A q-CATALAN-NUMBERS-DETERMINANT

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Abstract. We compute a determinant involving Carlitz's q-Catalan numbers.

Let $C_n(q)$ be defined by $C_0(q) = 1$ and

$$C_n(q) = \sum_{k=0}^{n-1} q^k C_k(q) C_{n-k-1}(q), \qquad n \ge 1. \tag{1}$$

From [2, Eq. (2.1)],[1] one can derive the expansion

$$1 = \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} C_n(q) z^n (-z; q^{-1})_{n+1}. \tag{2}$$

The purpose of this note is to prove the following theorem. It has recently been reproved by Cigler [1] using completely different means. In fact, Cigler is able to embed the result below into a wider framework and is thereby able to establish several similar theorems — some already known, some new.

Theorem 1. *For all non-negative integers n, we have*

$$\det_{0\le i,j\le n}\left(\sum_{m=0}^{i+j} C_{i+j-m}(q) x^m\right) = q^{n(n-1)(4n+1)/6} \sum_{k=0}^{n} (-1)^k q^{(n-k)^2} \begin{bmatrix} n+k \\ n-k \end{bmatrix} x^k. \tag{3}$$

Proof. We start by subtracting x times row $i-1$ from row i, $i = n, n-1, \dots, 1$ (in this order!), in the matrix of which we want to compute the determinant. As a result, we obtain that the determinant on the left-hand side of (3) is equal to

$$\det_{0\le i,j\le n}\begin{pmatrix} \sum_{m=0}^{j} C_{j-m}(q) x^m & i = 0 \\ C_{i+j}(q) & i \ge 1 \end{pmatrix}. \tag{4}$$

Next we replace column j by

$$(\text{column } j) - \sum_{s=0}^{j-1} q^{j-s-1} C_{j-s-1}(q) \cdot (\text{column } s),$$

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[1]Our $C_n(q)$ is denoted by $\tilde{C}_n(q)$ in [2]. The relation between these q-Catalan numbers and the ones in [2, Eq. (2.1)] is described in [2, Eq. (2.3)].

$j = 0, 1, \dots, n$. Clearly, this can be achieved by elementary column operations, which do not change the value of the determinant. Thereby, the determinant in (4) is transformed into

$$\begin{aligned}
&\det_{0\le i,j\le n}\begin{pmatrix}\sum_{m=0}^{j} C_{j-m}(q)x^m - \sum_{s=0}^{j-1} q^{j-s-1}C_{j-s-1}(q)\sum_{m=0}^{s} C_{s-m}(q)x^m & i=0\\ C_{i+j}(q) - \sum_{s=0}^{j-1} q^{j-s-1}C_{j-s-1}(q)C_{i+s}(q) & i\ge 1\end{pmatrix}\\
&=\det_{0\le i,j\le n}\begin{pmatrix}\sum_{m=0}^{j} C_{j-m}(q)x^m - \sum_{m=0}^{j-1} x^m\sum_{s=m}^{j-1} q^{j-s-1}C_{j-s-1}(q)C_{s-m}(q) & i=0\\ C_{i+j}(q) - \sum_{s=0}^{j-1} q^{j-s-1}C_{j-s-1}(q)C_{i+s}(q) & i\ge 1\end{pmatrix}\\
&=\det_{0\le i,j\le n}\begin{pmatrix} x^j & i=0\\ C_{i+j}(q) - \sum_{s=0}^{j-1} q^{j-s-1}C_{j-s-1}(q)C_{i+s}(q) & i\ge 1\end{pmatrix},
\end{aligned}$$

where we used (1) to obtain the last line. We may expand this determinant along the first row to arrive at

$$\sum_{k=0}^{n}(-1)^k x^k \det_{1\le i\le n,\ 0\le j\le n-1}\Bigg(C_{i+j+\chi(j\ge k)}(q) \\ - \sum_{s=0}^{j+\chi(j\ge k)-1} q^{j+\chi(j\ge k)-s-1}C_{j+\chi(j\ge k)-s-1}(q)C_{i+s}(q)\Bigg), \quad (5)$$

where $\chi(\mathcal{S}) = 1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S}) = 0$ otherwise.

We now focus our attention on the coefficient of $(-x)^k$ in this expansion, that is, on the determinant

$$\det_{0\le i,j\le n-1}\Bigg(C_{i+j+\chi(j\ge k)+1}(q) - \sum_{s=0}^{j+\chi(j\ge k)-1} q^{j+\chi(j\ge k)-s-1}C_{j+\chi(j\ge k)-s-1}(q)C_{i+s+1}(q)\Bigg). \quad (6)$$

(The reader should note that we did the replacement $i \to i+1$ compared to (5).) By using column operations, working from the left to the right, this determinant can be reduced to

$$\det_{0\le i,j\le n-1}\begin{pmatrix} C_{i+j+1}(q) & j<k\\ C_{i+j+2}(q) - \sum_{s=k}^{j} q^{j-s}C_{j-s}(q)C_{i+s+1}(q) & j\ge k\end{pmatrix}.$$

Now, for $j = k, k+1, n-1$, we replace column j by

$$\sum_{\ell=k}^{j} C_{j-\ell}(q)(\text{column } \ell).$$

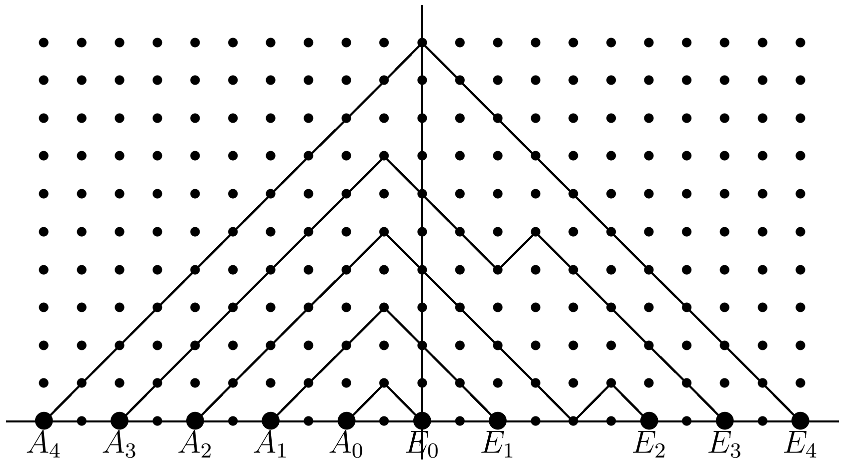

FIGURE 1

Again, this can be achieved by elementary column operations. In this manner, we see that the determinant in (6) is equal to

$$\det_{0\le i,j\le n-1}\begin{pmatrix} C_{i+j+1}(q) & j<k\\ \sum_{\ell=k}^{j} C_{j-\ell}(q)C_{i+\ell+2}(q) & \\ -\sum_{\ell=k}^{j} C_{j-\ell}(q)\sum_{s=k}^{\ell} q^{\ell-s}C_{\ell-s}(q)C_{i+s+1}(q) & j\ge k\end{pmatrix}$$
$$=\det_{0\le i,j\le n-1}\begin{pmatrix} C_{i+j+1}(q) & j<k\\ \sum_{\ell=k}^{j} C_{j-\ell}(q)C_{i+\ell+2}(q) & \\ -\sum_{s=k}^{j} C_{i+s+1}(q)\sum_{\ell=s}^{j} C_{j-\ell}(q)q^{\ell-s}C_{\ell-s}(q) & j\ge k\end{pmatrix}$$
$$=\det_{0\le i,j\le n-1}\begin{pmatrix} C_{i+j+1}(q) & j<k\\ \sum_{\ell=k}^{j} C_{j-\ell}(q)C_{i+\ell+2}(q)-\sum_{s=k}^{j} C_{i+s+1}(q)C_{j-s+1} & j\ge k\end{pmatrix}$$
$$=\det_{0\le i,j\le n-1}\begin{pmatrix} C_{i+j+1}(q) & j<k\\ C_{i+j+2}(q)-C_{i+k+1}(q)C_{j-k+1} & j\ge k\end{pmatrix},$$

where we used (1) to arrive at the next-to-last line.

The next step is to expand the last determinant by linearity in columns $k,k+1,\dots,n-1$. In theory, this produces 2^{n-k} terms. However, since choosing the term $C_{i+k+1}(q)C_{j-k+1}$ for two different j's will result in a vanishing determinant, only very few terms survive. To be precise, these are

$$\det_{0\le i,j\le n-1}\left(C_{i+j+\chi(j\ge k)+1}(q)\right)-\sum_{\ell=k}^{n-1}C_{\ell-k+1}(q)\det_{0\le i,j\le n-1}\begin{pmatrix} C_{i+j+\chi(j\ge k)+1}(q) & j\ne\ell\\ C_{i+k+1}(q) & j=\ell\end{pmatrix}$$
$$=\sum_{\ell=k-1}^{n-1}(-1)^{\ell-k+1}C_{\ell-k+1}(q)\det_{0\le i,j\le n-1}\left(C_{i+j+\chi(j\ge\ell+1)+1}(q)\right)$$
$$=\sum_{\ell=k}^{n}(-1)^{\ell-k}C_{\ell-k}(q)\det_{0\le i,j\le n-1}\left(C_{i+j+\chi(j\ge\ell)+1}(q)\right). \tag{7}$$

By the Lindström–Gessel–Viennot theorem ([4, Lemma 1], [3, Theorem 1]), the determinant in (7) can be given a combinatorial interpretation: it is equal to the generating function $\sum_{\mathbf{P}} w(\mathbf{P})$, where the sum is over all families $\mathbf{P}=(P_0,P_1,\dots,P_{n-1})$ of nonintersecting paths consisting of up-steps and down steps, where P_i is a path from $(-2i-2,0)$ to $(2i+2\chi(j\ge\ell))$ never running below the x-axis, $i=0,1,\dots,n-1$, and where the weight $w(\mathbf{P})$ is $q^{A(P_0)+\dots+A(P_{n-1})}$, with $A(P)$ denoting the area under the path P. An example with $n=5$ and $\ell=2$ is given in Figure 1.

By looking at the last up-step of paths $\ell,\ell+1,\dots,n-1$, it is not difficult to see that this generating function equals

$$q^{\sum_{t=1}^{n-1}\sum_{s=1}^{t}(4s-1)}q^{2\sum_{s=\ell}^{n-1}(s-\ell)}\sum_{0\le k_\ell\le\dots\le k_{n-1}}q^{k_\ell+\dots+k_{n-1}}\\
=q^{n(n-1)(4n+1)/6}q^{(n-\ell)(n-\ell-1)}\begin{bmatrix}n+\ell+1\\n-\ell\end{bmatrix}.$$

Therefore, if we substitute this in (7), and if we subsequently replace ℓ by $\ell+k$, then we obtain the expression

$$\sum_{\ell=0}^{n-k}(-1)^\ell q^{n(n-1)(4n+1)/6}q^{(n-k-\ell)(n-k-\ell-1)}C_\ell(q)\begin{bmatrix}n+k+\ell+1\\n-k-\ell\end{bmatrix}. \tag{8}$$

Using the q-binomial theorem, we have

$$\langle z^m\rangle\,(-z;q)_M=q^{\binom m2}\begin{bmatrix}M\\m\end{bmatrix},$$

where $(\alpha;q)_M:=(1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{M-1})$ for $M\ge1$ and $(\alpha;q)_0:=1$, and where $\langle z^m\rangle\,h(\mathbf{z})$ denotes the coefficient of z^m in the formal Laurent series $h(z)$. Hence, we can rewrite (8) as

$$\sum_{\ell=0}^{n-k}(-1)^\ell q^{n(n-1)(4n+1)/6}q^{\binom{n-k-\ell}{2}}C_\ell(q)\left\langle z^{n-k-\ell}\right\rangle(-z;q)_{n+k+\ell+1}\\
=\sum_{\ell=0}^{n-k}(-1)^\ell q^{n(n-1)(4n+1)/6}q^{\binom{n-k-\ell}{2}}C_\ell(q)\\
\cdot q^{(n-k-\ell)(n+k+\ell)}\left\langle z^{n-k-\ell}\right\rangle(-zq^{-n-k-\ell};q)_{n+k+\ell+1}\\
=\left\langle z^{n-k}\right\rangle(-z;q^{-1})_{n+k}q^{n(n-1)(4n+1)/6}q^{\binom n2+n(n-k)-\binom k2}\\
\times\sum_{\ell=0}^{n-k}(-1)^\ell q^{-\binom \ell2-\ell(n+k)}C_\ell(q)z^\ell(-zq^{-n-k};q^{-1})_\ell\\
=\left\langle z^{n-k}\right\rangle(-z;q^{-1})_{n+k}q^{n(n-1)(4n+1)/6}q^{\binom n2+n(n-k)-\binom k2}\\
=q^{n(n-1)(4n+1)/6}q^{(n-k)^2}\begin{bmatrix}n+k\\n-k\end{bmatrix},$$

where we used (2) to obtain the next-to-last line. To summarize: we have proved that the coefficient of $(-x)^k$ in (5) is equal to this last expression. In its turn, expression (5) is equal to our determinant on the left-hand side of (3). This completes the proof of the theorem. $\square$

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