GROWTH DIAGRAMS, AND INCREASING AND DECREASING CHAINS IN FILLINGS OF FERRERS SHAPES

C. Krattenthaler[†]

Fakultät für Mathematik, Universität Wien,
Nordbergstraße 15, A-1090 Vienna, Austria.
WWW: http://www.mat.univie.ac.at/~kratt

Dedicated to Amitai Regev on the occasion of his 65th birthday

Abstract. We put recent results by Chen, Deng, Du, Stanley and Yan on crossings and nestings of matchings and set partitions in the larger context of the enumeration of fillings of Ferrers shape on which one imposes restrictions on their increasing and decreasing chains. While Chen et al. work with Robinson–Schensted-like insertion/deletion algorithms, we use the growth diagram construction of Fomin to obtain our results. We extend the results by Chen et al., which, in the language of fillings, are results about 0-1-fillings, to arbitrary fillings. Finally, we point out that, very likely, these results are part of a bigger picture which also includes recent results of Jonsson on 0-1-fillings of stack polyominoes, and of results of Backelin, West and Xin and of Bousquet–Mélou and Steingrímsson on the enumeration of permutations and involutions with restricted patterns. In particular, we show that our growth diagram bijections do in fact provide alternative proofs of the results by Backelin, West and Xin and by Bousquet–Mélou and Steingrímsson.

1. Introduction

In the recent paper [6], Chen et al. used Robinson–Schensted like insertion/deletion processes to prove enumeration results about matchings and, more generally, set partitions with certain restrictions on their crossings and nestings. At the heart of their results, there is Greene's theorem [12] on the relation between increasing and decreasing subsequences in permutations and the shape of the tableaux which are obtained by the Robinson–Schensted correspondence. The purpose of this paper is to put these results in a larger context, namely the context of enumeration of fillings of shapes where one imposes restrictions on the increasing and decreasing chains of the fillings.

As we explain in Section 3, the results in [6] are equivalent to results about 0-1-fillings of triangular arrangements of cells. We show in Section 2 that (almost) all the

2000 *Mathematics Subject Classification.* Primary 05A15; Secondary 05A17 05A19 05E10.

Key words and phrases. Robinson–Schensted correspondence, Knuth correspondence, matchings, set partitions, integer partitions, growth diagrams.

[†] Research partially supported by EC's IHRP Programme, grant HPRN-CT-2001-00272, "Algebraic Combinatorics in Europe", by the Austrian Science Foundation FWF, grant S9607-N13, in the framework of the National Research Network "Analytic Combinatorics and Probabilistic Number Theory," and by the "Algebraic Combinatorics" Programme during Spring 2005 of the Institut Mittag–Leffler of the Royal Swedish Academy of Sciences.

theorems of [6] generalize to the setting of 0-1-fillings of arrangements of cells which have the form of a Ferrers diagram. In contrast to Chen et al., we work with the growth diagram construction of Fomin [8] which, in my opinion, leads to a more transparent presentation of the bijections between fillings and oscillating sequences of (integer) partitions which underlie these results.

In Section 4, we extend these constructions to the "Knuth setting," that is, to fillings where the constraint that in each row and in each column there is at most one 1 is dropped. The results that can be obtained in this way are presented in Theorems 7–13. They are based on four variations of the Robinson–Schensted–Knuth correspondence (presented here in the language of growth diagrams). These four variations are, in one form or another, well-known (see [11, App. A.4]), and their growth diagram versions have been worked out by Roby [21] (for the original Robinson–Schensted–Knuth correspondence) and van Leeuwen [19]. For the convenience of the reader, we recall these correspondences (in growth diagram version and in insertion version) in Section 4, and we use the opportunity to also work out the corresponding variations of Greene's theorem on increasing and decreasing chains, see Theorems 8, 10 and 12, which so far has only been done for the original Robinson–Schensted–Knuth correspondence.

The motivation to go beyond the setting of Chen et al. comes from recent results by Jonsson [15] on the enumeration of 0-1-fillings of stack polyominoes (shapes, which are more general than Ferrers diagrams) with restricted length of increasing and decreasing chains of 1's. The restriction that in each row and in each column of the shape there is at most one 1 is not present in the results by Jonsson. Unfortunately, the result on 0-1-fillings (see Theorem 13, Eq. (4.6)) which we are able to obtain from the growth diagram construction does not imply Jonsson's result, although it comes very close. Nevertheless, we believe that all these phenomena are part of a bigger picture that needs to be uncovered. To this bigger picture there belong certainly results by Backelin, West and Xin [1] and by Bousquet–Mélou and Steingrímsson [2] on the enumeration of permutations and involutions with restricted patterns. Indeed, we show that our growth diagram bijections provide alternative proofs for their results. (I owe this observation to Mireille Bousquet–Mélou.) In spite of this, apparently we do not yet have the right understanding for all these phenomena (not even regarding [1] and [2]). In particular, this paper seems to indicate that Robinson–Schensted–Knuth-like insertion/deletion, respectively growth diagram processes, do not seem to be right tools for proving the results from [15], say. All this is made more precise in Section 5, where we also formulate several open problems that should lead to an understanding of the "big picture."

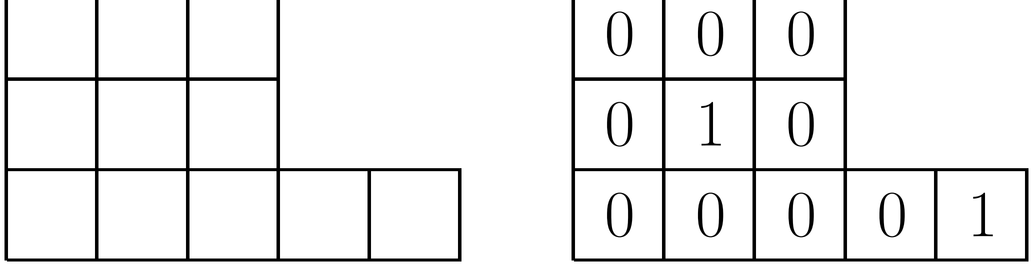
To conclude the introduction, we point out that, independently from [6], Robinson–Schensted like algorithms between set partitions and oscillating sequences of (integer) partitions have also been constructed by Halverson and Lewandowski [13], with the completely different motivation of explaining combinatorial identities arising from the representation theory of the partition algebra. Halverson and Lewandowski provide both the insertion/deletion and the growth diagram presentation of the algorithms. However, in their considerations, Greene's theorem does not play any role.

2. Growth diagrams

In this section we review the basic facts on growth diagrams (cf. [3, Sec. 3], [21, Ch. 2], [22], [23, Sec. 5.2], [25, Sec. 7.13]).

We start by fixing the standard partition notation (cf. e.g. [25, Sec. 7.2]). A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of positive integers. This also includes the *empty partition* (), denoted by $\emptyset$. For the sake of convenience, we shall often tacitly identify a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ with the infinite sequence $(\lambda_1, \lambda_2, \dots, \lambda_\ell, 0, 0, \dots)$, that is, the sequence which arises from λ by appending infinitely many 0's. To each partition λ, one associates its *Ferrers diagram* (also called *Ferrers shape*), which is the left-justified arrangement of squares with λ_i squares in the i-th row, $i = 1, 2, \dots$. We define a *partial order* $\subseteq$ on partitions by containment of their Ferrers diagrams. The *union* $\mu \cup \nu$ of two partitions μ and ν is the partition which arises by forming the union of the Ferrers diagrams of μ and ν. Thus, if $\mu = (\mu_1, \mu_2, \dots)$ and $\nu = (\nu_1, \nu_2, \dots)$, then $\mu \cup \nu$ is the partition $\lambda = (\lambda_1, \lambda_2, \dots)$, where $\lambda_i = \max\{\mu_i, \nu_i\}$ for $i = 1, 2, \dots$. The *intersection* $\mu \cap \nu$ of two partitions μ and ν is the partition which arises by forming the intersection of the Ferrers diagrams of μ and ν. Thus, if $\mu = (\mu_1, \mu_2, \dots)$ and $\nu = (\nu_1, \nu_2, \dots)$, then $\mu \cap \nu$ is the partition $\rho = (\rho_1, \rho_2, \dots)$, where $\rho_i = \min\{\mu_i, \nu_i\}$ for $i = 1, 2, \dots$. The *conjugate of* a partition λ is the partition $(\lambda'_1, \dots, \lambda'_{\lambda_1})$ where λ'_j is the length of the j-th column in the Ferrers diagram of λ.

The objects of consideration in the present paper are fillings of arrangements of cells which look like Ferrers shapes in French notation, that is, which have straight left side, straight bottom side, and which support a descending staircase, see Figure 1.a for an example. We shall encode such Ferrers shapes by sequences of D's and R's, where D stands for "down" and R stands for "right." To construct the sequence of D's and R's of a Ferrers shape we trace the right/up boundary of the Ferrers shape from top-left to bottom-right and write D whenever we encounter a down-step, respectively R whenever we encounter a right-step. For example, the Ferrers shape in Figure 1.a would be encoded by $RDRDDRDDRRD$.



a. A Ferrers shape in French notation b. A filling of the Ferrers shape

FIGURE 1

We fill the cells of such a Ferrers shape F with non-negative integers. In this section (and in the following section) the fillings will be restricted to 0-1-fillings such that every row and every column contains at most one 1. See Figure 1.b for an example.

Next, the corners of the cells are labelled by partitions such that the following two conditions are satisfied:

(C1) A partition is either equal to its right neighbour or smaller by exactly one square, the same being true for a partition and its top neighbour.

(C2) A partition and its right neighbour are equal if and only if in the column of cells of F below them there appears no 1 and if their bottom neighbours are also equal to each other. Similarly, a partition and its top neighbour are equal if and only if in the row of cells of F to the left of them there appears no 1 and if their left neighbours are also equal to each other.

See Figure 2 for an example. (More examples can be found in Figures 4–7.) There, we use a short notation for partitions. For example, 11 is short for $(1, 1)$. Moreover, we changed the convention of representing the filling slightly for better visibility, by suppressing 0's and by replacing 1's by X's. Indeed, the filling represented in Figure 2 is the same as the one in Figure 1.b.

Diagrams which obey the conditions (C1) and (C2) are called *growth diagrams.*

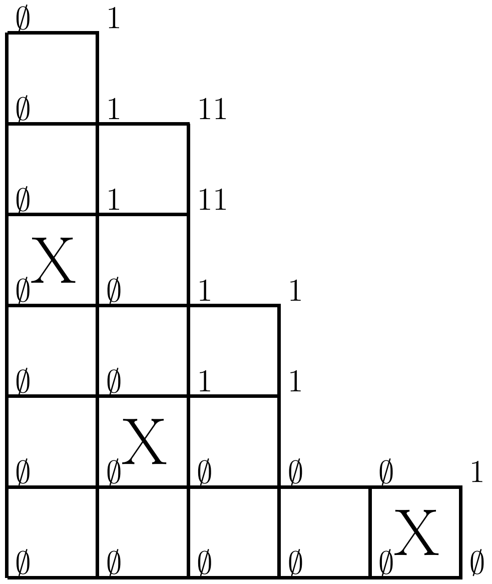


Figure 2. A growth diagram

We are interested in growth diagrams which obey the following (*forward*) *local rules* (see Figure 3).

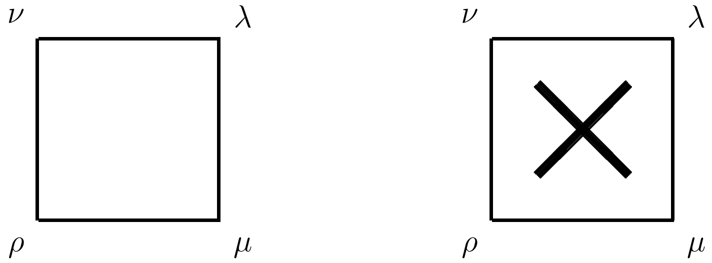
(F1) If $\rho = \mu = \nu$, and if there is no cross in the cell, then $\lambda = \rho$.

(F2) If $\rho = \mu \neq \nu$, then $\lambda = \nu$.

(F3) If $\rho = \nu \neq \mu$, then $\lambda = \mu$.

(F4) If ρ, μ, ν are pairwise different, then $\lambda = \mu \cup \nu$.

(F5) If $\rho \neq \mu = \nu$, then λ is formed by adding a square to the $(k+1)$-st row of $\mu = \nu$, given that $\mu = \nu$ and ρ differ in the k-th row.



a. A cell without cross   b. A cell with cross

FIGURE 3

(F6) If $\rho = \mu = \nu$, and if there is a cross in the cell, then λ is formed by adding a square to the first row of $\rho = \mu = \nu$.

Remarks. (1) Due to conditions (C1) and (C2), the rules (F1)–(F4) are forced. In particular, the uniqueness in rule (F4) is dictated by the constraint that neighbouring partitions can only differ by at most one square. Thus, the only "interesting" rules are (F5) and (F6).

(2) Given a 0-1-filling such that every row and every column contains at most one 1, and given a labelling of the corners along the left side and the bottom side of a Ferrers shape by partitions such that property (C1) and the additional property that two neighbouring partitions along the bottom side can only be different if there is no 1 in the column of cells of F above them, respectively two neighbouring partitions along the left side can only be different if there is no 1 in the row of cells of F to the right of them, the (forward) rules (F1)–(F6) allow one to algorithmically find the labels of the other corners of the cells by working one's way to the right and to the top.

(3) The rules (F5) and (F6) are carefully designed so that one can also work one's way the other direction, that is, given λ, μ, ν, one can reconstruct ρ *and* the filling of the cell. The corresponding (*backward*) *local rules* are:

(B1) If $\lambda = \mu = \nu$, then $\rho = \lambda$.
(B2) If $\lambda = \mu \neq \nu$, then $\rho = \nu$.
(B3) If $\lambda = \nu \neq \mu$, then $\rho = \mu$.
(B4) If λ, μ, ν are pairwise different, then $\rho = \mu \cap \nu$.
(B5) If $\lambda \neq \mu = \nu$, then ρ is formed by deleting a square from the $(k-1)$-st row of $\mu = \nu$, given that $\mu = \nu$ and λ differ in the k-th row, $k \geq 2$.
(B6) If $\lambda \neq \mu = \nu$, and if λ and $\mu = \nu$ differ in the first row, then $\rho = \mu = \nu$.

In case (B6) the cell is filled with a 1 (an X). In all other cases the cell is filled with a 0.

Thus, given a labelling of the corners along the right/up border of a Ferrers shape, one can algorithmically reconstruct the labels of the other corners of the cells *and* of the 0-1-filling by working one's way to the left and to the bottom.

In view of the above remarks, we have the following theorem. (See also [21, Theorem 2.6.7], [9, Theorem 3.6.3].)

Theorem 1. *Let F be a Ferrers shape given by the D-R-sequence $w = w_1 w_2 \dots w_k$. The* 0-1*-fillings of F with the property that every row and every column contains at* *most one* 1 *are in bijection with sequences* $(\emptyset=\lambda^0,\lambda^1,\dots,\lambda^k=\emptyset)$*, where* λ^{i-1} *and* λ^i *differ by at most one square, and* $\lambda^{i-1}\subseteq\lambda^i$ *if* $w_i=R$*, whereas* $\lambda^{i-1}\supseteq\lambda^i$ *if* $w_i=D$*. Moreover,* $\lambda^{i-1}\subsetneqq\lambda^i$ *if and only if there is a* 1 *in the column of cells of* F *below the corners labelled by* λ^{i-1} *and* λ^i*, and* $\lambda^{i-1}\supsetneqq\lambda^i$ *if and only if there is a* 1 *in the row of cells of* F *to the left of the corners labelled by* λ^{i-1} *and* λ^i*.*

Remarks. (1) In analogy to classical notions (cf. [21, 22, 26]), we call the sequences $(\lambda^0,\lambda^1,\dots,\lambda^k)$ in this theorem *oscillating tableaux of type* w *and shape* $\emptyset/\emptyset$.

(2) There are more general versions of Theorem 1 for *skew* Ferrers shapes. Since we have no need for these, we refrain from reproducing them here.

Proof of Theorem 1. To construct the mapping from the 0-1-fillings to the generalized oscillating tableaux, we label all the corners along the left side and the bottom side of F by $\emptyset$. Then we apply the forward local rules (F1)–(F6) to construct labels for all the other corners of F. The oscillating tableau $(\lambda^0,\lambda^1,\dots,\lambda^k)$ is read off as the labels along the right/up border of F. For example, the 0-1-filling in Figure 1.b is mapped to the oscillating tableau $(\emptyset,1,1,11,11,1,1,1,\emptyset,\emptyset,1,\emptyset)$ (see Figure 2).

That we obtain an oscillating tableau of type w is obvious from conditions (C1) and (C2). That the map is a bijection is obvious from the preceding discussion. □

It is a now well-known fact that, in the case that the Ferrers shape is a square and that we consider 0-1-fillings with *exactly* one 1 in each row and each column, the bijection in Theorem 1 is equivalent to the *Robinson–Schensted correspondence*. Namely, 0-1-fillings of an $n\times n$ square with exactly one 1 in each row and each column are in bijection with permutations. On the other hand, according to Theorem 1, the sequences $(\lambda^0,\lambda^1,\dots,\lambda^{2n})$ which we read off the top side and the right side of the square are sequences with $\emptyset=\lambda^0\subsetneqq\lambda^1\subsetneqq\cdots\subsetneqq\lambda^n\supsetneqq\cdots\supsetneqq\lambda^{2n-1}\supsetneqq\lambda^{2n}=\emptyset$. In their turn, these are in bijection with pairs (P,Q) of standard tableaux of the same shape, the common shape consisting of n squares. (See [25, Sec. 7.13]. The standard tableau Q is defined by the first half of the sequence, $\emptyset=\lambda^0\subsetneqq\lambda^1\subsetneqq\cdots\subsetneqq\lambda^n$, the entry i being put in the square by which λ^i and λ^{i-1} differ, and, similarly, the standard tableau P is defined by the second half of the sequence, $\emptyset=\lambda^{2n}\subsetneqq\lambda^{2n-1}\subsetneqq\cdots\subsetneqq\lambda^n$.) It is then a theorem (cf. [3, pp. 95–98], [25, Theorem 7.13.5]) that the bijection between permutations and pairs of standard tableaux defined by the growth diagram on the square coincides with the Robinson–Schensted correspondence, P being the *insertion tableau*, and Q being the *recording tableau*.

The special case where the Ferrers shape is triangular is discussed in more detail in the next section.

In addition to its local description, the bijection in Theorem 1 has also a *global* description. The latter is a consequence of a theorem of Greene [12] (see also [3, Theorems 2.1 and 3.2]). In order to formulate the result, we need the following definitions: a *NE-chain* of a 0-1-filling is a sequence of 1's in the filling such that any 1 in the sequence is above and to the right of the preceding 1 in the sequence. Similarly, a *SE-chain* of a 0-1-filling is a set of 1's in the filling such that any 1 in the sequence is below and to the right of the preceding 1 in the sequence.

Theorem 2. *Given a growth diagram with empty partitions labelling all the corners along the left side and the bottom side of the Ferrers shape, the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ labelling corner c satisfies the following two properties:*

(G1) *For any k, the maximal cardinality of the union of k NE-chains situated in the rectangular region to the left and below of c is equal to $\lambda_1 + \lambda_2 + \cdots + \lambda_k$.*

(G2) *For any k, the maximal cardinality of the union of k SE-chains situated in the rectangular region to the left and below of c is equal to $\lambda'_1 + \lambda'_2 + \cdots + \lambda'_k$, where λ' denotes the partition conjugate to λ.*

In particular, λ_1 is the length of the longest NE-chain in the rectangular region to the left and below of c, and λ'_1 is the length of the longest SE-chain in the same rectangular region.

In order to formulate the main theorem of this section, we introduce one more piece of notation. We write $N(F; n; NE = s, SE = t)$ for the number of 0-1-fillings of the Ferrers shape F with exactly n 1's, such that there is at most one 1 in each column and in each row, and such that the longest NE-chain has length s and the longest SE-chain, the smallest rectangle containing the chain being contained in F, has length t.

Theorem 3. *For any Ferrers shape F and positive integers s and t, we have*

$$N(F; n; NE = s, SE = t) = N(F; n; NE = t, SE = s).$$

Proof. We define a bijection between the 0-1-fillings counted by $N(F; n; NE = s, SE = t)$ and those counted by $N(F; n; NE = t, SE = s)$. Let the Ferrers shape F be given by the D-R-sequence $w = w_1 w_2 \dots w_k$. Given a 0-1-filling counted by $N(F; n; NE = s, SE = t)$ we apply the mapping of the proof of Theorem 1. Thus, we obtain an oscillating tableau $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^k = \emptyset)$. Since the 0-1-filling had n entries 1, in the oscillating tableau there are exactly n "rises" $\lambda^{i-1} \subsetneqq \lambda^i$ (and, hence, exactly n "falls" $\lambda^{i-1} \supsetneqq \lambda^i$). Moreover, by Theorem 2, we have $\lambda^i_1 \le s$ and $(\lambda^j)'_1 \le t$ for all i and j, with equality for at least one i and at least one j. Now we apply the inverse mapping to the sequence $(\emptyset = (\lambda^0)', (\lambda^1)', \dots, (\lambda^k)' = \emptyset)$ of conjugate partitions. Thus, we obtain a 0-1-filling counted by $N(F; n; NE = t, SE = s)$. □

If we specialize Theorem 3 to the case where F is a square and to 0-1-fillings which have exactly one entry 1 in each row and in each column, then we obtain a trivial statement: The number of permutations of $\{1, 2, \dots, n\}$ with longest increasing subsequence of length s and longest decreasing subsequence of length t is equal to the number of permutations of $\{1, 2, \dots, n\}$ with longest increasing subsequence of length t and longest decreasing subsequence of length s. This statement is indeed trivial because, given a former permutation $\pi_1 \pi_2 \dots \pi_n$, the reversal $\pi_n \pi_{n-1} \dots \pi_1$ will belong to the latter permutations.

However, for other Ferrers shapes F, there is no trivial explanation for Theorem 3. In particular, as we are going to show in the next section, if we specialize Theorem 3 to the case where F is triangular, then we obtain a non-obvious theorem, originally due to Chen et al. [6], about crossings and nestings in set partitions.

3. Crossings and nestings in set partitions and matchings

In this section we show that the main theorems in [6] are, essentially, special cases of the theorems in the previous section.

First of all, we have to recall the definitions and basic objects from [6]. The objects of consideration in [6] are *set partitions* of $\{1, 2, \dots, n\}$. A block $\{i_1, i_2, \dots, i_b\}$, $i_1 < i_2 < \cdots < i_b$, of such a set partition is represented by the set of pairs $\{(i_1, i_2), (i_2, i_3), \dots, (i_{b-1}, i_b)\}$. More generally, a set partition is represented by the union of all sets of pairs, the union being taken over all its blocks. This representation is called the *standard representation* of the set partition. For example, the set partition $\{\{1, 4, 5, 7\}, \{2, 6\}, \{3\}\}$ is represented as the set $\{(1, 4), (4, 5), (5, 7), (2, 6)\}$. Next, one defines a *$k$-crossing* of a set partition to be a subset $\{(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)\}$ of its standard representation where $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. Similarly, one defines a *k-nesting* of a set partition to be a subset $\{(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)\}$ of its standard representation where $i_1 < i_2 < \cdots < i_k < j_k < \cdots < j_2 < j_1$. (These notions have intuitive pictorial meanings if one connects a pair (i, j) in the standard representation of a set partition by an arc, cf. [6].) Finally, given a set partition P, we write $\operatorname{cross}(P)$ for the maximal number k such that P has a k-crossing, and we write $\operatorname{nest}(P)$ for the maximal number k such that P has a k-nesting,

In [6, Eq. (3)], the following theorem is proved by a Robinson–Schensted-like insertion/deletion process, which sets up a bijection between set partitions and "vacillating tableaux" (see below).

Theorem 4. *Let n, s, t be positive integers. Then the number of set partitions of $\{1, 2, \dots, n\}$ with $\operatorname{cross}(P) = s$ and $\operatorname{nest}(P) = t$ is equal to the number of set partitions of $\{1, 2, \dots, n\}$ with $\operatorname{cross}(P) = t$ and $\operatorname{nest}(P) = s$.*

As we now explain, this is just a special case of Theorem 3, where F is triangular. More precisely, let Δ_n be the triangular shape with $n-1$ cells in the bottom row, $n-2$ cells in the row above, etc., and 1 cell in the top-most row. See Figure 4 for an example in which $n = 7$. (The filling and labelling of the corners should be ignored at this point. For convenience, we also joined pending edges at the right and at the top of Δ_n.)

We represent a set partition of $\{1, 2, \dots, n\}$, given by its standard representation $\{(i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)\}$, as a 0-1-filling, by putting a 1 in the i_r-th column and the j_r-th row from above (where we number rows such that the row consisting of $j-1$ cells is the row numbered j), $r = 1, 2, \dots, m$. The filling corresponding to the set partition $\{\{1, 4, 5, 7\}, \{2, 6\}, \{3\}\}$ is shown in Figure 4, where we present again 1's by X's and suppress the 0's.

It is obvious that this defines a bijection between set partitions of $\{1, 2, \dots, n\}$ and 0-1-fillings of Δ_n in which every row and every column contains at most one 1. Moreover, a k-crossing corresponds to a SE-chain in the fillings, while a k-nesting corresponds to a NE-chain. Thus, Theorem 3 specialized to $F = \Delta_n$ yields Theorem 4 immediately.

Figure 4 shows as well the labelling of the corners by partitions if we apply the correspondence of Theorem 1. The "extra" corners created by the pending edges are labelled by empty partitions. The oscillating tableaux which one reads along the right/up border of Δ_n are sequences $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$ with the property that λ^{2i+1} is obtained from λ^{2i} by doing nothing or by deleting one square, and λ^{2i} is obtained from λ^{2i-1} by doing nothing or adding a square. Such oscillating tableaux are called *vacillating tableaux* in [6]. Again, from Theorem 1, it is obvious that set partitions of $\{1, 2, \dots, n\}$ are in bijection with vacillating tableaux $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$. Figure 4 does in fact work out the bijection for the set partition of Example 4 in [6]. One can verify that the resulting vacillating tableau is the same as in [6]. Indeed, in general, the bijection

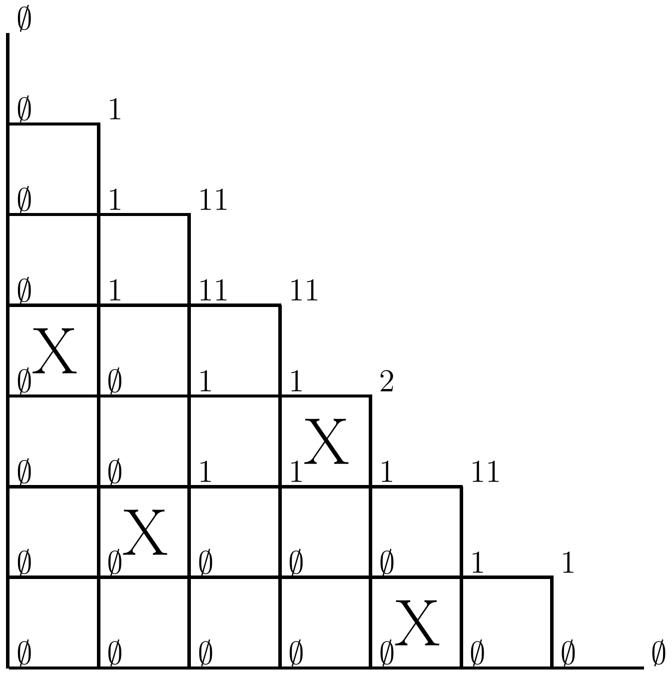


FIGURE 4. Bijection between $\{\{1,4,5,7\},\{2,6\},\{3\}\}$ and $(\emptyset,\emptyset,1,1,11,11,11,1,2,1,11,1,1,\emptyset,\emptyset)$

between set partitions and vacillating tableaux described in Section 3 of [6] is equivalent with our growth diagram bijection. This follows from the afore-mentioned fact that the growth diagrams model Robinson–Schensted insertion.

The main theorem in [6] is in fact a refinement of Theorem 4. This refinement takes also into account the minimal and the maximal elements in the blocks of a set partition P. While, from the growth diagram point of view, this refinement is special for triangular Ferrers shapes and cannot be extended in a natural way to arbitrary Ferrers shapes, it follows nevertheless with equal ease from the growth diagram point of view. Given a set partition P, let $\min(P)$ be the set of minimal elements of the blocks of P, and let $\max(P)$ be the set of maximal elements of the blocks of P. Then Theorem 1 from [6] reads as follows.

Theorem 5. *Let n,s,t be positive integers, and let S and T be two subsets of $\{1,2,\dots,n\}$. Then the number of set partitions of $\{1,2,\dots,n\}$ with $\operatorname{cross}(P)=s$, $\operatorname{nest}(P)=t$, $\min(P)=S$, $\max(P)=T$ is equal to the number of set partitions of $\{1,2,\dots,n\}$ with $\operatorname{cross}(P)=t$, $\operatorname{nest}(P)=s$, $\min(P)=S$, $\max(P)=T$.*

Proof. As in the proof of Theorem 3, we set up a bijection between the two different sets of set partitions. In fact, using the correspondence between set partitions and 0-1-fillings that we explained after the statement of Theorem 4, the bijection is exactly the same as the one in the proof of Theorem 3. We only have to figure out how one can detect the minimal and maximal elements in blocks in the vacillating tableau to which a set partition is mapped, and verify that these are kept invariant under conjugation of the partitions of the vacillating tableau.

Let $\{i_1, i_2, \dots, i_b\}$ be a block of the set partition P. Then, in the 0-1-filling corresponding to P, there is a 1 in *column* i_1 and row i_2, while there is no 1 in *row* i_1. Similarly, there is a 1 in column i_{b-1} and *row* i_b, while there is no 1 in *column* i_b. Consequently, the two corners on the right of row i_1 will be labelled by partitions λ^{2i_1-2} and λ^{2i_1-1} with $\lambda^{2i_1-2} = \lambda^{2i_1-1}$, and the two corners on the top of column i_1 will be labelled by partitions λ^{2i_1-1} and λ^{2i_1} with $\lambda^{2i_1-1} \subsetneq \lambda^{2i_1}$. On the other hand, the two corners on the right of row i_b will be labelled by partitions λ^{2i_b-2} and λ^{2i_b-1} with $\lambda^{2i_b-2} \supsetneq \lambda^{2i_b-1}$, and the two corners on the top of column i_b will be labelled by partitions λ^{2i_b-1} and λ^{2i_b} with $\lambda^{2i_b-1} = \lambda^{2i_b}$. Thus, in summary, the sets $\max(P)$ and $\max(P)$ can be detected from the growth properties of the subsequences $(\lambda^{2i-2}, \lambda^{2i-1}, \lambda^{2i})$ of the vacillating tableau corresponding to P. Clearly, these remain invariant under conjugation of the partitions. Hence, the theorem. $\square$

For the sake of completeness, we also review how one can realize the other Robinson–Schensted-like insertion/deletion mappings in [6].

To begin with, Theorem 5 in [6] describes a bijection between pairs (P, T) of set partitions P of $\{1, 2, \dots, n\}$ and (partial) standard tableaux T of shape λ with set of entries contained in $\max(P)$ and vacillating tableaux $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \lambda)$. In growth diagram language, the bijection can be realized by putting the 0-1-filling corresponding to P in the cells of Δ_n, labelling all the corners along the left side of Δ_n by $\emptyset$, and labelling the corners along the bottom side of Δ_n by the increasing sequence of partitions corresponding to T, that is, the ℓ-th corner is labelled by the partition corresponding to the shape that is covered by the entries of T contained in $\{1, 2, \dots, \ell\}$, $\ell = 0, 1, \dots, n$. Figure 5 shows the growth diagram describing the bijection in the case that

$$(P, T) = \left(\begin{matrix} 1 & 7 \\ 5 & \end{matrix} \;,\; \{\{1\}, \{2, 6\}, \{3\}, \{4, 7\}, \{5\}\} \right). \tag{3.1}$$

This is, in fact, the pair (P, T) of Example 3 in [6]. As the figure shows, this pair is mapped to the vacillating tableau

$$(\emptyset, \emptyset, 1, 1, 2, 2, 2, 2, 21, 21, 211, 21, 21, 11, 21), \tag{3.2}$$

in agreement with the result of the (differently defined) bijection in [6].

In Section 4 of [6], a variant of the bijection between set partitions and vacillating tableaux is discussed, namely a bijection between set partitions and *hesitating tableaux*. Here, a hesitating tableau is an oscillating tableau $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$ with the property that, for each i, either (1) $\lambda^{2i-2} = \lambda^{2i-1} \subsetneq \lambda^{2i}$, or (2) $\lambda^{2i-2} \supsetneq \lambda^{2i-1} = \lambda^{2i}$, or (3) $\lambda^{2i-2} \subsetneq \lambda^{2i-1} \supsetneq \lambda^{2i}$.

In terms of growth diagrams, this bijection can be again realized as a special case of the bijection described in the proof of Theorem 3. (This realization is also attributed to Michael Korn in [6].) Here, we do not transform set partitions into 0-1-fillings of a triangular shape, but into 0-1-fillings of a slightly modified shape that may vary depending on the original set partition. More precisely, given a set partition P of $\{1, 2, \dots, n\}$, we deform Δ_n by attaching an extra cell in the i-th row from above and the i-th column for every singleton block $\{i\}$ of P, and by attaching an extra cell in the j-th row from above and the j-th column whenever (i, j) and (j, k) are both in the standard representation of P, for some i and k. Then, as before, we transform P into a 0-1-filling of this modification of Δ_n, by placing a 1 in the i-th column and j-th row

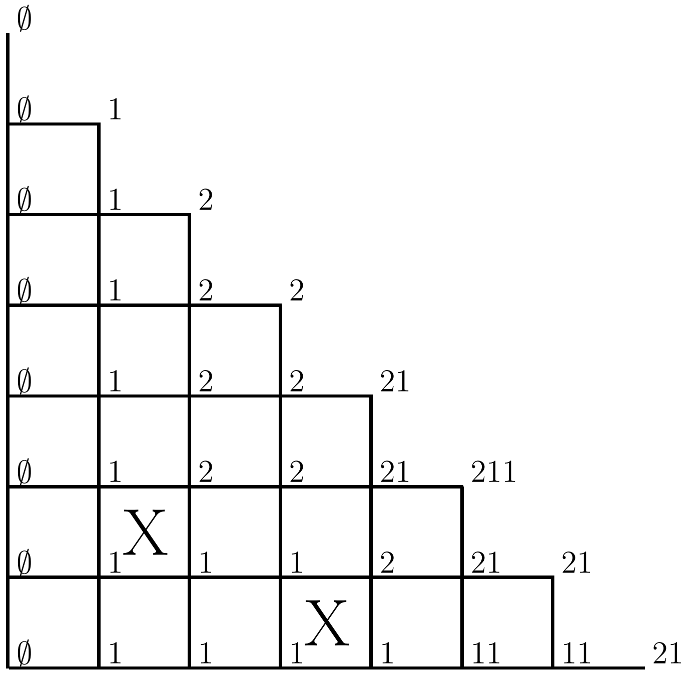


FIGURE 5. Bijection between the pair in (3.1) and the vacillating tableau in (3.2)

if (i, j) is a pair in the standard representation of P, and, in addition, by placing a 1 in the added cell in the i-th row and the i-th column for every singleton block $\{i\}$ of P. All other cells are filled with 0's. An example is shown in Figure 6. It shows the 0-1-filling corresponding to the partition $\{\{1, 4, 5, 7\}, \{2, 6\}, \{3\}\}$ under this modified rule. The added cells are indicated by dotted lines.

To realize the bijection to hesitating tableaux, we label all the corners along the left side and the bottom side by $\emptyset$, and then apply the forward local rules (F1)–(F6) to determine the labels for all other corners. Along the right/up border one reads off a hesitating tableau. It is easy to see that this defines a bijection. The example in Figure 6 maps the partition $\{\{1, 4, 5, 7\}, \{2, 6\}, \{3\}\}$ to the hesitating tableau $(\emptyset, \emptyset, 1, 1, 1, 11, 21, 11, 21, 2, 21, 11, 1, 1, \emptyset, \emptyset)$. This example corresponds to Example 6 in [6].

For the sake of completeness, we record the consequence of this bijection from [6, Theorem 11]. In the statement, we need the notion of *enhanced* k-crossings and k-nestings. To define these, one first defines the *enhanced representation* of a set partition P to be union of the standard representation of P with the set of pairs (i, i), where i ranges over all the singleton blocks $\{i\}$ of P. Then one defines an *enhanced k-crossing* of a set partition to be a subset $\{(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)\}$ of its enhanced representation where $i_1 < i_2 < \cdots < i_k \le j_1 < j_2 < \cdots < j_k$. Similarly, one defines an *enhanced k-nesting* of a set partition to be a subset $\{(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)\}$ of its enhanced representation where $i_1 < i_2 < \cdots < i_k \le j_k < \cdots < j_2 < j_1$. Finally, given a set partition P, we write $\overline{\mathrm{cross}}(P)$ for the maximal number k such that P has an enhanced k-crossing, and we write $\overline{\mathrm{nest}}(P)$ for the maximal number k such that P has an enhanced k-nesting,

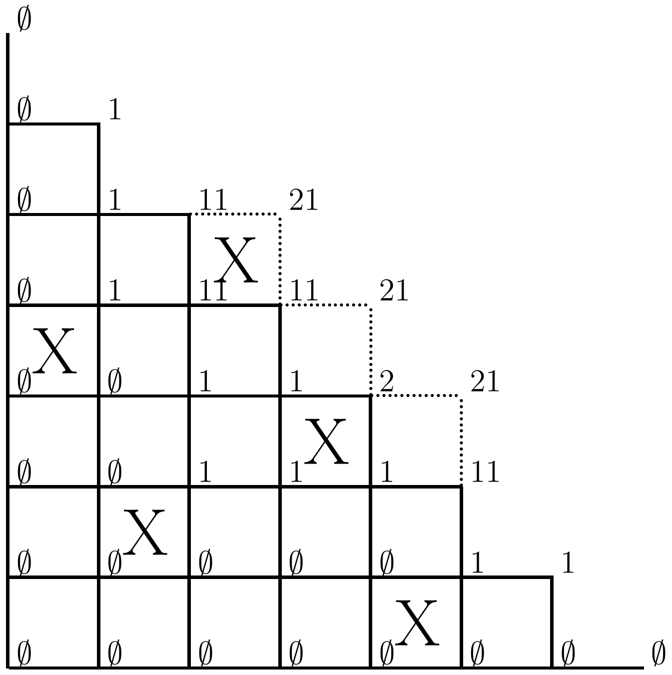


FIGURE 6. Bijection between $\{\{1,4,5,7\},\{2,6\},\{3\}\}$ and $(\emptyset,\emptyset,1,1,11,21,11,21,2,21,11,1,1,\emptyset,\emptyset)$

With the above notation, the following variant of Theorem 5 holds.

Theorem 6. *Let n,s,t be positive integers, and let S and T be two subsets of $\{1,2,\dots,n\}$. Then the number of set partitions of $\{1,2,\dots,n\}$ with $\overline{\operatorname{cross}}(P)=s$, $\overline{\operatorname{nest}}(P)=t$, $\min(P)=S$, $\max(P)=T$ is equal to the number of set partitions of $\{1,2,\dots,n\}$ with $\overline{\operatorname{cross}}(P)=t$, $\overline{\operatorname{nest}}(P)=s$, $\min(P)=S$, $\max(P)=T$.*

We conclude this section by recalling the growth diagram bijection between *matchings* and (ordinary) oscillating tableaux (cf. [21, Sec. 4] or [22]). Clearly, matchings of $\{1,2,\dots,2n\}$ can be alternatively seen as partitions of $\{1,2,\dots,2n\}$ all the blocks of which consist of two elements. In their turn, if we transform the latter to the corresponding 0-1-fillings of the triangular shape Δ_n, then these filling have the property that in the union of the i-th column and the i-th row (from above) there is exactly one 1. See Figure 7 for the 0-1-filling corresponding to the matching $\{\{1,4\},\{2,6\},\{3,5\}\}$. Consequently, if we apply the bijection from the proof of Theorem 3 we obtain a vacillating tableau $(\emptyset=\lambda^0,\lambda^1,\dots,\lambda^{4n}=\emptyset)$ with the property that, for each i, either (1) $\lambda^{2i-2}=\lambda^{2i-1}\subsetneqq\lambda^{2i}$, or (2) $\lambda^{2i-2}\supsetneqq\lambda^{2i-1}=\lambda^{2i}$. Thus, the information contained in the partitions λ^{2i-1} is superfluous and can be dropped. What remains is an (ordinary) oscillating tableau, that is, a sequence $(\emptyset=\lambda^0,\lambda^2,\lambda^4\dots,\lambda^{4n}=\emptyset)$ with the property that successive partitions in the sequence differ by exactly one square. Figure 7 shows that, under this bijection, the matching $\{\{1,4\},\{2,6\},\{3,5\}\}$ is mapped to the oscillating tableau $\{\emptyset,1,11,21,2,1,\emptyset\}$. This is in accordance with Example 7 in [6].

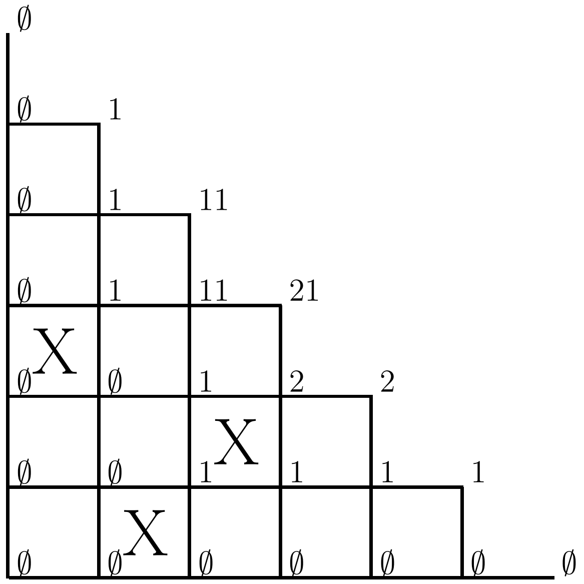


FIGURE 7. Bijection between $\{\{1,4\},\{2,6\},\{3,5\}\}$ and $\{\emptyset,1,11,21,2,1,\emptyset\}$

4. Growth diagrams for arbitrary fillings of Ferrers shapes

In this section we embed the considerations of the previous two sections into the larger context where we relax the conditions on the fillings that we imposed so far: namely, in this section, we do not insist anymore that our fillings have at most one 1 in each row and in each column and otherwise 0's. That is, we shall allow more than one 1 in rows and columns and we shall also allow arbitrary non-negative entries in our fillings. As it turns out, there are now four variants how to define growth diagrams for these more general fillings which lead to (different) extensions of Theorem 3. (All of these are, of course, special instances of the general set-up in [10, Theorem 3.6].) They have been described in some detail in [21] and [19], albeit without giving the analogues of Greene's theorem in all four cases, the latter being exactly what we need for our purposes. We use the opportunity here to provide thorough descriptions of all four of these correspondences in this section, including the analogues of Greene's theorem. The corresponding extensions of Theorems 1 and 2 are given in Theorems 7–12, while the corresponding extensions of Theorem 3 are given in Theorem 13.

To sketch the idea, let us consider the filling of the 2×2 square on the left of Figure 8. (At this point, the labellings of the corners of the cells should be ignored.) We cannot apply the forward local rules to such a diagram since the entries in the cells are not just 1's and 0's, and, even if they should be (such as in the filling of the rectangle on the left of Figure 10), there could be several 1's in a column or in a row. To remedy this, we "separate" the entries. That is, we construct a diagram with more rows and columns so that entries which are originally in the same column or in the same row are put in different columns and rows in the larger diagram, and that an entry m is replaced by m 1's in the new diagram, all of which placed in different rows and columns. For this "separation" we have two choices for the columns and two choices for the rows: either we "separate" entries in a row by putting them into a chain from bottom/left to top/right, or we "separate" them by putting them into a chain from top/left to bottom/right,

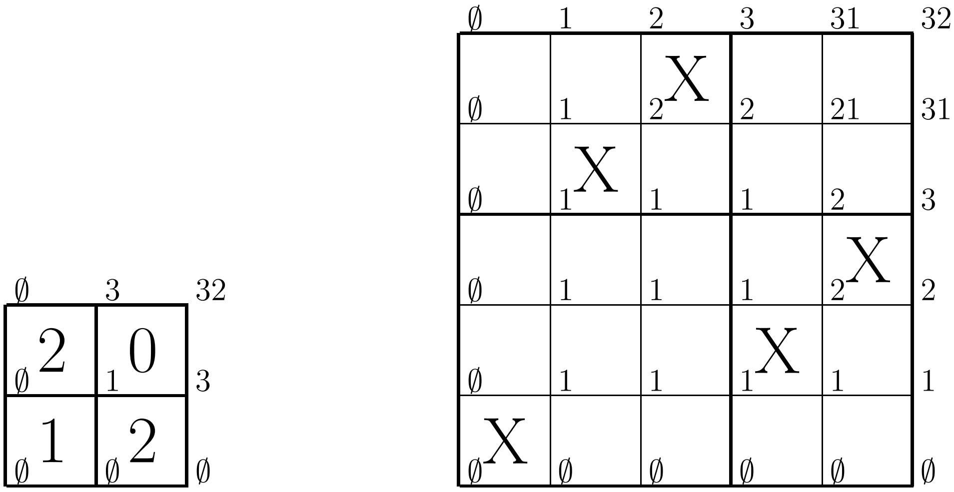


FIGURE 8. The "blow-up" of an arbitrary filling

the same being true for entries in a column. In total, this gives $2 \times 2 = 4$ choices of separation. As we shall see, two out of these four (the Second and Third Variant below) are, in fact, equivalent modulo a reflection of growth diagrams. Interestingly, this equivalence can only be seen directly from the growth diagram algorithms, but not from the insertion algorithms that go with them. (The First and Fourth Variant are also related, but in a much more subtle way. See [19, Sec. 3.2].)

4.1. **First variant: RSK.** The first variant (described in detail in [21, Sec. 4.1]) generalizes the Robinson–Schensted–Knuth (RSK) correspondence. It is defined for arbitrary fillings of a Ferrers shape with non-negative integers.

Let us consider the filling of the 2×2 square on the left of Figure 8. (The labellings of the corners of the cells should be ignored at this point.) The filling is now converted into the 0-1-filling of a larger shape (where, again, 1's are represented by X's and 0's are suppressed). If a cell is filled with entry m, we replace m by a chain of m X's arranged from bottom/left to top/right. If there should be several entries in a column then we arrange the chains coming from the entries of the column as well from bottom/left to top/right. We do the same for the rows. The diagram on the right of Figure 8 shows the result of this conversion when applied to the filling on the left of Figure 8. (Still, the labellings of the corners of the cells in the augmented diagram should be ignored at this point.) In the figure, the original columns and rows are indicated by thick lines, whereas the newly created columns and rows are indicated by thin lines.

Now we can apply the forward rules (F1)–(F6) to the augmented diagram. That is, we label all the corners of the cells on the left side and the bottom side of the augmented diagram by $\emptyset$, and then we apply (F1)–(F6) to determine the labels of all the other corners. Subsequently, we "shrink back" the augmented diagram, that is, we record only the labels of the corners located at the intersections of thick lines. This yields the labels on the left of Figure 8. For a different example see Figure 10.

The labellings by partitions that one obtains in this manner have again the property that a partition is contained in its right neighbour and in its top neighbour. In addition, two neighbouring partitions differ by a *horizontal strip*, that is, by a set of squares no two of which are in the same column.

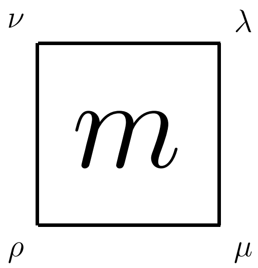


FIGURE 9. A cell filled with a non-negative integer m

Clearly, we could also define local forward and backward rules *directly* on the original (smaller) diagram. This has been done in detail in [19, Sec. 3.1]. For the sake of completeness, we recall these rules here in a slightly different, but of course equivalent, fashion. Consider the cell in Figure 9, filled by a non-negative integer m, and labelled by the partitions ρ, μ, ν, where $\rho \subseteq \mu$ and $\rho \subseteq \nu$, μ and ρ differ by a horizontal strip, and ν and ρ differ by a horizontal strip. Then λ is determined by the following algorithm:

($F^1 0$) Set CARRY := m and $i := 1$.
($F^1 1$) Set $\lambda_i := \max\{\mu_i, \nu_i\} + \text{CARRY}$.
($F^1 2$) If $\lambda_i = 0$, then stop. The output of the algorithm is $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{i-1})$. If not, then set CARRY := $\min\{\mu_i, \nu_i\} - \rho_i$ and $i := i + 1$. Go to ($F^1 1$).

Conversely, given μ, ν, λ, where $\mu \subseteq \lambda$ and $\nu \subseteq \lambda$, where λ and μ differ by a horizontal strip, and where λ and ν differ by a horizontal strip, the backward algorithm works in the following way:

($B^1 0$) Set $i := \max\{j : \lambda_j \text{ is positive}\}$, and CARRY := 0.
($B^1 1$) Set $\rho_i := \min\{\mu_i, \nu_i\} - \text{CARRY}$.
($B^1 2$) Set CARRY := $\lambda_i - \max\{\mu_i, \nu_i\}$ and $i := i - 1$. If $i = 0$, then stop. The output of the algorithm is $\rho = (\rho_1, \rho_2, \dots)$ and $m = \text{CARRY}$. If not, go to ($B^1 1$).

The extension of Theorem 1 then reads as follows. In the statement of the theorem below (and also later), for two partitions with $\nu \subseteq \mu$ we write μ/ν for the difference of the Ferrers diagrams corresponding to μ and ν, respectively, that is, for the set of squares which belong to the Ferrers diagram of μ but not to the Ferrers diagram of ν.

Theorem 7. *Let F be a Ferrers shape given by the D-R-sequence $w = w_1 w_2 \dots w_k$. Fillings of F with non-negative integers are in bijection with sequences $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^k = \emptyset)$, where λ^i/λ^{i-1} is a horizontal strip if $w_i = R$, whereas λ^{i-1}/λ^i is a horizontal strip if $w_i = D$.*

It is a now well-known fact (cf. [21, Sec. 4.1]) that, in the case that the Ferrers shape is a rectangle, the bijection in Theorem 7 is equivalent to the *Robinson–Schensted–Knuth correspondence*. Namely, assuming that F is a $p \times q$ rectangle, according to Theorem 7, the sequences $(\lambda^0, \lambda^1, \dots, \lambda^{p+q})$ which we read off the top side and the right side of the square are sequences with $\emptyset = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^q \supseteq \cdots \supseteq \lambda^{p+q-1} \supseteq \lambda^{p+q} = \emptyset$, where λ^i/λ^{i-1} is a horizontal strip for $i = 1, 2, \dots, q$, and where λ^i/λ^{i+1} is a horizontal strip for $i = q, q+1, \dots, p+q-1$. In their turn, these are in bijection with pairs (P, Q) of

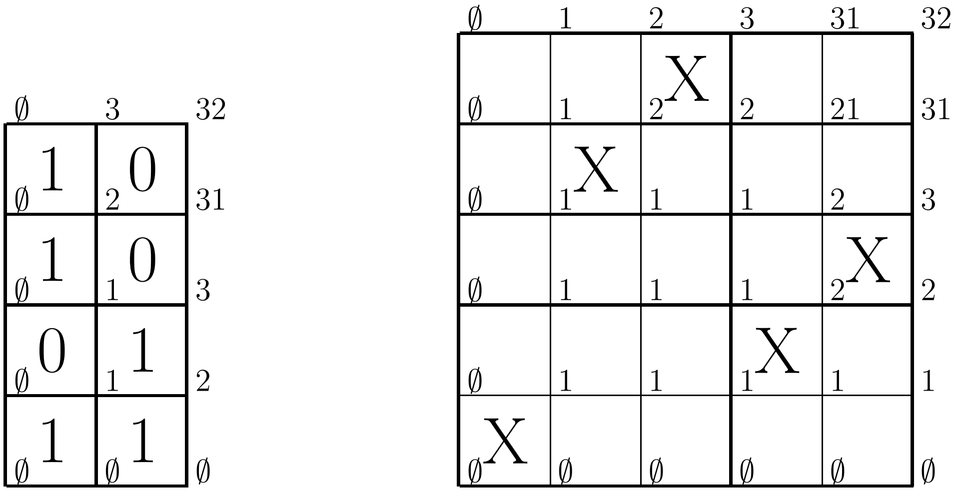


FIGURE 10. Growth diagrams and RSK

semistandard tableaux of the same shape, the common shape consisting of n squares, where n is the sum of all the entries of the filling. (See [21]. The semistandard tableau Q is defined by the first half of the sequence, $\emptyset = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^q$, the entry i being put in the squares by which λ^i and λ^{i-1} differ, and, similarly, the semistandard tableau P is defined by the second half of the sequence, $\emptyset = \lambda^{p+q} \subseteq \lambda^{p+q-1} \subseteq \cdots \subseteq \lambda^q$.) It is then a theorem that the bijection between fillings and pairs of semistandard tableaux defined by the growth diagram on the rectangle coincides with the Robinson–Schensted–Knuth correspondence, P being the *insertion tableau*, and Q being the *recording tableau*. (The reader is referred to [11, Ch. 4], [20], [23, Sec. 4.8], [25, Sec. 7.11] for extensive information on the Robinson–Schensted–Knuth correspondence.)

Figure 10 is meant to illustrate this, using an example that will serve as a running example here as well as for the other three variants. Let us recall that the RSK correspondence starts with a rectangular filling as on the left of Figure 10, transforms the filling into a two-rowed array, and then transforms the two-rowed array into a pair of semistandard tableaux by an insertion procedure. The two-rowed array is obtained from the filling, by considering the entry, m say, in the i-th row (from below) and j-th column (from the left), and recording m pairs $\binom{j}{i}$. These pairs are then ordered into a two-rowed array such that the entries in the top row are weakly increasing, and, in the bottom row, entries must be weakly increasing below equal entries in the top row. Thus, the filling in Figure 8 corresponds to the two-rowed array

$$\begin{pmatrix} 1\ 1\ 1\ 2\ 2 \\ 1\ 2\ 2\ 1\ 1 \end{pmatrix},$$

while the filling in Figure 10 corresponds to the two-rowed array

$$\begin{pmatrix} 1\ 1\ 1\ 2\ 2 \\ 1\ 3\ 4\ 1\ 2 \end{pmatrix}. \tag{4.1}$$

Now the bottom entries are inserted according to *row insertion*, in which an element bumps the first entry in a row which is strictly larger. The top entries keep track of where the individual insertions stop. If this is applied to the two-rowed array in (4.1), one obtains

$$(P,Q)=\begin{pmatrix} \begin{matrix}1&1&2\\3&4&\end{matrix} & , & \begin{matrix}1&1&1\\2&2&\end{matrix}\end{pmatrix},$$

which is indeed in agreement with the increasing sequences of partitions along the right side and the top side of the rectangle, respectively.

Again, in addition to its local description, the bijection in Theorem 7 has also a *global* description. It is again a consequence of Greene's theorem (stated here as Theorem 2) and the description of the bijection based on "separation" of entries along columns and rows. In order to formulate the result, we adapt our previous definitions: a *NE-chain* of a filling is a sequence of non-zero entries in the filling such that any entry in the sequence is weakly above and weakly to the right of the preceding entry in the sequence. The *length* of such a NE-chain is defined as the *sum* of all the entries in the chain. On the other hand, a *se-chain* of a filling is a sequence of non-zero entries in the filling such that any entry in the sequence is strictly below and strictly to the right of the preceding entry in the sequence. In contrast to NE-chains, we define the *length* of a se-chain as the *number* of entries in the chain. (These definitions can be best motivated as weak, respectively strict, chains of balls in the matrix–ball model for fillings described in [11, Sec. 4.2].)

Theorem 8. *Given a diagram with empty partitions labelling all the corners along the left side and the bottom side of the Ferrers shape, which has been completed according to RSK, the partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_\ell)$ labelling corner c satisfies the following two properties:*

(G^11) *For any k, the maximal sum of all the entries in a collection of k pairwise disjoint NE-chains situated in the rectangular region to the left and below of c is equal to $\lambda_1+\lambda_2+\dots+\lambda_k$.*

(G^12) *Fix a positive integer k and consider collections of k se-chains with the property that no entry e can be in more than e of these se-chains. Then the maximal cardinality of the multiset* (!) *union of the se-chains in such a collection of se-chains situated in the rectangular region to the left and below of c is equal to $\lambda_1'+\lambda_2'+\dots+\lambda_k'$, where λ' denotes the partition conjugate to λ.*

In particular, λ_1 is the length of the longest NE-chain in the rectangular region to the left and below of c, and λ_1' is the length of the longest se-chain in the same rectangular region.

4.2. **Second variant: dual RSK.** The second variant generalizes the "dual correspondence" of Knuth [20], which we abbreviate as *dual RSK*. It is only defined for 0-1-fillings of a Ferrers shape. (In contrast to Sections 2 and 3, these can, however, be arbitrary, that is, there can be several 1's in a row or a column.)

As an illustration for this second variant, we use again the filling on the left of Figure 10, which is reproduced on the left of Figure 11. (The labellings of the corners of the cells should be ignored at this point.) The filling is now converted into a 0-1-filling of a larger shape, but in a different way than before (where, again, 1's will be represented by X's and 0's are suppressed). Namely, if there should be several 1's in a column then, as in the first variant, we arrange them from bottom/left to top/right. However, we do the "opposite" for the rows, that is, if there should be several 1's in a row then we arrange them from top/left to bottom/right. The diagram on the right of

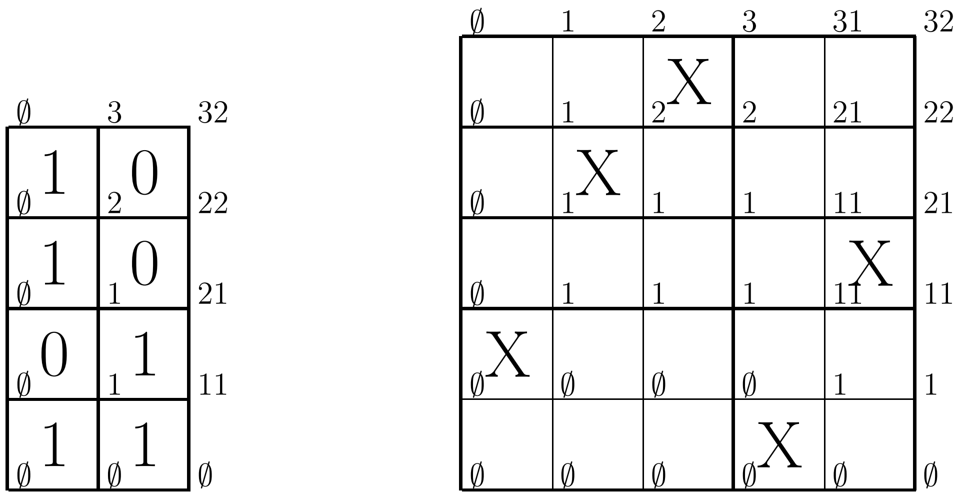


Figure 11. Growth diagrams and dual RSK

Figure 11 shows the result of this conversion when applied to the filling on the left of Figure 11.

Now, as in the first variant, we apply the forward rules (F1)–(F6) to the augmented diagram, and, once this is done, we "shrink back" the augmented diagram, as before. This yields the labels on the left of Figure 11.

Of course, the labellings by partitions that one obtains in this manner have again the property that a partition is contained in its right neighbour and in its top neighbour. However, now two partitions which are horizontal neighbours differ by a *horizontal strip*, whereas two partitions which are *vertical* neighbours differ by a *vertical* strip, that is, by a set of squares no two of which are in the same row.

Direct local forward and backward rules on the original (smaller) diagram are also available for this variant, see [19, Sec. 6]. Our presentation is again slightly different, but equivalent. Namely, consider the cell in Figure 9, filled by $m = 0$ or $m = 1$ and labelled by the partitions ρ, μ, ν, where $\rho \subseteq \mu$ and $\rho \subseteq \nu$, μ and ρ differ by a horizontal strip, and ν and ρ differ by a vertical strip. Then λ is determined by the following algorithm:

(F^{2}0) Set CARRY $:= m$ and $i := 1$.
(F^{2}1) Set $\lambda_i := \max\{\mu_i + \text{CARRY}, \nu_i\}$.
(F^{2}2) If $\lambda_i = 0$, then stop. The output of the algorithm is $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{i-1})$. If not, then set CARRY $:= \min\{\mu_i + \text{CARRY}, \nu_i\} - \rho_i$ and $i := i+1$. Go to (F^{2}1).

Conversely, given μ, ν, λ, where $\mu \subseteq \lambda$ and $\nu \subseteq \lambda$, where λ and μ differ by a vertical strip, and where λ and ν differ by a horizontal strip, the backward algorithm works in the following way:

(B^{2}0) Set $i := \max\{j : \lambda_j \text{ is positive}\}$ and CARRY $:= 0$.
(B^{2}1) Set $\rho_i := \min\{\mu_i, \nu_i - \text{CARRY}\}$.
(B^{2}2) Set CARRY $:= \lambda_i - \max\{\mu_i, \nu_i - \text{CARRY}\}$ and $i := i - 1$. If $i = 0$, then stop. The output of the algorithm is $\rho = (\rho_1, \rho_2, \dots)$ and $m = $ CARRY. If not, go to (B^{2}1).

The corresponding extension of Theorem 1 then reads as follows.

Theorem 9. *Let F be a Ferrers shape given by the D-R-sequence $w = w_1 w_2 \dots w_k$. Then 0-1-fillings of F with non-negative integers are in bijection with sequences $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^k = \emptyset)$, where λ^i/λ^{i-1} is a horizontal strip if $w_i = R$, whereas λ^{i-1}/λ^i is a vertical strip if $w_i = D$.*

In the case that the Ferrers shape is a rectangle, the bijection in Theorem 9 is equivalent to dual RSK. Namely, assuming that F is a $p \times q$ rectangle, according to Theorem 9, the sequences $(\lambda^0, \lambda^1, \dots, \lambda^{p+q})$ which we read off the top side and the right side of the square are sequences with $\emptyset = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^q \supseteq \cdots \supseteq \lambda^{p+q-1} \supseteq \lambda^{p+q} = \emptyset$, where λ^i/λ^{i-1} is a horizontal strip for $i = 1, 2, \dots, q$, and where λ^i/λ^{i+1} is a vertical strip for $i = q, q+1, \dots, p+q-1$. In their turn, these are in bijection with pairs (P, Q), where Q and the transpose of P are semistandard tableaux of the same shape, the common shape consisting of n squares, where n is the sum of all the entries of the filling. The semistandard tableau Q is defined by the first half of the sequence, $\emptyset = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^q$, as before, while P is defined by the second half of the sequence, $\emptyset = \lambda^{p+q} \subseteq \lambda^{p+q-1} \subseteq \cdots \subseteq \lambda^q$, as before. Since, in the latter chain of partitions, successive partitions differ by *vertical* strips, P is not a semistandard tableau, but its transpose is. This bijection between fillings and pairs (P, Q) coincides with dual RSK. (We refer the reader to [11, App. A.4.3], [20, Sec. 5], [23, Sec. 4.8], [25, Sec. 7.14] for more information on the dual correspondence.)

Figure 11 illustrates this with our running example. Let us recall that dual RSK starts with a rectangular filling, consisting of 0's and 1's, as on the left of Figure 11. The filling is now transformed into a two-rowed array, by recording a pair $\binom{j}{i}$ for a 1 in the i-th row (from below) and j-th column (from the left). Subseqently, the pairs are ordered into a two-rowed array as before, so that we obtain again the two-rowed array (4.1). Now the bottom entries are inserted according to *column insertion* (cf. [20, Sec. 5, Algorithm INSERT*; the tableaux there must be transposed to obtain our version here]), in which an element bumps the first entry in a column which is larger than or equal to it. Again, the top entries keep track of where the individual insertions stop. If this is applied to the two-rowed array in (4.1), one obtains

$$(P^t, Q^t) = \left(\begin{matrix} 1 & 1 \\ 2 & 3 \\ 4 & \end{matrix} \; , \; \begin{matrix} 1 & 2 \\ 1 & 2 \\ 1 & \end{matrix} \right), \tag{4.2}$$

which is indeed in agreement with the increasing sequences of partitions along the right side and the top side of the rectangle, respectively, if one transposes both arrays in (4.2).

Again, in addition to its local description, the bijection in Theorem 9 has also a *global* description. It is again a consequence of Greene's theorem (stated here as Theorem 2) and the description of the bijection based on "separation" of entries along columns and rows. In order to formulate the global description of the bijection in terms of increasing and decreasing chains, we need to define a *nE-chain* of a filling to be a sequence of 1's in the filling such that any 1 in the sequence is strictly above and weakly to the right of the preceding 1 in the sequence. Furthermore, we define a *Se-chain* of a filling to be a sequence of 1's in the filling such that any 1 in the sequence is weakly below and strictly to the right of the preceding 1 in the sequence. The *length* of a nE-chain or a Se-chain is defined as the number of 1's in the chain. (Again, these definitions can be best motivated as chains of balls in the matrix–ball model for fillings described in [11, Sec. 4.2].)

Theorem 10. *Given a diagram with empty partitions labelling all the corners along the left side and the bottom side of the Ferrers shape, which has been completed according to dual RSK, the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ labelling corner c satisfies the following two properties:*

(G^{2}1) *For any k, the maximal cardinality of the union of k nE-chains situated in the rectangular region to the left and below of c is equal to $\lambda_1 + \lambda_2 + \dots + \lambda_k$.*

(G^{2}2) *For any k, the maximal cardinality of the union of k Se-chains situated in the rectangular region to the left and below of c is equal to $\lambda_1' + \lambda_2' + \dots + \lambda_k'$, where λ' denotes the partition conjugate to λ.*

In particular, λ_1 is the length of the longest nE-chain in the rectangular region to the left and below of c, and λ_1' is the length of the longest Se-chain in the same rectangular region.

4.3. **Third variant: RSK'.** The third variant which we have in mind is, in growth diagram terms, the "reflection" of the second variant. By "reflection," we mean reflection of growth diagrams in a diagonal. More precisely, given a 0-1-filling of a Ferrers diagram F, we separate 1's in the same column by arranging them from top/left to bottom/right, while we separate 1's in the same row by arranging them from bottom/left to top/right. What we obtain when we apply this to our running example, is shown in Figure 12.

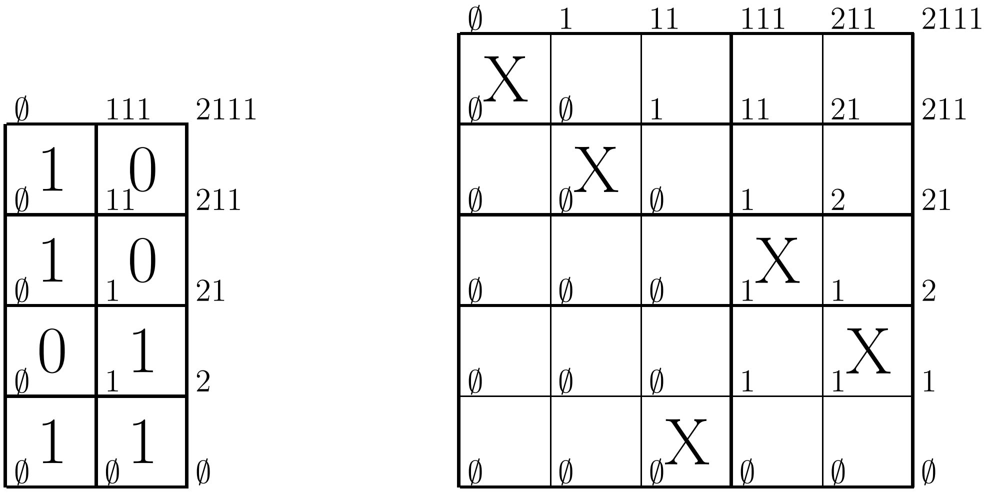


Figure 12. Growth diagrams and RSK'

We abbreviate this algorithm as *RSK'*. Since, as we said, in growth diagram terms, this is just the second variant, but reflected in a diagonal, the relevant facts have already been told when discussing dual RSK, with one exception: we have to explain to which insertion algorithm RSK' is equivalent.

In order to do so, we transform again our 0-1-filling in Figure 12 into a two-rowed array. This is again done by constructing pairs from the 1's in the filling, as before. However, the pairs are now ordered in a different way. Namely, we order the pairs such that the entries in the top row are weakly increasing, in the bottom row, however, entries must be *decreasing* below equal entries in the top row. Thus, the filling in Figure 12 corresponds to the two-rowed array

$$\begin{pmatrix} 1\;1\;1\;2\;2 \\ 4\;3\;1\;2\;1 \end{pmatrix}. \tag{4.3}$$

Now we apply row insertion to the bottom entries to construct a semistandard tableau P and use the top entries to record the insertions in the array Q. In general, Q will not be a semistandard tableau, but the transpose of Q will be. If we apply this procedure to our two-rowed array in (4.3), then we obtain the pair

$$(P,Q) = \left(\begin{matrix} 1\;1 \\ 2 \\ 3 \\ 4 \end{matrix} \; , \; \begin{matrix} 1\;2 \\ 1 \\ 1 \\ 2 \end{matrix} \right),$$

which is indeed in agreement with the increasing sequences of partitions along the right side and the top side of the rectangle in Figure 12, respectively. (An equivalent insertion algorithm is described in [11, App. A.4.3, (1d), (2d)].)

4.4. **Fourth variant: dual RSK'.** Our last variant is, as the first variant, defined for arbitrary fillings of a Ferrers shape with non-negative integers. We abbreviate it by *dual RSK'*.

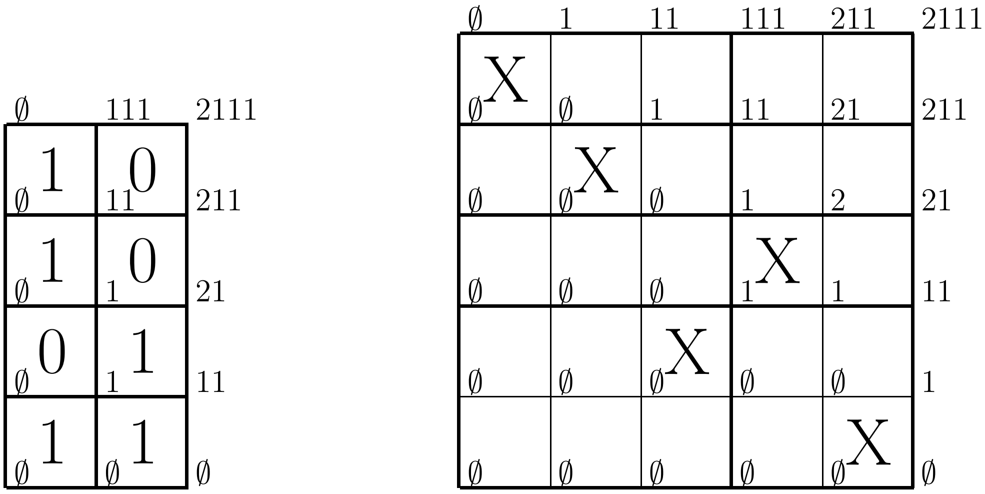


Figure 13. Growth diagrams and dual RSK'

Our running example serves once more to illustrate this fourth variant, see Figure 13. Here, we separate entries in the following way. If a cell is filled with entry m, we replace m by a chain of m X's arranged from top/left to bottom/right. If there should be several entries in a column then we arrange the chains coming from the entries of the column as well from top/left to bottom/right. We do the same for the rows. (In brief, everything is reversed compared to the first variant.)

Finally, as always, we apply the forward rules (F1)–(F6) to the augmented diagram, and, once this is done, we "shrink back" the augmented diagram. See Figure 13.

The labellings by partitions that one obtains in this manner have again the property that a partition is contained in its right neighbour and in its top neighbour. However, here, two neighbouring partitions differ by a *vertical strip*.

The corresponding *direct* local forward and backward rules have been worked out in [19, Sec. 3.2]. Again, for the sake of completeness, we recall them here in a slightly different fashion. Let us again consider the cell in Figure 9, filled by a non-negative integer m, and labelled by the partitions ρ, μ, ν, where $\rho \subseteq \mu$ and $\rho \subseteq \nu$, μ and ρ differ by a vertical strip, and ν and ρ differ by a vertical strip. Then, with the usual truth function $\chi(\mathcal{A}) = 1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A}) = 0$ otherwise, λ is determined by the following algorithm:

(F^{4}0) Set CARRY $:= m$ and $i := 1$.
(F^{4}1) Set $\lambda_i := \max\{\mu_i, \nu_i\} + \min\{\chi(\rho_i = \mu_i = \nu_i), \mathrm{CARRY}\}$.
(F^{4}2) If $\lambda_i = 0$, then stop. The output of the algorithm is $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{i-1})$. If not, then set CARRY $:=$ CARRY $- \min\{\chi(\rho_i = \mu_i = \nu_i), \mathrm{CARRY}\} + \min\{\mu_i, \nu_i\} - \rho_i$ and $i := i + 1$. Go to (F^{4}1).

Conversely, given μ, ν, λ, where $\mu \subseteq \lambda$ and $\nu \subseteq \lambda$, where λ and μ differ by a vertical strip, and where λ and ν differ by a vertical strip, the backward algorithm works in the following way:

(B^{4}0) Set $i := \max\{j : \lambda_j \text{ is positive}\}$ and CARRY $:= 0$.
(B^{4}1) Set $\rho_i := \min\{\mu_i, \nu_i\} - \min\{\chi(\mu_i = \nu_i = \lambda_i), \mathrm{CARRY}\}$.
(B^{4}2) Set CARRY $:=$ CARRY $- \min\{\chi(\mu_i = \nu_i = \lambda_i), \mathrm{CARRY}\} + \lambda_i - \max\{\mu_i, \nu_i\}$ and $i := i - 1$. If $i = 0$, then stop. The output of the algorithm is $\rho = (\rho_1, \rho_2, \dots)$ and $m =$ CARRY. If not, go to (B^{4}1).

The corresponding extension of Theorem 1 then reads as follows.

Theorem 11. *Let F be a Ferrers shape given by the D-R-sequence $w = w_1 w_2 \dots w_k$. Fillings of F with non-negative integers are in bijection with sequences ($\emptyset = \lambda^0, \lambda^1, \dots, \lambda^k = \emptyset$), where λ^i/λ^{i-1} is a vertical strip if $w_i = R$, whereas λ^{i-1}/λ^i is a vertical strip if $w_i = D$.*

In the case that the Ferrers shape is a rectangle, the bijection in Theorem 11 is equivalent to the following insertion algorithm. In order to describe this algorithm, we transform again our filling in Figure 13 into a two-rowed array. We do this in the same way as in the insertion procedure modelling the third variant. Namely, for an entry m in the i-th row (from below) and j-th column (from the left), we record m pairs $\binom{j}{i}$. The pairs are then ordered such that the entries in the top row are weakly increasing, in the bottom row, however, entries must be *decreasing* below equal entries in the top row. Clearly, in the case of our running example, this yields again the two-rowed array (4.3).

Now we apply column insertion to the bottom entries to construct the semistandard tableau P^t and use the top entries to record the insertions in the semistandard tableau Q^t. If we apply this procedure to our two-rowed array in (4.3), then we obtain the pair

$$(P^t, Q^t) = \left(\begin{matrix} 1\,1\,3\,4 \\ 2 \end{matrix} \;,\; \begin{matrix} 1\,1\,1\,2 \\ 2 \end{matrix} \right), \tag{4.4}$$

which is indeed in agreement with the increasing sequences of partitions along the right side and the top side of the rectangle in Figure 13, respectively, if one transposes both arrays in (4.4). (An equivalent insertion algorithm is described in [11, App. A.4.1], and is called "Burge correspondence" there since it appears as an aside in [5]. The same terminology is used in [19, Sec. 3.2].)

As before, the bijection in Theorem 11 has also a *global* description, as a consequence of Greene's theorem. In view of previous definitions, it should be clear what we mean by *ne-chains* and *SE-chains*. We are then ready to formulate the corresponding result.

Theorem 12. *Given a diagram with empty partitions labelling all the corners along the left side and the bottom side of the Ferrers shape, which has been completed according to dual RSK', the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ labelling corner c satisfies the following two properties:*

($G^4$1) *Fix a positive integer k and consider collections of k ne-chains with the property that no entry e can be in more than e of these ne-chains. Then the maximal cardinality of the multiset* (!) *union of the ne-chains in such a collection of ne-chains situated in the rectangular region to the left and below of c is equal to $\lambda_1 + \lambda_2 + \cdots + \lambda_k$.*

($G^4$2) *For any k, the maximal sum of all the entries in a collection of k pairwise disjoint SE-chains situated in the rectangular region to the left and below of c is equal to $\lambda_1' + \lambda_2' + \cdots + \lambda_k'$, where λ' denotes the partition conjugate to λ.*

In particular, λ_1 is the length of the longest ne-chain in the rectangular region to the left and below of c, and λ_1' is the length of the longest SE-chain in the same rectangular region.

4.5. **Consequences on the enumeration of fillings with restrictions on their increasing and decreasing chains.** We are now ready to state and prove the two extensions of Theorem 3 which result from Theorems 8, 10 and 12. In the statement, in analogy to previous notation, we write $N^*(F;n;NE = s, se = t)$ for the number of fillings of the Ferrers shape F with non-negative integers with sum of entries equal to n such that the longest NE-chain has length s and the longest se-chain, the smallest rectangle containing the chain being contained in F, has length t. The notation $N^*(F;n;ne = s, SE = t)$ has the obvious analogous meaning. Furthermore, we write $N^{01}(F;n;nE = s, Se = t)$ for the number of 0-1-fillings of the Ferrers shape F with sum of entries equal to n such that the longest nE-chain has length s and the longest Se-chain, the smallest rectangle containing the chain being contained in F, has length t. The notation $N^{01}(F;n;Ne = s, sE = t)$ has the obvious analogous meaning. We then have the following extensions of Theorem 3.

Theorem 13. *For any Ferrers shape F and positive integers s and t, we have*

$$N^*(F;n;NE = s, se = t) = N^*(F;n;ne = t, SE = s) \tag{4.5}$$

and

$$N^{01}(F;n;nE = s, Se = t) = N^{01}(F;n;Ne = t, sE = s). \tag{4.6}$$

Proof. We sketch the proof of (4.5). We define a bijection between the fillings counted by $N^*(F;n;NE = s, se = t)$ and those counted by $N^*(F;n;ne = t, SE = s)$. Let the Ferrers shape F be given by the D-R-sequence $w = w_1w_2\dots w_k$. Given a filling counted by $N^*(F;n;NE = s, se = t)$ we apply the mapping of the proof of Theorem 7. Thus, we obtain a sequence $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^k = \emptyset)$, where λ^i/λ^{i-1} is a horizontal strip if $w_i = R$, whereas λ^{i-1}/λ^i is a horizontal strip if $w_i = D$. Since the sum of the entries in the filling was n, in the sequence the sum of the amounts of "rises" $\lambda^{i-1} \subseteq \lambda^i$ equals n (the amount is the number of squares by which λ^i and λ^{i-1} differ; equivalently, the sum of the amounts of "falls" $\lambda^{i-1} \supseteq \lambda^i$ is exactly n). Moreover, by Theorem 8, we have $\lambda_1^i \le s$ and $(\lambda^j)_1' \le t$ for all i and j, with equality for at least one i and at least one j. Now we apply the inverse mapping from Theorem 11 to the sequence $(\emptyset = (\lambda^0)', (\lambda^1)', \dots, (\lambda^k)' = \emptyset)$ of conjugate partitions. Thus, due to Theorem 12, we obtain a filling counted by $N^*(F; n; ne = t, SE = s)$.

The proof of (4.6) is completely analogous, but uses the mapping from Theorem 9 to go from fillings counted by $N^{01}(F; n; nE = s, Se = t)$ to oscillating sequences of partitions, and the inverse "reflected" mapping from Section 4.3 to go from the sequence of conjugate partitions to fillings counted by $N^{01}(F; n; Ne = s, sE = t)$. Theorem 10 is used to see how the lengths of the nE-chains, Se-chains, Ne-chains, and sE-chains are related to the partitions in the oscillating sequences of partitions. □

In the special case that F is triangular, it is straight-forward to use the bijections of this section to formulate extensions of Theorems 4–6 to partitions of *multisets*. We omit the details here for the sake of brevity. The reader will have no difficulty to work them out.

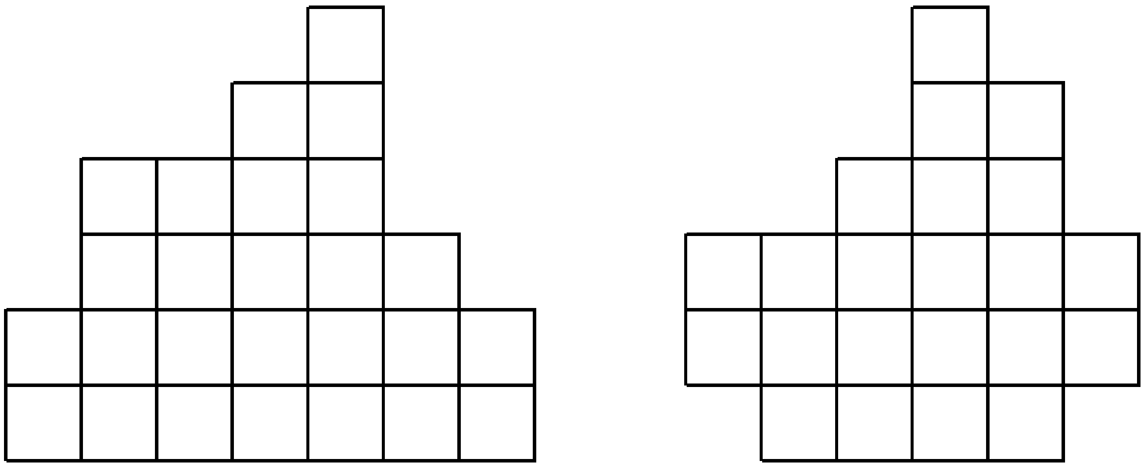
5. The big picture?

In this paper we have been investigating which results on fillings, where length restrictions are imposed on their chains, can be obtained by Robinson–Schensted-like correspondences. While I believe that the corresponding analysis here is (more or less) complete, I believe at the same time that the obtained results constitute just a small section in a much larger field of phenomena that are concerned with fillings which avoid certain patterns. The evidence that I can put forward are recent results by Backelin, West and Xin [1], Bousquet–Mélou and Steingrímsson [2] (generalizing previous results of Jaggard [14]), and Jonsson [15]. This, speculative, last section is devoted to a comparison of these results with ours, and to posing some problems that suggest themselves in this context.

To begin with, in [15] Jonsson considers 0-1-fillings of *stack polyominoes*. Here, a stack polyomino is the concatenation of a Ferrers shape in French notation that has been reflected in a vertical line with another (unreflected) Ferrers shape in French notation. See Figure 14.a for an example. Extending previously introduced notation, we write $N^{01}(F; n; ne = s, se = t)$ for the number of 0-1-fillings of a stack polyomino F with exactly n 1's such that the longest ne-chain, the smallest rectangle containing the chain being contained in F, has length s and the longest se-chain, the smallest rectangle containing the chain being contained in F, has length t. Then Jonsson proves in [15, Theorem 14] (Theorem 13 of [15] contains even a refinement) that

$$N^{01}(F; n; ne = s, se = *) = N^{01}(F'; n; ne = s, se = *) \tag{5.1}$$

if n is maximal so that 0-1-configurations exist, the ne-chains of which are of length at most s, where F' is the Ferrers shape which arises by permuting the columns of F so that they are ordered from the longest column to the shortest. ($se = *$ means that there is no restriction on the length of the se-chains.) Moreover, he and Welker [16] extended this result to arbitrary n. Not only that, he also says that he expects these results to remain true if F is a moon polyomino, i.e., an arrangement of cells such that along any row of cells and along any column of cells there is no hole, and such that any two columns of F have the property that one column can be embedded in the other by applying a horizontal shift. See Figure 14.b for an example.



a. A stack polyomino b. A moon polyomino

FIGURE 14

Jonsson proves his result by an involved inductive argument which does not shed any light why his result is true. Jonsson and Welker, on the other hand, use machinery from commutative algebra to prove their generalization. However, the most natural proof that one could think of is, of course, a bijective one. This leads us to our first problem.

Problem 1. *Find a bijective proof of* (5.1).

Where is the (at least, potential) connection to the material of our paper? If we would apply Jonsson and Welker's results to a stack polyomino F and its reflection in a vertical line, we would obtain that

$$N^{01}(F;n;ne=s,se=*)=N^{01}(F;n;ne=*,se=s). \tag{5.2}$$

Clearly, this is very close to the assertions of Theorems 3 and 13, and it brings us to our next problem.

Problem 2. *Is it true that for any Ferrers shape* (*stack polyomino, moon polyomino*) *F and positive integers s and t, we have*

$$N^{01}(F;n;ne=s,se=t)=N^{01}(F;n;ne=t,se=s) \quad ? \tag{5.3}$$

The reader should observe that, although Theorems 3 and 13 are very similar to (5.3), the bijections used in their proofs cannot be used to prove (5.3). Certainly, Theorem 3 addresses 0-1-fillings, but 0-1-fillings with the *additional condition* that in each row and in each column there is at most one 1. Also (4.6) is about 0-1-fillings, and indeed about 0-1-fillings without further restrictions. However, the bijection does not keep track of lengths of ne-chains and se-chains, but instead of lengths of nE-chains, Se-chains, Ne-chains and sE-chains. On the other hand, (4.5) does address ne-chains and se-chains.

Indeed, it would imply

$$N^*(F;n;ne=s,se=*)=N^*(F;n;ne=*,se=s). \tag{5.4}$$

As Jakob Jonsson (private communication) pointed out to me, although this is not an assertion about 0-1-fillings but about *arbitrary* fillings, and although it does not imply (5.3), it implies at least (5.2). Namely, consider the simplicial complex $\Delta_{ne\le s}$ of 0-1-fillings of F with ne-chains being of length at most s. (We refer the reader to [24, Ch. II] or [4, Ch. 5] for terminology and background on simplicial complexes and Stanley–Reisner rings.) Similarly, define the simplicial complex $\Delta_{se\le s}$. There is an obvious bijection between monomials in the Stanley–Reisner ring of $\Delta_{ne\le s}$ and *arbitrary* fillings of F with ne-chains being of length at most s, and there is an analogous, equally obvious bijection between monomials in the Stanley–Reisner ring of $\Delta_{se\le s}$ and *arbitrary* fillings of F with se-chains being of length at most s. Using this language, (5.4) says that there are as many monomials of degree n in the Stanley–Reisner ring of $\Delta_{ne\le s}$ as there are monomials of degree n in the Stanley–Reisner ring of $\Delta_{se\le s}$. But this says that the Hilbert functions for these two Stanley–Reisner rings are the same, which implies that the corresponding simplicial complexes have the same h-vector and dimension, and hence the same f-vector. The latter means that, for any positive integer n, there are as many 0-1-fillings with n 1's and ne-chains being of length at most s as there are 0-1-fillings with n 1's and se-chains being of length at most s, which is equivalent to (5.2).

Problem 3. *Are there extensions of Theorems* 3 *and* 13 *to stack polyominoes? To moon polyominoes? Are there analogues of* (5.1) *for nE-chains, Se-chains, Ne-chains, sE-chains, for arbitrary fillings?*

The papers by Backelin, West and Xin [1] and Bousquet–Mélou and Steingrímsson [2], on the other hand, consider 0-1-fillings of Ferrers shapes with the property that there is *exactly* one 1 in every row and in every column of the Ferrers shape. As Mireille Bousquet–Mélou pointed out to me, the bijection behind Theorem 3 provides in fact alternative proofs of the results in these two papers. In the case of the main result in [2] this alternative proof is actually considerably simpler.

To be more specific, it is shown in [1, Prop. 2.3] that the main theorem in that paper, [1, Theorem 2.1], is implied by the key assertion (see [1, Prop. 2.2]) that there are as many such fillings of a Ferrers shape F with longest NE-chain of length s as there are such fillings with longest SE-chain of length s. This is indeed covered by Theorem 3, by choosing n to be equal to the maximum of width and height of F. (Clearly, if width and height of a Ferrers shape F are not equal, then there are no 0-1-fillings with exactly one 1 in every row and in every column of F.)

Bousquet–Mélou and Steingrímsson [2], on the other hand, prove an analogue of the main result of Backelin, West and Xin for *involutions* (see [2, Theorem 1]). Their proof is based on a result due to Jaggard [14, Theorem 3.1] (see also [2, Prop. 3]) which shows that it is enough to prove that, for any *symmetric* Ferrers shape F, there are as many *symmetric* 0-1-fillings of F (as above, that is, with the property that there is exactly one 1 in every row and in every column of F) with longest NE-chain of length s as there are such symmetric 0-1-fillings with longest SE-chain of length s. (Here, "symmetric" means that the filling, respectively the Ferrers diagram, remains invariant under a reflection with respect to the main diagonal. In the case of the Ferrers diagram, a more familiar term is "self-conjugate.") It is now not too difficult to see that the growth diagram bijection from Section 2 proving Theorems 1 and 3 provides a proof of the latter key assertion. Namely, under the bijection in the proof of Theorem 1, *symmetric* 0-1-fillings correspond to *symmetric* sequences $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^k = \emptyset)$ as described in the theorem, that is, to sequences with the property $\lambda^i = \lambda^{k-i}$ for all i. Hence, the bijection in Theorem 3 restricts to a bijection between *symmetric* 0-1-fillings.

The above arguments show that our bijection proves actually a refined version of [1, Prop. 2.2], in which one can keep track of the length of NE-chains and SE-chains *at the same time*. More generally, these arguments prove a version of Theorem 3 for *symmetric* 0-1-fillings. Namely, for a symmetric (= self-conjugate) Ferrers diagram F let us write $N_{\text{sym}}(F; n; NE = s, SE = t)$ for the number of *symmetric* 0-1-fillings of the Ferrers shape F with exactly n 1's, such that there is at most one 1 in each column and in each row, and such that the longest NE-chain has length s and the longest SE-chain, the smallest rectangle containing the chain being contained in F, has length t. Then we have the following theorem.

Theorem 14. *For any symmetric Ferrers shape F and positive integers s and t, we have*

$$N_{sym}(F; n; NE = s, SE = t) = N_{sym}(F; n; NE = t, SE = s).$$

Although the above arguments treat the results from [1] and [2] in a uniform manner, and although it seems that they elucidate them to a considerable extent, there remain still some mysteries that seem worth to be investigated further. These mysteries concern the transformation due to Backelin, West and Xin on which the proofs in [1] and [2] are based. In order to explain this, let us recall that neither in [1] nor in [2] are the above enumerative key assertions proved by a direct bijection, but in a round-about way which, in the end, is again based on a bijection, but a bijection between different sets of objects. This bijection is described in Section 3 of [1] and is called "transformation ϕ" in [2]. Let us call it the "BWX transformation." As is explained in [1, Prop. 2.4], the BWX transformation induces a bijection for the first of the two above key assertions, and vice versa, any bijection for the key assertion induces a bijection that could replace the BWX transformation. Thus, the question arises whether the BWX transformation and our growth diagram bijection have something to do which each other, via the link in [1, Prop. 2.4]. This question is of particular interest in connection with the article [2]: the key result in the latter paper is the proof that the BWX transformation commutes with reflection. This turns out to be extremely hard. On the other hand, that the growth diagram bijection in Theorem 3 commutes with reflection is a triviality.

Problem 4. *What is the relation between the BWX transformation and the growth diagram bijections in Section 2?*

Interestingly, in a not at all equally precise, but somewhat speculative form, this question was already posed at the end of the Introduction of [2]. Now, in the light of the above remarks and observations, this speculation seems to get firm ground.

For the convenience of the reader, we also state the "symmetric" version of Theorem 13 explicitly, which follows in a manner completely analogous to the way Theorem 14 followed from the bijection in Theorem 3. In the statement, $N^*_{\text{sym}}(F; n; NE = s, se = t)$ denotes the number of *symmetric* fillings of the symmetric Ferrers shape F

with non-negative integers with sum of entries equal to n such that the longest NE-chain has length s and the longest se-chain, the smallest rectangle containing the chain being contained in F, has length t. The notation $N^*_{\text{sym}}(F;n;ne=s,SE=t)$ has the obvious analogous meaning. Furthermore, $N^{01}_{\text{sym}}(F;n;nE=s,Se=t)$ denotes the number of *symmetric* 0-1-fillings of the symmetric Ferrers shape F with sum of entries equal to n such that the longest nE-chain has length s and the longest Se-chain, the smallest rectangle containing the chain being contained in F, has length t. The notation $N^{01}_{\text{sym}}(F;n;Ne=s,sE=t)$ has the obvious analogous meaning.

Theorem 15. *For any symmetric Ferrers shape F and positive integers s and t, we have*

$$N^*_{sym}(F;n;NE=s,se=t)=N^*_{sym}(F;n;ne=t,SE=s) \tag{5.5}$$

and

$$N^{01}_{sym}(F;n;nE=s,Se=t)=N^{01}_{sym}(F;n;Ne=t,sE=s). \tag{5.6}$$

If one goes back to the papers by Backelin, West and Xin, and by Bousquet–Mélou and Steingrímsson, which are about permutations with forbidden patterns, then one may wonder if there are also theorems in the case of arbitrary fillings for more general patterns than just increasing or decreasing patterns, or in the case of 0-1-fillings where one relaxes the condition that every row and column must contain exactly one 1. On the other hand, if one recalls the results by Jonsson and Welker, then one is tempted to ask whether the above theorems may be extended, in one form or another, to more general shapes. Thus, we are led to the following problem.

Problem 5. *Are there extensions of Theorems* 3, 13–15*, or of* (5.3) *to more general patterns? To more general shapes?*

Finally one may also ask the question whether there is anything special with 0-1-fillings as opposed to arbitrary fillings, or whether one can also obtain results in the spirit of this paper for fillings where the size of the entries is at most m, for an *arbitrary* fixed m.

Problem 6. *Can one extend the results for* 0-1*-fillings to fillings with entries from* $\{0,1,2,\dots,m\}$*?*

Further papers which should be considered in the present context are [7, 17, 18].

Acknowledgments

The idea for this article arose in discussions with Donald Knuth at the Institut Mittag–Leffler during the "Algebraic Combinatorics" programme in Spring 2005, which are herewith gratefully acknowledged. I would like to thank Anders Björner and Richard Stanley, and the Institut Mittag–Leffler, for giving me the opportunity to work in a relaxed and inspiring atmosphere in the framework of this programme. Furthermore, I am indebted to Mireille Bousquet–Mélou, Jakob Jonsson and Marc van Leeuwen for many useful suggestions for improvements, corrections, and for pointing out short-comings in a first version, which helped to improve the contents of this paper considerably.

References

[1] J. Backelin, J. West and G. Xin, *Wilf-equivalence for singleton classes*, Adv. Appl. Math. (to appear).

[2] M. Bousquet–Mélou and E. Steingrímsson, *Decreasing subsequences in permutations and Wilf equivalence for involutions*, J. Algebraic Combin. **22** (2005), 383–409.

[3] T. Britz and S. Fomin, *Finite posets and Ferrers shapes*, Adv. Math. **158** (2001), 86–127.

[4] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge University Press, Cambridge, 1993.

[5] W. H. Burge, *Four correspondences between graphs and generalized Young tableaux*, J. Combin. Theory A **17** (1974), 12–30.

[6] W. Y. C. Chen, E. Y. P. Deng, R. R. X. Du, R. P. Stanley and C. H. Yan, *Crossings and nestings of matchings and partitions*, Trans. Amer. Math. Soc. (to appear); `ar`χ`iv:math.CO/0501230`.

[7] W. Y. C. Chen, T. Mansour and S. H. F. Yan, *Matchings avoiding partial patterns*, preprint, `ar`χ`iv:math.CO/0504342`.

[8] S. V. Fomin, *Generalized Robinson–Schensted–Knuth correspondence*, (Russian), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **155** (1986); translation in J. Soviet Math. **41** (1988), 979–991.

[9] S. Fomin, *Schensted algorithms for graded graphs*, J. Alg. Combin. **4** (1995), 5–45.

[10] S. Fomin, *Schur operators and Knuth correspondences*, J. Combin. Theory Ser. A **72** (1995), 277–292.

[11] W. Fulton, *Young tableaux*, Cambridge University Press, Cambridge, 1997.

[12] C. Greene, *An extension of Schensted's theorem*, Adv. in Math. **14** (1974), 254–265.

[13] T. Halverson and T. Lewandowski, *RSK insertion for set partitions and diagram algebras*, Electron. J. Combin. **11**(2) (2005), Article #R24, 24 pages.

[14] A. D. Jaggard, *Prefix exchanging and pattern avoidance by involutions*, Electron. J. Combin. **9** (2002/03), Article #R16, 24 pp.

[15] J. Jonsson, *Generalized triangulations and diagonal-free subsets of stack polyominoes*, J. Combin. Theory Ser. A **112** (2005), 117–142.

[16] J. Jonsson and V. Welker, *A spherical initial ideal for Pfaffians*, preprint, `ar`χ`iv:math.CO/0601335`.

[17] M. Klazar, *Bell numbers, their relatives, and algebraic differential equations*, J. Combin. Theory Ser. A **102** (2003), 63–87.

[18] M. Klazar, *On identities concerning the numbers of crossings and nestings of two edges in matchings*, preprint, `ar`χ`iv:math.CO/0503012`.

[19] M. van Leeuwen, *Spin-preserving Knuth correspondences for ribbon tableaux*, Electron. J. Combin. **12**(1), Article #R10, 65 pp.

[20] D. E. Knuth, *Permutations, matrices, and generalized Young tableaux*, Pacific J. Math. **34** (1970), 709–727.

[21] T. W. Roby, *Applications and extensions of Fomin's generalization of the Robinson–Schensted correspondence to differential posets*, Ph.D. thesis, M.I.T., Cambridge, Massachusetts, 1991.

[22] T. W. Roby, *The connection between the Robinson–Schensted correspondence for skew oscillating tableaux and graded graphs*, Discrete Math. **139** (1995), 481–485.

[23] B. E. Sagan, *The symmetric group*, 2nd edition, Springer–Verlag, New York, 2001.

[24] R. P. Stanley, *Combinatorics and commutative algebra*, 2nd edition, Birkhäuser, Boston, 1996.

[25] R. P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, Cambridge, 1999.

[26] S. Sundaram, *Tableaux in the representation theory of the classical Lie groups*, in: Invariant Theory and Tableaux (Dennis Stanton, ed.), Springer Verlag, 1990, pp. 191–225.

Fakultät für Mathematik, Universität Wien, Nordbergstrasze 15, A-1090 Vienna, Austria. WWW: `http://www.mat.univie.ac.at/~kratt`