# A DETERMINANT OF GENERALIZED FIBONACCI NUMBERS

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Abstract. We evaluate a determinant of generalized Fibonacci numbers, thus providing a common generalization of several determinant evaluation results that have previously appeared in the literature, all of them extending Cassini's identity for Fibonacci numbers.

## 1. Introduction

The well-known *Fibonacci sequence* is given by $f_n = f_{n-1} + f_{n-2}$ with $f_0 = f_1 = 1$. Numerous properties of this sequence are known. We refer the reader to the monograph [9] for a wealth of information on this sequence. One of these properties is the so called Cassini identity, given by

$$f_n f_{n+2} - f_{n+1}^2 = (-1)^n,$$

which can be written in matrix form as

$$\det \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix} = (-1)^n. \tag{1.1}$$

Miles [6] introduced *k-generalized Fibonacci numbers* $f_n^{(k)}$ by

$$f_n^{(k)} = \sum_{i=1}^{k} f_{n-i}^{(k)},$$

with $f_n^{(k)} = 0$ for every $0 \le n \le k-2$, $f_{k-1}^{(k)} = 1$, and he gave the following generalization of (1.1):

$$\det \begin{pmatrix} f_n^{(k)} & f_{n+1}^{(k)} & \cdots & f_{n+k-1}^{(k)} \\ f_{n+1}^{(k)} & f_{n+2}^{(k)} & \cdots & f_{n+k}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+k-1}^{(k)} & f_{n+k}^{(k)} & \cdots & f_{n+2k-2}^{(k)} \end{pmatrix} = (-1)^{\frac{(2n+k)(k-1)}{2}}. \tag{1.2}$$

More recently, Stakhov [8] has generalized Cassini's identity for sequences of the form $f_n = f_{n-1} + f_{n-p-1}$.

Hoggat and Lind [4] consider the so called "dying rabbit problem", previously introduced in [1] and studied in [2] or [3], which modifies the original Fibonacci setting by letting rabbits die. In previous work by one of the authors [7], the sequence arising in

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this setting was studied in detail. For instance, the recurrence relation for this sequence depends on two parameters $k, \ell \geq 2$ and is given by

$$C_n^{(k,\ell)} = C_{n-\ell}^{(k,\ell)} + C_{n-\ell-1}^{(k,\ell)} + \cdots + C_{n-k-\ell+1}^{(k,\ell)},$$

where $C_0^{(k,\ell)}, \ldots, C_{k+\ell-2}^{(k,\ell)}$ are initial values which will be specified below. It was also proved that, if $r_1, \ldots, r_{k+\ell-1}$ are the distinct roots of $g_{k,\ell}(x) = x^{k+\ell-1} - \frac{x^k-1}{x-1}$, then the general term of the sequence is given by $C_n^{(k,\ell)} = \sum_{i=1}^{k+\ell-1} a_i r_i^n$, with

$$a_i = \frac{(-1)^{k+\ell+i-1}}{\prod_{j>i}(r_j - r_i) \prod_{j<i}(r_i - r_j)} \\ \times \left( \sum_{l=0}^{k-2} C_l^{(k,\ell)} \frac{r_i^{l+1} - 1}{r_i^{l+1}(r_i - 1)} + \sum_{l=k-1}^{k+\ell-3} C_l^{(k,\ell)} \frac{r_i^k - 1}{r_i^{l+1}(r_i - 1)} + C_{k+\ell-2}^{(k,\ell)} \right). \tag{1.3}$$

Given the previous sequence, for every $j \geq 0$ we can define a matrix $A_{j,k,\ell}$ by

$$A_{j,k,\ell} = \begin{pmatrix} C_j^{(k,\ell)} & C_{j+\ell}^{(k,\ell)} & C_{j+\ell+1}^{(k,\ell)} & \cdots & C_{j+k+2\ell-3}^{(k,\ell)} \\ C_{j+1}^{(k,\ell)} & C_{j+\ell+1}^{(k,\ell)} & C_{j+\ell+2}^{(k,\ell)} & \cdots & C_{j+k+2\ell-2}^{(k,\ell)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{j+k+\ell-2}^{(k,\ell)} & C_{j+k+2\ell-2}^{(k,\ell)} & C_{j+k+2\ell-1}^{(k,\ell)} & \cdots & C_{j+2k+3\ell-5}^{(k,\ell)}. \end{pmatrix}$$

The main goal of this paper will be to find an explicit expression for $\det(A_{j,k,\ell})$, thus extending (1.1) and (1.2).

## 2. Extending Cassini's identity

Before we proceed, we have to fix our initial conditions. In the original setting [7], when we start with a pair of rabbits that become mature $\ell$ months after their birth and die $k$ months after their matureness, the $k + \ell - 1$ initial conditions are given by $C_0^{(k,\ell)} = \cdots = C_{\ell-1}^{(k,\ell)} = 1$ and $C_n^{(k,\ell)} = C_{n-1}^{(k,\ell)} + C_{n-\ell}^{(k,\ell)}$ for every $\ell \leq n \leq k + \ell - 2$. Instead, in what follows we will consider the following initial conditions:

$$\widetilde{C}_0^{(k,\ell)} = 1,$$
$$\widetilde{C}_1^{(k,\ell)} = \cdots = \widetilde{C}_{k-1}^{(k,\ell)} = 0,$$
$$\widetilde{C}_k^{(k,\ell)} = \cdots = \widetilde{C}_{k+\ell-2}^{(k,\ell)} = 1.$$

Note that this change in the initial conditions results only in a shift of indices. Namely, if $C_n^{(k,\ell)}$ denotes the original sequence and $\widetilde{C}_n^{(k,\ell)}$ denotes the sequence given by the same recurrence relation and these new initial conditions, then for every $n \geq 0$ we have

$$C_n^{(k,\ell)} = \widetilde{C}_{n+k+1}^{(k,\ell)}.$$

Thus, if $\widetilde{A}_{j,k,\ell}$ is the corresponding matrix (defined in the obvious way), we have $A_{j,k,\ell} = \widetilde{A}_{j+k+1,k,\ell}$. Hence, we can focus on finding a formula for $\det(\widetilde{A}_{j,k,\ell})$.

First of all, observe that $\det(\widetilde{A}_{j,k,\ell}) = (-1)^{k+\ell-2}\det(\widetilde{A}_{j-1,k,\ell})$ because $\widetilde{A}_{j,k,\ell}$ is obtained from $\widetilde{A}_{j-1,k,\ell}$ by replacing the first row by the sum of the first $k$ rows of the matrix, and then permuting the rows so that the first row becomes the last one. If we apply this idea repeatedly, we obtain that $\det(\widetilde{A}_{j,k,\ell}) = (-1)^{j(k+\ell-2)}\det(\widetilde{A}_{0,k,\ell})$. Hence, it is sufficient to compute this latter determinant.

We shall focus now on computing this determinant, which explicitly is

$$\det(\widetilde{A}_{0,k,\ell}) = \det\begin{pmatrix} \widetilde{C}_0^{(k,\ell)} & \widetilde{C}_\ell^{(k,\ell)} & \widetilde{C}_{\ell+1}^{(k,\ell)} & \cdots & \widetilde{C}_{k+2\ell-3}^{(k,\ell)} \\ \widetilde{C}_1^{(k,\ell)} & \widetilde{C}_{\ell+1}^{(k,\ell)} & \widetilde{C}_{\ell+2}^{(k,\ell)} & \cdots & \widetilde{C}_{k+2\ell-2}^{(k,\ell)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{C}_{k+\ell-2}^{(k,\ell)} & \widetilde{C}_{k+2\ell-2}^{(k,\ell)} & \widetilde{C}_{k+2\ell-1}^{(k,\ell)} & \cdots & \widetilde{C}_{2k+3\ell-5}^{(k,\ell)} \end{pmatrix}.$$

To do so, recall that we have $\widetilde{C}_n^{(k,\ell)} = \sum_{s=1}^{k+\ell-1} a_s r_s^n$, where the $a_i$'s are given by (1.3). We substitute this in the above determinant and use multilinearity in the columns to expand it into the sum

$$\sum_{1\le s_1,\dots,s_{k+\ell-1}\le k+\ell-1}\left(\prod_{j=1}^{k+\ell-1} a_{s_j}\right)\det_{1\le i\le k+\ell-1}\left(r_{s_1}^{i-1}\; r_{s_2}^{i+\ell-1}\; r_{s_3}^{i+\ell}\;\cdots\; r_{s_{k+\ell-1}}^{i+k+2\ell-4}\right).$$

Now, if in this sum two of the $s_j$'s should equal each other, then the corresponding two columns in the determinant would be dependent so that the determinant would vanish. We can therefore restrict the sum to permutations of $\{1,2,\dots,k+\ell-1\}$. With $S_{k+\ell-1}$ denoting the set of these permutations, this leads to

$$\begin{aligned}
\det(\widetilde{A}_{0,k,\ell}) &= \sum_{\sigma\in S_{k+\ell-1}}\left(\prod_{j=1}^{k+\ell-1} a_{\sigma(j)}\right)\det_{1\le i\le k+\ell-1}\left(r_{\sigma(1)}^{i-1}\; r_{\sigma(2)}^{i+\ell-1}\; r_{\sigma(3)}^{i+\ell}\;\cdots\; r_{\sigma(k+\ell-1)}^{i+k+2\ell-4}\right)\\
&= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\sum_{\sigma\in S_{k+\ell-1}}\left(\prod_{j=2}^{k+\ell-1} r_{\sigma(j)}^{\ell+j-2}\right)\det_{1\le i,j\le k+\ell-1}\left(r_{\sigma(j)}^{i-1}\right)\\
&= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\sum_{\sigma\in S_{k+\ell-1}}(\operatorname{sgn}\sigma)\left(\prod_{j=2}^{k+\ell-1} r_{\sigma(j)}^{\ell+j-2}\right)\det_{1\le i,j\le k+\ell-1}\left(r_j^{i-1}\right)\\
&= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\left(\prod_{1\le i<j\le k+\ell-1}(r_j-r_i)\right)\sum_{\sigma\in S_{k+\ell-1}}(\operatorname{sgn}\sigma)\left(\prod_{j=2}^{k+\ell-1} r_{\sigma(j)}^{\ell+j-2}\right)\\
&= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\left(\prod_{1\le i<j\le k+\ell-1}(r_j-r_i)\right)\det_{1\le i\le k+\ell-1}\left(1\; r_i^{\ell}\; r_i^{\ell+1}\;\cdots\; r_i^{k+2\ell-3}\right)\\
&= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\left(\prod_{1\le i<j\le k+\ell-1}(r_j-r_i)\right)\left(\prod_{i=1}^{k+\ell-1} r_i\right)^{k+2\ell-3}\\
&\qquad\times\det_{1\le i\le k+\ell-1}\left(r_i^{-k-2\ell+3}\; r_i^{-k-\ell+3}\; r_i^{-k-\ell+4}\;\cdots\; 1\right)
\end{aligned}$$

$$= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\left(\prod_{1\le i<j\le k+\ell-1} (r_j-r_i)(r_i^{-1}-r_j^{-1})\right)\left(\prod_{i=1}^{k+\ell-1} r_i\right)^{k+2\ell-3}$$
$$\times h_{\ell-1}(r_1^{-1},\dots,r_{k+\ell-1}^{-1}). \tag{2.1}$$

In the last line we have used the following notations and facts: first of all, $h_m(x_1,\dots,x_N)$ denotes the $m$-th complete homogeneous symmetric function in $N$ variables $x_1,\dots,x_N$, explicitly given by

$$h_m(x_1,\dots,x_N) = \sum_{1\le i_1\le\dots\le i_m\le N} x_{i_1}\cdots x_{i_m}.$$

Furthermore, the Schur function indexed by a partition $\lambda=(\lambda_1,\dots,\lambda_N)$ in the variables $x_1,\dots,x_N$ is defined by

$$s_\lambda(x_1,\dots,x_N) = \frac{\det_{1\le i,j\le N}\left(x_i^{\lambda_j+N-j}\right)}{\det_{1\le i,j\le N}\left(x_i^{N-j}\right)} = \frac{\det_{1\le i,j\le N}\left(x_i^{\lambda_j+N-j}\right)}{\prod_{1\le i<j\le N}(x_i-x_j)}.$$

It is well-known (cf. [5, p. 41, Eq. (3.4)]) that for $\lambda=(m,0,\dots,0)$ the Schur function $s_\lambda(x_1,\dots,x_N)$ reduces to $h_m(x_1,\dots,x_N)$. These facts together explain the last line in the above computation.

To proceed further, let us first observe that, by reading off the constant coefficient of $g_{k,\ell}(x)$, we obtain

$$\prod_{i=1}^{k+\ell-1} r_i = (-1)^{k+\ell}.$$

Furthermore, we have

$$g_{k,\ell}(x) = x^{k+\ell-1} - \frac{x^k-1}{x-1} = \prod_{i=1}^{k+\ell-1}(x-r_i) = (-1)^{k+\ell-1}\prod_{i=1}^{k+\ell-1} r_i(1-r_i^{-1}x)$$
$$= -\prod_{i=1}^{k+\ell-1}(1-r_i^{-1}x).$$

Hence, we obtain

$$\begin{aligned}\sum_{m=0}^{\infty} h_m(r_1^{-1},\dots,r_{k+\ell-1}^{-1})x^m &= \frac{1}{\prod_{i=1}^{k+\ell-1}(1-r_i^{-1}x)}\\ &= \frac{1}{\frac{x^k-1}{x-1}-x^{k+\ell-1}}\\ &= \frac{1-x}{1-x^k-x^{k+\ell-1}+x^{k+\ell}}\\ &= 1-x+x^k-x^{k+1}+\dots+O(x^{k+\ell-1}).\end{aligned}$$

In order to evaluate $h_{\ell-1}(r_1^{-1},\dots,r_{k+\ell-1}^{-1})$, we just have to extract the coefficient of $x^{\ell-1}$ in the expansion on the right-hand side. This is easy: if $\ell-1$ equals a multiple of $k$

then we obtain 1, if $\ell - 2$ equals a multiple of $k$ then we obtain $-1$, and in all other cases we obtain 0.

We continue evaluating the other factors in (2.1). We have

$$\prod_{1\le i<j\le k+\ell-1} (r_j - r_i)(r_i^{-1} - r_j^{-1}) = \prod_{1\le i<j\le k+\ell-1} \frac{(r_j - r_i)^2}{r_i r_j} = \frac{\displaystyle\prod_{1\le i<j\le k+\ell-1} (r_j - r_i)^2}{\left(\displaystyle\prod_{i=1}^{k+\ell-1} r_i\right)^{k+\ell-2}} = \frac{\displaystyle\prod_{1\le i<j\le k+\ell-1} (r_j - r_i)^2}{(-1)^{k+\ell}}.$$

Furthermore, we must compute $\displaystyle\prod_{j=1}^{k+\ell-1} a_j$. To begin with, recall the formula (1.3) and the fact that $\widetilde{C}_0^{(k,\ell)} = \widetilde{C}_k^{(k,\ell)} = \cdots = \widetilde{C}_{k+\ell-2}^{(k,\ell)} = 1$ and $\widetilde{C}_1^{(k,\ell)} = \cdots = \widetilde{C}_{k-1}^{(k,\ell)} = 0$. With this in mind, we get

$$\begin{aligned}\prod_{j=1}^{k+\ell-1} a_j &= \frac{\displaystyle\prod_{j=1}^{k+\ell-1} (-1)^{k+\ell+j-1}}{\displaystyle\prod_{1\le i<j\le k+\ell-1} (r_j - r_i)^2} \prod_{j=1}^{k+\ell-1} \left( \frac{r_j - 1}{r_j(r_j - 1)} + \sum_{i=1}^{\ell-2} \frac{r_j^k - 1}{r_j^{k+i}(r_j - 1)} + 1 \right) \\ &= \frac{(-1)^{\frac{(3k+3\ell-2)(k+\ell-1)}{2}}}{\displaystyle\prod_{1\le i<j\le k+\ell-1} (r_j - r_i)^2} \prod_{j=1}^{k+\ell-1} \left( \frac{1}{r_j} + \sum_{i=1}^{\ell-2} \frac{r_j^k - 1}{r_j^{k+i}(r_j - 1)} + 1 \right).\end{aligned}$$

Moreover, observe that

$$\begin{aligned}\frac{1}{r_j} + \sum_{i=1}^{\ell-2} \frac{r_j^k - 1}{r_j^{k+i}(r_j - 1)} + 1 &= \frac{1}{r_j} + \frac{r_j^k - 1}{r_j - 1} \sum_{i=1}^{\ell-2} \frac{1}{r_j^{k+i}} + 1 \\ &= \frac{1}{r_j} + r_j^{k+\ell-1} \sum_{i=1}^{\ell-2} \frac{1}{r_j^{k+i}} + 1 \\ &= \frac{r_j^\ell - 1}{r_j(r_j - 1)}.\end{aligned}$$

Here, to obtain the second line, we have used the fact that $1 \neq r_j$ is a root of $x^{k+\ell-1} - \frac{x^k - 1}{x - 1}$.

Now, to conclude we must compute $\prod_{j=1}^{k+\ell-1} \frac{r_j^\ell - 1}{r_j(r_j-1)}$. To do so, let $\omega$ be a primitive $\ell$-th root of unity. Then

$$\begin{aligned}\prod_{j=1}^{k+\ell-1} (r_j^\ell - 1) &= \prod_{j=1}^{k+\ell-1} \prod_{i=1}^{\ell} (r_j - \omega^i) = \prod_{i=1}^{\ell} \prod_{j=1}^{k+\ell-1} (r_j - \omega^i)\\ &= \left(\prod_{j=1}^{k+\ell-1} (r_j - 1)\right) \left(\prod_{i=1}^{\ell-1} \prod_{j=1}^{k+\ell-1} (r_j - \omega^i)\right)\\ &= \left(\prod_{j=1}^{k+\ell-1} (r_j - 1)\right) (-1)^{(k+\ell-1)(\ell-1)} \left(\prod_{i=1}^{\ell-1} g_{k,\ell}(\omega^i)\right).\end{aligned}$$

Furthermore, $g_{k,\ell}(\omega^i) = \omega^{i(k+\ell-1)} - \frac{\omega^{ik}-1}{\omega^i-1} = -\frac{\omega^{i(k-1)}-1}{\omega^i-1}$. Consequently, we have

$$\prod_{j=1}^{k+\ell-1} \frac{r_j^\ell - 1}{r_j(r_j-1)} = (-1)^{(k+\ell)\ell} \left(\prod_{i=1}^{\ell-1} \frac{\omega^{i(k-1)} - 1}{\omega^i - 1}\right).$$

Finally observe that

$$\prod_{i=1}^{\ell-1} \frac{\omega^{i(k-1)} - 1}{\omega^i - 1} = \begin{cases} 1, & \text{if } \gcd(\ell, k-1) = 1;\\ 0, & \text{otherwise.}\end{cases}$$

We can now collect all the work done to obtain the following result.

**Theorem.** *For all integers $k$ and $\ell$ with $k, \ell \geq 2$, we have*

$$\det(\widetilde{A}_{0,k,\ell}) = \begin{cases} (-1)^{\frac{(k+\ell)(k+\ell-1)}{2}+1}, & \text{if } \ell - 1 = \alpha k \text{ and } \gcd(\ell, k-1) = 1;\\ (-1)^{\frac{(k+\ell)(k+\ell-1)}{2}}, & \text{if } \ell - 2 = \beta k \text{ and } \gcd(\ell, k-1) = 1;\\ 0, & \text{otherwise.}\end{cases}$$

**Corollary.** *Let $k_0, \ell_0 \geq 2$ be any integers. Then the following hold:*

i) *The sequence $\{\alpha_\ell\}_{\ell \geq 2}$ given by $\alpha_\ell = |\det(\widetilde{A}_{0,k_0,\ell})|$ is periodic, and its period is a divisor of $k_0 \cdot \mathrm{rad}(k_0 - 1)$.*

ii) *The sequence $\{\beta_k\}_{k \geq 2}$ given by $\beta_k = |\det(\widetilde{A}_{0,k,\ell_0})|$ is eventually zero.*

*Proof.* Clearly $\gcd(\ell, k-1) > 1$ implies that $\gcd(\ell + k \cdot \mathrm{rad}(k-1), k-1) > 1$. In the same way, if $\ell - 1$ and $\ell - 2$ are not multiples of $k$, then neither are $\ell + k \cdot \mathrm{rad}(k-1) - 1$ or $\ell + k \cdot \mathrm{rad}(k-1) - 2$. Consequently, if $a_\ell = 0$, also $a_{\ell + k \cdot \mathrm{rad}(k-1)} = 0$ as claimed. □

## 3. A general setting for further work

The computations that have been carried out in the previous section make sense in a more general setting. Indeed, let $m$ be a natural number and consider two families of integers $\mathcal{F}_1 = \{\alpha_0, \dots, \alpha_{m-1}\}$ and $\mathcal{F}_2 = \{\beta_1, \dots, \beta_m\}$. We can now define a sequence in the following recursive way:

$$s_n = \begin{cases} \alpha_n, & \text{if } n < m,\\ \sum_{j=1}^m \beta_j s_{n-j}, & \text{otherwise.}\end{cases}$$

Given a third family of integers $\mathcal{F}_3 = \{\gamma_1, \ldots, \gamma_m\}$ we define a sequence $\{A(n)\}_{n\geq 0}$ of $m \times m$ matrices by:

$$A(n)_{i,j} = s_{n+i+\gamma_j},\ 1 \leq i, j \leq m.$$

In this setting, we have the following easy lemma.

**Lemma 1.** *Up to a constant, the sequence* $\{\det A(n)\}_{n\geq 0}$ *is a geometric progression defined by powers of* $\beta_m$*; i.e.,* $\det A(n) = C\beta_m$ *for some constant* $C$.

*Proof.* This is an easy consequence of the definition of $\{s_n\}$, together with elementary properties of determinants. Namely, the last row of $A(n)$ is given by

$$A(n)_{m,j} = s_{n+m+\gamma_j} = \sum_{i=1}^{m} \beta_i s_{n+m+\gamma_j-i}.$$

Hence, it follows that

$$\det A(n) = (-1)^{m-1}\beta_m \det A(n-1),$$

as claimed. $\square$

As a consequence of the lemma above, the only point of interest is to compute the constant $C$ which, of course, will depend on the three families $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$. In Section 2, we have considered the case when $m = k + l - 1$ and:

$$\mathcal{F}_1 = \{C_0^{(k,l)}, \ldots, C_{k+k-2}^{(k,l)}\},$$
$$\mathcal{F}_2 = \{1, 1, 1, \ldots, 1\},$$
$$\mathcal{F}_3 = \{0, l, l+1, \ldots, k+2l-3\}.$$

A good source for future work would be to try to understand the dependency of $C$ as a function of the families $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ in this general setting.

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