Q$, the inequality (2.6) implies that this expression is at least $d = i - cQ$.

To prove inequality (4.2), we compute the difference of the 2-adic valuations of $\theta_{q,q+2i_1}$ and $\theta_{q,q+2i}$ for $i \in [cQ, i_1]$: let again $i = cQ + d$, $0 \leq d \leq (Q-q)/2$. Then, using (4.3), we have

$$\begin{aligned}
\nu_2(\theta_{q,q+2i_1}) - \nu_2(\theta_{q,q+2i}) &= \nu_2(\theta_{q,q+2(cQ+Q/2-q_1)}) - \nu_2(\theta_{q,q+2(cQ+d)}) \\
&= -\sum_{j=0}^{\frac{Q}{2}-q_1-1} s(j) + \sum_{j=\frac{Q}{2}+q_1}^{Q-1} s(j) - \left(\frac{Q}{2} - q_1\right) \\
&\qquad + \sum_{j=0}^{d-1} s(j) - \sum_{j=2q_1+d}^{2q_1+2d-1} s(j) + d \\
&= -\sum_{j=d}^{\frac{Q}{2}-q_1-1} s(j) - \sum_{j=2q_1+d}^{\frac{Q}{2}+q_1-1} s(j) + \sum_{j=2q_1+2d}^{Q-1} s(j) - \left(\frac{Q}{2} - q_1 - d\right).
\end{aligned}$$

Now we apply the reflection formula (2.4) to all the sum of digits functions. This yields

$$\begin{aligned}
\nu_2(\theta_{q,q+2i_1}) - \nu_2(\theta_{q,q+2i}) &= \sum_{j=q_1}^{\frac{Q}{2}-d-1} s(j) + \sum_{j=\frac{Q}{2}-q_1}^{Q-2q_1-d-1} s(j) - \sum_{j=0}^{Q-2q_1-2d-1} s(j) \\
&= \sum_{j=q_1}^{\frac{Q}{2}-d-1} s(j) + \sum_{j=\frac{Q}{2}-q_1}^{Q-2q_1-d-1} s(j) - \sum_{j=0}^{\frac{Q}{2}-q_1-d-1} (s(2j) + s(2j+1)) \\
&= \sum_{j=q_1}^{\frac{Q}{2}-d-1} s(j) + \sum_{j=\frac{Q}{2}-q_1}^{Q-2q_1-d-1} s(j) - 2\sum_{j=0}^{\frac{Q}{2}-q_1-d-1} s(j) - \left(\frac{Q}{2} - q_1 - d\right).
\end{aligned}$$

Since $q_1 \geq \frac{Q}{2} - q_1 \geq \frac{Q}{2} - q_1 - d$, we may apply (2.6) twice to obtain finally

$$\nu_2(\theta_{q,q+2i_1}) - \nu_2(\theta_{q,q+2i}) \geq \left(\frac{Q}{2} - q_1 - d\right) = i_1 - i,$$

as desired. $\square$

Corollary 4.3. *For a fixed even q, the values of the sequence $(\nu_2(\theta_{q,q+2i}))_{i\geq 0}$ are never 1 for $i \in [cQ, (c+1)Q - q]$.*

Proof. By the symmetry of the values around the center of the interval, and by the inequality (4.1), the only possible candidates are $i = cQ+1$ and $i = (c+1)Q-q-1$. Formula (4.4) for $\nu_2(\theta_{q,q+2i})$ implies that in that case we must have

$$s\left(q+1-\frac{Q}{2}\right) = 1.$$

This is only possible if $q+1-\frac{Q}{2}$ is a power of 2, and this, in its turn, implies that q must be odd, a contradiction. $\square$

Theorem 4.4. *Let q be a fixed positive even integer, $Q = 2^{\lceil \log_2 q \rceil}$, and let c be any nonnegative integer. Then the values of the sequence $(\nu_2(\theta_{q,q+2i}))_{i\geq 0}$ for $i \in [(c+1)Q-q,(c+1)Q]$ are symmetric around its center $i_2 = (c+1)Q-q/2$. The values at the extreme points of the interval $i = (c+1)Q-q$ and $i = (c+1)Q$ are 0, and the value at the center is*

$$\nu_2(\theta_{q,q+2i_2}) = \frac{q}{2}\log_2 Q - 2\sum_{j=0}^{\frac{q-2}{2}} s(j) + \frac{q}{2}(s(c)-s(c+1)+1).$$

Furthermore, for $i \in [i_2,(c+1)Q]$ the inequalities

$$\nu_2(\theta_{q,q+2i}) \geq (c+1)Q - i \tag{4.5}$$

and

$$\nu_2(\theta_{q,q+2i}) \leq \nu_2(\theta_{q,q+2i_2}) - (i-i_2) \tag{4.6}$$

hold. $\square$

Proof. We use again the notation $q_1 = q/2$. To show the symmetry write $i = cQ+d$ with $Q-2q_1 \leq d \leq Q$. In particular, we have $2q_1+d \geq Q$, and, hence, if $d \leq Q-q_1$, the 2-adic valuation of $\theta_{q,q+2i}$ is given by (cf. (3.6), which also holds if $2q_1+d = Q$ as the quantity vanishes in this case)

$$\begin{aligned}\nu_2(\theta_{q,q+2i}) &= \nu_2(\theta_{q,q+2(cQ+d)}) \\ &= -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d-Q}^{2q_1+2d-Q-1} s(j) + (2q_1+d-Q)(s(c)-s(c+1)+1),\end{aligned} \tag{4.7}$$

while, if $d > Q-q_1$, it is given by (cf. (3.7))

$$\begin{aligned}&\nu_2(\theta_{q,q+2i}) = \nu_2(\theta_{q,q+2(cQ+d)}) \\ &-\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d-Q}^{2q_1+2d-Q-1} s(j) - (2q_1+2d-2Q) + (Q-d)(s(c)-s(c+1)+1).\end{aligned} \tag{4.8}$$

Now let $i > i_2$, that is, $d > Q-q_1$. Symmetry around the center $i_2 = (c+1)Q-q_1$ means to show that

$$\nu_2(\theta_{q,q+2(cQ+2Q-q-d)}) = \nu_2(\theta_{q,q+2(cQ+d)}).$$

Using (4.7), the left-hand term is given by

$$
\begin{aligned}
&\nu_2(\theta_{q,q+2((c+2)Q-2q_1-d)}) \\
&\quad = -\sum_{j=0}^{2Q-2q_1-d-1} s(j) + \sum_{j=Q-d}^{3Q-2q_1-2d-1} s(j) + (Q-d)(s(c)-s(c+1)+1) \\
&\quad = -\sum_{j=0}^{Q-1} s(j) + \sum_{j=2Q-2q_1-d}^{Q-1} s(j) + \sum_{j=0}^{2Q-1} s(j) - \sum_{j=0}^{Q-d-1} s(j) - \sum_{j=3Q-2q_1-2d}^{2Q-1} s(j) \\
&\qquad\qquad + (Q-d)(s(c)-s(c+1)+1) \\
&\quad = \sum_{j=2Q-2q_1-d}^{Q-1} s(j) + \sum_{j=Q}^{2Q-1} s(j) - \sum_{j=0}^{Q-d-1} s(j) - \sum_{j=3Q-2q_1-2d}^{2Q-1} s(j) \\
&\qquad\qquad + (Q-d)(s(c)-s(c+1)+1) \\
&\quad = \sum_{j=2Q-2q_1-d}^{Q-1} s(j) + \sum_{j=Q-d}^{Q-1} s(j) + Q - \sum_{j=3Q-2q_1-2d}^{2Q-1} s(j) \\
&\qquad\qquad + (Q-d)(s(c)-s(c+1)+1).
\end{aligned}
$$

Now we apply the reflection identity (2.4) to all the sum of digit functions. Thus, we obtain

$$
\begin{aligned}
\nu_2(\theta_{q,q+2((c+2)Q-2q_1-d)}) &= (2q_1+2d-Q)\log_2 Q - \sum_{j=0}^{2q_1+d-Q-1} s(j) - \sum_{j=0}^{d-1} s(j) \\
&\quad - (2q_1+2d-Q)\log_2(2Q) + \sum_{j=0}^{2q_1+2d-Q-1} s(j) \\
&\quad + Q + (Q-d)(s(c)-s(c+1)+1) \\
&= -\sum_{j=0}^{d-1} s(j) + \sum_{j=2q_1+d-Q}^{2q_1+2d-Q-1} s(j) - (2q_1+2d-2Q) + (Q-d)(s(c)-s(c+1)+1) \\
&= \nu_2(\theta_{q,q+2(cQ+d)}),
\end{aligned}
$$

by (4.8), as desired. That the values at the extreme points of the interval $i = (c+1)Q - q$ and $i = (c+1)Q$ are 0, was already shown in Theorem 3.1, Case 1.

Next we determine the 2-adic value of $\theta_{q,q+2i}$ at the center $i_2 = (c+1)Q - q/2$ of the interval. By (4.7), we have

$$
\begin{aligned}
\nu_2(\theta_{q,q+2i_2}) &= \nu_2(\theta_{q,q+2((c+1)Q-q_1)}) \\
&= -\sum_{j=0}^{Q-q_1-1} s(j) + \sum_{j=q_1}^{Q-1} s(j) + q_1(s(c)-s(c+1)+1) \\
&= -\sum_{j=0}^{q_1-1} s(j) + \sum_{j=Q-q_1}^{Q-1} s(j) + q_1(s(c)-s(c+1)+1) \\
&= q_1\log_2 Q - 2\sum_{j=0}^{q_1-1} s(j) + q_1(s(c)-s(c+1)+1).
\end{aligned}
$$

Inequality (4.5) was already implicitly proved in the proof of Theorem 3.1, Case 1d. Namely, if $i = cQ + d$, with $d \ge Q - q_1$, then the conditions of Case 1d are satisfied, and there it was shown (see the paragraph after (3.7)) that

$$\nu_2(\theta_{q,q+2i}) = \nu_2(\theta_{q,q+2(cQ+d)}) \ge Q - d = (c+1)Q - i,$$

as desired.

To prove inequality (4.6), we compute the difference of the 2-adic valuations of $\theta_{q,q+2i_2}$ and $\theta_{q,q+2i}$ for $i \in [i_2, (c+1)Q]$: let again $i = cQ + d$, $Q - q_1 \le d \le Q$. Then, using (4.8) again, we have

$$\begin{aligned}
\nu_2(\theta_{q,q+2i_2}) - \nu_2(\theta_{q,q+2i}) &= \nu_2(\theta_{q,q+2((c+1)Q-q_1)}) - \nu_2(\theta_{q,q+2(cQ+d)}) \\
&= -\sum_{j=0}^{Q-q_1-1} s(j) + \sum_{j=q_1}^{Q-1} s(j) + q_1(s(c) - s(c+1) + 1) + \sum_{j=0}^{d-1} s(j) \\
&\qquad - \sum_{j=2q_1+d-Q}^{2q_1+2d-Q-1} s(j) + (2q_1 + 2d - 2Q) - (Q-d)(s(c) - s(c+1) + 1) \\
&= \sum_{j=Q-q_1}^{d-1} s(j) + \sum_{j=q_1}^{2q_1+d-Q-1} s(j) - \sum_{j=Q}^{2q_1+2d-Q-1} s(j) \\
&\qquad\qquad + (q_1 + d - Q)(s(c) - s(c+1) + 3) \\
&= \sum_{j=Q-q_1}^{d-1} s(j) + \sum_{j=q_1}^{2q_1+d-Q-1} s(j) - \sum_{j=0}^{2q_1+2d-2Q-1} s(j) \\
&\qquad\qquad + (q_1 + d - Q)(s(c) - s(c+1) + 1) \\
&= \sum_{j=Q-q_1}^{d-1} s(j) + \sum_{j=q_1}^{2q_1+d-Q-1} s(j) \\
&\qquad\qquad - 2\sum_{j=0}^{q_1+d-Q-1} s(j) + (q_1 + d - Q)(s(c) - s(c+1)).
\end{aligned}$$

Since $Q - q_1 \ge q_1 + d - Q$ and $q_1 \ge q_1 + d - Q$, we may apply the inequality (2.6) twice to obtain

$$\nu_2(\theta_{q,q+2i_2}) - \nu_2(\theta_{q,q+2i}) \ge (q_1 + d - Q)(s(c) - s(c+1) + 2).$$

As we used already quite often, $s(c) - s(c+1) + 1 \ge 0$. Therefore,

$$\nu_2(\theta_{q,q+2i_2}) - \nu_2(\theta_{q,q+2i}) \ge q_1 + d - Q = i - i_2,$$

as desired. ☐

Corollary 4.5. *For a fixed even q, the values of the sequence $(\nu_2(\theta_{q,q+2i}))_{i\ge0}$ are never 1 for $i \in [(c+1)Q - q, (c+1)Q]$, except if q is a power of 2 and c is even. In the latter case, $\nu_2(\theta_{q,q+2i})$ is equal to 1 for $i = (c+1)Q - q - 1$ and for $i = (c+1)Q - 1$.*

Proof. By the symmetry of the values around the center of the interval, and by the inequality (4.5), the only possible candidates are $i = (c+1)Q - q + 1$ and $i = (c+1)Q - 1$. Let us concentrate on $i = (c+1)Q - 1$, which, as before, we write using the parameter d as $i = (c+1)Q - 1 = cQ + d$. The formula (3.7) in the proof of Theorem 3.1, CASE 1D, (which we used to prove (4.5)), yields that $\nu_2(\theta_{q,q+2i})$ is equal to

$$-\sum_{j=0}^{Q-2} s(j) + \sum_{j=q-1}^{q+Q-3} s(j) - (q-2) + (s(c) - s(c+1) + 1)$$

$$\begin{aligned} &= -\sum_{j=0}^{Q-1} s(j) + s(Q-1) + \sum_{j=q-1}^{q+Q-2} s(j) - s(q+Q-2) \\ &\qquad - (q-2) + (s(c) - s(c+1) + 1) \\ &= \lceil \log_2 q \rceil - s(q+Q-2) + 1 + (s(c) - s(c+1) + 1). \end{aligned}$$

For this expression to be equal to 1, we must have $s(q+Q-2) = \lceil \log_2 q \rceil$ and $s(c) - s(c+1) + 1 = 0$. The former is the case if and only if $q = Q$, that is, if q is a power of 2, and the latter is the case if and only if c is even. $\square$

In an analogous manner, one can prove the following two theorems, with accompanying corollaries, covering the case where q is odd.

Theorem 4.6. *Let q be a fixed positive odd integer, $Q = 2^{\lceil \log_2 q \rceil}$, and let c be any nonnegative integer. Then the values of the sequence $(\nu_2(\theta_{q,q+2i}))_{i\geq 0}$ for $i \in [cQ, (c+1)Q - q]$ are symmetric around its center $cQ + (Q-q)/2$. The values at the extreme points of the interval $i = cQ$ and $i = (c+1)Q - q$ are 0, and the value at the central points $i_3 = cQ + (Q-q-1)/2$ and $cQ + (Q-q+1)/2$ is*

$$\nu_2(\theta_{q,q+2i_3}) = \frac{Q-q-1}{2} \log_2 \frac{Q}{2} - 2 \sum_{j=0}^{\frac{Q-q-3}{2}} s(j) - s\left(\frac{Q-q-1}{2}\right),$$

independent of c. Furthermore, for $i \in [cQ, i_3]$ the inequalities

$$\nu_2(\theta_{q,q+2i}) \geq i - cQ \tag{4.9}$$

and

$$\nu_2(\theta_{q,q+2i}) \leq \nu_2(\theta_{q,q+2i_3}) - (i_3 - i) \tag{4.10}$$

hold.

Corollary 4.7. *For a fixed odd q, the values of the sequence $(\nu_2(\theta_{q,q+2i}))_{i\geq 0}$ are never 1 for $i \in [cQ, (c+1)Q - q]$, except if q has the form $2^M + 2^m - 1$, for some positive integers m and M, $m < M$. In the latter case, $\nu_2(\theta_{q,q+2i})$ is equal to 1 for $i = cQ + 1$ and for $i = (c+1)Q - q - 1$.*

Proof. The arguments from the proof of Corollary 4.3 apply also here. Thus, again, the only possible candidates are $i = cQ + 1$ and $i = (c+1)Q - q - 1$. Furthermore, we must have

$$s\left(q + 1 - \frac{Q}{2}\right) = 1.$$

This is only possible if $q + 1 - \frac{Q}{2}$ is a power of 2, which means that q has the form given in the statement of the corollary. $\square$

Theorem 4.8. *Let q be a fixed positive odd integer, $Q = 2^{\lceil \log_2 q \rceil}$, and let c be any nonnegative integer. Then the values of the sequence $(\nu_2(\theta_{q,q+2i}))_{i\geq 0}$ for $i \in [(c+1)Q - q, (c+1)Q]$ are symmetric around its center $(c+1)Q - q/2$. The values at the extreme points of the interval $i = (c+1)Q - q$ and $i = (c+1)Q$ are 0, and the value at the central points $i_4 = (c+1)Q - (q+1)/2$ and $(c+1)Q - (q-1)/2$ is*

$$\nu_2(\theta_{q,q+2i_4}) = \frac{q-1}{2}\log_2 Q - 2\sum_{j=0}^{\frac{q-3}{2}} s(j) - s\left(\frac{q-1}{2}\right) + \frac{q-1}{2}(s(c) - s(c+1) + 1).$$

Furthermore, for $i \in [i_4, (c+1)Q]$ the inequalities

$$\nu_2(\theta_{q,q+2i}) \geq (c+1)Q - i \tag{4.11}$$

and

$$\nu_2(\theta_{q,q+2i}) \leq \nu_2(\theta_{q,q+2i_4}) - (i - i_4) \tag{4.12}$$

hold.

Corollary 4.9. *For a fixed odd q, the values of the sequence $(\nu_2(\theta_{q,q+2i}))_{i\geq 0}$ are never 1 for $i \in [(c+1)Q - q, (c+1)Q]$.*

Proof. The arguments from the proof of Corollary 4.5 apply also here. The conclusion was that we can have $\nu_2(\theta_{q,q+2i}) = 1$, for some i, only if q is a power of 2. This is a contradiction to our assumption that q is odd. $\square$

5 Skew symmetric matrices and the parity of $\varepsilon_{2p,n}$

Let $\mathbb{F} = \mathbb{C}, \mathbb{R}$ and denote by $\mathrm{GL}(n, \mathbb{F}) \subset \mathrm{M}_n(\mathbb{F})$ the group of $n \times n$ invertible matrices. Recall that $\mathrm{A}_n(\mathbb{F})$ is the linear space of $n \times n$ skew symmetric matrices A of order n over $\mathbb{F}$, i.e., $A^\top = -A$. Clearly $\dim \mathrm{A}_n(\mathbb{F}) = \binom{n}{2}$. Two matrices $A, B \in \mathrm{A}_n(\mathbb{F})$ are called *congruent* if $A = TBT^\top$ for some $T \in \mathrm{GL}(n, \mathbb{F})$. Let $S_2 := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The following result is well-known in the real case, but its complex version does not seem to appear in standard modern books on linear algebra.

Proposition 5.1. *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $A \in \mathrm{A}_n(\mathbb{F})$. Then A has even rank, $2p$ say, and A is congruent over $\mathbb{F}$ to a direct sum of p copies of S_2 and the $(n-2p)\times(n-2p)$ zero matrix. In particular, $B \in \mathrm{A}_n(\mathbb{F})$ is congruent to A over $\mathbb{F}$ if and only if* $\mathrm{rank}\ A = \mathrm{rank}\ B$.

Proof. We first prove the fact that any $A \in \mathrm{A}_n(\mathbb{F})$ is congruent to the direct sum of copies of S_2 and 0. The result is trivial if $A = 0$. Let $n = 2$ and $\mathrm{rank}\ A = 2$. Then $A = aS_2$ for some $0 \neq a \in \mathbb{F}$. For $\mathbb{F} = \mathbb{C}$ we have $A = (\sqrt{a}I_2)S_2(\sqrt{a}I_2)^\top$. For $\mathbb{F} = \mathbb{R}$ and $a > 0$ the above formula holds. For $a < 0$ we have $A = (\sqrt{-a}P)S_2(\sqrt{-a}P)^\top$, where $P := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Assume by induction that any $A \in \mathrm{A}_n(\mathbb{F})$ is congruent to the direct sum of copies of S_2 and $0 \in \mathrm{M}_n(\mathbb{F})$ for $n = m \geq 2$. Let $n = m+1$ and $A \in \mathrm{A}_{m+1}(\mathbb{F})$. Suppose first that $\det A = 0$. Let $\mathbf{0} \neq \mathbf{x} \in \mathbb{F}^n$ and $A\mathbf{x} = 0$. Let $Q \in \mathrm{GL}(m+1, F)$ such that the last column of $Q^\top$ is $\mathbf{x}$. Then $QAQ^\top = A_1 \oplus 0$, and $A_1 \in \mathrm{A}_m(\mathbb{F})$. Use the induction hypothesis to deduce that A is conjugate to the direct sum of copies of S_2 and 0. It remains to study the case where $m+1$ is even and $\det A \neq 0$. Let $A = (a_{ij})_{i,j=1}^{m+1}$ and $A_1 = (a_{ij})_{i,j=1}^{m} \in \mathrm{A}_m(\mathbb{F})$. Since A has rank $m+1$, A_1 has at least rank $m-1$. Since m is odd A_1 has exactly rank $m-1$. So A_1 is conjugate to a direct sum of $\frac{m-1}{2}$ copies of S_2 and one copy of 0. Using the corresponding congruence on A, we may assume without loss of generality that $A_1 = \left(\bigoplus_{i=1}^{(m-1)/2} S_2\right) \oplus 0$. For each $i = 1, \ldots, m-1$ subtract from column $m+1$ of A the corresponding multiple of column i to obtain the zero element for the $(i, m+1)$-entry. Repeat these elementary operations with the rows of A to eliminate the $(m+1, i)$-entry for $i = 1, \ldots, m-1$. The resulting matrix is of the form $B = \left(\bigoplus_{i=1}^{(m-1/2)} S_2\right) \oplus bS_2$. Clearly A is congruent to B. Hence $b \neq 0$. Since bS_2 is congruent to S_2, we deduce that A is congruent to a direct sum of copies of S_2.

Since a direct sum of copies of S_2 and 0 has an even rank we deduce that any $A \in \mathrm{A}_n(\mathbb{F})$ has even rank. □

The following result is known to the experts. We bring its proof for completeness.

Proposition 5.2. *Let $\mathbb{F} = \mathbb{C}, \mathbb{R}$, $n \geq 2$, $p \geq 1$ be integers and assume that $p \leq \lfloor \frac{n}{2} \rfloor$. Let $\mathbb{P}\mathrm{W}_{2p,n}(\mathbb{F}) \subset \mathbb{P}\mathrm{A}_n(\mathbb{F})$ be the projective variety of all (nonzero) skew symmetric matrices of rank at most $2p$. Then $\mathbb{P}\mathrm{W}_{2p,n}(\mathbb{F})$ is an irreducible projective variety in $\mathbb{P}\mathrm{A}_n(\mathbb{F})$ of codimension $\binom{n-2p}{2}$. The variety of its singular points is $\mathbb{P}\mathrm{W}_{2(p-1),n}(\mathbb{F})$.*

Proof. Let $\mathrm{W}^o_{2p,n}(\mathbb{F}) \subset \mathrm{A}_n(\mathbb{F})$ be the quasi-variety of all $n \times n$ skew symmetric matrices of rank $2p$. Proposition 5.1 yields that $\mathrm{GL}(n, \mathbb{F})$ acts transitively on $\mathrm{W}^o_{2p,n}(\mathbb{F})$. Hence $\mathrm{W}^o_{2p,n}(\mathbb{F})$ is a homogeneous space and a manifold. Since $\mathrm{W}_{2(p-1),n}(\mathbb{F})$ is a strict affine subvariety of $\mathrm{W}_{2p,n}(\mathbb{F})$ it follows that $\mathrm{W}^o_{2p,n}(\mathbb{F})$ is a subset of smooth points and $\dim \mathrm{W}^o_{2p,n}(\mathbb{F}) = \dim \mathrm{W}_{2p,n}(\mathbb{F})$. The neighborhood of each point is obtained by the corresponding action of the neighborhood of $I_n \in \mathrm{GL}(n, \mathbb{F})$. Proposition 5.1 yields that the orbit of any $B \in \mathrm{W}_{2(p-1),n}(\mathbb{F})$ does not contain any matrix in $\mathrm{W}^o_{2p,n}(\mathbb{F})$. Hence $\mathrm{W}_{2(p-1),n}(\mathbb{F})$ is the variety of singular points in $\mathrm{W}_{2p,n}(\mathbb{F})$.

We now find the dimension and codimension of $\mathrm{W}_{2p,n}(\mathbb{F})$. For $p = \lfloor \frac{n}{2} \rfloor$, we have $\mathrm{W}_{2p,n}(\mathbb{F}) = \mathrm{A}_n(\mathbb{F})$. Hence $\dim \mathrm{W}_{2\lfloor \frac{n}{2} \rfloor,n}(\mathbb{F}) = \binom{n}{2}$ and $\operatorname{codim} \mathrm{W}_{2\lfloor \frac{n}{2} \rfloor,n}(\mathbb{F}) = \binom{n-2\lfloor \frac{n}{2} \rfloor}{2} = 0$. Let $1 \leq p < \lfloor \frac{n}{2} \rfloor$. Let $A = (a_{ij})_{i,j=1}^{n} \in \mathrm{W}^o_{2p,n}(\mathbb{F})$. Then A has $2p$ independent rows. Assume for simplicity that the first $2p$ rows are linearly independent. Hence the first $2p$ columns of A are linearly independent. Hence $A_1 = (a_{ij})_{i,j=1}^{2p} \in \mathrm{A}_{2p}(\mathbb{F})$ is nonsingular. Therefore there exists a unique block lower triangular matrix $T = \begin{pmatrix} I_{2p} & 0 \\ R & I_{n-2p} \end{pmatrix}$ such that $TAT^\top = A_1 \oplus 0$. Equivalently $A = T^{-1}(A_1 \oplus 0)(T^\top)^{-1}$. Hence $\dim \mathrm{W}_{2p,n} = \binom{2p}{2} + 2p(n-2p)$ and $\operatorname{codim} \mathrm{W}_{2p,n} = \binom{n-2p}{2}$. Thus, $\operatorname{codim} \mathbb{P}\mathrm{W}_{2p,n} = \binom{n-2p}{2}$. □

Theorem 5.3. *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $4 \leq n, 1 \leq p < \lfloor \frac{n}{2} \rfloor$. Let $\mathbb{P}\mathrm{W}_{2p,n}(\mathbb{F})$ be the irreducible variety of all (projectivized) nonzero skew symmetric $n \times n$ matrices of rank at most $2p$ in the projective space $\mathbb{P}\mathrm{A}_n(\mathbb{F})$ of all nonzero $n \times n$ skew symmetric matrices over $\mathbb{F}$. Then the degree of $\mathbb{P}\mathrm{W}_{2p,n}(\mathbb{C})$ is odd if and only if either p or $n-p$ is divisible by $2^{\lceil \log_2(n-2p) \rceil}$. Furthermore, if either p or $n-p$ is divisible by $2^{\lceil \log_2(n-2p) \rceil}$, then any $\left(\binom{n-p}{2} + 1\right)$-dimensional subspace of $n \times n$ real skew symmetric matrices contains a nonzero matrix of rank at most $2p$. For these values of n and p the dimensions of subspaces are best possible, i.e., (1.6) holds.*

Proof. Recall from (1.3) that $\varepsilon_{2p,n} := \deg \mathbb{P}\mathrm{W}_{2p,n} = \delta_{2p+1,n}/2^{n-2p-1}$. The definition (2.1) of $\theta_{q,n}$ yields that $\nu_2(\varepsilon_{2p,n}) = \nu_2(\theta_{n-2p-1,n}) - (n-2p-1)$. Proposition 2.1 yields that $\nu_2(\varepsilon_{2p,n}) = \nu_2(\theta_{n-2p,n})$. Use Theorem 3.1 to deduce that $\varepsilon_{2p,n}$ is odd if and only if either p or $n-p$ is divisible by $2^{\lceil \log_2(n-2p)\rceil}$.

Assume that either p or $n-p$ is divisible by $2^{\lceil \log_2(n-2p)\rceil}$. Then the discussion in §1 implies that any $\left(\binom{n-p}{2}+1\right)$-dimensional subspace of $n\times n$ real skew symmetric matrices contains a nonzero matrix of rank at most $2p$. The sharpness of these dimensions follows from the fact that a complex subspace L of $\mathrm{A}_n(\mathbb{C})$ of dimension $\binom{n-p}{2}$ in general position will not contain a nonzero $A \in \mathrm{A}_n(\mathbb{C})$ of rank at most $2p$. ∎

Corollary 5.4. *Let $n \equiv 2 \pmod 4$. Then any two-dimensional real subspace of $n\times n$ skew symmetric matrices contains a nonzero singular matrix.*

6 Rectangular matrices and the parity of $\gamma_{k,m,n}$

In this section we consider the parity problem for $\gamma_{k,m,n}$, the latter being defined in (1.2). It is more convenient to introduce the following symmetric quantity. For $n \in \mathbb{N}$, let $\mathbf{H}(n)$ be the *hyperfactorial* $\prod_{k=0}^{n-1} k!$. Let $a,b,c \in \mathbb{N}$. Then a straightforward calculation shows:

$$\begin{aligned} B(a,b,c) &= \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{(i+j+k-1)}{(i+j+k-2)} \\ &= \prod_{i=1}^{a}\frac{(b+c+i-1)!\,(i-1)!}{(b+i-1)!\,(c+i-1)!} \\ &= \frac{\mathbf{H}(a)\,\mathbf{H}(b)\,\mathbf{H}(c)\,\mathbf{H}(a+b+c)}{\mathbf{H}(a+b)\,\mathbf{H}(b+c)\,\mathbf{H}(c+a)}. \end{aligned} \tag{6.1}$$

$B(a,b,c)$ is a symmetric function on $\mathbb{N}^3$. We remark that $B(a,b,c)$ is the number of plane partitions which are contained in an $a\times b\times c$ box (see e.g. [4]). From the definition (1.2) of $\gamma_{k,m,n}$ it is obvious that

$$\gamma_{k,m,n} = B(n-k, m-k, k), \quad 1 \le k \le \min(m,n). \tag{6.2}$$

For simplicity of notation we let

$$S(a) := \sum_{i=0}^{a-1} s(i) \quad \text{and} \quad \nu(a,b,c) := \nu_2(B(a,b,c)) \quad \text{for any } a,b,c \in \mathbb{N}.$$

Then, by Proposition 2.3, we have $\nu_2(\mathbf{H}(a)) = \frac{(a-1)a}{2} - S(a)$ and

$$\nu(a,b,c) = -S(a+b+c) - S(a) - S(b) - S(c) + S(a+b) + S(b+c) + S(a+c). \tag{6.3}$$

Lemma 6.1. *Let $a,b,c \in \mathbb{N}$. Then the following identities hold:*

$$\nu(2a,2b,2c) = 2\nu(a,b,c), \tag{6.4}$$
$$\nu(2a,2b+1,2c+1) = \nu(a,b+1,c) + \nu(a,b,c+1), \tag{6.5}$$
$$\nu(2a+1,2b,2c) = \nu(a,b,c) + \nu(a+1,b,c), \tag{6.6}$$

$$\nu(2a+1,2b+1,2c+1) = \nu(a,b+1,c+1) + \nu(a+1,b,c) + s(b+c) - s(b+c+1) + 2. \tag{6.7}$$

In particular,

(1) $B(2a,2b,2c)$ *is odd if and only if* $B(a,b,c)$ *is odd.*

(2) $B(2a,2b+1,2c+1)$ *is odd if and only if both* $B(a,b+1,c)$ *and* $B(a,b,c+1)$ *are odd.*

(3) $B(2a+1,2b,2c)$ *is odd if and only if both* $B(a,b,c)$ *and* $B(a+1,b,c)$ *are odd.*

(4) $B(2a+1,2b+1,2c+1)$ *is always even.*

Proof. Equality (2.8) is equivalent to

$$S(2p) = 2S(p) + p,\ S(2p+1) = S(p+1) + S(p) + p, \quad \text{for any } p \in \mathbb{N}. \tag{6.8}$$

Use (6.3) and the above equalities to deduce (6.4)–(6.6) straightforwardly. In particular, these equalities yield the corresponding claims about the oddness of $B(u,v,w)$.

To obtain (6.7), we use in addition to (6.3) and the above equalities the obvious equality $S(p+1) = S(p) + s(p)$ for $p = b+c, b+c+1$. Since $s(p+1) \le s(p) + 1$ it follows that $B(2a+1,2b+1,2c+1)$ is always even. □

Remark 6.2. (1) Lemma 6.1 provides us with an algorithm to compute the parity of $B(a,b,c)$ from the binary expansions of a,b,c directly, without having to actually compute $B(a,b,c)$. Namely, given a,b,c, one determines the parities of a,b,c. If all of a,b,c are odd, then Conclusion (4) in Lemma 6.1 says that $B(a,b,c)$ is even. If all of a,b,c are even, then one uses Conclusion (1) to reduce the problem to the problem of determining the parity of $B(a/2,b/2,c/2)$. If exactly two of a,b,c should be odd, then one uses Conclusion (2) for a similar reduction, and if only one of a,b,c is odd, then one uses Conclusion (3). One sees quickly that this yields an algorithm which can be most conveniently run on the binary expansions of a,b,c. See Proposition 6.5 for an attempt to turn this algorithm into a concrete characterization of those a,b,c for which $B(a,b,c)$ is odd.

(2) For the interested reader, we remark that the inspiration for Lemma 6.1 comes from results on plane partitions due to Stembridge and Eisenkölbl. More precisely, Stembridge showed in [20] that a certain (-1)-enumeration (for our purposes it suffices to say that this means a weighted enumeration in which some plane partitions count as 1, as in ordinary enumeration, and others count as -1) of plane partitions contained in an $a \times b \times c$-box is equal (up to sign) to the number of self-complementary plane partitions contained in the same box. Since, by definition of self-complementary plane partitions, there cannot exist any if all of a,b,c are odd, this result implies immediately Conclusion (4) in Lemma 6.1. Subsequently, Eisenkölbl [5] has embarked on the (-1)-enumeration of self-complementary plane partitions. Her result is that a certain (-1)-enumeration of self-complementary plane partitions contained in an $a \times b \times c$-box is equal to

$$\begin{aligned} B\left(\tfrac{a}{2},\tfrac{b}{2},\tfrac{c}{2}\right)^2 \quad & \text{for } a,b,c \text{ even},\\ B\left(\tfrac{a}{2},\tfrac{b+1}{2},\tfrac{c-1}{2}\right) B\left(\tfrac{a}{2},\tfrac{b-1}{2},\tfrac{c+1}{2}\right) \quad & \text{for } a \text{ even and } b,\, c \text{ odd},\end{aligned}$$

$$B\left(\tfrac{a+1}{2},\tfrac{b}{2},\tfrac{c}{2}\right)B\left(\tfrac{a-1}{2},\tfrac{b}{2},\tfrac{c}{2}\right)\quad \text{for } a \text{ odd and } b,\, c \text{ even.}$$

The first of the three cases implies Conclusion (1) in Lemma 6.1, the second implies Conclusion (2), and the third implies Conclusion (3). The relations (6.4)–(6.7) refine these conclusions on the level of 2-adic valuations.

Let b and c be two nonnegative integers. Furthermore, let

$$b=\sum_{i=0}^{\infty} b_i 2^i,\ c=\sum_{i=0}^{\infty} c_i 2^i,\quad b_i, c_i \in \{0,1\},\ i=0,\dots,$$

be the binary expansion of b and c, respectively. We say that the pair (b,c) has a *disjoint binary expansion* if $b_i c_i = 0$ for $i=0,\dots$. The following proposition is straightforward to establish.

Proposition 6.3. *Let b and c be nonnegative integers. Then $s(b+c) \le s(b)+s(c)$, and $s(b+c)=s(b)+s(c)$ if and only the pair (b,c) has a disjoint binary expansion.*

To find out under which conditions on the parameters a,b,c the number $B(a,b,c)$ is odd, it is enough to consider the case $a \le \min(b,c)$.

Theorem 6.4. *Let $b,c \in \mathbb{N}$. Then*

1. *$B(1,b,c)$ is odd if and only if (b,c) has a disjoint binary expansion. In particular, for any $q \ge 0$, $B(2^q,2^q b,2^q c)$ is odd if and only if $(2^q b,2^q c)$ has a disjoint binary expansion.*

2. *Let $\min(b,c) \ge 2$.*
   - *If b and c are even then $B(2,b,c)$ is odd if and only if the pair (b,c) has a disjoint binary expansion.*
   - *If b is even and c is odd then $B(2,b,c)$ is odd if and only if the pairs (b,c) and $(b,c+1)$ have disjoint binary expansions.*
   - *If b and c are odd then $B(2,b,c)$ is odd if and only if the pairs $(b,c+1)$ and $(b+1,c)$ have disjoint binary expansions.*

3. *Let $\min(b,c) \ge 3$.*
   - *Assume that b and c are even.*
     If $b,c \equiv 0 \bmod 4$ then $B(3,b,c)$ is odd if and only if the pair (b,c) has a disjoint binary expansion.
     If $b,c+2 \equiv 0 \bmod 4$ then $B(3,b,c)$ is odd if and only if the pairs (b,c) and $(b,c+2)$ have disjoint binary expansions.
     If $b,c \equiv 2 \bmod 4$ then $B(3,b,c)$ is odd if and only if the pairs (b,c), $(b,c+2)$ and $(b+2,c)$ have disjoint binary expansions.
   - *Assume that b is even and c is odd.*
     If $b \equiv 0 \bmod 4$ then $B(3,b,c)$ is odd if and only if the pairs (b,c) and $(b,c+1)$ have disjoint binary expansions.
     If $b \equiv 2 \bmod 4$ then $B(3,b,c)$ is odd if and only if the pairs $(b,c+1)$ and $(b+2,c)$ have disjoint binary expansions.

Proof. 1. The expression (6.1) yields that $B(1,b,c) = \binom{b+c}{b}$. Hence $\nu(1,b,c) = s(b)+s(c)-s(b+c)$. Thus $\nu(1,b,c) = 0$ if and only if (b,c) has a disjoint binary expansion. The last assertion follows from (6.4) and from the observation that (b,c) has a disjoint binary expansion if and only if $(2^q b, 2^q c)$ has a disjoint binary expansion.

2.
- If b and c are even then Conclusion (1) in Lemma 6.1 and item 1, which we just established, yield that $B(2,b,c)$ is odd if and only the pair (b,c) has a disjoint binary expansion.
- Assume that $b = 2b'$, $c = 2c' + 1$, $b', c' \in \mathbb{N}$. Since $\nu(a,b,c)$ is a symmetric function in a,b,c, (6.6) yields that $\nu(2, 2b', 2c' + 1) = \nu(1, b', c') + \nu(1, b', c' + 1)$. Hence $B(2,b,c)$ is odd if and only if (b', c') and $(b', c' + 1)$ have disjoint binary expansions. This is equivalent to the assumption that the two even pairs $(b, c-1)$ and $(b, c+1)$ have disjoint binary expansions. Since b is even and c odd the assumption that $(b, c-1)$ has disjoint binary expansions is equivalent to the assumption that (b,c) has a disjoint binary expansion.
- Assume that $b = 2b' + 1$, $c = 2c' + 1$, $b', c' \in \mathbb{N}$. Then (6.5) yields $\nu(2,b,c) = \nu(1, b'+1, c) + \nu(1, b', c'+1)$. Hence $B(2,b,c)$ is odd if and only if $(b'+1, c')$ and $(b', c'+1)$ have disjoint binary expansions. This is equivalent to the assumption that the two even pairs $(b+1, c-1)$ and $(b-1, c+1)$ have disjoint binary expansions. This is also equivalent to to the assumption that $(b+1, c)$ and $(b, c+1)$ have disjoint binary expansions.

3.
- Assume that $b = 2b'$, $c = 2c'$, $a', b' \geq 2$. Relation (6.6) implies that $B(3,b,c) = B(1,b',c') + B(2,b',c')$. Hence $B(3,b,c)$ is odd if an only if $B(1,b',c')$ and $B(2,b',c')$ are odd. Recall that $B(1,b',c')$ is odd if and only if (b',c') has a disjoint binary expansion. This is equivalent to the assumption that (b,c) has a disjoint binary expansion.

  Assume that b' and c' are even. Then $B(2,a',b')$ is odd if and only (b',c') has a disjoint binary expansion.

  Assume now that b' is even and c' are is odd. Then $B(2,b',c')$ is odd if (b',c') and $(b',c'+1)$ have disjoint binary expansions. This is equivalent to the assumption that (b,c) and $(b,c{+}2)$ have disjoint binary expansions.

  Assume now that b' and c' is odd. Then $B(2,b',c')$ is odd if and only if the pairs $(b',c'+1)$ and $(b'+1,c')$ have disjoint binary expansions. This is equivalent to the assumption that the pairs $(b,c{+}2)$ and $(b{+}2,c)$ have disjoint binary expansions.
- Assume that $b = 2b'$, $c = 2c' + 1$, $b' \geq 2$, $c' \geq 1$. Then $\nu(3, 2b', 2c' + 1) = \nu(1, b', c'+1) + \nu(2, b', c')$. Hence $B(3,b,c)$ is odd if and only if $B(1,b',c'+1)$ and $B(2,b',c')$ are odd. $B(1,b',c'+1)$ is odd if and only if $(b', c'+1)$ has a disjoint binary expansion. This is equivalent to the assumption that $(b, c+1)$ has a disjoint binary expansion.

  Assume that b' is even, i.e., $4 \mid b$. Suppose first that c' is even, i.e., $4 \mid (c-1)$. Then $B(2,b',c')$ is odd if and only if (b',c') has a disjoint binary expansion. This is equivalent to the assumption that (b,c) has a disjoint binary expansion.

Assume second that c' is odd, i.e., $4 \mid (c+1)$. Then $B(2,b',c')$ is odd if and only if (b',c') and $(b',c'+1)$ have disjoint binary expansions. This is equivalent to the assumption that (b,c) and $(b,c+1)$ have disjoint binary expansions.

Assume now that b' is odd, i.e., $4 \mid (b+2)$. Suppose first that c' is even. Then $B(2,b',c')$ is odd if and only if (b',c') and $(b'+1,c')$ have disjoint binary expansions. This is equivalent to the assumption that (b,c) and $(b+2,c)$ have disjoint binary expansions.

Suppose second that c' is odd. Assume first that $c'=1$, i.e., $c=3$. Then $B(2,b',1)$ is odd if and only if $(b',2)$ has a disjoint binary expansion. This is equivalent to the assumption that $(b,4)=(b,c+1)$ has a disjoint binary expansion. Since $4 \mid (b+2)$ it follows that $(b+2,3)$ has a disjoint binary expansion.

Assume second that $c' \geq 3$. Then $B(2,b',c')$ is odd if and only if $(b'+1,c')$ and $(b',c'+1)$ have disjoint binary expansions. This is equivalent to the assumption that $(b+2,c)$ and $(b,c+1)$ have disjoint binary expansions.

□

The above theorem can be generalized schematically as follows:

Proposition 6.5. *Let $a,b,c \in \mathbb{N}$ and assume that $\min(b,c) \geq a$. Let $q := \lceil \log_2 a \rceil$ and assume that $b \equiv b_r$, $c \equiv c_r \mod 2^q$ for some $b_r, c_r \in [0, 2^q-1]$. Then there exists a sequence of nonnegative integers $d_i, e_i \in [0, 2^q-1]$, $i=1,\dots,N(a,b_r,c_r)$, depending only on a,b_r,c_r, such that $B(a,b,c)$ is odd if and only $(b+d_i, c+e_i)$ has a disjoint binary expansion for $i=1,\dots,N(a,b_r,c_r)$.*

Proof. We prove the proposition by induction on q. For $q=0,1$ the proposition holds in view of Theorem 6.4. Assume that the proposition holds for any $q \leq p-1$, where $p \geq 2$ and any b,c such that $\min(b,c) \geq a$. Assume that $\lceil \log_2 a \rceil = p$.

- Let $a=2a'$, $a' \in \mathbb{N}$. Then $\lceil \log_2 a \rceil = \lceil \log_2 a' \rceil + 1$. Assume first that b and c are even. Then $\nu(a,b,c) = 2\nu(\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$ and the proposition follows straightforwardly from the induction hypothesis.

  Assume now that $b=2b'$, $c=2c'+1$, $a' \leq b', c' \in \mathbb{N}$. Then $\nu(2a',2b',2c'+1) = \nu(a',b',c') + \nu(a',b',c'+1)$. Hence $B(a,b,c)$ is odd if and only $B(a',b',c')$ and $B(a',b',c'+1)$ are odd. Use the induction hypothesis and the observation that (u,v) has a disjoint binary expansion if and only if $(2u,2v+1)$ has one for the case $B(a',b',c')$ to deduce the proposition. (For the case $B(a',b',c'+1)$ note that if $d_i \leq 2^{p-1}-1$ then $2d_i+1 \leq 2^p-1$.)

  Use the equality $B(a,b,c) = B(a,c,b)$ to deduce the proposition in the case that b is odd and c is even.

  Assume now that $b=2b'+1$, $c=2c'+1$, $a' \leq b', c' \in \mathbb{N}$. Then $\nu(2a',2b'+1,2c'+1) = \nu(a',b'+1,c') + \nu(a',b',c'+1)$. Hence $B(a,b,c)$ is odd if and only $B(a',b'+1,c')$ and $B(a',b',c'+1)$ are odd. Use the induction hypothesis and the above remarks to deduce the proposition in this case.

- Assume that $a=2a'+1$, $a' \in \mathbb{N}$. Then $\lceil \log_2 a \rceil = \lceil \log_2 a+1 \rceil = \lceil \log_2 a'+1 \rceil + 1$.

Since $B(a,b,c) = B(a,c,b)$ is even if a,b,c are odd, it is enough to consider the case $b = 2b'$, $a' < b' \in \mathbb{N}$. Assume first that $c = 2c'$, $a' < c' \in \mathbb{N}$. Then $\nu(2a'+1, 2b', 2c') = \nu(a', b', c') + \nu(a'+1, b', c')$. Hence $B(a,b,c)$ is odd if and only $B(a', b', c')$ and $B(a'+1, b', c')$ are odd. Use the induction hypothesis and the above arguments to deduce the proposition.

Assume finally that $c = 2c'+1$, $a' \le c'$. Then $\nu(2a'+1, 2b', 2c'+1) = \nu(a', b', c'+1) + \nu(a'+1, b', c')$. Hence $B(a,b,c)$ is odd if and only $B(a', b', c'+1)$ and $B(a'+1, b', c')$ are odd. The case where $B(a', b', c'+1)$ is odd is done by induction on a'. For $c' > a'$ the case $B(a'+1, b'c')$ is odd is done by induction on $a'+1$. The case $c' = a'$, which is equivalent to the case where $B(a'+1, b', a') = B(a', b', a'+1)$ is odd is done by induction on a'. (As in the case of $B(3, 2b', 3)$ in Theorem 6.4.)

□

In principle the proof of Proposition 6.5 can be used to find the sequences d_i, c_i, $i = 1, \ldots, N(a, b_r, c_r)$, recursively. However, the explicit construction of all such sequences seems complicated even in the simple case where $a = 2^q$, $q = 2, \ldots$. Note that Case 1 of Theorem 6.4 finds the sequence for $a = 2^q$ and $b_r = c_r = 0$. The cases $b_r = 0$, $c_r = 1$ and $b_r = c_r = 1$ have simple results.

Proposition 6.6. *Let $q \in \mathbb{N}$ and $2^q \le b, c \in \mathbb{N}$. If $2^q \mid b$, $2^q \mid (c-1)$, then $B(2^q, b, c)$ is odd if and only if (b,c) and $(b, c+2^q-1)$ have disjoint binary expansions. If $2^q \mid (b-1)$, $2^q \mid (c-1)$, then $B(2^q, b, c)$ is odd if and only if $(b+1, c)$, $(b, c+1)$, $(b+2^q-1, c)$, and $(b, c+2^q-1)$ have disjoint binary expansions.*

The proof of this proposition is left to the reader.
The results of §1 yield the following theorem.

Theorem 6.7. *Let $k, m, n \in \mathbb{N}$, assume that $k < \min(m,n)$, and let $\gamma_{k,m,n}$ be the positive integer given by (1.2). Let $\mathrm{M}_{m,n}(\mathbb{R})$ be the space of all $m \times n$ real valued matrices. Let $L \subset \mathrm{M}_{m,n}(\mathbb{R})$ be a subspace of dimension $(m-k)(n-k)+1$. If $\gamma_{k,m,n}$ is odd then L contains a nonzero matrix rank at most k.*

Corollary 6.8. *For the following positive integers $1 \le k < n \le m$ any $((m-k)(n-k)+1)$-dimensional subspace of $\mathrm{M}_{m,n}(\mathbb{R})$ contains a nonzero matrix rank at most k:*

1. *$k = n-1$ and $(m-n+1, n-1)$ has a disjoint binary expansion.*
2. *$2 \le k = n-2$.*
   - *n and m are even and $(n-2, m-n+2)$ has a disjoint binary expansion.*
   - *n is even, m is odd, $(n-2, m-n+2)$ and $(n-2, m-n+3)$ have disjoint binary expansions.*
   - *n is odd, m is even, $(n-2, m-n+3)$ and $(n-1, m-n+2)$ have disjoint binary expansions.*
3. *$3 \le k = n-3$.*
   - *$4 \mid (n-3)$, $4 \mid m$, and $(n-3, m-n+3)$ has a disjoint binary expansion.*

- $4 \mid (n-3)$, $4 \mid (m+2)$, $(n-3, m-n+3)$ *and* $(n-3, m-n+5)$ *have disjoint binary expansions.*
- $4 \mid (n-1)$, $4 \mid (m+2)$, $(n-3, m-n+3)$ *and* $(n-1, m-n+3)$ *have disjoint binary expansions.*
- $4 \mid (n-1)$, $4 \mid m$, $(n-1, m-n+3)$ *and* $(n-3, m-n+5)$ *have disjoint binary expansions.*
- $4 \mid (n-3)$, m *odd,* $(n-3, m-n+3)$ *and* $(n-3, m-n+4)$ *have disjoint binary expansions.*
- $4 \mid (n-1)$, m *odd,* $(n-3, m-n+4)$ *and* $(n-1, m-n+3)$ *have disjoint binary expansions.*
- $4 \mid (m-n+3)$, n *is even,* $(n-3, m-n+3)$ *and* $(n-2, m-n+3)$ *have disjoint binary expansions.*
- $4 \mid (m-n+5)$, n *is even,* $(n-2, m-n+3)$ *and* $(n-3, m-n+5)$ *have disjoint binary expansions.*

4. *Let* $q \in \mathbb{N}$.
   - $n = k + 2^q$, $2^q \mid k$, $2^q \mid m$, *and* $(k, m-k)$ *has a disjoint binary expansion.*
   - $n = k + 2^q$, $2^q \mid k$, $2^q \mid (m-1)$, $(k, m-k)$ *and* $(k, m-k+2^q-1)$ *have disjoint binary expansions.*
   - $2^{q+1} < n = k+2^q$, $2^q \mid (k-1)$, $2^q \mid (m-1)$, $(k, m-k)$ *and* $(k+2^q-1, m-k)$ *have disjoint binary expansions.*
   - $2^{q+1} < n = k+2^q$, $2^q \mid (k-1)$, $2^q \mid (m-2)$, *and the pairs* $(k+1, m-k)$, $(k, m-k+1)$, $(k+2^q-1, m-k)$, *and* $(k, m-k+2^q-1)$ *have disjoint binary expansions.*

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