HYPERGEOMETRIC PROOF OF A CURIOUS IDENTITY OF LYONS, PAULE AND RIESE

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Dedicated to Ratan Prakash Agarwal

Abstract. We give a classical hypergeometric proof of an identity of Lyons, Paule and Riese, which arose in work of Lyons and Steif on determinantal processes, and which was originally proved by the former authors by the use of computer algorithms.

1. Introduction

In recent work of Lyons and Steif [1] on determinantal processes, in order to do the estimation of the entropy of a certain process, it turned out to be necessary to prove the following identity between a double sum and a linear combination of two harmonic numbers:

$$\sum_{n=1}^{\infty}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{1}{2^{2n-1}\,(2n-1)\,(k-j-n+1/2)} = -\frac{2}{j}\sum_{k=1}^{j}\frac{1}{2k-1} = -\frac{2}{j}\left(H_{2j}-\tfrac{1}{2}H_j\right), \qquad (1)$$

where H_m denotes the m-th harmonic number $\sum_{i=1}^{m}\frac{1}{i}$. This identity was proved by Lyons, Paule and Riese in [2] by the use of computer algorithms for the summation of binomial and hypergeometric series. More precisely, it is the *Mathematica* implementations for the Gosper–Zeilberger algorithm [4, 8, 9] developed by Paule and Schorn [3], and the *Mathematica* implementation of (a significantly enhanced version of) Sister Celine's algorithm (see [4]) developed by Riese, Wegschaider and Zimmermann [7, 5] which was used there for the proof of (1).

On the other hand, if one is interested to "go to the roots" of an identity, then one is (usually) obliged to look for a "classical" proof, based on summation and transformation formulas for hypergeometric series. This is what we do in this note. As a result, we

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present a proof of identity (1) which makes use of Kummer's ${}_2F_1$-summation formula (see (3) below), of Gauß' (first) ${}_2F_1$-summation formula (see (4) below), and, as an unexpected ingredient, of a limiting case of the non-terminating extension of Whipple's transformation due to Bailey (see (6) below).

2. Proof of the identity

Using the standard hypergeometric notation

$$ {}_pF_q\left[\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z\right]=\sum_{m=0}^{\infty}\frac{(a_1)_m\cdots(a_p)_m}{m!\,(b_1)_m\cdots(b_q)_m}z^m\ , $$

where the Pochhammer symbol $(\alpha)_m$ is defined by $(\alpha)_m:=\alpha(\alpha+1)\cdots(\alpha+m-1)$, $m\ge 1$, $(\alpha)_0:=1$, we may rewrite the sum over k on the left-hand side of (1) in the form

$$ -\sum_{n\ge1}\frac{1}{2^{2n-2}\,(2n-1)\,(2n+2j-1)}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}-j-n,1-2n\\ \frac{3}{2}-j-n\end{matrix};-1\right]. \tag{2} $$

Next we apply the contiguous relation

$$ {}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};z\right]={}_2F_1\left[\begin{matrix}a+1,b\\ c+1\end{matrix};z\right]-\frac{b\,(c-a)\,z}{c\,(c+1)}\,{}_2F_1\left[\begin{matrix}a+1,b+1\\ c+2\end{matrix};z\right] $$

in its iterated form

$$ {}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};z\right]=\sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{z^r\,(b)_r\,(c-a)_r}{(c)_r\,(c+m)_r}\,{}_2F_1\left[\begin{matrix}a+m,b+r\\ c+m+r\end{matrix};z\right]. $$

With $m=j$ this yields

$$ -\sum_{n\ge1}\sum_{r=0}^{j}\binom{j}{r}\frac{r!\,(1-2n)_r}{2^{2n}\,(\frac{1}{2}-n)_{r+1}\,(\frac{1}{2}-j-n)_{r+1}}\,{}_2F_1\left[\begin{matrix}1-2n+r,\frac{1}{2}-n\\ \frac{3}{2}-n+r\end{matrix};-1\right] $$

for the expression (2). The ${}_2F_1$-series can be summed by means of Kummer's summation (cf. [6, (1.7.1.6), corrected, (2.3.2.9); Appendix (III.5)])

$$ {}_2F_1\left[\begin{matrix}a,b\\ 1+a-b\end{matrix};-1\right]=\frac{\Gamma(1+\frac{a}{2})\,\Gamma(1+a-b)}{\Gamma(1+a)\,\Gamma(1+\frac{a}{2}-b)}. \tag{3} $$

In fact, since we can restrict r to $r\le 2n-1$ because of the term $(1-2n)_r$ in the numerator of the summand, the ${}_2F_1$-series is a terminating series, in which case actually a limit case of Kummer's summation must be considered. It turns out that the ${}_2F_1$-series is 0 for even r. The result of the summation for odd r, $r=2s-1$, yields the expression

$$ -\sum_{n\ge1}\sum_{s=1}^{\lfloor (j+1)/2\rfloor}\binom{j}{2s-1}\frac{(2s-1)!\,(1-2n)_{2s-1}\,(n-s)_{n-s}}{2^{2n-1}\,(\frac{1}{2}-n)_{2s}\,(\frac{1}{2}-j-n)_{2s}\,(\frac{1}{2}-s)_{n-s}}. $$

Now we interchange the summations over n and s, and we write the (now inner) sum over n in hypergeometric notation. It should be noted that the term $(1-2n)_{2s-1}$ implies that the sum over n may be started at s (instead of at 1). The result is

$$\sum_{s=1}^{\lfloor (j+1)/2\rfloor} (-1)^s \frac{2^{2s-2}(s-1)!^2\,(2+j-2s)_{2s-1}}{(2s-1)!\,(\frac{1}{2}-j-s)_{2s}}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}+j-s,s\\ \frac{1}{2}+j+s\end{matrix};1\right].$$

The ${}_2F_1$-series can be evaluated by means of Gauß' ${}_2F_1$-summation formula (cf. [6, (1.7.6); Appendix (III.3)])

$${}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};1\right]=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}. \tag{4}$$

If we subsequently write the remaining sum over s in hypergeometric notation, then we arrive at

$$\frac{j}{\frac{1}{2}-j}\,{}_4F_3\left[\begin{matrix}1,1,\frac{1}{2}-\frac{j}{2},1-\frac{j}{2}\\ \frac{3}{2},\frac{3}{2}-j,\frac{3}{2}\end{matrix};1\right]. \tag{5}$$

This is not exactly the final result. We now appeal to one of the bigger identities of hypergeometric theory, the non-terminating Whipple transformation due to Bailey (cf. [6, (2.4.4.3), reversed]),

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix}a,b,c,d\\ e,f,1+a+b+c+d-e-f\end{matrix};1\right]\\
&=\frac{\Gamma(e+f-a)\,\Gamma(e+f-a-b-c)\,\Gamma(e+f-a-b-d)\,\Gamma(e+f-a-c-d)}{\Gamma(e+f-a-b)\,\Gamma(e+f-a-c)\,\Gamma(e+f-a-d)\,\Gamma(e+f-a-b-c-d)}\\
&\quad\times{}_7F_6\left[\begin{matrix}-1-a+e+f,\frac{1}{2}-\frac{a}{2}+\frac{e}{2}+\frac{f}{2},f-a,e-a,b,c,d\\ -\frac{1}{2}-\frac{a}{2}+\frac{e}{2}+\frac{f}{2},e,f,e+f-a-b,e+f-a-c,e+f-a-d\end{matrix};1\right]\\
&-\frac{\Gamma(e)\,\Gamma(f)\,\Gamma(a+b+c+d-e-f)\,\Gamma(e+f-a-b-c)\,\Gamma(e+f-a-b-d)}{\Gamma(e+f-a-b-c-d)\,\Gamma(2e+f-a-b-c-d)\,\Gamma(e+2f-a-b-c-d)}\\
&\quad\times\frac{\Gamma(e+f-a-c-d)\,\Gamma(e+f-b-c-d)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(d)}\,{}_4F_3\left[\begin{matrix}e+f-a-b-c,\\ 2e+f-a-b-c-d,\end{matrix}\right.\\
&\qquad\left.\begin{matrix}e+f-a-b-d,e+f-a-c-d,e+f-b-c-d\\ e+2f-a-b-c-d,1-a-b-c-d+e+f\end{matrix};1\right].
\end{aligned} \tag{6}$$

In this identity we substitute $a=b=1$, $c=\frac{1}{2}-\frac{j}{2}$, $d=1-\frac{j}{2}$, $e=\frac{3}{2}+\varepsilon$, and $f=\frac{3}{2}+\varepsilon-j$. With this choice, the second summand on the right-hand side vanishes, and the result is

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix}1,1,\frac{1}{2}-\frac{j}{2},1-\frac{j}{2}\\ \frac{3}{2}+\varepsilon,\frac{3}{2}+\varepsilon-j,\frac{3}{2}-2\varepsilon\end{matrix};1\right]\\
&\qquad=\frac{\Gamma(\frac{1}{2}+2\varepsilon)\,\Gamma(2+2\varepsilon-j)\,\Gamma(2\varepsilon-\frac{j}{2})\,\Gamma(\frac{1}{2}+2\varepsilon-\frac{j}{2})}{\Gamma(-\frac{1}{2}+2\varepsilon)\,\Gamma(1+2\varepsilon-j)\,\Gamma(1+2\varepsilon-\frac{j}{2})\,\Gamma(\frac{3}{2}+2\varepsilon-\frac{j}{2})}\\
&\qquad\quad\times{}_6F_5\left[\begin{matrix}\frac{3}{2}+\varepsilon-\frac{j}{2},\frac{1}{2}+\varepsilon-j,\frac{1}{2}+\varepsilon,\frac{1}{2}-\frac{j}{2},1-\frac{j}{2},1\\ \frac{1}{2}+\varepsilon-\frac{j}{2},\frac{3}{2}+\varepsilon,\frac{3}{2}+\varepsilon-j,\frac{3}{2}+2\varepsilon-\frac{j}{2},1+2\varepsilon-\frac{j}{2}\end{matrix};1\right].
\end{aligned}$$

Now we let $\varepsilon\to 0$. As a result we obtain that the expression (5) is equal to

$$\frac{2}{\frac{1}{2}-j}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}-j,\frac{1}{2},1\\ \frac{3}{2},\frac{3}{2}-j\end{matrix};1\right].$$

By the definition of the hypergeometric series, this is

$$\sum_{k=0}^{\infty} \frac{1}{\left(\frac{1}{2}+k\right)\,\left(\frac{1}{2}-j+k\right)}.$$

Upon partial fraction expansion this becomes

$$-\lim_{M\to\infty}\frac{2}{j}\sum_{k=0}^{M}\left(\frac{1}{2k+1}+\frac{1}{2j-2k-1}\right)=-\frac{2}{j}\sum_{k=0}^{j-1}\left(\frac{1}{2k+1}+\frac{1}{2j-2k-1}\right)$$
$$-\lim_{M\to\infty}\frac{2}{j}\sum_{k=j}^{M}\left(\frac{1}{2k+1}+\frac{1}{2j-2k-1}\right)$$
$$=-\frac{2}{j}\sum_{k=1}^{j}\frac{1}{2k-1}-\frac{2}{j}\lim_{M\to\infty}\sum_{k=M-j+1}^{M}\frac{1}{2k+1}.$$

Clearly, this expression is equal to the desired result

$$-\frac{2}{j}\sum_{k=1}^{j}\frac{1}{2k-1}.$$

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