# Oscillating tableaux and nonintersecting lattice paths

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Abstract. We give alternative proofs of determinantal formulas for certain up-down tableaux and down-up tableaux generating functions, which were first given in another paper by the author. These new proofs are based on interpreting up-down and down-up tableaux as certain families of nonintersecting lattice paths with three types of steps.

**1. Introduction and definitions.** We provide proofs by nonintersecting lattice paths for two formulas in [20, 19] (see (1.1) and (1.2) below) about the generating functions for certain oscillating tableaux. However, before we are able to reproduce the formulas we have to recall some definitions. A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers which are in decreasing order, i.e. $\lambda_1 \ge \lambda_2 \ge \cdots$, where only finitely many entries differ from zero. Zeros in the "tail" of a partition are frequently omitted. For instance, $(6,4,4,1,0,0,\dots) = (6,4,4,1,0) = (6,4,4,1)$. Partitions can be viewed pictorially as left-justified arrays of cells where the number of cells in the $i$-th row of the array is given by $\lambda_i$. This array of cells is called *Ferrers diagram* of the partition $\lambda$. For example, the Ferrers diagram of $(6,4,4,1)$ is displayed in Figure 1.

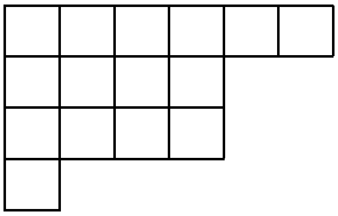


Figure 1

The partition *conjugate* to a partition $\lambda$ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ where $\lambda'_i$ is the number of cells in the $i$-th *column* of $\lambda$. Equivalently, the Ferrers diagram of $\lambda'$ is the Ferrers diagram of $\lambda$ reflected in its main diagonal. For example, the partition conjugate to $(6,4,4,1)$ is $(4,3,3,3,1,1)$. On the set of partitions a *partial order* $\subseteq$ is

1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 05E10; Secondary 05A15, 20C33, 22E45.

*Key words and phrases.* Oscillating tableaux, nonintersecting lattice paths.

defined by: $\mu \subseteq \lambda$ if and only if $\mu_i \le \lambda_i$ for all $i$. Equivalently, $\mu \subseteq \lambda$ if and only if the Ferrers diagram of $\mu$ is contained in the Ferrers diagram of $\lambda$.

A *(generalized) up-down tableau from $\mu$ to $\lambda$* [13, 31, 32, 37] is a sequence $T = (\tau_0, \tau_1, \dots, \tau_{2n-1}, \tau_{2n})$ of partitions (Ferrers diagrams), for some $n$, such that

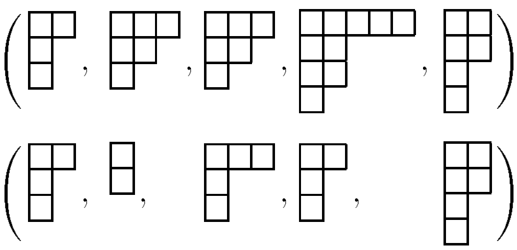
$$\mu = \tau_0 \subseteq \tau_1 \supseteq \tau_2 \subseteq \cdots \subseteq \tau_{2n-1} \supseteq \tau_{2n} = \lambda,$$

and $\tau_{i-1}$ and $\tau_i$ differ by a *horizontal strip* for each $i = 1, 2, \dots, 2n$. A horizontal strip (cf. [25, p. 4]) is an arrangement of cells with at most one cell in each column. The number $2n$ is called the *length* of the up-down tableau $T$.

Similarly, a *(generalized) down-up tableau from $\mu$ to $\lambda$* is a sequence $T = (\tau_0, \tau_1, \dots, \tau_{2n-1}, \tau_{2n})$ of partitions, for some $n$, such that

$$\mu = \tau_0 \supseteq \tau_1 \subseteq \tau_2 \supseteq \cdots \supseteq \tau_{2n-1} \subseteq \tau_{2n} = \lambda,$$

and $\tau_{i-1}$ and $\tau_i$ differ by a *horizontal strip* for each $i = 1, 2, \dots, 2n$. Again, the number $2n$ is called the *length* of the down-up tableau $T$. Examples for an up-down tableau and a down-up tableau from $(2, 1, 1)$ to $(2, 2, 1, 1)$ are given in Figure 2. Both have length 4.



up-down tableau

down-up tableau

Figure 2

We say that an up-down tableau (respectively down-up tableau) $T = (\tau_0, \tau_1, \dots, \tau_{2n-1}, \tau_{2n})$ *has at most $r$ columns* (*respectively rows*) if each $\tau_i$, $i = 0, 1, \dots, 2n$, has at most $r$ columns (rows). So for example, the up-down tableau in Figure 2 has at most 5 (6, 7, ... ) columns and the down-up tableau in Figure 2 has at most 4 (5, 6, ... ) rows.

The *weight* $w(T)$ of an up-down or down-up tableau $T$ is defined by

$$w(T) := x_1^{\left||\tau_0|-|\tau_1|\right|} x_1^{\left||\tau_1|-|\tau_2|\right|} x_2^{\left||\tau_2|-|\tau_3|\right|} x_2^{\left||\tau_3|-|\tau_4|\right|} \cdots .$$

Thus the weight of the up-down tableau in Figure 2 is given by $x_1^2 x_1^0 x_2^4 x_2^4 = x_1^2 x_2^8$, while the weight of the down-up tableau in Figure 2 is given by $x_1^2 x_1^3 x_2^1 x_2^2 = x_1^5 x_2^3$. Let us write $\mathbf{x}$ for the sequence of variables $x_1, x_2, \dots$.

Now we are able to state the formulas of [20, 19]. Let $ud^{(r)}_{\mu\to\lambda}(\mathbf{x})$ denote the generating function $\sum_T w(T)$ summed over all up-down tableaux $T$ from $\mu$ to $\lambda$ with at most $r$ columns. Then for any partitions $\alpha = (\alpha_r, \alpha_{r-1}, \dots, \alpha_1)$ and $\beta = (\beta_r, \beta_{r-1}, \dots, \beta_1)$ we have

$$ud^{(r)}_{\beta'\to\alpha'}(\mathbf{x}) = \det_{1\le s,t\le r} \Big( \sum_k e_{\alpha_s+s-\beta_t-t+k}(\mathbf{x}) e_k(\mathbf{x}) - \sum_k e_{\alpha_s+s+\beta_t+t+k}(\mathbf{x}) e_k(\mathbf{x}) \Big), \tag{1.1}$$

where $e_m(\mathbf{x})$ denotes the elementary symmetric function of order $m$ in the variables $x_1, x_2, \dots$ (cf. [25, p. 12]). Note that we definitely want the entries of $\alpha$ and $\beta$ to be indexed in reverse order.

Likewise, let $du^{(r)}_{\mu\to\lambda}(\mathbf{x})$ denote the generating function $\sum_T w(T)$ summed over all down-up tableaux $T$ from $\mu$ to $\lambda$ with at most $r$ rows. Then for partitions $\alpha$ and $\beta$ as above we have

$$du^{(r)}_{\beta\to\alpha}(\mathbf{x}) = \det_{1\le s,t\le r}\Big(\sum_k h_{\alpha_s+s-\beta_t-t+k}(\mathbf{x})h_k(\mathbf{x}) - \sum_k h_{\alpha_s+s+\beta_t+t+k}(\mathbf{x})h_k(\mathbf{x})\Big), \quad (1.2)$$

where $h_m(\mathbf{x})$ denotes the complete homogeneous symmetric function of order $m$ in the variables $x_1, x_2, \dots$ (cf. [25, p. 14]).

The significance of the formulas (1.1) and (1.2) is elaborated in [20]. In particular, from these formulas enumeration results for standard oscillating tableaux with a bounded number of rows can be derived. These standard oscillating tableaux play an important role in Berele's decomposition formula [1] for powers of defining representations of the complex symplectic groups. Also, if $\alpha$ or $\beta$ are rectangular shapes the determinant in (1.1) basically is an irreducible symplectic character. This led to a new combinatorial description of irreducible symplectic characters in terms of oscillating tableaux [20].

The formulas (1.1) and (1.2) were proved in [20] by interpreting the right-hand side determinants of (1.1) and (1.2) as generating functions for certain families of two-rowed arrays and then setting up a bijection between these families of two-rowed arrays and up-down and down-up tableaux. These bijections were based on Robinson–Schensted type algorithms. In the next section we shall prove (1.1) and (1.2) by using nonintersecting lattice paths and variations of the so-called reflection principle (cf. e.g. [6, p. 22]). The method of nonintersecting lattice paths, due to Gessel and Viennot [15, 16] (cf. also [34, sec. 1]), has been successfully utilized in numerous papers [2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 14, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 33, 34, 35, 36, 38]. So this paper might be considered as a further contribution to this list of papers (which very likely is incomplete). Combinations of the nonintersecting lattice path method and variants of the reflection principle earlier appeared in [12, 19, 20, 21].

**2. Nonintersecting lattice paths.** To begin with, we introduce some lattice path notation. Given a set $S$ of steps we denote the set of all lattice paths from $A$ to $E$ with steps from the set $S$ by $\mathcal{P}(A \to E; S)$. We frequently shall write $P : A \to E$ to indicate that the lattice path $P$ runs from $A$ to $E$. A set of paths is said to be *nonintersecting* if no two paths of this set have a point in common. Otherwise it is called *intersecting*. If to each step $s \in S$ a weight $w(s)$ is assigned, the weight $w(P)$ of a path $P$ is defined to be the product of the weights of all its steps. The weight $w(\mathbf{P})$ of a family $\mathbf{P} = (P_1, \dots, P_r)$ of paths is defined to be the product $\prod_{i=1}^r w(P_i)$ of the weights of all the paths in the family. Given any weight function $w$ defined on a set $\mathcal{M}$, by the generating function $GF(\mathcal{M}; w)$ we mean $\sum_{x\in\mathcal{M}} w(x)$.

*Proof of (1.1).* Here we consider lattice paths in the plane integer lattice with steps from the set $S_1 = \{(i,j) \to (i,j+1),\ (i,2j-2) \to (i+1,2j-1),\ (i,2j-1) \to (i-1,2j),$

where $i, j$ run through the integers}. In words, the set of steps consists of vertical steps north, forward diagonal steps northeast from even height to odd height, and backward diagonal steps northwest from odd height to even height. In the sequel, when we say 'lattice path' we always mean a lattice path with steps from the set $S_1$.

First we set up a one-to-one correspondence between *up-down tableaux from $\beta'$ to $\alpha'$ with at most $r$ columns* and *families $\mathbf{P} = (P_1, P_2, \ldots, P_r)$ of nonintersecting lattice paths where $P_i$ runs from $A_i = (\beta_i + i, 0)$ to $E_i = (\alpha_i + i, \infty)$ and never touches the $y$-axis, $i = 1, 2, \ldots, r$.* Let $T = (\tau_0, \tau_1, \ldots, \tau_{2n})$ be an up-down tableau from $\beta'$ to $\alpha'$ with at most $r$ columns. The bijection is defined in a way such that the $i$-th path encodes the growth and shrink of the $(r - i + 1)$-st column of $T$. To be more precise, starting with $A_i = (\beta_i + i, 0)$ for $j = 1, 2, \ldots, n$ do the following procedure: If $\tau_{2j-1}$ has one cell more in the $(r - i + 1)$-st column than $\tau_{2j-2}$ then add a forward diagonal step to $P_i$, otherwise a vertical step. If $\tau_{2j}$ has one cell less in the $(r - i + 1)$-st column than $\tau_{2j-1}$ then add a backward diagonal step, otherwise a vertical step. Finally, an infinite number of vertical steps is added to each path to connect to the end points $(\alpha_i + i, \infty)$. Thus, given $r = 5$, the up-down tableau in Figure 2 corresponds to the set of nonintersecting lattice paths displayed in Figure 3. Note that in this example we have $\alpha = (4, 2, 0, 0, 0)$ and $\beta = (3, 1, 0, 0, 0)$, i.e. $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 2$, $\alpha_5 = 4$, $\beta_1 = \beta_2 = \beta_3 = 0$, $\beta_4 = 1$, $\beta_5 = 3$.

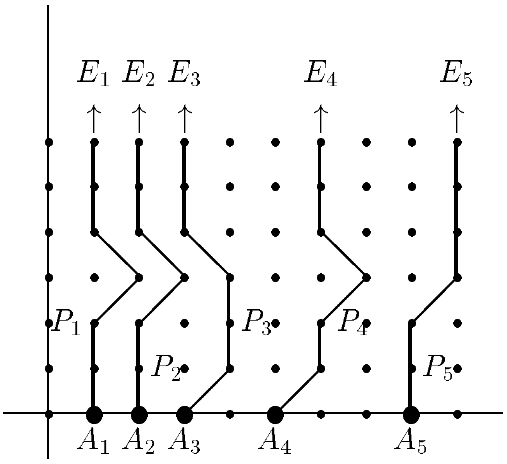



Figure 3

Clearly this operation sets up the claimed bijection. It becomes weight-preserving if we define the weight $w(v)$ of a vertical step $v$ to be 1, the weight $w(f)$ of a forward diagonal step $f : (i, 2j - 2) \to (i + 1, 2j - 1)$ as well as the weight $w(b)$ of a backward diagonal step $b : (i, 2j - 1) \to (i - 1, 2j)$ to be $x_j$. Recall that this weight function $w$ on the step set $S_1$ induces a weight $w$ on families of lattice paths as was explained at the beginning of this section.

Next we give a combinatorial description of the determinant in (1.1). Because of

$$GF(\mathcal{P}((a, 0) \to (b, \infty); S_1); w) = \sum_k e_{b-a+k}(\mathbf{x}) e_k(\mathbf{x})$$

we may write the determinant in (1.1) as

$$\det\left(GF(\mathcal{P}(A_t\to E_s;S_1);w)-GF(\mathcal{P}(\mathfrak{R}A_t\to E_s;S_1);w)\right), \tag{1.3}$$

where $\mathfrak{R}$ is the reflection in the $y$-axis. By expanding the determinant, (1.3) can be written in the form

$$\sum_{\substack{\sigma\in\mathfrak{S}_r\\ \eta\in\{1,-1\}^r}} \operatorname{sgn}\sigma\prod_{i=1}^{r}\eta_i\prod_{i,\eta_i=1}GF(\mathcal{P}(A_{\sigma(i)}\to E_i;S_1);w)$$
$$\prod_{i,\eta_i=-1}GF(\mathcal{P}(\mathfrak{R}A_{\sigma(i)}\to E_i;S_1);w),$$

or

$$\sum_{\substack{\sigma\in\mathfrak{S}_r,\ \eta\in\{0,1\}^r\\ \mathbf{P}=(P_1,\dots,P_r),\ P_i:\mathfrak{R}^{\eta_i}A_{\sigma(i)}\to E_i}} (-1)^{\|\eta\|}\operatorname{sgn}\sigma\, w(\mathbf{P}), \tag{1.4}$$

where $\|\eta\|=\sum_{i=1}^r\eta_i$. Of course, $\mathfrak{R}^0$ means the identity map.

By (1.4) we have written the determinant in (1.1) as a generating function for families $\mathbf{P}=(P_1,P_2,\dots,P_r)$ of lattice paths, $P_i:\mathfrak{R}^{\eta_i}A_{\sigma(i)}\to E_i$, for some $\sigma\in\mathfrak{S}_r$ and $\eta\in\{0,1\}^r$. Let us call $\sigma$ the *permutation associated to* $\mathbf{P}$ and $\eta$ the *reflection indicator associated to* $\mathbf{P}$. In the sum (1.4) many terms will cancel. We shall construct a *sign-reversing* (with respect to $(-1)^{\|\eta\|}\operatorname{sgn}\sigma$) and *weight-preserving* (with respect to $w$) *involution* on the set of all families $\mathbf{P}=(P_1,P_2,\dots,P_r)$ of lattice paths that were described above, which are *either intersecting or contain a path that touches the $y$-axis.* Suppose that this had been done. Then in the sum (1.4) only the terms which come from *nonintersecting families of lattice paths that do not touch the $y$-axis* will survive. In particular, none of the starting points of such a family $\mathbf{P}$ could be located to the left of the $y$-axis. Hence, the reflection indicator associated to $\mathbf{P}$ must equal $(1,1,\dots,1)$. Besides, because $\mathbf{P}$ is nonintersecting the location of the starting and end points of the paths of $\mathbf{P}$ implies that the permutation associated to $\mathbf{P}$ must be the identity permutation. But these families $\mathbf{P}$ of lattice paths exactly correspond to the up-down tableaux under consideration, as was shown above. This would establish (1.1).

Now we define the involution, which we denote by $\Theta_1$. Let $\mathbf{P}=(P_1,P_2,\dots,P_r)$ be a family of lattice paths, $P_i:\mathfrak{R}^{\eta_i}A_{\sigma(i)}\to E_i$, for some $\sigma\in\mathfrak{S}_r$ and $\eta\in\{0,1\}^r$, which is either intersecting or contains a path that touches the $y$-axis. First consider all meeting points of a path of $\mathbf{P}$ and the $y$-axis. Let $M$ be the highest of these meeting points. Among the paths which go through $M$ choose $P_I$ with $I$ being minimal. Note that the $y$-coordinate of $M$ must be even. Now, to $P_I$'s initial portion up to $M$ we apply the following modified reflection in the $y$-axis (see Figure 4). Beginning from the starting point of $P_I$ up to $M$ consider the steps of $P_I$ in groups of two. There are four different cases. Replace two vertical steps by two vertical steps, a vertical step and a backward diagonal step by a forward diagonal step and a vertical step, a forward diagonal step and a vertical step by a vertical step and a backward diagonal step, a forward diagonal step and a backward diagonal step by a forward diagonal step and a backward diagonal step. What is obtained by this procedure is joined with the terminal portion of $P_I$ beginning from $M$. In Figure 4 the circled points indicate the grouping of the steps into groups of two that was described above. Let the image of $P_I$ under application of this modified reflection be denoted by $P_I'$. We then obtain the new family $\Theta_1(\mathbf{P}) = (P_1, \dots, P_I', \dots, P_r)$. Since the new starting point of $P_I'$ is the reflection of the starting point of $P_I$ in the $y$-axis, the permutation associated to $\Theta_1(\mathbf{P})$ in this case is still $\sigma$ while the reflection indicator associated to $\Theta_1(\mathbf{P})$ is given by $(\eta_1, \dots, 1-\eta_I, \dots, \eta_r)$.

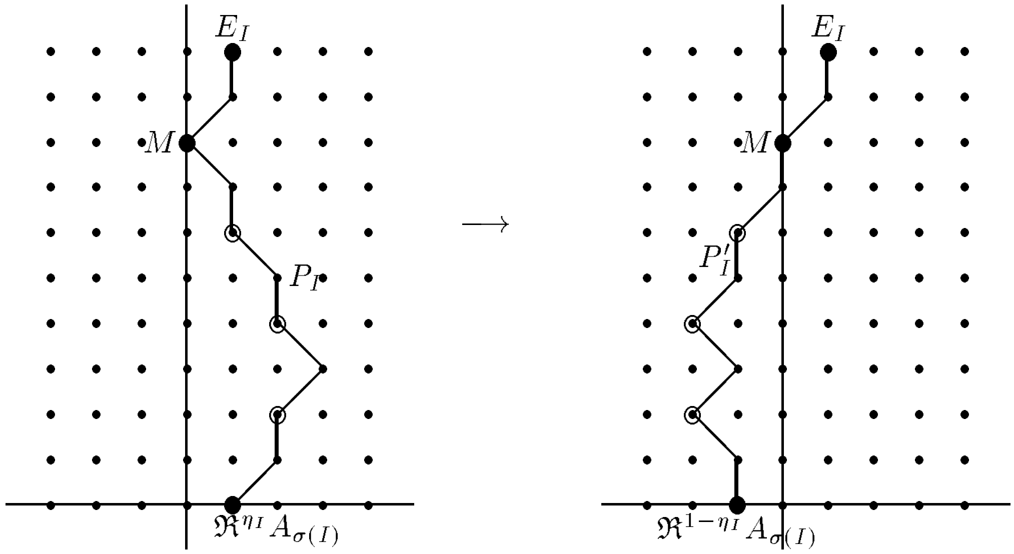



Figure 4

If none of the paths of $\mathbf{P}$ meet the $y$-axis consider all meeting points between any two paths of $\mathbf{P}$. Choose the highest meeting points, and among those the left-most. Denote this meeting point by $M$. Now we apply the usual Gessel–Viennot interchanging procedure. To be more precise, among the pairs of paths which have $M$ in common, choose $(P_I, P_J)$, with $(I, J)$ being minimal with respect to the lexicographic order of pairs of integers. Next we decompose $P_I$ into $P_I^{(1)}$ and $P_I^{(2)}$, where $P_I^{(1)}$ is the portion of $P_I$ going from $\mathfrak{R}^{\eta_I}\mathcal{A}_{\sigma(I)}$ to $M$, and where $P_I^{(2)}$ is the portion of $P_I$ between $M$ and $E_I$. Analogously, $P_J$ is decomposed by $M$ into $P_J^{(1)}$ and $P_J^{(2)}$. Then by concatenation form the new paths $P_I' = P_J^{(1)} P_I^{(2)}$ and $P_J' = P_I^{(1)} P_J^{(2)}$. In other terms, the initial portions of $P_I$ and $P_J$ up to their last meeting point are interchanged, the terminal portions are preserved. Thus we obtain the new family $\Theta_1(\mathbf{P}) = (P_1, \dots, P_I', \dots, P_J', \dots, P_r)$. Obviously, since the $I$-th and $J$-th starting point have been interchanged, the permutation associated to $\Theta_1(\mathbf{P})$ is $\sigma \circ (IJ)$ ($(IJ)$ is the transposition that interchanges $I$ and $J$), and the reflection indicator associated to $\Theta_1(\mathbf{P})$ is $(\eta_1, \dots, \eta_J, \dots, \eta_I, \dots, \eta_r)$.

It is easy to see that in both cases, by application of $\Theta_1$ to $\Theta_1(\mathbf{P})$, one arrives at $\mathbf{P}$ again. Consequently $\Theta_1$ is an involution. Besides, by construction it is weight-preserving with respect to $w$ and sign-reversing with respect to $(-1)^{\|\eta\|}\operatorname{sgn}\sigma$. $\square$

*Proof of (1.2).* Here we consider lattice paths in the plane integer lattice with steps from the set $S_2 = \{(i,j) \to (i,j+1),\ (i,2j-1) \to (i-1,2j-1),\ (i,2j) \to (i+1,2j)$, where $i,j$ run through the integers$\}$. In words, the set of steps consists of vertical steps north, backward horizontal steps (west) at odd height, and forward horizontal steps (east) at even height. In the sequel, when we say 'lattice path' we always mean a lattice path with steps from the set $S_2$.

First we set up a one-to-one correspondence between *down-up tableaux from $\beta$ to $\alpha$ with at most $r$ rows* and *families $\mathbf{P} = (P_1, P_2, \dots, P_r)$ of nonintersecting lattice paths where $P_i$ runs from $A_i = (\beta_i + i, 0)$ to $E_i = (\alpha_i + i, \infty)$ and never touches the $y$-axis,* $i = 1, 2, \dots, r$. Let $T = (\tau_0, \tau_1, \dots, \tau_{2n})$ be a down-up tableau from $\beta$ to $\alpha$ with at most $r$ rows. The bijection is defined in a way such that the $i$-th path encodes the growth and shrink of the $(r-i+1)$-st row of $T$. To be more precise, starting with $A_i = (\beta_i + i, 0)$ for $j = 1, 2, \dots, n$ do the following procedure: If $\tau_{2j-1}$ has $n$ cells less in the $(r-i+1)$-st row than $\tau_{2j-2}$ then add a vertical step followed by $n$ backward horizontal steps to $P_i$. If $\tau_{2j}$ has $n$ cells more in the $(r-i+1)$-st row than $\tau_{2j-1}$ then add a vertical step followed by $n$ forward horizontal steps to $P_i$. Finally, an infinite number of vertical steps is added to each path to connect to the end points $(\alpha_i + i, \infty)$. Thus, given $r = 4$, the down-up tableau in Figure 2 corresponds to the set of nonintersecting lattice paths displayed in Figure 5. Note that in this example we have $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\alpha_4 = 2$, $\beta_1 = 0$, $\beta_2 = 1$, $\beta_3 = 1$, $\beta_4 = 2$.

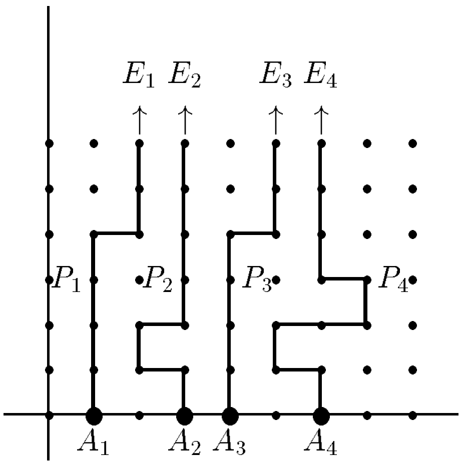



Figure 5

Clearly this operation sets up the claimed bijection. It becomes weight-preserving if we define the weight $w(v)$ of a vertical step $v$ to be 1, the weight $w(f)$ of a backward horizontal step $f : (i, 2j-1) \to (i-1, 2j-1)$ as well as the weight $w(b)$ of a forward horizontal step $b : (i, 2j) \to (i+1, 2j)$ to be $x_j$. Recall that this weight function $w$ on the step set $S_2$ induces a weight $w$ on families of lattice paths as was explained at the beginning of this section.

Next we give a combinatorial description of the determinant in (1.2). Because of

$$GF(\mathcal{P}((a,0)\to(b,\infty);S_2);w)=\sum_k h_{b-a+k}(\mathbf{x})h_k(\mathbf{x})$$

we may write the determinant in (1.2) as

$$\det\big(GF(\mathcal{P}(A_t\to E_s;S_2);w)-GF(\mathcal{P}(\mathfrak{R}A_t\to E_s;S_2);w)\big), \tag{1.5}$$

where $\mathfrak{R}$ is again the reflection in the $y$-axis. By expanding the determinant, (1.5) can be written in the form

$$\sum_{\substack{\sigma\in\mathfrak{S}_r\\ \eta\in\{1,-1\}^r}} \operatorname{sgn}\sigma\prod_{i=1}^r\eta_i\prod_{i,\eta_i=1}GF(\mathcal{P}(A_{\sigma(i)}\to E_i;S_2);w)$$

$$\prod_{i,\eta_i=-1}GF(\mathcal{P}(\mathfrak{R}A_{\sigma(i)}\to E_i;S_2);w),$$

or

$$\sum_{\substack{\sigma\in\mathfrak{S}_r,\ \eta\in\{0,1\}^r\\ \mathbf{P}=(P_1,\dots,P_r),\ P_i:\mathfrak{R}^{\eta_i}A_{\sigma(i)}\to E_i}}(-1)^{\|\eta\|}\operatorname{sgn}\sigma\, w(\mathbf{P}), \tag{1.6}$$

where again $\|\eta\|=\sum_{i=1}^r\eta_i$.

By (1.6) we have written also the determinant in (1.2) as a generating function for families $\mathbf{P}=(P_1,P_2,\dots,P_r)$ of lattice paths, $P_i:\mathfrak{R}^{\eta_i}A_{\sigma(i)}\to E_i$, for some $\sigma\in\mathfrak{S}_r$ and $\eta\in\{0,1\}^r$. The only difference to (1.4) is that the set of steps now is $S_2$ instead of $S_1$. Again, let us call $\sigma$ the *permutation associated to* $\mathbf{P}$ and $\eta$ the *reflection indicator associated to* $\mathbf{P}$. Also here, we shall construct a *sign-reversing* (with respect to $(-1)^{\|\eta\|}\operatorname{sgn}\sigma$) and *weight-preserving* (with respect to $w$) *involution* on the set of all families $\mathbf{P}=(P_1,P_2,\dots,P_r)$ of lattice paths that were described above, which are *either intersecting or contain a path that touches the $y$-axis.* Once this is done, similar arguments as those in the proof of (1.1) show that (1.2) would be established.

Now we define the involution, which we denote by $\Theta_2$. Let $\mathbf{P}=(P_1,P_2,\dots,P_r)$ be a family of lattice paths, $P_i:\mathfrak{R}^{\eta_i}A_{\sigma(i)}\to E_i$, for some $\sigma\in\mathfrak{S}_r$ and $\eta\in\{0,1\}^r$, which is either intersecting or contains a path that touches the $y$-axis. As before, first consider all meeting points of any path of $\mathbf{P}$ with the $y$-axis. Let $M$ be the highest of these meeting points. Among the paths which go through $M$ choose $P_I$ with $I$ being minimal. Again, note that the $y$-coordinate of $M$ must be even. Now, to $P_I$'s initial portion up to $M$ we apply the following modified reflection in the $y$-axis (see Figure 6). Beginning from the starting point of $P_I$ up to $M$ consider the portions of $P_I$ which lie between the horizontal lines $y=2j-2$ and $y=2j$, portions of $P_I$ on the line $y=2j-2$ excluded while portions on $y=2j$ included. If in this portion there is a vertical step followed by $n$ backward horizontal steps and then a vertical step followed by $m$ forward horizontal steps (note that the vertical steps are obligatory) replace it by a vertical step followed by $m$ backward horizontal steps and a vertical step followed by $n$ forward horizontal steps. What is obtained by this procedure is joined with the terminal portion of $P_I$ beginning from $M$. In Figure 6 the circled points indicate where the paths are cut into portions according to the above description.

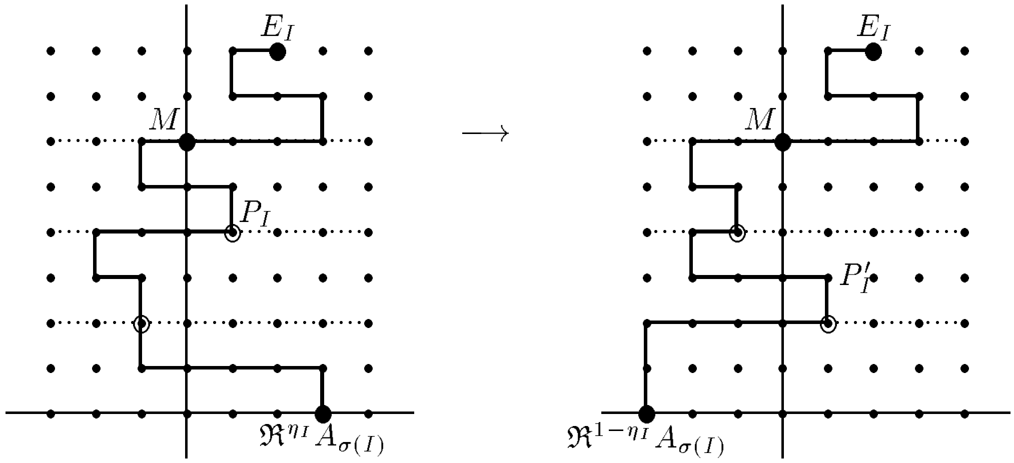



Figure 6

If none of the paths of $\mathbf{P}$ meet the $y$-axis consider all meeting points between any two paths of $\mathbf{P}$. Choose the highest meeting points, and among those the left-most meeting point, and then apply the Gessel–Viennot interchanging procedure, as was described before in the proof of (1.1).

Again it is easy to see that in both cases, by application of $\Theta_2$ to $\Theta_2(\mathbf{P})$, one arrives at $\mathbf{P}$ again. Consequently $\Theta_2$ is a sign-reversing (with respect to $(-1)^{\|\eta\|} \operatorname{sgn} \sigma$) and weight-reversing (with respect to $w$) involution. $\square$

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