SOME NEW FORMULAS FOR π

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Abstract. We show how to find series expansions for π of the form $\pi = \sum_{n=0}^{\infty} S(n)/\binom{mn}{pn}a^n$, where $S(n)$ is some polynomial in n (depending on m, p, a). We prove that there exist such expansions for $m = 8k$, $p = 4k$, $a = (-4)^k$, for any k, and give explicit examples for such expansions for small values of m, p and a.

1. Introduction

Using the formula (due to Bill Gosper [3])

$$\pi = \sum_{n=0}^{\infty} \frac{50n-6}{\binom{3n}{n}2^n},$$

Fabrice Bellard [1, file pi1.c] found an algorithm for computing the n-th decimal of π without computing the earlier ones. Thus he improved an earlier algorithm due to Simon Plouffe [6].

This formula can be proved in the following way. We have

$$\frac{1}{\binom{3n}{n}} = (3n+1)\int_0^1 x^{2n}(1-x)^n dx.$$

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Hence the right hand side of the formula will be

$$\int_0^1 \sum_{n=0}^{\infty} (50n-6)(3n+1) \left(\frac{x^2(1-x)}{2} \right)^n dx.$$

But

$$\sum_{n=0}^{\infty} (50n-6)(3n+1) y^n = \frac{2(56y^2+97y-3)}{(1-y)^3},$$

so we get

$$\begin{aligned} RHS &= 8 \int_0^1 \frac{28x^6 - 56x^5 + 28x^4 - 97x^3 + 97x^2 - 6}{(x^3 - x^2 + 2)^3} dx \\ &= \left[\frac{4x(x-1)(x^3 - 28x^2 + 9x + 8)}{(x^3 - x^2 + 2)^2} + 4 \arctan(x-1) \right]_0^1 = \pi. \end{aligned}$$

We then asked if there are other such formulas of type

$$\pi = \sum_{n=0}^{\infty} \frac{S(n)}{\binom{mn}{pn} a^n},$$

where $S(n)$ is a polynomial in n. Using the same trick as before we have

$$\frac{1}{\binom{mn}{pn}} = (mn+1) \int_0^1 x^{pn} (1-x)^{(m-p)n} \, dx,$$

and hence

$$RHS = \int_0^1 \sum_{n=0}^{\infty} (mn+1) S(n) \left(\frac{x^p (1-x)^{m-p}}{a} \right)^n dx. \tag{1.1}$$

If $S(n)$ has degree d then

$$\sum_{n=0}^{\infty} (mn+1) S(n) y^n = \frac{T(y)}{(1-y)^{d+2}}, \tag{1.2}$$

where T has degree $d+1$. It follows with

$$y = \frac{x^p (1-x)^{m-p}}{a}$$

that

$$RHS = \int_0^1 \frac{P(x)}{(x^p (1-x)^{m-p} - a)^{d+2}} \, dx, \tag{1.3}$$

where $P(x)$ is a polynomial in x of degree $m(d+1)$. We want this integral to be equal to π. A good way to get π is to have $\arctan(x)$ or $\arctan(x-1)$ after integration. This means that

$$x^p(1-x)^{m-p} - a$$

must have the factor x^2+1 or $(x-1)^2+1$, that is have a zero at i or $1+i$. This restricts m and p and gives the value of a. After experimenting with the LLL-algorithm we found formulas for π in the following cases (there could be many more):

m	p	a	$\deg(S)$	
3	1	2	1	Gosper [3]
7	2	2	5	Bellard [1][1]
8	4	-4	4	
10	4	4	8	
12	4	-4	8	
16	8	16	8	
24	12	-64	12	
32	16	256	16	
40	20	-4^5	20	
48	24	4^6	24	
56	28	-4^7	28	
64	32	4^8	32	
72	36	-4^9	36	
80	40	4^{10}	40	

E.g., to find the formula in the last case we computed

$$s(k) = \sum_{n=0}^{\infty} \frac{n^k}{\binom{80n}{40n} 4^{10n}}$$

for $k = 0, 1, 2, \ldots, 40$ with 6000 digits. Then the LLL-algorithm found (in about three days) the linear combination between π and $s(0), s(1), \ldots, s(40)$. A good check is that the coefficient of π is a product of small primes (actually all less than 80).

[1] This formula was used in Bellard's world record setting computation of the 1000 billionth *binary* digit of π, being based on the algorithm in [2].

2. Proving the formulas

For $m \leqslant 16$ the integrals can be computed by brute force using *Maple*. The higher cases are all symmetric, i.e., $m = 2p$. Using the symmetry

$$x \longleftrightarrow 1 - x$$

one can assume that $\arctan(x)$ and $\arctan(x-1)$ each contribute $\pi/2$. So we make the wild assumption that

$$\int \frac{P(x)}{Q(x)^{d+2}}\,dx = \frac{R(x)}{Q(x)^{d+1}} + 2\arctan(x) + 2\arctan(x-1),$$

where $R(x)$ is a polynomial with $R(0) = R(1) = 0$. Differentiation with respect to x yields

$$\frac{P}{Q^{d+2}} = \frac{R'}{Q^{d+1}} - (d+1)\frac{Q'R}{Q^{d+2}} + 2\left(\frac{1}{x^2+1} + \frac{1}{x^2-2x+2}\right),$$

or

$$QR' - (d+1)Q'R = P - 2Q^{d+2}\left(\frac{1}{x^2+1} + \frac{1}{x^2-2x+2}\right).$$

This is a differential equation of first order in R with polynomial coefficients. Taking the case $\binom{32n}{16n}$, we have $\deg(P) = 544$ and $d = 16$, so the denominator Q^{d+2} has degree 576. We solve the differential equation using *Maple*'s

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dsolve(*,R(0)=0,series)
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setting the order to 600. Then we find that R is a polynomial of degree 543. In practice one replaces the coefficients of $\frac{1}{x^2+1}$ and $\frac{1}{x^2-2x+2}$ by 2 times the denominator in the formula found for π. Then R will have (huge) integer coefficients.

Remark 1. Computing the integral in the case $\binom{8n}{4n}$, *Maple* got the terms

$$\arctan\left(\frac{2x^3 - 3x^2 + 7x - 3}{5}\right) + \arctan\left(\frac{x}{2} - \frac{1}{4}\right),$$

which can be shown to be equal to

$$\arctan(x) + \arctan(x-1).$$

In the nonsymmetric case $\binom{12n}{4n}$ one gets $\arctan(x-1)$ and $\arctan(x^3 - 2x^2 + x - 1)$, but the latter term makes no contribution since it has the same values at 0 and 1.

Remark 2. When we set this up then, in principle, we could let *Maple* do the evaluation of $T(y)$ (and thus of $P(x)$) through (1.2), i.e., let *Maple*'s formal summation routines compute the sum

$$\sum_{n=0}^{\infty}(mn+1)S(n)y^n,$$

or, to make the point clearly visible, compute a sum of the form

$$\sum_{n=0}^{\infty}\tilde{S}(n)y^n,$$

where $\tilde{S}(n)$ is a certain polynomial in n. However, it turns out that this is very time-consuming. It is a much better idea to expand $\tilde{S}(n)$ as $\tilde{S}(n)=\sum_{j=0}^{d+1}a(j)\binom{n+j}{d+1}$, and then use that $\sum_{n=0}^{\infty}\binom{n+j}{d+1}y^n=y^{d+1-j}/(1-y)^{d+2}$ for $j=0,1,\dots,d+1$. See also the approach that we follow in Section 4 when we prove our main result.

3. The case $\binom{8kn}{4kn}$, $k=1,2,3,\dots$

Solving the differential equation for $R(x)$ one finds that $R(x)$ has the factor $x(1-x)(2x-1)$. If one divides out the factor $2x-1$, the remaining factor is invariant under the substitution $x\longleftrightarrow 1-x$. Hence we make the substitution

$$R(x)=(2x-1)\check{R}(z),$$

where

$$z=x(1-x).$$

Then

$$\frac{dz}{dx}=1-2x \text{ and } (1-2x)^2=1-4z.$$

Now

$$Q=z^{4k}-(-4)^k,$$

so

$$Q'=4kz^{4k-1}(1-2x)$$

and

$$R'=2\check{R}(z)+(2x-1)\frac{d\check{R}}{dz}(1-2x)=2\check{R}-(1-4z)\frac{d\check{R}}{dz}.$$

It follows that the left hand side of the differential equation is

$$\begin{aligned} QR' - (4k+1)Q'R &= Q\left(2\check{R} - (1-4z)\frac{d\check{R}}{dz}\right) + (4k+1)4kz^{4k-1}(1-2x)^2\check{R} \\ &= (2Q + 4k(4k+1)(1-4z)z^{4k-1})\check{R} - (1-4z)Q\frac{d\check{R}}{dz}. \end{aligned}$$

On the right hand side we have

$$\frac{1}{x^2+1} + \frac{1}{x^2-2x+2} = \frac{3-2z}{z^2-2z+2},$$

and Q is for all k divisible by

$$z^4 + 4 = (z^2-2z+2)(z^2+2z+2).$$

Hence P and Q and

$$Q\left(\frac{1}{x^2+1} + \frac{1}{x^2-2x+2}\right)$$

are all polynomials in z. Finally we get the following differential equation for $\check{R}$:

$$-(1-4z)Q\frac{d\check{R}}{dz} + (2Q + 4k(4k+1)(1-4z)z^{4k-1})\check{R} - P + 2(3-2z)\frac{Q^{4k+2}}{z^2-2z+2} = 0. \quad (3.1)$$

If P is known then one solves this equation for $\check{R}$ just as before (but the degrees are cut in half).

But there is a possibility to *find* the formula for π *and prove* it *in one stroke*. Let

$$N(k) = 4k(4k+1),$$

and assume that

$$\check{R}(z) = \sum_{j=1}^{N(k)-1} a(j)z^j \quad (3.2)$$

and

$$\pi = \sum_{n=0}^{\infty} \frac{S_k(n)}{\binom{8kn}{4kn}(-4)^{kn}}, \quad (3.3)$$

with

$$S_k(n) = \sum_{j=0}^{4k} a(N(k)+j)n^j. \quad (3.4)$$

Substituting this into the differential equation we will get a system of linear equations for the $a(j)$'s of size $(N(k)+4k)\times(N(k)+4k)$. If this system is nonsingular, we can be sure to be able to solve it and thus *find and prove* an expansion for π of the form (3.3).

We went (again) to the computer and generated the system of equations for small values of k. Let $A(k)$ be the matrix of the system. Let further $r(k)$ be the least common denominator of the coefficients of $S_k(n)$. We obtained the following tables:

k	$\det(A(k))$
1	$2^{91}3^{8}5^{7}7^{2}$
2	$-2^{523}3^{52}5^{17}7^{14}11^{4}13^{3}$
3	$2^{1367}3^{177}5^{41}7^{25}11^{20}13^{19}17^{5}19^{4}23^{2}$
4	$-2^{3231}3^{167}5^{83}7^{53}11^{28}13^{27}17^{25}19^{8}23^{6}29^{3}31^{2}$
5	$2^{5399}3^{290}5^{345}7^{93}11^{41}13^{37}17^{33}19^{32}23^{10}29^{7}31^{6}37^{3}$

k	$r(k)$
1	$3^{2}5^{2}7^{2}$
2	$3^{6}5^{3}7^{2}11^{2}13^{2}$
3	$2^{5}3^{3}5^{2}7^{2}11^{2}13^{2}17^{2}19^{2}23^{2}$
4	$2^{3}3^{10}5^{6}7^{3}11\cdot 13^{2}17^{2}19^{2}23^{2}29^{2}31^{2}$
5	$2^{11}3^{13}5^{3}7^{4}11^{3}13^{3}17^{2}19^{2}23^{2}29^{2}31^{2}37^{2}$
6	$2^{10}3^{7}5^{8}7^{3}11^{3}13^{3}17^{2}19^{2}23^{2}29^{2}31^{2}37^{2}41^{2}43^{2}47^{2}$
7	$2^{17}3^{16}5^{9}7^{3}11^{4}13^{3}17^{3}19^{2}23^{2}29^{2}31^{2}37^{2}41^{2}43^{2}47^{2}53^{2}$
8	$2^{10}3^{19}5^{8}7^{8}11^{4}13^{3}17^{3}19^{3}23^{2}29^{2}31^{2}37^{2}41^{2}43^{2}47^{2}53^{2}59^{2}61^{2}$
9	$2^{23}3^{4}5^{10}7^{8}11^{4}13^{4}17^{3}19^{3}23^{3}29^{2}31^{2}37^{2}41^{2}43^{2}47^{2}53^{2}59^{2}61^{2}67^{2}71^{2}$
10	$2^{20}3^{20}5^{4}7^{9}11^{5}13^{4}17^{3}19^{3}23^{2}29^{2}31^{2}37^{2}41^{2}43^{2}47^{2}53^{2}59^{2}61^{2}67^{2}71^{2}73^{2}79^{2}$

From the table it seems "obvious" that the determinant of $A(k)$ will never vanish as it grows rather quickly in absolute value with k. We also see that the same primes occur in the factorizations of $r(k)$ and $\det(A(k))$. Even more striking, it seems that the largest prime factors occurring grow only slowly (namely approximately linearly) when k increases. The last fact strongly indicates that there may even be a closed form formula for $\det(A(k))$. As it turns out, this is indeed the case. In the next section we will explicitly compute the determinant of an equivalent system of equations, from which it follows that (see the remark after Theorem 6)

$$\det(A(k)) = (-1)^{k-1}2^{32k^3+24k^2+2k-1}k^{8k^2+2k}((4k+1)!)^{4k}\frac{(8k)!}{(4k)!}\prod_{j=1}^{4k}\frac{(2j)!}{j!}. \tag{3.5}$$

Hence we have the following theorem, which is the main theorem of our paper.

Theorem 1. *For all $k \geq 1$ there is a formula*

$$\pi = \sum_{n=0}^{\infty} \frac{S_k(n)}{\binom{8kn}{4kn}(-4)^{kn}},$$

where $S_k(n)$ is a polynomial in n of degree $4k$ with rational coefficients. The polynomial $S_k(n)$ can be found by solving the system of linear equations generated by (3.1) *and the Ansatz* (3.2) *and* (3.4).

The denominators of all $a(j)$'s divide $r(k)$ which is much smaller than $\det(A(k))$. This means that there must be some miracle occurring at the end when solving the system. E.g., when $k = 5$ then $\det(A(5))$ has about 2400 digits but $r(5)$ only 40. Unfortunately, we are not able to offer an explanation for that.

In practice we are only interested in the coefficients of S_k so we try to eliminate $a(1), a(2), \ldots, a(N(k) - 1)$ first. This can be done by first avoiding all equations containing $a(N(k)), \ldots, a(N(k)+4k)$, i.e., the equations coming from the coefficients of t^{4kv}, $v = 0, 1, \ldots, 4k + 1$. This looks very nice theoretically, in particular as the system for $a(1), \ldots, a(N(k))$ is triangular, but in practice the computer breaks down since the rational numbers occuring become very large.

We close this section by listing a few explicit examples.

Example 2. *We have*

$$\pi = \frac{1}{r} \sum_{n=0}^{\infty} \frac{S(n)}{\binom{8n}{4n}(-4)^n},$$

where

$$r = 3^2 5^2 7^2$$

and

$$S(n) = -89286 + 3875948n - 34970134n^2 + 110202472n^3 - 115193600n^4.$$

Example 3. *We have*

$$\pi = \frac{1}{r} \sum_{n=0}^{\infty} \frac{S(n)}{\binom{16n}{8n}16^n},$$

where

$$r = 3^6 5^3 7^2 11^2 13^2$$

and

$$\begin{aligned} S(n) = &-869897157255 - 3524219363487888n + 112466777263118189n^2 \\ &- 1242789726208374386n^3 + 6693196178751930680n^4 - 19768094496651298112n^5 \\ &+ 32808347163463348736n^6 - 28892659596072587264n^7 + 10530503748472012800n^8. \end{aligned}$$

Example 4. *We have*

$$\pi = \frac{1}{r} \sum_{n=0}^{\infty} \frac{S(n)}{\binom{32n}{16n} 256^n},$$

where

$$r = 2^3 3^{10} 5^6 7^3 11 \cdot 13^2 17^2 19^2 23^2 29^2 31^2$$

and

$$\begin{aligned} S(n) = &- 2062111884756347479085709280875 \\ &+ 15054917403028390237535697172618820919000n \\ &- 112401149404087658213839386716211975291975n^2 \\ &+ 3257881651942682891818557726225840674110002n^3 \\ &- 51677309510890630500607898599463036267961280n^4 \\ &+ 517337977987354819322786909541179043148522720n^5 \\ &- 3526396494329560718758086392841258152390245120n^6 \\ &+ 17114576623599516622750121611007480594379936358 4n^7 \\ &- 60739416613228219940886539658145904402068029440n^8 \\ &+ 159935882563435860391195903248596461569183580160n^9 \\ &- 313951952615028230229958218839819183812205608960n^{10} \\ &+ 457341091673257198565533286493831205566468325376n^{11} \\ &- 486846784774707448105420279985074159657397780480n^{12} \\ &+ 367314505118245777241612044490633887668208926720n^{13} \\ &- 185647326591648164598342857319777582801297080320n^{14} \\ &+ 56224688035707015687999128994324690418467340288n^{15} \\ &- 7687255778816557786073977795149360408612044800n^{16}. \end{aligned}$$

Example 5. *(nonsymmetric). We have*

$$\pi = \frac{1}{r} \sum_{n=0}^{\infty} \frac{S(n)}{\binom{10n}{4n} 4^n},$$

where

$$r = 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 17 \cdot 19$$

and

$$
\begin{aligned}
S(n) &= -4843934523072 - 1008341177146848n + 23756198610834352n^2 \\
&\quad - 242873913552020704n^3 + 1195813551184400032n^4 - 3272960363556054592n^5 \\
&\quad + 4909379167837011328n^6 - 3816399750842818816n^7 + 1190182007407360000n^8.
\end{aligned}
$$

4. Proof of the Theorem

We want to prove that, by making the Ansatz (3.2) and (3.4) and substituting this into the differential equation (3.1) (the polynomial P being given by S_k through (1.1)–(1.3), Q being given by $(x(1-x))^{4k} - (-4)^k)$, the resulting system of linear equations will always have a solution. In fact, we aim at finding an explicit formula for the determinant of the corresponding matrix of coefficients that allows us to conclude that it can never vanish.

It turns out that for that purpose it is more convenient to set up the system of linear equations in a different, but equivalent way. This equivalent system will have coefficient matrix M (see (4.4)). The evaluation of its determinant will be accomplished through Eqs. (4.6), (4.7), (4.8), (4.10), and Theorem 6.

To be precise, we encode the polynomial $S_k(x)$ (and, thus, $T(y)$ and $P(x)$) differently. We claim that $T(y)$ has an expansion of the form

$$T(y) = \sum_{j=0}^{4k+1} c_j y^j, \tag{4.1}$$

subject to the single constraint

$$\prod_{i=1}^{4k+1} (4ik-1)c_0 + \sum_{j=1}^{4k+1} (-1)^j \left(\prod_{i=1}^{4k+1-j} (4ik-1) \right) \left(\prod_{i=1}^{j-1} (4ik+1) \right) c_j \;=\; 0. \tag{4.2}$$

(As usual, empty poducts have to be interpreted as 1.) This is seen as follows. The polynomial $S_k(x)$ can be written in the form

$$S_k(n) = \sum_{j=0}^{4k} \binom{n+j}{4k} s_j,$$

for some coefficients s_j. Hence, we have

$$\begin{aligned}
\frac{T(y)}{(1-y)^{4k+2}} &= \sum_{n=0}^{\infty}(4kn+1)S_k(n)y^n \\
&= \sum_{n=0}^{\infty}(4kn+1)\sum_{j=0}^{4k}\binom{n+j}{4k}s_jy^n \\
&= \sum_{j=0}^{4k}s_j\sum_{n=0}^{\infty}(4k(n+j+1)-4k(j+1)+1)\binom{n+j}{4k}y^n \\
&= \sum_{j=0}^{4k}s_j\sum_{n=0}^{\infty}\left(4k(4k+1)\binom{n+j+1}{4k+1}y^n-(4k(j+1)-1)\binom{n+j}{4k}y^n\right) \\
&= \sum_{j=0}^{4k}s_j\left(4k(4k+1)\frac{y^{4k-j}}{(1-y)^{4k+2}}-(4k(j+1)-1)\frac{y^{4k-j}}{(1-y)^{4k+1}}\right) \\
&= \frac{1}{(1-y)^{4k+2}}\sum_{j=0}^{4k+1}y^j\left((4jk+1)s_{4k-j}+(4k(4k-j+2)-1)s_{4k-j+1}\right).
\end{aligned}$$

In the last line, s_{-1} and s_{4k+1} have to be read as 0. It is now a trivial exercise to substitute the coefficients of y^j in the sum in the last line into the left-hand side of (4.2) and verify the truth of (4.2).

The above implies that

$$P(z)=\sum_{j=0}^{4k+1}c_jz^{4jk}/(-4)^{jk},$$

where the coefficients c_j obey (4.2).

Now we are ready to set up the system of linear equations. We make again the Ansatz (3.2), but we replace (3.4) by

$$P(z)=\sum_{j=0}^{4k+1}a(N(k)+j)z^{4jk}/(-4)^{jk}, \tag{4.3}$$

where the $a(N(k)+j)$, $j=0,1,\dots,4k+1$, are subject to (4.2) (i.e., the relation (4.2) holds when c_j is replaced by $a(N(k)+j)$). Clearly, we have to add (4.2) to the set of equations that result from the differential equation (3.1).

The coefficient matrix of the system looks a follows:

$$M=\begin{pmatrix} x & y \\ U & V\end{pmatrix}, \tag{4.4}$$

where x is a line vector of $N(k)-1$ zeroes, $y=(y_0,y_1,\dots,y_{4k+1})$ is the vector of coefficients of (4.2), i.e., $y_0=\prod_{i=1}^{4k+1}(4ik-1)$, and

$$y_\ell=(-1)^\ell\left(\prod_{i=1}^{4k+1-\ell}(4ik-1)\right)\left(\prod_{i=1}^{\ell-1}(4ik+1)\right),$$

$\ell=1,2,\dots,4k+1$, where U is an $(N(k)+4k)\times(N(k)-1)$ matrix and V is an $(N(k)+4k)\times(4k+2)$ matrix, both of which we define below.

We consider the top-most line of M (which is formed out of x and y) as row 0 of M. We label the rows of U and V by i running from 1 to $N(k)+4k$. Furthermore, we label the columns of M by j running from 1 to $N(k)+4k+1$.

Following this labelling scheme, the matrix U has nonzero entries only in the four diagonals $i=j$, $i=j+1$, $i=j+4k$, $i=j+4k+1$. We denote the entries in column j on these four diagonals in order $f_0(j)$, $f_1(j)$, $g_0(j)$, and $g_1(j)$, where

$$\begin{aligned}
f_0(j)&=j(-4)^k,\\
f_1(j)&=-(4j+2)(-4)^k,\\
g_0(j)&=(N(k)-j),\\
g_1(j)&=-(4N(k)-4j-2).
\end{aligned}$$

To be precise, the (j,j)-entry is $f_0(j)$, the $(j+1,j)$-entry is $f_1(j)$, the $(j+4k,j)$-entry is $g_0(j)$, the $(j+4k+1,j)$-entry is $g_1(j)$, $j=1,2,\dots,N(k)-1$.

On the other hand, the matrix V is composed out of columns, labelled $N(k)$, $N(k)+1,\dots,N(k)+4k+1$, each of which containing just one nonzero entry. To be precise, the nonzero entry of column $N(k)+j$ is located in the $(4jk+1)$-st row (according to our labelling scheme), and it is equal to $(-4)^{-jk}$, $j=0,1,\dots,4k+1$.

We will now compute the determinant of M and show that it does not vanish for any k.

We perform some row operations on M, with the effect that the entries of y get eliminated. This is achieved by subtracting $\prod_{i=1}^{4k+1}(4ik-1)$ times row 1 from row 0, and

$$(-1)^\ell(-4)^{\ell k}\left(\prod_{i=1}^{4k+1-\ell}(4ik-1)\right)\left(\prod_{i=1}^{\ell-1}(4ik+1)\right)$$

times row $4\ell k+1$ from row 0, $\ell=1,2,\dots,4k+1$. Doing this, we must expect changes in row 0 in columns 1, $4k$, $4k+1$, $8k$, $8k+1$, …,$N(k)-4k=16k^2$. However, at this point a miracle occurs: the new entries in row 0 in columns $4\ell k+1$, $\ell=0,1,\dots,4k-1$, are still 0. On the other hand, the values of the new entries in row 0 in columns $4\ell k$,

$\ell = 1, 2, \dots, 4k$ are

$$(-1)^{\ell-1}(-4)^{(\ell+1)k} 8k(4k+1) \left(\prod_{i=1}^{4k-\ell} (4ik-1) \right) \left(\prod_{i=1}^{\ell-1} (4ik+1) \right). \tag{4.5}$$

After these manipulations we obtain a matrix of the form

$$M' = \begin{pmatrix} x' & y' \\ U & V \end{pmatrix},$$

where x' and y' have the same dimensions as before x and y, respectively, where the only nonzero entries of x' are in columns labelled by numbers which are divisible by $4k$, with the entry in column $4\ell k$ given by (4.5), and where y' consists only of zeroes. We have

$$\det M = \det M'. \tag{4.6}$$

The next step consists in expanding the determinant of M' with respect to columns $N(k), N(k)+1, \dots, N(k)+4k+1$ (i.e., the last $4k+2$ columns). Since each of these columns contains just one nonzero entry (which is a power of $(-4)^{-k}$), we have

$$\det M' = \pm(-4)^{-k(2k+1)(4k+1)} \det M'', \tag{4.7}$$

where M'' is the matrix arising from M' by deleting the last $4k+2$ columns and the rows $4\ell k+1$, $\ell = 0, 1, \dots, 4k+1$. More precisely, the matrix M'' has the following form:

$$\begin{pmatrix} x_1 & x_2 & x_3 & \dots\dots\dots & & x_{4k+1} \\ F_1 & 0 & 0 & \dots\dots\dots & & 0 \\ G_1 & F_2 & 0 & \dots\dots\dots & & 0 \\ 0 & G_2 & F_3 & \dots\dots\dots & & 0 \\ 0 & 0 & G_3 & \ddots & & \vdots \\ & & \ddots & \ddots & & 0 \\ & & & 0 & G_{4k} & F_{4k+1} \\ & & & \dots & 0 & G_{4k+1} \end{pmatrix},$$

where x_ℓ, $\ell = 1, 2, \dots, 4k$, is a line vector with $4k$ entries, all of them being zero except for the last, which is equal to (4.5), where x_{4k+1} is a line vector of $4k-1$ zeroes, where F_ℓ and G_ℓ, $\ell = 1, 2, \dots, 4k$, are $(4k-1) \times (4k)$ matrices with nonzero entries only in the two main diagonals, and where F_{4k+1} and G_{4k+1} are $(4k-1) \times (4k-1)$ matrices, G_{4k+1} being upper triangular. To be precise, for $\ell = 1, 2, \dots, 4k$ we have

$$F_\ell = \begin{pmatrix} f_1(4(\ell-1)k+1) & f_0(4(\ell-1)k+2) & 0 & \dots & & & \\ 0 & f_1(4(\ell-1)k+2) & f_0(4(\ell-1)k+3) & 0 & \dots & & \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & 0 & f_1(4\ell k-2) & f_0(4\ell k-1) & 0 & \\ & & & 0 & f_1(4\ell k-1) & f_0(4\ell k) & \end{pmatrix}$$

and

$$G_\ell = \begin{pmatrix} g_1(4(\ell-1)k+1) & g_0(4(\ell-1)k+2) & 0 & \dots & & & \\ 0 & g_1(4(\ell-1)k+2) & g_0(4(\ell-1)k+3) & 0 & \dots & & \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & 0 & g_1(4\ell k-2) & g_0(4\ell k-1) & 0 & \\ & & & 0 & g_1(4\ell k-1) & g_0(4\ell k) & \end{pmatrix},$$

and we have

$$G_{4k+1} = \begin{pmatrix} g_1(16k^2+1) & g_0(16k^2+2) & 0 & \dots & \\ 0 & g_1(16k^2+2) & g_0(16k^2+3) & \dots & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & g_1(16k^2+4k-2) & g_0(16k^2+4k-1) \\ & & & 0 & g_1(16k^2+4k-1) \end{pmatrix}.$$

The precise form of F_{4k+1} is without relevance for us. We do a Laplace expansion with respect to the last $4k-1$ rows. Because of the triangular form of G_{4k+1} we obtain

$$\det M'' = \left(\prod_{i=16k^2+1}^{16k^2+4k-1} g_1(i) \right) \det M''', \tag{4.8}$$

where

$$M''' = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_{4k} \\ F_1 & 0 & 0 & \dots & \\ G_1 & F_2 & 0 & \dots & \\ 0 & G_2 & F_3 & \dots & \\ 0 & 0 & G_3 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & 0 \\ & & & G_{4k-1} & F_{4k} \\ & & & 0 & G_{4k} \end{pmatrix}.$$

Instead of M''' we consider a more general matrix. Define the functions

$$\begin{aligned} f_0(t,j) &= ((N(k)+j)Y_t - X_{2,t})(-4)^k, \\ f_1(t,j) &= -((4N(k)+4j+2)Y_t - 4X_{1,t})(-4)^k, \\ g_0(t,j) &= (X_{2,t} - jY_t), \\ g_1(t,j) &= -(4X_{1,t} - (4j+2)Y_t). \end{aligned}$$

It should be noted that these functions specialize to $f_0(j), f_1(j), g_0(j), g_1(j)$, respectively, if $X_{1,t} = X_{2,t} = N(k)$ and $Y_t = 1$. Now we define the matrix M^X by

$$M^X = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_{4k} \\ F_1^X & 0 & 0 & \dots & \\ G_1^X & F_2^X & 0 & \dots & \\ 0 & G_2^X & F_3^X & \dots & \\ 0 & 0 & G_3^X & \ddots & \vdots \\ & \ddots & \ddots & \ddots & 0 \\ & & & G_{4k-1}^X & F_{4k}^X \\ & & & 0 & G_{4k}^X \end{pmatrix}, \tag{4.9}$$

where

$$F_\ell^X = \begin{pmatrix} f_1(1,4(\ell-1)k+1) & f_0(1,4(\ell-1)k+2) & 0 & \dots & \\ 0 & f_1(2,4(\ell-1)k+2) & f_0(2,4(\ell-1)k+3) & 0 & \dots \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & f_1(4k-2,4\ell k-2) & f_0(4k-2,4\ell k-1) & 0 \\ & & 0 & f_1(4k-1,4\ell k-1) & f_0(4k-1,4\ell k) \end{pmatrix}$$

and

$$G_\ell^X = \begin{pmatrix} g_1(1,4(\ell-1)k+1) & g_0(1,4(\ell-1)k+2) & 0 & \dots & \\ 0 & g_1(2,4(\ell-1)k+2) & g_0(2,4(\ell-1)k+3) & 0 & \dots \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & g_1(4k-2,4\ell k-2) & g_0(4k-2,4\ell k-1) & 0 \\ & & 0 & g_1(4k-1,4\ell k-1) & g_0(4k-1,4\ell k) \end{pmatrix},$$

Clearly, we have

$$M''' = M^X\big|_{X_{1,t}=X_{2,t}=N(k),\, Y_t=1}. \tag{4.10}$$

The evaluation of $\det M^X$ is given in the theorem below. From the result it is obvious that $\det M^X\big|_{X_{1,t}=X_{2,t}=N(k),\, Y_t=1}$ is nonzero, and, thus, also $\det M$.

Theorem 6. *We have*

$$\begin{aligned} \det M^X = (-1)^{k-1} 4^{2k(4k^2+7k+2)} k^{2k(4k+1)} \prod_{i=1}^{4k} (i+1)_{4k-i+1} \\ \times \prod_{a=1}^{4k-1} \big(2X_{1,a} - (32k^2+2a-1)Y_a\big) \\ \times \prod_{1\le a\le b\le 4k-1} (2X_{2,b}Y_a - 2X_{1,a}Y_b - (2b-2a+1)Y_aY_b), \end{aligned} \tag{4.11}$$

where $(\alpha)_k$ *is the standard notation for shifted factorials,* $(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1)$, $k \geq 1$, *and* $(\alpha)_0 := 1$.

Remark 3. Once having found this theorem, it is not difficult to prove (3.5), by working out how the coefficients of $P(z)$ resulting from the Ansatz (4.1)–(4.3) are related to the coefficients of $P(z)$ resulting from the Ansatz (3.4). Since this is not essential for the proof of Theorem 1, we leave the details to the reader.

Proof of Theorem 6. We follow the "identification of factors" method as described in Section 2.4 in [5].

First we show that $(2X_{1,a} - (32k^2 + 2a - 1)Y_a)$ divides $\det M^X$, $a = 1, 2, \ldots, 4k-1$. What has to be proved is that $\det M^X$ vanishes for $X_{1,a} = (32k^2 + 2a - 1)Y_a/2$. This can be done by showing that for this choice of $X_{1,a}$ there is a nontrivial linear combination of the rows of M^X. Indeed, if $X_{1,a} = (32k^2 + 2a - 1)Y_a/2$ we have

$$\frac{2(X_{2,4k-1} - (N(k)-1)Y_{4k-1})}{(-4)^{k(4k+1)+1}(16k^2+1)\prod_{t=1}^{4k-1}(4tk+1)} \cdot (\text{row } 0 \text{ of } M^X)$$
$$+ \sum_{r=0}^{4k} \sum_{s=0}^{4k-a-1} \left(\frac{(-1)^{r(k-1)}}{4^{rk}} \prod_{t=0}^{r-1} \frac{4k-1+4tk}{16k^2+1-4tk} \right.$$
$$\left. \cdot\, 2^s \prod_{t=4k-s}^{4k-1} \frac{2X_{1,t} - (32k^2+2t-1)Y_t}{X_{2,t-1} - (16k^2+t-1)Y_{t-1}} \right)$$
$$\cdot\, (\text{row } (16k^2 - (4k-1)r - s - 1) \text{ of } M^X) = 0,$$

as is easy to verify.

Next we claim that $(2X_{2,b}Y_a - 2X_{1,a}Y_b - (2b-2a+1)Y_aY_b)$ divides $\det M^X$, $1 \leq a \leq b \leq 4k-1$. Let us first impose the additional restriction that $a < b$. Using the above reasoning, the claim then follows from the fact that if $X_{2,b} = \frac{Y_b}{Y_a}X_{1,a} + (2b-2a+1)Y_b/2$ we have

$$\sum_{r=0}^{4k} \sum_{s=4k-b-1}^{4k-a-1} \frac{1}{(-4)^{rk}} \left(\prod_{t=1}^{r} \frac{2X_{1,a} - (32k^2 - 8kt + 2a + 1)Y_a}{2X_{1,a} - (64k^2 + 8k - 8kt + 2a + 1)Y_a} \right)$$
$$\cdot\, 4^{b-4k+s+1} \left(\prod_{t=4k-s}^{b} \frac{2X_{1,t}Y_a - 2X_{1,a}Y_t - (2t-2a)Y_aY_t}{2X_{2,t-1}Y_a - 2X_{1,a}Y_{t-1} - (2t-2a-1)Y_aY_{t-1}} \right)$$
$$\cdot\, (\text{row } (16k^2 - (4k-1)r - s - 1) \text{ of } M^X) = 0,$$

as is again easy to verify. On the other hand, if $a = b$, then the same argument shows that $(2X_{2,a} - 2X_{1,a} - Y_a)$ divides $\det M^X$. It remains to be checked that also

Y_a divides $\det M^X$. Indeed, if $Y_a = 0$ then we have

$$\sum_{r=0}^{4k} \frac{1}{(-4)^{rk}} \cdot (\text{row } (16k^2 - (4k-1)r - 4k + a) \text{ of } M^X) = 0,$$

whence Y_a divides $\det M^X$ for $a = 1, 2, \dots, 4k-1$.

These arguments show that the product on the right-hand side of (4.11) divides $\det M^X$ as a polynomial in the $X_{1,a}$'s, $X_{2,a}$'s, and Y_a's.

Clearly, the degree in the $X_{1,a}$'s, $X_{2,a}$'s, and Y_a's of $\det M^X$ is at most $16k^2 - 1$. But the degree of the right-hand side of (4.11) is exactly $16k^2 - 1$. Therefore we have proved that

$$\det M^X = C_1 \prod_{i=1}^{4k} (i+1)_{4k-i+1} \prod_{a=1}^{4k-1} \big(2X_{1,a} - (32k^2 + 2a - 1)Y_a\big) \\ \times \prod_{1 \le a \le b \le 4k-1} (2X_{2,b}Y_a - 2X_{1,a}Y_b - (2b - 2a + 1)Y_aY_b), \quad (4.12)$$

where C_1 is a constant independent of the $X_{1,a}$'s, $X_{2,a}$'s, and Y_a's.

In order to determine C_1, we compare coefficients of

$$X_{1,1}^{4k} X_{1,2}^{4k-1} \cdots X_{1,4k-1}^{2} Y_1^1 Y_2^2 \cdots Y_{4k-1}^{4k-1} \quad (4.13)$$

on both sides of (4.12). We claim that the coefficient of this monomial in $\det M^X$ is equal to $\det M^C$, where M^C is defined exactly in the same way as M^X (see (4.9)), except that the definitions of the functions f_0, f_1, g_0, g_1 are replaced by

$$f_0(t,j) = (N(k) + j)(-4)^k, \quad (4.14a)$$
$$f_1(t,j) = 4(-4)^k, \quad (4.14b)$$
$$g_0(t,j) = -j, \quad (4.14c)$$
$$g_1(t,j) = -4. \quad (4.14d)$$

This is seen as follows. The monomial (4.13) does not contain any $X_{2,a}$. Therefore, for finding its coefficient in $\det M^X$, we may set $X_{2,a} = 0$ in M^X for all a.

In which way may the monomial (4.13) appear in $\det M^X$ (with all $X_{2,a}$ equal to 0)? A typical term in the expansion of $\det M^X$ is the product of $16k^2$ entries of M^X, each from a different row and column. The monomial (4.13) contains $X_{1,1}^{4k}$. The variable $X_{1,1}$ is only found in columns $4\ell k + 1$, $\ell = 0, 1, \dots, 4k-1$ (and rows labelled by numbers $\equiv 1 \bmod 4k-1$, according to our labelling scheme). Therefore in a product of entries (each from a different row and column) which produces a term containing $X_{1,1}^{4k}$ *all* the entries from columns $4\ell k+1$ must be ones containing $X_{1,1}$. This explains the above definitions (4.14b) and (4.14d) of $f_1(1,4\ell k+1)$ and $g_1(1,4\ell k+1)$, $\ell=0,1,\dots,4k-1$, respectively. Moreover, we *must* generate the Y_1 in (4.13) from an entry in a column $4\ell k+2$, for some ℓ. (The variable Y_1 is also found in entries in columns $4\ell k+1$, but these columns are already taken by our choice of entries which contain the $X_{1,1}$'s.) This explains the definitions (4.14a) and (4.14c) of $f_0(1,4\ell k+2)$ and $g_0(1,4\ell k+2)$, $\ell=0,1,\dots,4k-1$, respectively. Next we ask how we can find (in the remaining columns and rows) entries which contain $X_{1,2}^{4k-1}$. Arguing in an analogous manner, the variable $X_{1,2}$ only appears in columns $4\ell k+2$, $\ell=0,1,2\dots,4k-1$. One of these columns is already taken by the entry from which we picked Y_1. Therefore in *all* the remaining ones we must choose entries containing $X_{1,2}$. This explains the definitions (4.14b) and (4.14d) of $f_1(2,4\ell k+2)$ and $g_1(2,4\ell k+2)$, $\ell=0,1,\dots,4k-1$, respectively. Next we consider the term Y_2^2 in (4.13). It must come from two entries in columns $4\ell k+3$, for two different ℓ's. This explains the definitions (4.14a) and (4.14c) of $f_0(2,4\ell k+3)$ and $g_0(2,4\ell k+3)$, $\ell=0,1,\dots,4k-1$, respectively. Etc.

The evaluation of $\det M^C$ follows from Lemma 7 below with $X_a=1$ and $Z_a=N(k)$ for $a=1,2,\dots,4k-1$. □

We consider now a more general determinant than $\det M^C$, the latter having been defined through the functions in (4.14). Replace these functions by

$$\begin{aligned}
f_0(t,j)&=(Z_t+j)(-4)^k,\\
f_1(t,j)&=4(-4)^kX_t,\\
g_0(t,j)&=-j,\\
g_1(t,j)&=-4X_t.
\end{aligned}$$

Let us denote the matrix defined by these functions in the same way as before M^C by M^Z. Clearly, M^Z specializes to M^C if all X_t are set equal to 1 and all Z_t to $N(k)$.

The determinant of M^Z evaluates as follows.

Lemma 7. *We have*

$$\det M^Z=(-1)^{k-1}2^{16k^3+20k^2+14k-1}k^{4k}(4k+1)!\prod_{a=1}^{4k-1}\left(X_a^{4k+1-a}\prod_{b=0}^{a-1}(Z_a-4bk)\right). \tag{4.15}$$

Proof. We proceed in a similar way as in the proof of Theorem 6. In the first step we show that the product on the right-hand side of (4.15) divides $\det M^Z$ as a polynomial in the X_a's and Z_a's. Then, in the second step, we compare the degrees of the product and $\det M^Z$. Since the degree of $\det M^Z$ turns out to be at most the degree of the product, it then follows that $\det M^Z$ is equal to the product times some constant which is independent of the X_t's and Z_t's. Finally, in the third step, this constant is found by computing the leading coefficient of $\det M^Z$.

Step 1. The product $\prod_{a=1}^{4k-1}\left(X_a^{4k+1-a}\prod_{b=0}^{a-1}(Z_a-4bk)\right)$ *divides* $\det M^Z$. We start by applying several row and column operations to $\det M^Z$, with the final goal of reducing the size of the determinant. First, for $i=16k^2-1,16k^2-2,\dots,4k$, in this order, we add $(-4)^k$ times row i to row $i-4k+1$. (It should be recalled that, according to our labelling scheme, we number the rows of M^Z from 0 to $16k^2-1$.) Thus, we obtain the determinant of the following matrix:

$$\begin{pmatrix} x_1 & x_2 & x_3 & \dots\dots\dots\dots & & x_{4k}\\ F'_1 & F'_2 & F'_3 & \dots\dots\dots\dots & & F'_{4k}\\ G_1 & F'_1 & F'_2 & \dots\dots\dots\dots & & F'_{4k-1}\\ 0 & G_2 & F'_1 & \dots\dots\dots\dots & & F'_{4k-2}\\ 0 & 0 & G_3 & \ddots & & \vdots\\ & & \ddots & \ddots & \ddots & F'_2\\ & & & 0 & G_{4k-1} & F'_1\\ & & & \dots & 0 & G_{4k} \end{pmatrix},$$

where the x_ℓ's, $\ell=1,2,\dots,4k$, are defined as earlier, and where the (F'_ℓ)'s and G_ℓ's, $\ell=1,2,\dots,4k$, are the $(4k-1)\times(4k)$ matrices

$$F'_\ell=\begin{pmatrix} 0 & (-4)^{\ell k}Z_1 & 0 & \dots & & & \\ 0 & 0 & (-4)^{\ell k}Z_2 & 0 & \dots & & \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & & 0 & (-4)^{\ell k}Z_{4k-2} & 0 \\ & & & 0 & 0 & (-4)^{\ell k}Z_{4k-1} \end{pmatrix}$$

and

$$G_\ell=\begin{pmatrix} -4X_1 & -4(\ell-1)k-2 & 0 & \dots & & & \\ 0 & -4X_2 & -4(\ell-1)k-3 & 0 & \dots & & \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & 0 & -4X_{4k-2} & -4\ell k-1 & 0 \\ & & & 0 & -4X_{4k-1} & -4\ell k \end{pmatrix}.$$

Next we "make" the submatrices G_ℓ, $\ell=1,2,\dots,4k$, to diagonal matrices, by subtracting $(j+1)/4X_j$ times column j from column $j+1$, $j=1,2,\dots,4k-1$, $(j+1)/4X_{j-4k}$ times column j from column $j+1$, $j=4k+1,4k+2,\dots,8k-1,\dots$, and $(j+1)/4X_{j-16k^2+4k}$ times column j from column $j+1$, $j=16k^2-4k+1,16k^2-$

$4k+2,\dots,16k^2-1$. After these operations we obtain the determinant of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots\cdots & & x_{4k} \\ F_{1,1} & F_{2,2} & F_{3,3} & \cdots\cdots & & F_{4k,4k} \\ G & F_{1,2} & F_{2,3} & \cdots\cdots & & F_{4k-1,4k} \\ 0 & G & F_{1,3} & \cdots\cdots & & F_{4k-2,4k} \\ 0 & 0 & G & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & F_{2,4k} \\ & & & 0 & G & F_{1,4k} \\ & & & \dots & 0 & G \end{pmatrix},$$

where G and the $F_{\alpha,\beta}$'s, $1\le\alpha\le\beta\le 4k$, are the $(4k-1)\times(4k)$ matrices

$$G=\begin{pmatrix} -4X_1 & 0 & 0 & \dots & & & \\ 0 & -4X_2 & 0 & 0 & \dots & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & -4X_{4k-2} & 0 & 0 \\ & & & & 0 & -4X_{4k-1} & 0 \end{pmatrix}$$

and

$$F_{\alpha,\beta}=\begin{pmatrix} 0 & f^{(\alpha,\beta)}_{1,2} & f^{(\alpha,\beta)}_{1,3} & \dots & & f^{(\alpha,\beta)}_{1,4k} \\ 0 & 0 & f^{(\alpha,\beta)}_{2,3} & f^{(\alpha,\beta)}_{2,4} & \dots & f^{(\alpha,\beta)}_{2,4k} \\ & \ddots & \ddots & & & \vdots \\ & & \ddots & \ddots & & \vdots \\ & & & 0 & f^{(\alpha,\beta)}_{4k-2,4k-1} & f^{(\alpha,\beta)}_{4k-2,4k} \\ & & & 0 & 0 & f^{(\alpha,\beta)}_{4k-1,4k} \end{pmatrix},$$

where

$$f^{(\alpha,\beta)}_{rs}=\frac{(-4)^{\alpha k-s+r+1}Z_r\,(4(\beta-1)k+r+2)_{s-r-1}}{X_{r+1}X_{r+2}\cdots X_{s-1}}.$$

Now we eliminate the last columns in $F_{\alpha\beta}$ for $1\le\alpha<\beta\le 4k$. We start by eliminating the last column of $F_{1,4k}$. We do this by adding $f^{(1,4k)}_{r,4k}/4X_r$ times column $16k^2-8k+r$ to column $16k^2$, $r=1,2,\dots,4k-1$. This makes all the entries in the last column which are in rows $16k^2-8k+2,\dots,16k^2-4k-1,16k^2-4k$ zero, whereas the entries in the last column in rows $16k^2-12k+3,\dots,16k^2-8k+1$ are modified. Next we eliminate these entries in a similar fashion, by using the columns $16k^2-12k+r$, $r=1,2,\dots,4k-1$, etc. In the end all the entries in the last column in rows $4k,4k+1,\dots,16k^2-1$ will be zero, whereas the entries in the last column in rows $1,2,\dots,4k-1$ will have been (significantly) modified. An analogous procedure is applied to eliminate the entries in the last columns of $F_{\alpha,4k-1}$, $\alpha=1,2,\dots,4k-2$. Just to mention the first step: We add $f^{(1,4k-1)}_{r,4k}/4X_r$ times column $16k^2-12k+r$ to column $16k^2-4k$, $r=1,2,\dots,4k-1$. This makes all the entries in column $16k^2-4k$ which are in rows $16k^2-12k+3,\dots,16k^2-8k,16k^2-8k+1$ zero, whereas the entries in the last column in rows $16k^2-16k+4,\dots,16k^2-12k+2$ are modified. Etc.

The advantage after having done all this is that now all the entries in columns $4\ell k$, $\ell=1,2,\dots,4k$, are zero except for entries in rows $0,1,\dots,4k-1$. This fact, and the fact that the submatrices G are diagonal matrices (of rectangular form) with last column consisting entirely of zeroes, makes it possible to reduce the determinant of the (new) matrix significantly. For $i=16k^2-1,16k^2-2,\dots,4k$ we may expand the determinant with respect to row i, in this order. If the details are worked out, then we see that our original determinant $\det M^Z$ is equal to

$$(-4)^{16k^2-4k}\prod_{a=1}^{4k-1}X_a^{4k}\det\begin{pmatrix} u_1 & u_2 & \dots & u_{4k}\\ e_{1,1} & e_{1,2} & \dots & e_{1,4k}\\ \dots & \dots & \dots & \dots\\ e_{4k-1,1} & e_{4k-1,2} & \dots & e_{4k-1,4k}\end{pmatrix},$$

where u_ℓ is given by (4.5), and where e_{ij} is given by

$$\sum_{r=1}^{j}\sum_{\substack{4k=t_0>t_1>\dots>t_r=i\\ j+1=n_0>n_1>n_2>\dots>n_r=1}}(-1)^{r-1}(-4)^{jk-4k+i+1}Z_{t_1}Z_{t_2}\cdots Z_{t_r}\\ \times\frac{\prod_{\nu=0}^{r-1}(4k(n_\nu-1)-4k+t_{\nu+1}+2)_{t_\nu-t_{\nu+1}-1}}{X_{i+1}X_{i+2}\cdots X_{4k-1}}.$$

Clearly, we may extract $(-4)^{jk}$ from column j, $j=1,2,\dots,4k$, and $(-4)^{-4k+i+1}/X_{i+1}X_{i+2}\cdots X_{4k-1}$ from row i, $i=1,2,\dots,4k-1$ (still using our non-standard labelling scheme where the rows are numbered $0,1,\dots,4k-1$), so that we obtain the expression

$$(-4)^{16k^2-4k+2k^2(4k+1)-(2k-1)(4k-1)}\prod_{a=1}^{4k-1}X_a^{4k+1-a}\\ \times\det\begin{pmatrix} \tilde u_1 & \tilde u_2 & \dots & \tilde u_{4k}\\ \tilde e_{1,1} & \tilde e_{1,2} & \dots & \tilde e_{1,4k}\\ \dots & \dots & \dots & \dots\\ \tilde e_{4k-1,1} & \tilde e_{4k-1,2} & \dots & \tilde e_{4k-1,4k}\end{pmatrix}, \quad (4.16)$$

where $\tilde u_\ell$ is given by

$$(-1)^{\ell-1}(-4)^k 8k(4k+1)\left(\prod_{i=1}^{4k-\ell}(4ik-1)\right)\left(\prod_{i=1}^{\ell-1}(4ik+1)\right),$$

and where $\tilde{e}_{ij}$ is given by

$$\sum_{r=1}^{j} \sum_{\substack{4k=t_0>t_1>\cdots>t_r=i\\ j+1=\tilde{n}_0>\tilde{n}_1>\tilde{n}_2>\cdots>\tilde{n}_r=1}} (-1)^{r-1} Z_{t_1} Z_{t_2} \cdots Z_{t_r} \\ \times \prod_{\nu=0}^{r-1} (4k(\tilde{n}_\nu - 1) - 4k + t_{\nu+1} + 2)_{t_\nu - t_{\nu+1} - 1}. \quad (4.17)$$

From (4.16) it is abundantly clear that $\prod_{a=1}^{4k-1} X_a^{4k+1-a}$ divides $\det M^Z$ (which, after all, is equal to (4.16)). It remains to show that also $\prod_{a=1}^{4k-1} \prod_{b=0}^{a-1} (Z_a - 4bk)$ divides $\det M^Z$.

Let a and b, $1 \le a \le 4k-1$, $0 \le b \le a-1$, be given. We want to show that $Z_a - 4bk$ divides $\det M^Z$. We will show the equivalent fact that the rows of $\det M^Z \big|_{Z_a = 4bk}$ are linearly dependent. The crucial observation, from which this claim follows easily, is that the entries $\tilde{e}_{ij}$, $j = 1, 2, \dots, 4k$, in row i of the determinant in (4.16) are given by a polynomial in j, $p_i(j)$ say, of degree $(4k - i - 1)$ and with leading coefficient (i.e., coefficient of j^{4k-i-1}) equal to

$$\sum_{r=1}^{j} \sum_{4k=t_0>t_1>\cdots>t_r=i} (-1)^{r-1} Z_{t_1} Z_{t_2} \cdots Z_{t_r} \frac{(4k)^{4k-i-r}}{\prod_{\nu=0}^{r-1} (t_\nu - i)}. \quad (4.18)$$

This is seen as follows. The summand in (4.17) is a polynomial in $j, \tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_{r-1}$ of multidegree $(4k - t_1 - 1, t_1 - t_2 - 1, t_2 - t_3 - 1, \dots, t_{r-1} - t_r - 1)$ (i.e., the degree in j is $4k - t_1 - 1$, the degree in $\tilde{n}_1$ is $t_1 - t_2 - 1$, etc.). Because of the fact that (for fixed u and varying v) $\sum_{\gamma=u}^{v} \gamma^e$ is a polynomial in v of degree $e+1$ with leading coefficient (i.e., coefficient of v^{e+1}) equal to $1/(e+1)$, successive summation over $\tilde{n}_{r-1}$, $\tilde{n}_{r-2}$, …, $\tilde{n}_1$ yields the claimed facts.

Under the specialization $Z_a = 4bk$, it is seen by "inspection" that (4.18) with i replaced by $a - b$ vanishes, because the summand corresponding to $t_0 > \cdots > t_{\omega-1} > t_\omega = a > t_{\omega+1} > \cdots > t_r$ cancels with the summand corresponding to $t_0 > \cdots > t_{\omega-1} > t_{\omega+1} > \cdots > t_r$. Hence, the polynomial $p_{a-b}(j)$ has degree (at most) $4k - a + b - 2$ (instead of $4k - a + b - 1$). Consequently, if $Z_a = 4bk$ then the entries in rows $a - b$, $a - b + 1$, …, $4k - 1$ are given by polynomials in j (to wit: j denoting the column index of the entries) of respective degrees $4k - a + b - 2$ (!), $4k - a + b - 2$, $4k - a + b - 3$, …, 2, 1. These are $4k - a + b$ polynomials, all of degree at most $4k - a + b - 2$. It follows that there must be a nontrivial linear combination of these polynomials that vanishes. Hence, the rows $a - b$, $a - b + 1$, …, $4k - 1$ are linearly dependent, which, in turn, implies that the determinant in (4.16) (and, thus, also $\det M^Z$) vanishes for $Z_a = 4bk$, $1 \le a \le 4k - 1$, $0 \le b \le a - 1$.

Step 2. Comparison of degrees. Clearly, the degree of $\det M^Z$ as a polynomial in the X_t's and Z_t's is at most $16k^2-1$, whereas the degree of the product on the right-hand side of (4.15) is exactly $16k^2-1$. Hence, we have

$$\det M^Z = C_2 \prod_{a=1}^{4k-1} \big(X_a^{4k+1-a} \prod_{b=0}^{a-1} (Z_a - 4bk)\big),$$

where C_2 is a constant independent of the X_t's and Z_t's.

Step 3. Computation of the leading coefficient. In order to determine C_2, we determine the coefficient of $\prod_{a=1}^{4k-1} X_a^{4k+1-a} Z_a^a$ in the expansion of $\det M^Z$. By arguments similar to those at the end of the proof of Theorem 6, it is seen that this coefficient is given by the determinant of the following matrix, which we denote by M^L. It is defined exactly in the same way as M^X (see (4.9)), except that the definitions of the functions f_0, f_1, g_0, g_1 are replaced by

$$\begin{aligned} f_0(t,j) &= (-4)^k, \\ f_1(t,j) &= 4(-4)^k, \\ g_0(t,j) &= 0, \\ g_1(t,j) &= -4. \end{aligned}$$

By expanding this determinant with respect to row 0, we obtain

$$\det M^L = \sum_{\ell=1}^{4k} (-1)^{\ell-1} u_\ell \det M_\ell^L, \tag{4.19}$$

where u_ℓ is, as earlier, given by (4.5), and M_ℓ^L is the matrix arising from M^L by deleting row 0 and column $4\ell k$.

Let ℓ, $1 \le \ell \le 4k$, be fixed. We will next compute $\det M_\ell^L$. When we built M_ℓ^L from M^L, we deleted in particular row 0. Therefore we will now switch to the usual labelling scheme for rows and columns of a matrix, i.e., we will subsequently not only label the columns by $1, 2, \dots$ but also the rows.

If $\ell < 4k$, then we expand $\det M_\ell^L$ with respect to the last $4k-1$ rows and then with respect to the last column. Since these rows and this column contain only one nonzero entry, we obtain some multiple of the determinant of a $(16k^2-4k-1)\times(16k^2-4k-1)$ matrix. If $\ell < 4k-1$, then we continue by expanding the (now reduced) determinant with respect to the last $4k-2$ rows and then with respect to the last 2 columns. We continue in the same manner until we have reduced $\det M_\ell^L$ to the determinant of a $(4\ell k-1)\times(4\ell k-1)$ matrix, more precisely, until we arrive at

$$\det M_\ell^L = 4^{\binom{4k}{2}-\binom{\ell}{2}} (-4)^{k\binom{4k-\ell+1}{2}} \det M_\ell',$$

where M'_ℓ is the matrix

$$\begin{pmatrix} F & 0 & 0 & \dots\dots & 0 \\ G & F & 0 & \dots\dots & 0 \\ 0 & G & F & \dots\dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & G\quad F & 0 \\ 0 & \dots\dots & 0 & G & F' \\ 0 & \dots\dots\dots & & 0 & U \end{pmatrix}, \qquad (\ell \text{ occurrences of } F),$$

with F and G the $(4k-1)\times(4k)$ matrices

$$F = \begin{pmatrix} 4(-4)^k & (-4)^k & 0 & \dots & & & \\ 0 & 4(-4)^k & (-4)^k & 0 & \dots & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & 4(-4)^k & (-4)^k & 0 \\ & & & & 0 & 4(-4)^k & (-4)^k \end{pmatrix}$$

and

$$G = \begin{pmatrix} -4 & 0 & 0 & \dots & & & \\ 0 & -4 & 0 & 0 & \dots & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & -4 & 0 & 0 \\ & & & & 0 & -4 & 0 \end{pmatrix},$$

F' the $(4k-1)\times(4k-1)$ matrix which arises from F by deleting its last column, and U the $(\ell-1)\times(4k-1)$ matrix

$$U = \begin{pmatrix} -4 & 0 & 0 & \dots\dots\dots\dots & 0 \\ 0 & -4 & 0 & \dots\dots\dots\dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots\dots & -4 & 0\ \dots & 0 \end{pmatrix}.$$

We continue by expanding M'_ℓ with respect to the last $\ell-1$ rows. Thus we obtain

$$\det M_\ell^L = 4^{\binom{4k}{2}-\binom{\ell}{2}+\ell-1}(-4)^{k\binom{4k-\ell+1}{2}}\det \overline{M}_\ell \tag{4.20}$$

for $\det M_\ell^L$, where $\overline{M}_\ell$ is the matrix

$$\begin{pmatrix} F & 0 & 0 & \dots\dots & 0 \\ G & F & 0 & \dots\dots & 0 \\ 0 & G & F & \dots\dots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & G & F & 0 \\ 0 & \dots\dots & & 0 & G & V \end{pmatrix}, \qquad (\ell - 1 \text{ occurrences of } F),$$

with V the $(4k-1) \times (4k-\ell)$ matrix

$$V = \begin{pmatrix} 0 & 0 & \dots\dots\dots\dots & & & 0 \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ 0 & 0 & \dots\dots\dots\dots & & & 0 \\ (-4)^k & 0 & \dots & & & \\ 4(-4)^k & (-4)^k & 0 & \dots & & \\ 0 & 4(-4)^k & (-4)^k & 0 & \dots & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 4(-4)^k & (-4)^k \\ & & & & 0 & 4(-4)^k \end{pmatrix}. \tag{4.21}$$

Next we prepare for a reduction from the top of $\overline{M}_\ell$. We subtract 4 times column j from column $j-1$, $j = 4(\ell-1)k, 4(\ell-1)k-1, \dots, 4k(\ell-2)+2$, $j = 4k(\ell-2), 4k(\ell-2)-1, \dots, 4k(\ell-3)+2, \dots, j = 4k, 4k-1, \dots, 2$, in this order. Thus $\det M'_\ell$ is converted to $\det M''_\ell$, where M''_ℓ is the matrix

$$\begin{pmatrix} F'' & 0 & 0 & \dots\dots\dots & & 0 \\ G'' & F'' & 0 & \dots\dots\dots & & 0 \\ 0 & G'' & F'' & \dots\dots\dots & & 0 \\ 0 & 0 & G'' & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots\dots & 0 & G'' & F'' & 0 \\ 0 & \dots\dots\dots & & 0 & G'' & V \end{pmatrix},$$

with F'' and G'' the $(4k-1)\times(4k)$ matrices

$$F''=\begin{pmatrix} 0 & (-4)^k & 0 & \dots \\ 0 & 0 & (-4)^k & 0 & \dots \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & 0 & 0 & (-4)^k & 0 \\ & & & & 0 & 0 & (-4)^k \end{pmatrix}$$

and

$$G''=\begin{pmatrix} -4 & 0 & 0 & \dots \\ (-4)^2 & -4 & 0 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots & \ddots \\ (-4)^{4k-2} & \dots\dots\dots & (-4)^2 & -4 & 0 & 0 \\ (-4)^{4k-1} & \dots\dots\dots & (-4)^3 & (-4)^2 & -4 & 0 \end{pmatrix}.$$

Our next goal is to "push" the (nonzero) entries in columns $4tk+1$, $t=0,1,\dots,\ell-2$, down to rows $(4k-1)(\ell-1)+1,\dots,(4k-1)\ell-1,(4k-1)\ell$. (This is similar to what we did in Step 1 when we "pushed" all the nonzero entries in columns $4tk$, $t=1,2,\dots,4k$ up to rows $0,1,\dots,4k-1$.) In order to achieve this for the 1-st column, we add

$$\sum_{r=1}^{\ell-2}\sum_{s=2}^{4k}(-1)^r(-4)^{-rk+s-1}\binom{s-1}{r-1}\cdot(\text{column }(4kr+s))$$

to column 1. Similarly, in order to achieve this for the $(4k+1)$-st column, we add

$$\sum_{r=2}^{\ell-2}\sum_{s=2}^{4k}(-1)^{r-1}(-4)^{-(r-1)k+s-1}\binom{s-1}{r-2}\cdot(\text{column }(4kr+s))$$

to column $4k+1$. Etc. As a result, the determinant $\det M''_\ell$ is converted to the determinant of the matrix

$$\begin{pmatrix} F'' & 0 & 0 & \dots\dots\dots\dots & 0 \\ G' & F'' & 0 & \dots\dots\dots\dots & 0 \\ 0 & G' & F'' & \dots\dots\dots\dots & 0 \\ 0 & 0 & G' & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & 0 \\ 0 & \dots\dots & 0 & G' & F'' & 0 \\ H_1 & H_2 & \dots\dots & H_{\ell-2} & G'' & V \end{pmatrix}, \tag{4.22}$$

where F'' and G'' are as before, G' is the $(4k-1)\times(4k)$ matrix

$$G' = \begin{pmatrix} 0 & 0 & 0 & \dots & & & \\ 0 & -4 & 0 & 0 & \dots & & \\ \vdots & \ddots & \ddots & \ddots & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ 0 & (-4)^{4k-3} & \dots & (-4)^2 & -4 & 0 & 0 \\ 0 & (-4)^{4k-2} & \dots & (-4)^3 & (-4)^2 & -4 & 0 \end{pmatrix},$$

and H_t, $t=1,2,\dots,\ell-2$, is a $(4k-1)\times(4k)$ matrix with all entries equal to 0, except for the entries in the first column. To be precise, the entry in the first column and row s of H_t is given by

$$(-1)^{\ell-1-t}(-4)^{-(\ell-1-t)k+s}\binom{s-1}{\ell-1-t}. \tag{4.23}$$

It should be noted that the entries in the first column of G'' (G'' appearing at the bottom of the matrix (4.22), as do the matrices H_t) is given by (4.23) with $t=\ell-1$.

Now everything is prepared for the reduction. We expand the determinant of (4.22) with respect to rows $1,2,\dots,(4k-1)(\ell-1)$. This reduces the determinant of (4.22) to

$$(-1)^{\binom{\ell}{2}}(-4)^{(\ell-1)(4k-1)k}\det \tilde M_\ell,$$

where $\tilde M_\ell$ is a $(4k-1)\times(4k-1)$ matrix of the form

$$\begin{pmatrix} N & V \end{pmatrix},$$

with the (s,t)-entry of N being given by (4.23), $s=1,2,\dots,4k-1$, $t=1,2,\dots,\ell-1$, and V the $(4k-1)\times(4k-\ell)$ matrix from above. If we substitute all this in (4.20), we obtain that $\det M_\ell^L$ is equal to

$$(-1)^{\binom{\ell}{2}}4^{\binom{4k}{2}-\binom{\ell}{2}+\ell-1}(-4)^{k\binom{4k-\ell+1}{2}+(\ell-1)(4k-1)k}\det\tilde M_\ell. \tag{4.24}$$

The submatrix V of $\tilde M_\ell$ is almost diagonal. Subtraction of 4 times row s from row $s+1$ in $\tilde M_\ell$, $s=\ell-1,\ell,\dots,4k-2$, will transform it into a completely diagonal matrix (namely into the matrix on the right-hand side of (4.21) with all entries $4(-4)^k$ replaced by 0). As a side effect, this will turn the $(4k-1,1)$-entry of $\tilde M_\ell$ into

$$(-1)^{\ell-2}(-4)^{-(\ell-2)k+4k-1}\binom{4k-1}{\ell-1}.$$

As is easily seen, the determinant of the in this way modified matrix, M_ℓ^* say, is

$$(-1)^{4k-\ell+\binom{\ell-1}{2}}(-4)^{(4k-\ell)k}\cdot((4k-1,1)\text{-entry of } M_\ell^*)\\ \times\prod_{s=1}^{\ell-2}((s,\ell-s)\text{-entry of } M_\ell^*),$$

or, explicitly,

$$(-1)^{\ell}(-4)^{(4k-\ell)k-(\ell-2)k+4k-1-\binom{\ell-2}{2}k+\binom{\ell-1}{2}}\binom{4k-1}{\ell-1}.$$

Substitution of the above in (4.24) yields that the determinant $\det M_\ell^L$ is equal to

$$(-1)^{\ell k}4^{8k^3+10k^2-\ell k+2k-1}\binom{4k-1}{\ell-1}.$$

Now we substitute this in (4.19). We obtain that $\det M^L$ is equal to

$$\sum_{\ell=1}^{4k}(-1)^k 4^{8k^3+10k^2+3k}\binom{4k-1}{\ell-1}2k(4k+1)\left(\prod_{i=1}^{4k-\ell}(4ik-1)\right)\left(\prod_{i=1}^{\ell-1}(4ik+1)\right)\\ =(-1)^k2^{16k^3+20k^2+14k-1}k^{4k}(4k+1)(4k-1)!\sum_{\ell=1}^{4k}\binom{\frac{1}{4k}-1}{4k-\ell}\binom{-\frac{1}{4k}-1}{\ell-1}.$$

The sum is readily evaluated by means of the Chu–Vandermonde summation (see e.g. [4, Sec. 5.1, (5.27)]), so that we obtain

$$(-1)^k2^{16k^3+20k^2+14k-1}k^{4k}(4k+1)(4k-1)!\binom{-2}{4k-1}\\ =(-1)^{k-1}2^{16k^3+20k^2+14k-1}k^{4k}(4k+1)!\,.$$

Since the coefficient of $\prod_{a=1}^{4k-1}X_a^{4k+1-a}Z_a^a$ in the expression on the right-hand side of (4.15) is exactly the same, we have completed the proof of the lemma. $\square$

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