POSITIVE m-DIVISIBLE NON-CROSSING PARTITIONS AND THEIR KREWERAS MAPS

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ABSTRACT. We study positive m-divisible non-crossing partitions and their positive Kreweras maps. In classical types, we describe their combinatorial realisations as certain non-crossing set partitions. We also realise these positive Kreweras maps as pseudo-rotations on a circle, respectively on an annulus. We enumerate positive m-divisible non-crossing partitions in classical types that are invariant under powers of the positive Kreweras maps with respect to several parameters. In order to cope with the exceptional types, we develop a different combinatorial model in general type describing positive m-divisible non-crossing partitions that are invariant under powers of the positive Kreweras maps. We finally show that altogether these results establish several cyclic sieving phenomena.

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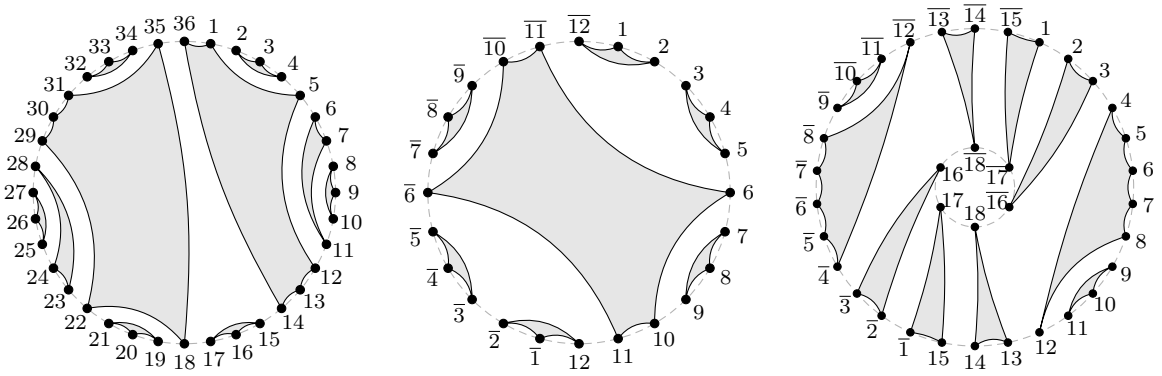

FIGURE 1. Examples of positive 3-divisible non-crossing set partitions of the respective types A_{11}, B_4 and D_6.

1. INTRODUCTION

Non-crossing set partitions have been introduced into combinatorial mathematics by Kreweras [Kre72] in 1972. Biane [Bia97] presented an algebraic realisation in terms of certain factorisations of a long cycle in the symmetric group. These classical non-crossing partitions were lifted to all finite Coxeter groups and a given Coxeter element by Bessis [Bes03], and independently by Brady and Watt [Bra01, BW02]. The earlier combinatorial construction of non-crossing set partitions for type B due to Reiner [Rei97] turned out to be equivalent to the type B situation of the construction of Bessis and of Brady and Watt. The (algebraic) type D non-crossing partitions of Bessis and of Brady and Watt were translated into (combinatorial) non-crossing set partitions for type D by Reiner and Athanasiadis in [AR04].

Inspired by a refinement of Kreweras' non-crossing set partitions due to Edelman [Ede80a], Armstrong [Arm06] introduced a positive integer parameter m and defined *m-divisible non-crossing partitions* for any finite irreducible Coxeter group W and any Coxeter element $c \in W$, together with a natural partial order. Armstrong showed that, in type A_{n-1} these objects can be realised as Edelman's refinement, namely as non-crossing set partitions of $\{1, 2, \dots, mn\}$ where all block sizes are divisible by m. In type B_n they can be realised as non-crossing set partitions of $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ which remain invariant under the simultaneous replacement $i \mapsto -i$, and where again all block sizes are divisible by m [Arm06, Thm. 4.5.6]. The first author and Müller [KM10] (with a missing piece reported in [Kim11]) presented a realisation in type D_n as non-crossing set partitions of $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ which are arranged on an annulus subject to similar, but more technical conditions. In all these realisations as set partitions (see Figure 1 for examples), the above mentioned partial order corresponds to the classical refinement order on set partitions.

As observed for example in [Cha04, Prop. 9], one of the remarkable facts about these generalised non-crossing partitions is that their total number is given by the uniform formula

$$\mathrm{Cat}^{(m)}(W) := \prod_{i=1}^{n} \frac{mh + d_i}{d_i}, \tag{1.1}$$

where n denotes the *rank* of W, $d_1 \leq \ldots \leq d_n$ denote the *degrees* of W, and where $h = d_n$ is the *Coxeter number*; see Subsection 2.1 for precise definitions. This number is called *Fuß–Catalan number of type W*. These numbers also count various other combinatorial objects associated with W. Among these are the generalised non-nesting partitions of Athanasiadis [Ath04], the generalised clusters of Fomin and Reading [FR05], and the generalised Coxeter-sortable elements of the second author, Thomas and Williams [STW25]. We refer to the latter article for a detailed discussion of these various Fuß-analogues counted by the Fuß–Catalan numbers.

In the present article, we study a subset of the m-divisible non-crossing partitions associated with W, namely *positive m-divisible non-crossing partitions of type W*; see Definition 2.3. For $m = 1$, these have been introduced by Reading [Rea07a, Cor. 9.2 plus the following paragraph] (implicitly) and by Simões [Sim12, Thm. 1.1] (explicitly; only for crystallographic types). For generic m, positive m-divisible non-crossing partitions appear in [BRT12, Cor. 7.4 plus the following paragraph] (again, only for crystallographic types), and in [STW25, Ch. 7].

The term "positive" here stems from the observation that the bijection between generalised non-crossing partitions and generalised clusters in [STW25] restricts to a bijection between positive non-crossing partitions and positive clusters for the generalised cluster complex of [FR05], where a cluster is *positive* if it consists entirely of (coloured) positive roots.

The total number of positive m-divisible non-crossing partitions is given by the *positive Fuß–Catalan number of type W* (see [BRT12, Cor. 7.4] for crystallographic types and [STW25, Thm. 7.2.1] for all types) defined by

$$\mathrm{Cat}_{+}^{(m)}(W) := \prod_{i=1}^{n} \frac{mh + d_i - 2}{d_i}. \tag{1.2}$$

This expression can be found in work by Athanasiadis [Ath04] where he shows that it counts certain bounded regions in the extended Shi arrangement. It was then shown in [FR05, Cor. 12.2] that positive generalised clusters are also counted by this formula, and both [BRT12, Cor. 7.4] and [STW25, Thm. 7.2.1] use the earlier mentioned bijective correspondence to derive the same counting formula for positive m-divisible non-crossing partitions.

Our first focus is on describing positive m-divisible non-crossing partitions for the classical types A_{n-1}, B_n, and D_n as subsets of the above mentioned realisations as set partitions[1]. We show in Theorem 3.4 that in type A_{n-1} being positive means that the block containing 1 also contains mn, in Theorem 4.4 that in type B_n being positive means that the block containing 1 also contains an element from $\{-1, -2, \dots, -mn\}$, and in Theorem 5.4 that in type D_n being positive means that the block containing 1 also contains an element from $\{-1, -2, \dots, -m(n-1)\}$.

Our next set of results consists of counting formulae for types A_{n-1}, B_n, and D_n that refine the total count (1.2). More precisely, we provide formulae for the number of (multichains of) positive m-divisible non-crossing partitions with restrictions on the block structure; see Subsections 3.4, 4.4 and 5.4.

Our final set of results concerns a uniformly defined action on positive m-divisible non-crossing partitions. This action is given by two variants of a map introduced by Armstrong [Arm06, Def. 3.4.15]. Armstrong's map is an extension of a map originally introduced by Kreweras [Kre72, Sec. 3] commonly known as *"Kreweras complement."* For this reason, we shall call Armstrong's map (*generalised*) *"Kreweras map"*.

Armstrong's map does not restrict to an action on *positive* m-divisible non-crossing partitions. Inspired by a representation-theoretic functor in the work of Simões [Sim12] and of Buan, Reiten and Thomas [BRT12], we introduce "positive analogues" of Armstrong's (generalised) Kreweras map on positive m-divisible non-crossing partitions in Definition 2.12. One of these *"positive Kreweras maps"* was already described in [AST13, Sec. 7], where also its order was computed; see Theorem 2.17.

Armstrong's original Kreweras map translates into ordinary rotation on the realisations of m-divisible non-crossing set partitions in classical types. It turns out that our variants are slightly more complicated to describe. They translate into certain "pseudo-rotations," meaning that they are realised as a "deformation" of rotation; see Theorems 3.6, 4.6 and 5.6.

In all types, we provide counting formulae for the number of positive m-divisible non-crossing partitions that are invariant under a fixed power of either of the two positive Kreweras maps, including refinements; see Subsections 3.6, 4.6 and 5.6.

We finally use the above described results to address cyclic sieving phenomena for the positive m-divisible non-crossing partitions. The phenomenon of cyclic sieving has been introduced by Reiner, Stanton and White in [RSW04]. It refers to a certain enumerative property of combinatorial objects under the action of a cyclic group. Armstrong [Arm06, Conj. 5.4.7] went on to conjecture a cyclic sieving phenomenon for his Kreweras map. A related conjecture on cyclic sieving of m-divisible non-crossing partitions was made by Bessis and Reiner [BR11, Conj. 6.5]. Both conjectures were proved by the first author and Müller [Kra13, KM13].

[1]As usual, we call a partition π of the (ordered) set $\{a_1, a_2, \dots, a_k\}$ *non-crossing*, if the graphical representation of π, where the elements $a_1, a_2, \dots, a_k$ are placed, in this order, around a circle, and successive elements in the blocks of π are connected by an arc, can be realised so that the arcs do not cross each other.

In this paper, we prove that the positive m-divisible non-crossing partitions together with either of the two variants of the positive Kreweras map also satisfy a cyclic sieving phenomenon; see Theorem 2.19. This is in fact a cumulative theorem, resulting from Theorems 3.23, 4.19 and 5.20 for the classical types, and Theorem 6.15 for the exceptional types.

This article is organised as follows. In Section 2, we recall the definitions of m-divisible non-crossing partitions associated with a Coxeter group W, and their Kreweras map(s). There, we also define positive m-divisible non-crossing partitions and the positive analogue of the Kreweras map(s). Moreover, we collect several properties of these objects (see Theorem 2.8 and Lemma 2.9), and we state one of the main results, namely the cyclic sieving phenomena in Theorem 2.19.

This is followed by Sections 3–5, in which the positive m-divisible non-crossing partitions and their positive Kreweras maps are discussed in depth for the classical types. Each of these sections starts with a translation of the algebraic definition of these objects into the language of set partitions; see Theorems 3.4, 4.4 and 5.4. These translations motivate purely combinatorial definitions of positive m-divisible non-crossing partitions and their positive Kreweras maps (in terms of a pseudo-rotation) for the classical types; see Subsections 3.2, 4.2 and 5.2. Subsequently, a subsection is devoted to the enumeration of positive m-divisible non-crossing partitions in each type subject to various restrictions on their block structure; see Subsections 3.4, 4.4 and 5.4. In a further subsection, we characterise, for each type, the positive m-divisible non-crossing partitions that are invariant under a given power of either of the two positive Kreweras maps; see Subsections 3.5, 4.5 and 5.5. Then, in Subsections 3.6, 4.6 and 5.6, we enumerate those that are invariant. Finally, we use the obtained formulae to deduce corresponding cyclic sieving phenomena; see Theorems 3.23, 4.19 and 5.20.

In Section 6, we treat the exceptional types with respect to invariance and cyclic sieving under the action of the positive Kreweras maps. We present several problems that were left open by our analyses in Section 7.

In order to streamline the presentation, we have "outsourced" most of the (frequently technical) proofs of our results to appendices. More precisely, our combinatorial proofs of the order of the (combinatorial version of) the Kreweras maps acting on positive m-divisible non-crossing partitions in the classical types are contained in Appendix A.

The various proofs of our enumeration formulae for positive m-divisible non-crossing partitions of classical type, without or with invariance under powers of the positive Kreweras maps, from Subsections 3.4, 3.6, 4.4, 4.6, 5.4 and 5.6 can be found in Appendix B.

This is followed by Appendix C in which we have collected the proofs of our characterisations of positive m-divisible non-crossing partitions that are invariant under the action of a given power of the positive Kreweras maps, again in the classical types.

In Appendix D we provide the proofs for some auxiliary results for our cyclic sieving results in classical types from Subsections 3.7, 4.7 and 5.7.

Finally, the required computational details for the verification of our cyclic sieving result for positive m-divisible non-crossing partitions in the exceptional types in Theorem 6.15 are carried out in Appendix E.

Acknowledgement This project was initiated in 2012 when, at the occasion of the 24th International Conference on "Formal Power Series and Algebraic Combinatorics" in Nagoya, Japan, Christian innocently asked Christian: "Willst Du ein weiteres zyklisches Sieben beweisen?"[2] Over time, this turned out to be slightly more involved than originally anticipated by Christian. It took multiple meetings at each of the Universität Wien, the Freie Universität Berlin, the Technische Universität Berlin, and the Ruhr-Universität Bochum, until this article finally materialised in 2025 just before the 37th International Conference on "Formal Power Series and Algebraic Combinatorics" in Sapporo, Japan.

2. A general setup for positive m-divisible non-crossing partitions

This section provides the setup for m-divisible non-crossing partitions in their various incarnations for general finite irreducible Coxeter systems. In particular, we provide two equivalent descriptions of *positive* m-divisible non-crossing partitions. Most of the background material is standard, though we refer to [Arm06, STW25] for further detail. After a brief reminder of the notions for finite Coxeter systems, we discuss (positive) m-divisible non-crossing partitions in Subsections 2.1 and 2.2, and then introduce our two positive Kreweras maps in Subsection 2.3. Since they cannot be fully used for $m = 1$, we provide an alternative description in Subsection 2.4 that makes sense for general $m \geq 1$. After having established the needed structures, we conclude with Subsection 2.5, in which we first recall the definition of the cyclic sieving phenomenon of Reiner, Stanton and White [RSW04], and then present two uniformly stated cyclic sieving phenomena for our positive m-divisible non-crossing partitions in Theorem 2.19. (More refined cyclic sieving phenomena which are specific for type A_{n-1} respectively for type B_n are contained in Theorems 3.23 and 4.19.)

The following can all be found in standard references such as [Hum90]. Let $(W, \mathcal{S})$ denote a finite irreducible Coxeter system of rank $n = |\mathcal{S}|$ acting on a Euclidean vector space $V \cong \mathbb{R}^n$ with inner product $\langle \cdot, \cdot \rangle$. Let $\Phi \subseteq V$ be a root system for $(W, \mathcal{S})$ with simple roots $\Delta = \{\alpha_s \mid s \in \mathcal{S}\}$, positive roots Φ^+, and negative roots $\Phi^- = -\Phi^+$. The elements in $\mathcal{S} = \{r_\alpha \mid \alpha \in \Delta\}$ are called *simple reflections*, and the *reflections* in W are their conjugates,

$$\mathcal{R} = \{r_\beta \mid \beta \in \Phi^+\} = \{wsw^{-1} \mid w \in W, s \in \mathcal{S}\},$$

where r_β denotes the reflection sending β to its negative while fixing pointwise its orthogonal complement $\beta^\perp = \{v \in V \mid \langle \beta, v \rangle = 0\}$. The length function $\ell_{\mathcal{S}} : W \to \mathbb{N}$ sends $w \in W$ to the length of any reduced $\mathcal{S}$-word for w. There exists a unique element $w_\circ \in W$ of longest length $N = |\Phi^+| = |\mathcal{R}|$. Since all reduced $\mathcal{S}$-words for an element are related by braid relations and, moreover, the two simple reflections on both sides of a braid relation coincide, we may define the *support* of an element $w \in W$ by the set of simple reflections

[2] "Do you want to prove another cyclic sieving phenomenon?"

appearing in any of its reduced $\mathcal{S}$-words. An element $w \in W$ is *of full support* if its support is all of $\mathcal{S}$, or, equivalently, if it does not live in any proper standard parabolic subgroup.

A *(standard) Coxeter element* $c \in W$ is given by the product of all generators $\mathcal{S}$ in any order, $c = s_1 \dots s_n \in W$ for $\mathcal{S} = \{s_1, \dots, s_n\}$. Observe that all standard Coxeter elements are conjugate to each other and that their order is given by the *Coxeter number* h, which moreover features in the formula $N = nh/2$. The eigenvalues of a Coxeter element are given by $\{e^{2\pi i(d_1-1)/h}, \dots, e^{2\pi i(d_n-1)/h}\}$, where $d_1, \dots, d_n$ are the *degrees* of W.

Example 2.1 (Running example in type A_2)**.** The finite irreducible Coxeter system of type A_2 can be represented by the symmetric group acting on three letters. In this case, we have

$$\mathcal{S} = \big\{(12), (23)\big\} \subset \big\{(12), (13), (23)\big\} = \mathcal{R}\,,$$

where (ij) is the transposition interchanging the letters i and j. The unique longest element $w_\circ = (12)(23)(12) = (23)(12)(23)$ has length $N = 3$. One choice of a Coxeter element is the long cycle $c = (12)(23) = (123)$ and we see that the Coxeter number is $h = 3$. The degrees are moreover $d_1 = 2$ and $d_2 = 3$. This yields the (positive) Fuß–Catalan numbers

$$\mathrm{Cat}^{(m)}(W) = \tfrac{1}{2}(m+1)(3m+2), \quad \mathrm{Cat}_+^{(m)}(W) = \tfrac{1}{2}m(3m+1)\,. \tag{2.1}$$

For $m = 0, 1, 2$, these evaluate to $(1, 5, 12)$ and to $(0, 2, 7)$, respectively.

2.1. **m-divisible non-crossing partitions.** It is well established that there is also a *dual* viewpoint on Coxeter groups by generating such a group by all its reflections. Non-crossing partitions play a crucial role in this approach. The *reflection length* $\ell_{\mathcal{R}} : W \to \mathbb{N}$ is defined analogously as the length with respect to reduced $\mathcal{R}$-words. Analogously to the (left or right) weak order on W, one defines the *absolute order* $\leq_{\mathcal{R}}$ by letting $u \leq_{\mathcal{R}} v$ if $\ell_{\mathcal{R}}(u) + \ell_{\mathcal{R}}(u^{-1}v) = \ell_{\mathcal{R}}(v)$. Thus, $u \leq_{\mathcal{R}} v$ if and only if there is a reduced $\mathcal{R}$-word for v which begins with a reduced $\mathcal{R}$-word for u.

For a Coxeter element c, one defines the set of *non-crossing partitions* $NC(W, c)$ as the set of all elements $w \in W$ with $w \leq_{\mathcal{R}} c$. $NC(W, c)$ with the partial order $\leq_{\mathcal{R}}$ forms a lattice, cf. [BW08]. Since $\leq_{\mathcal{R}}$ is invariant under conjugation, this lattice is, abstractly, independent of the choice of c. We refer to [RRS17] for an in-depth discussion of this independence of the chosen Coxeter element.

For a finite irreducible Coxeter group W of rank n, Armstrong's *m-divisible non-crossing partitions* [Arm06] are then defined by

$$NC^{(m)}(W, c) = \big\{(w_0, w_1, \dots, w_m) \mid w_0 w_1 \cdots w_m = c \text{ and } \ell_{\mathcal{R}}(w_0) + \ell_{\mathcal{R}}(w_1) + \cdots + \ell_{\mathcal{R}}(w_m) = n\big\}.$$

For $m = 1$, this definition reduces to the definition of non-crossing partitions $NC(W, c)$, by identifying $(w_0, w_1) \in NC^{(1)}(W, c)$ with the element $w_0 \in NC(W, c)$. The set $NC^{(m)}(W, c)$ becomes a partially ordered set by letting

$$(u_0, u_1, \dots, u_m) \leq_{\mathcal{R}} (v_0, v_1, \dots, v_m) \quad \text{if } u_i \geq_{\mathcal{R}} v_i, \text{ for } i = 1, 2, \dots, m.$$

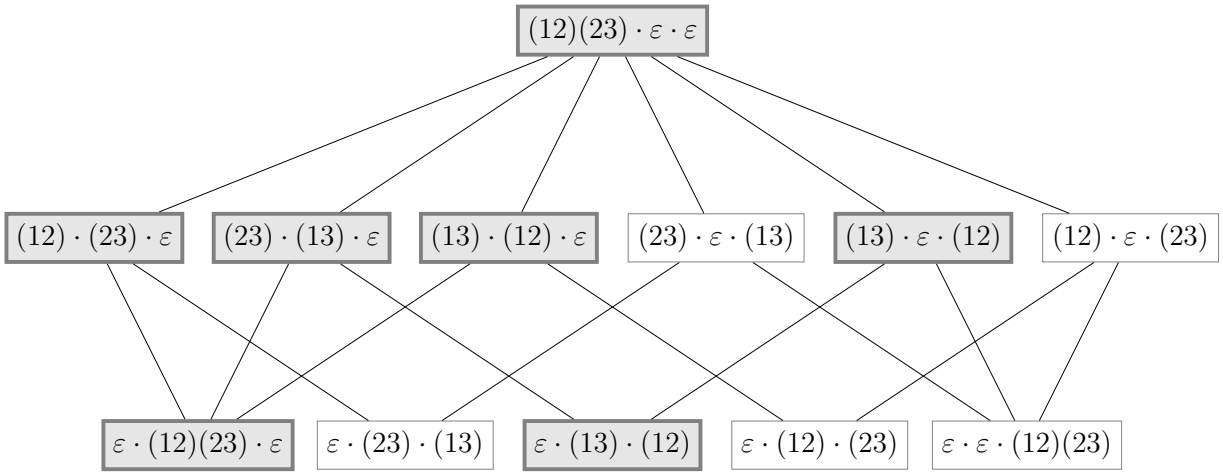

FIGURE 2. The 2-divisible non-crossing partitions $NC^{(2)}(A_2, (12)(23))$.

The elements of $NC^{(m)}(W, c)$ are well-known to be counted by the Fuß–Catalan numbers

$$|NC^{(m)}(W, c)| = \mathrm{Cat}^{(m)}(W)\,.$$

Example 2.2. Following the running example from Example 2.1, we illustrate the twelve 2-divisible non-crossing partitions $NC^{(2)}(A_2, (12)(23))$ in Figure 2 where we separate the parts by dots.

As an abstract partially ordered set, $NC^{(m)}(W, c)$ does again not depend on the choice of the Coxeter element c. In general, it is not a lattice but a graded join-semilattice.

The main object of study in this paper is the following subset of m-divisible non-crossing partitions.

Definition 2.3. An m-divisible non-crossing partition $(w_0, w_1, \ldots, w_m) \in NC^{(m)}(W, c)$ is called *positive* if the product $w_0 \cdots w_{m-1} \in NC(W, c)$ has full support. The set of *positive m-divisible non-crossing partitions* is denoted by $NC^{(m)}_+(W, c)$.

The seven positive elements in $NC^{(2)}(A_2, (12)(23))$ are highlighted in Figure 2. The above definition is motivated by a bijective correspondence between m-divisible non-crossing partitions and the generalised cluster complex defined by Fomin and Reading in [FR05]. In [BRT12], Buan, Reiten, and Thomas provided a representation-theoretic bijection between facets in the m-generalised cluster complex and m-divisible non-crossing partitions for crystallographic types. This bijection was made combinatorial and uniform for all types by the second author, Thomas and Williams in [STW25]. Under this bijection, positive m-divisible non-crossing partitions correspond exactly to "positive clusters" which are counted by the *positive Fuß–Catalan numbers* (1.2); see [FR05].

2.2. **An alternative description of m-divisible non-crossing partitions.** Following [STW25, Ch. 7], we next provide an alternative description of m-divisible non-crossing

partitions and their positive analogues. We first need to recall some background on *Coxeter-sortable elements* in Coxeter groups as introduced by Reading in [Rea07b] and further studied in various places; see for example [RS11, CLS14, STW25].

Fix a reduced word $\mathrm{c} = \mathrm{s}_1 \cdots \mathrm{s}_n$ for the Coxeter element c, and define the *c-sorting word* $\mathrm{w}(\mathrm{c})$ for $w \in W$ to be the lexicographically first reduced word for w when written as a subword of $\mathrm{c}^\infty := c \cdot c \cdot c \cdots$. (Here and below, we mildly distinguish between the letter s in the word c from the simple reflection $s \in \mathcal{S}$, and similarly from a letter r and the corresponding reflection $r \in W$.) Observe that all reduced words for c are obtained from each other by transpositions of consecutive commuting letters, or *up to commutations* for short. Thus, given the Coxeter element c, the various c-sorting words for $w \in W$ all coincide up to commutations. Let furthermore $\mathrm{w}_\circ(\mathrm{c}) = s_1 \cdots s_N$ be the c-sorting word of the longest element $w_\circ \in W$, starting with an initial copy of $\mathrm{c} = \mathrm{s}_1 \cdots \mathrm{s}_n$. It was shown for example in [CLS14, Prop. 7.1] that, up to commutations, this word ends in a twisted version of the reduced word of the Coxeter element, namely

$$\mathrm{s}_{N-n+1} \cdots \mathrm{s}_N = \psi(\mathrm{c}) = \psi(\mathrm{s}_1) \cdots \psi(\mathrm{s}_n),$$

where $\psi(s) = w_\circ s w_\circ$ for $s \in \mathcal{S}$. Depending on the given Coxeter element c, the word $\mathrm{w}_\circ(\mathrm{c})$ induces a *reflection ordering*

$$r_1 <_c r_2 <_c \cdots <_c r_N, \tag{2.2}$$

on $\mathcal{R}$ with $r_i = s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1 \in \mathcal{R}$. For a definition (which is not further used in this paper) and further properties of reflection orderings; see [Dye93]. For later reference, we refer to this reflection ordering by the word $\mathbf{inv}_c = \mathrm{r}_1 \cdots \mathrm{r}_N$. We denote the final n letters of $\mathbf{inv}_c$ (i.e., the word $\psi(\mathrm{c})$) by $\mathbf{inv}_c^R$ so that we obtain $\mathbf{inv}_c = \mathbf{inv}_c^L \mathbf{inv}_c^R$.

Example 2.4. Continuing the above example with $\mathrm{c} = (12)(23)$, the c-sorting word for the longest element is $\mathrm{w}_\circ(\mathrm{c}) = (12)(23)(12)$, while the corresponding reflection ordering is

$$(12) <_c (13) <_c (23)\,.$$

We also see that $\mathrm{w}_\circ(\mathrm{c}) = (12)(23)(12)$ starts with an initial copy of $\mathrm{c} = (12)(23)$ and ends with $\psi(\mathrm{c}) = (23)(12)$. This gives

$$\mathbf{inv}_c = (12)(13)(23) = (12) \cdot (13)(23) = \mathbf{inv}_c^L \cdot \mathbf{inv}_c^R\,.$$

Note that the middle reflection (23) in $\mathrm{w}_\circ(\mathrm{c})$ appears both in the initial copy of c and also in $\psi(\mathrm{c})$. In general, we cannot guarantee that there is a single word equal to $\mathrm{w}_\circ(\mathrm{c})$ up to commutations starting with c and ending with $\psi(\mathrm{c})$. For example, in type A_3 with $\mathrm{c} = (12)(23)(34)$ we have

$$\mathrm{w}_\circ(\mathrm{c}) = (12)(23)(34)(12)(23)(12)$$

and the third letter (34) appears both in the initial copy of c and in $\psi(\mathrm{c}) = (34)(23)(12)$. This has no effect on the constructions below.

The following property of non-crossing partitions is crucial, we refer to [Rea07a, Rem. 6.8] and to [STW25, Prop. 4.1.3] for details.

Proposition 2.5. *Each $w \in NC(W,c)$ has a unique reduced $\mathcal{R}$-word $r_1 r_2 \cdots r_p$ with $r_1 <_c r_2 <_c < \cdots <_c r_p$. Moreover, every reduced $\mathcal{R}$-word $r_1 r_2 \cdots r_p$ with $r_1 <_c r_2 <_c < \cdots <_c r_p$ yields an element in $NC(W,c)$.*

Indeed, this factorisation is the unique factorisation of w into reflections induced by the EL-labelling of the non-crossing partition lattice in [ABW07]. Together with the definition of m-divisible non-crossing partitions, we obtain the following two alternative descriptions. Define $NC_\Delta^{(m)}(W,c)$ to be the set of subwords of $\mathbf{inv}_c^{m+1}$ that are reduced $\mathcal{R}$-words for the Coxeter element c, in which we distinguish the letters in different copies of $\mathbf{inv}_c$,

$$NC_\Delta^{(m)}(W,c) = \{\mathrm{w}_0 \cdot \mathrm{w}_1 \cdot \ldots \cdot \mathrm{w}_m \mid \mathrm{w}_i = \mathrm{t}_1^{(i)} \cdots \mathrm{t}_{k_i}^{(i)} \subset \mathbf{inv}_c \text{ and } c = t_1^{(0)} \cdots t_{k_0}^{(0)} \cdots t_1^{(m)} \cdots t_{k_m}^{(m)} \text{ is reduced}\}, \quad (2.3)$$

where we write $\mathrm{u} \subset \mathrm{v}$ if u is a subword of v. See Figure 3 for this set of subwords in the running example of type A_2 with $m = 2$. As before, we use t_i for the letter of $\mathbf{inv}_c$ corresponding to the reflection $t_i \in \mathcal{R}$. Analogously, the word $\mathrm{w}_i = \mathrm{t}_1^{(i)} \cdots \mathrm{t}_{k_i}^{(i)}$ is the subword of $\mathbf{inv}_c$ corresponding to the element $w_i = t_1^{(i)} \cdots t_{k_i}^{(i)} \in NC(W,c)$.

We then have the following theorem.

Theorem 2.6 ([STW25, Thm. 4.2.5])**.** *Sending a reduced subword in $NC_\Delta^{(m)}(W,c)$ to the tuple $(w_0, \ldots, w_m) \in NC^{(m)}(W,c)$, where w_i is the product of the reflections in the subword inside the $(i+1)$-st copy of $\mathbf{inv}_c$, is a bijection.*

Example 2.7. The bijection between $NC^{(2)}(A_2, (12)(23))$ and $NC_\Delta^{(2)}(A_2, (12)(23))$ is given in Figure 3.

For an element $(w_0, w_1, \ldots, w_m) \in NC^{(m)}(W,c)$ with corresponding $\mathrm{w}_0 \cdot \mathrm{w}_1 \cdot \ldots \cdot \mathrm{w}_m \in NC_\Delta^{(m)}(W,c)$, we have $w_0 \in NC(W,c)$. Furthermore, the word $\mathrm{w}_0 \subset \mathbf{inv}_c$ is redundant since $w_0 = c(w_1 \cdots w_m)^{-1}$ with unique reduced increasing word w_0. Therefore, sending $\mathrm{w}_0 \cdot \mathrm{w}_1 \cdot \ldots \cdot \mathrm{w}_m$ to $\mathrm{w}_1 \cdot \ldots \cdot \mathrm{w}_m$ is a bijection between $NC_\Delta^{(m)}(W,c)$ and

$$NC_\nabla^{(m)}(W,c) := \{\mathrm{w}_1 \cdot \ldots \cdot \mathrm{w}_m \mid \mathrm{w}_i = \mathrm{t}_1^{(i)} \cdots \mathrm{t}_{k_i}^{(i)} \subset \mathbf{inv}_c \text{ and } t_1^{(1)} \cdots t_{k_1}^{(1)} \cdots t_1^{(m)} \cdots t_{k_m}^{(m)} \in NC(W,c) \text{ is reduced}\}. \quad (2.4)$$

Moreover, $NC_\Delta^{(m-1)}(W,c) \subset NC_\nabla^{(m)}(W,c)$ is a proper subset.

These two descriptions can be used to determine positive m-divisible non-crossing partitions as follows. Denote by

$$w = w^L \cdot w^R \quad (2.5)$$

the decomposition of $w \in NC(W,c)$ such that w^L and w^R are the parts of the unique subword for w of $\mathbf{inv}_c$ that lie inside $\mathbf{inv}_c^L$ and inside $\mathbf{inv}_c^R$, respectively. The following theorem gives an alternative description of positive m-divisible non-crossing partitions. It was first observed in crystallographic types in [BRT12, Thm. 7]. The general case was established in [STW25, Ch. 7].

$NC^{(2)}(A_2,(12)(23))$					$NC^{(2)}_{\Delta}(A_2,(12)(23))$				
$(12)(23)$	$\cdot$	ε	$\cdot$	ε	$\mathbf{(12)}(13)\mathbf{(23)}$	$\cdot$	$(12)(13)(23)$	$\cdot$	$(12)(13)(23)$
ε	$\cdot$	ε	$\cdot$	$(12)(23)$	$(12)(13)(23)$	$\cdot$	$(12)(13)(23)$	$\cdot$	$\mathbf{(12)}(13)\mathbf{(23)}$
ε	$\cdot$	(13)	$\cdot$	(12)	$(12)(13)(23)$	$\cdot$	$(12)\mathbf{(13)}(23)$	$\cdot$	$\mathbf{(12)}(13)(23)$
(23)	$\cdot$	(13)	$\cdot$	ε	$(12)(13)\mathbf{(23)}$	$\cdot$	$(12)\mathbf{(13)}(23)$	$\cdot$	$(12)(13)(23)$
(13)	$\cdot$	(12)	$\cdot$	ε	$(12)\mathbf{(13)}(23)$	$\cdot$	$\mathbf{(12)}(13)(23)$	$\cdot$	$(12)(13)(23)$
(12)	$\cdot$	ε	$\cdot$	(23)	$\mathbf{(12)}(13)(23)$	$\cdot$	$(12)(13)(23)$	$\cdot$	$(12)(13)\mathbf{(23)}$
ε	$\cdot$	(23)	$\cdot$	(13)	$(12)(13)(23)$	$\cdot$	$(12)(13)\mathbf{(23)}$	$\cdot$	$(12)\mathbf{(13)}(23)$
ε	$\cdot$	$(12)(23)$	$\cdot$	ε	$(12)(13)(23)$	$\cdot$	$\mathbf{(12)}(13)\mathbf{(23)}$	$\cdot$	$(12)(13)(23)$
(12)	$\cdot$	(23)	$\cdot$	ε	$\mathbf{(12)}(13)(23)$	$\cdot$	$(12)(13)\mathbf{(23)}$	$\cdot$	$(12)(13)(23)$
(23)	$\cdot$	ε	$\cdot$	(13)	$(12)(13)\mathbf{(23)}$	$\cdot$	$(12)(13)(23)$	$\cdot$	$(12)\mathbf{(13)}(23)$
(13)	$\cdot$	ε	$\cdot$	(12)	$(12)\mathbf{(13)}(23)$	$\cdot$	$(12)(13)(23)$	$\cdot$	$\mathbf{(12)}(13)(23)$
ε	$\cdot$	(12)	$\cdot$	(23)	$(12)(13)(23)$	$\cdot$	$\mathbf{(12)}(13)(23)$	$\cdot$	$(12)(13)\mathbf{(23)}$

FIGURE 3. The bijection between $NC^{(2)}(A_2,(12)(23))$ and $NC^{(2)}_{\Delta}(A_2,(12)(23))$.

Theorem 2.8. *The tuple $(w_0,w_1,\dots,w_m)\in NC^{(m)}(W,c)$ is positive if and only if $w_m = w_m^L$.*

We thus call the corresponding elements $\mathrm{w}_0\cdot\ldots\cdot\mathrm{w}_m\in NC^{(m)}_{\Delta}(W,c)$ and $\mathrm{w}_1\cdot\ldots\cdot\mathrm{w}_m\in NC^{(m)}_{\nabla}(W,c)$ positive if $\mathrm{w}_m\subset\mathbf{inv}_c^L$. In Figure 2, we highlight the seven positive elements in $NC^{(2)}(A_2,(12)(23))$ and observe in Figure 3 that these are exactly those where the final component of $NC^{(m)}_{\Delta}(W,c)$ is a subword of $\mathbf{inv}_c^L=(12)$. We note that this theorem has the following immediate consequence, which we record for later use.

Lemma 2.9. *Let $w\in NC(W,c)$, and decompose w in the form $w=uv$ such that $\ell_{\mathcal{R}}(u)+\ell_{\mathcal{R}}(v)=\ell_{\mathcal{R}}(w)$. Suppose $v=r_1\cdots r_k$ for $\mathrm{r}_1\cdots\mathrm{r}_k\subset\mathbf{inv}_c^R$ and cu^{-1} has full support. Then $w^L=u$ and $w^R=v$.*

Proof. Since cu^{-1} has full support, we have $(cu^{-1},u)\in NC^{(1)}_{+}(W,c)$ by Definition 2.3. Theorem 2.8 then implies that $u^L=u$. By definition, u^L is formed by determining the subword of $\mathcal{R}$ that is a reduced $\mathcal{R}$-word for u, and then disregarding those reflections which lie inside $\mathbf{inv}_c^R$. Since $u^L=u$, nothing had to be disregarded. We conclude by observing that $w^L=(uv)^L=u^L=u$ and $w^R=(uv)^R=u^Rv=v$. □

The description of (positive) m-divisible non-crossing partitions in terms of $NC^{(m)}_{\nabla}(W,c)$ also has the following enumerative implications, namely formulae for the (positive) Fuß–Catalan numbers in terms of a sum of binomial coefficients; see [STW25, Thm. 4.5.1].

Corollary 2.10. *Let W be a finite irreducible Coxeter group of rank n. Then*

$$\mathrm{Cat}^{(m)}(W) = \sum \binom{m+1+\mathrm{asc}(\mathrm{r}_1,\dots,\mathrm{r}_n)}{n},$$
$$\mathrm{Cat}^{(m)}_+(W) = \sum \binom{m+\mathrm{des}(\mathrm{r}_1,\dots,\mathrm{r}_n)}{n} = \sum \binom{m+\mathrm{asc}(\mathrm{r}_1,\dots,\mathrm{r}_n,\mathrm{r}_{\mathrm{fix}})}{n},$$

where all sums range over the factorisations $\mathrm{r}_1\cdots\mathrm{r}_n \subset \mathbf{inv}_c$ of the Coxeter element c into reflections $r_i \in \mathcal{R}$. Here, $\mathrm{asc}(\mathrm{r}_1,\dots,\mathrm{r}_n)$ and $\mathrm{des}(\mathrm{r}_1,\dots,\mathrm{r}_n)$ are the number of indices i such that $r_i <_c r_{i+1}$ and such that $r_i >_c r_{i+1}$, respectively. Moreover, $\mathrm{r}_{\mathrm{fix}}$ in the last sum is the first letter in the word $\mathbf{inv}^R_c$.

Indeed, the description of the positive Fuß–Catalan numbers in terms of descents does not directly follow from the analysis above. We record it here for its beauty. On the other hand, the other description using the letter $\mathrm{r}_{\mathrm{fix}}$ will be crucial in the final section where we establish a similar count for positive elements that are invariant under either of the two *positive Kreweras maps*.

Example 2.11. In our ongoing example, we have $c = (12)(23) = (23)(13) = (13)(12)$ for the ordering $(12) <_c (13) <_c (23)$. Moreover, $\mathrm{r}_{\mathrm{fix}} = (13)$. This yields

$$\mathrm{asc}((12),(23)) = 1, \quad 0 = \mathrm{asc}((13),(12)) = \mathrm{asc}((23),(13)),$$
$$\mathrm{asc}((12),(23),(13)) = \mathrm{asc}((13),(12),(13)) = 1, \quad 0 = \mathrm{asc}((23),(13),(13)),$$

implying

$$\mathrm{Cat}^{(m)}(W) = 2\binom{m+1}{2} + \binom{m+2}{2}, \quad \mathrm{Cat}^{(m)}_+(W) = 2\binom{m+1}{2} + \binom{m}{2},$$

in accordance with (2.1).

2.3. **The positive Kreweras maps, I.** Beside refined counting formulae, we study in this paper a "positive analogue" of the *(generalised) Kreweras map* on m-divisible non-crossing partitions. The two variants $\mathsf{K}^{(m)}$ and $\widetilde{\mathsf{K}}^{(m)}$ below were introduced by Armstrong in [Arm06, Def. 3.4.15]. For $m \geq 1$, these are given by

$$\mathsf{K}^{(m)} : \begin{array}{ccc} NC^{(m)}(W,c) & \xrightarrow{\sim} & NC^{(m)}(W,c) \\ (w_0,w_1,\dots,w_m) & \mapsto & (v, cw_mc^{-1}, w_1\dots,w_{m-1}), \end{array}$$

$$\widetilde{\mathsf{K}}^{(m)} : \begin{array}{ccc} NC^{(m)}(W,c) & \xrightarrow{\sim} & NC^{(m)}(W,c) \\ (w_0,w_1,\dots,w_m) & \mapsto & (cw_mc^{-1}, w_0\dots,w_{m-1}), \end{array}$$

where $v = c(cw_mc^{-1}w_1\dots w_{m-1})^{-1} = (cw_mc^{-1})w_0(cw_mc^{-1})^{-1}$. Both maps are close relatives. More precisely, the embedding

$$\varphi : \begin{array}{ccc} NC^{(m)}(W,c) & \hookrightarrow & NC^{(m+1)}(W,c) \\ (w_0,w_1,\dots,w_m) & \mapsto & (\varepsilon, w_0, w_1,\dots,w_m) \end{array}$$

has the property that it maps $\widetilde{\mathsf{K}}^{(m)}$ to $\mathsf{K}^{(m+1)}$. In symbols, $\varphi \circ \widetilde{\mathsf{K}}^{(m)} = \mathsf{K}^{(m+1)} \circ \varphi$. Moreover, these two variants can easily be interpreted in terms of $NC_{\Delta}^{(m)}(W,c)$ and $NC_{\nabla}^{(m)}(W,c)$, and the relation comes from the mentioned embedding of $NC_{\Delta}^{(m)}(W,c)$ into $NC_{\nabla}^{(m+1)}(W,c)$. We drop the superscripts in $\mathsf{K}^{(m)}$ and in $\widetilde{\mathsf{K}}^{(m)}$ if m is clear from the context and only write K and $\widetilde{\mathsf{K}}$, respectively.

The Kreweras maps do not stabilise the set of positive m-divisible non-crossing partitions. One of the main motivations for this paper is to define and study positive versions of both maps. Indeed, it turns out that these positive variants can naturally be seen on $NC_{\nabla}^{(m)}(W,c)$ and on $NC_{\Delta}^{(m)}(W,c)$, and that they are combinatorial descriptions of a representation-theoretic functor on the positive cluster category.

Definition 2.12. For $m \geq 2$, the *positive Kreweras map* $\mathsf{K}_{+}^{(m)}$ is given by

$$\begin{array}{rccc} \mathsf{K}_{+}^{(m)}: & NC_{+}^{(m)}(W,c) & \xrightarrow{\sim} & NC_{+}^{(m)}(W,c) \\ & (w_0, w_1, \ldots, w_m) & \mapsto & (v_+, cw_{m-1}^R w_m c^{-1}, w_1, \ldots, w_{m-1}^L), \end{array}$$

where $v_+ = c(cw_{m-1}^R w_m c^{-1} w_1 \cdots w_{m-1}^L)^{-1} = (cw_{m-1}^R w_m c^{-1}) w_0 (cw_{m-1}^R w_m c^{-1})^{-1}$. For $m \geq 1$, its variant $\widetilde{\mathsf{K}}_{+}^{(m)}$ is given by

$$\begin{array}{rccc} \widetilde{\mathsf{K}}_{+}^{(m)}: & NC_{+}^{(m)}(W,c) & \xrightarrow{\sim} & NC_{+}^{(m)}(W,c) \\ & (w_0, w_1, \ldots, w_m) & \mapsto & (cw_{m-1}^R w_m c^{-1}, w_0, w_1, \ldots, w_{m-1}^L). \end{array}$$

As for the (original) Kreweras maps, the embedding φ of $NC^{(m)}(W,c)$ into $NC^{(m+1)}(W,c)$ given above embeds $NC_{+}^{(m)}(W,c)$ into $NC_{+}^{(m+1)}(W,c)$, and we have $\varphi \circ \widetilde{\mathsf{K}}_{+}^{(m)} = \mathsf{K}_{+}^{(m+1)} \circ \varphi$. We again usually drop the superscripts in $\mathsf{K}_{+}^{(m)}$ and $\widetilde{\mathsf{K}}_{+}^{(m)}$ and write K_{+} and $\widetilde{\mathsf{K}}_{+}$, respectively. We also emphasise that this version of $\mathsf{K}_{+}^{(m)}$ is only defined for $m \geq 2$. In Subsection 2.4, we will present an alternative description that also covers the case where $m = 1$.

Example 2.13. We consider $\mathsf{K}_{+}^{(2)}$ and both $\widetilde{\mathsf{K}}_{+}^{(1)}$ and $\widetilde{\mathsf{K}}_{+}^{(2)}$ in our running example. We start with $\mathsf{K}_{+}^{(2)}$ acting on $NC_{+}^{(2)}(A_2,c)$:

$$\overset{\curvearrowright}{(13)\cdot\varepsilon\cdot(12)} \longmapsto (12)\cdot(23)\cdot\varepsilon \longmapsto (23)\cdot(13)\cdot\varepsilon \longmapsto \overset{\curvearrowright}{(13)\cdot(12)\cdot\varepsilon}$$

$$\overset{\curvearrowright}{\varepsilon\cdot(13)\cdot(12)} \longmapsto \varepsilon\cdot(12)(23)\cdot\varepsilon$$

$$\overset{\curvearrowright}{(12)(23)\cdot\varepsilon\cdot\varepsilon}$$

Considering only the second orbit and the above inclusion

$$\varphi : NC^{(1)}(A_2,c) \longrightarrow NC^{(2)}(A_2,c),$$

we get $\widetilde{\mathsf{K}}_+^{(1)}$. The action of $\widetilde{\mathsf{K}}_+^{(2)}$ on $NC_+^{(2)}(A_2, c)$ is finally

$$\begin{array}{c} \longrightarrow (13)\cdot\varepsilon\cdot(12) \mapsto (23)\cdot(13)\cdot\varepsilon \mapsto (12)\cdot(23)\cdot\varepsilon \mapsto (13)\cdot(12)\cdot\varepsilon \longmapsto \\ \longmapsto \varepsilon\cdot(12)(23)\cdot\varepsilon \longleftarrow (12)(23)\cdot\varepsilon\cdot\varepsilon \longleftarrow \varepsilon\cdot(13)\cdot(12) \longleftarrow \end{array}$$

2.4. **The positive Kreweras maps, II.** The goal of this subsection is to present alternative descriptions of the positive Kreweras maps $\mathsf{K}_+^{(m)}$ and $\widetilde{\mathsf{K}}_+^{(m)}$ introduced in Definition 2.12 which enable us to make sense of $\mathsf{K}_+^{(m)}$ also for $m = 1$ (see Definition 2.16). Starting point is the observation that these positive Kreweras maps can be seen as Coxeter-theoretic versions of the *shift functor* in the *positive generalised cluster category*, compare [BRT12, Sim12]. Without going into the details of the involved homological algebra, we give here its combinatorial incarnation as given in [CLS14, STW25]. Up to commutations, it is known that $\mathrm{c}^h = \mathrm{w}_\circ(\mathrm{c})\psi(\mathrm{w}_\circ(\mathrm{c}))$. This implies the following equality of bi-infinite words, ${}^\infty\mathrm{c}^\infty = {}^\infty\big(\mathrm{w}_\circ(\mathrm{c})\psi(\mathrm{w}_\circ(\mathrm{c}))\big)^\infty$. In our ongoing example (with $c = (12)(23)$ and $h = 3$), this is

$$(12)(23)\cdot(12)(23)\cdot(12)(23) = (12)(23)(12)\cdot(23)(12)(23),$$

and the bi-infinite word is either

$$\cdots|\ (12)(23)\ |\ (12)(23)\ |\ (12)(23)\ |\cdots$$

or

$$\cdots|\ (12)(23)(12)\ |\ (23)(12)(23)\ |\cdots,$$

respectively. These bi-infinite words serve as a model for the bounded derived category of finite-dimensional modules over the path algebra of the Dynkin quiver associated with the Coxeter element $c \in W$. There are two natural operations on letters in this word:

- the *shift functor* Σ sending a letter in a copy of $\mathrm{w}_\circ(\mathrm{c})$ (or of $\psi(\mathrm{w}_\circ(\mathrm{c}))$) to the corresponding letter in the next copy of $\psi(\mathrm{w}_\circ(\mathrm{c}))$ (or of $\mathrm{w}_\circ(\mathrm{c})$, respectively);
- the *Auslander–Reiten translate* τ sending a letter in a copy of c to the corresponding letter in the previous copy of c.

The objects in the positive generalised cluster category are now orbits in this bi-infinite word under the action of $\Sigma^m \circ \tau$. By construction, a fundamental domain is given by the letters inside the word $\mathbf{inv}_c^{m-1}\mathbf{inv}_c^L$. Denote by $\mathsf{K}_\circ = \mathsf{K}_\circ^{(m)}$ this action on letters in $\mathbf{inv}_c^{m-1}\mathbf{inv}_c^L$ and extend it letter-wise to subwords. The following proposition can be found in [STW25, Ch. 7].

Proposition 2.14. *For $m \geq 1$, $\mathsf{K}_\circ^{(m)}$ acts both on the positive elements in $NC_\nabla^{(m)}(W, c)$ and on the positive elements in $NC_\Delta^{(m-1)}(W, c)$. Restricting the above identification between $NC^{(m)}(W, c)$ and $NC_\nabla^{(m)}(W, c)$ to positive elements, we obtain $\mathsf{K}_+^{(m)} = \mathsf{K}_\circ^{(m)}$. Restricting further to $NC_\Delta^{(m-1)}(W, c) \subset NC_\nabla^{(m)}(W, c)$, we obtain also $\widetilde{\mathsf{K}}_+^{(m-1)} = \mathsf{K}_\circ^{(m)}$.*

Example 2.15. In Example 2.13, the letters for $NC_\nabla^{(2)}(A_2, c)$ are

$$\mathbf{inv}_c\mathbf{inv}_c^L = (12)^{(1)}(13)^{(1)}(23)^{(1)}\cdot(12)^{(2)}(13)(23),$$

where the final two letters (13)(23) are identified with the initial two letters (12)(13) in this order, and where we indicated by superscripts for each reflection in which copy of $\mathbf{inv}_c$ it occurs. The action of $\mathsf{K}_\circ$ on these letters is

$$\curvearrowright (12)^{(1)} \longmapsto (12)^{(2)} \longmapsto (23)^{(1)} \longmapsto (13)^{(1)} \curvearrowright$$

This yields that the order of $\mathsf{K}_\circ$ is four in this case, and extending this action to subwords yields the above orbit structures.

Following Proposition 2.14, we may consider both versions of the positive Kreweras maps $\mathsf{K}_+^{(m)}$ and $\widetilde{\mathsf{K}}_+^{(m)}$ as the action of $\mathsf{K}_\circ$ on the positive elements in $NC_\nabla^{(m)}(W,c)$, respectively in $NC_\Delta^{(m)}(W,c)$. This has the additional advantage that this definition also allows to consider the action of $\mathsf{K}_+^{(1)}$ on positive elements in $NC_\nabla^{(1)}(W,c)$. This situation was not covered using the above version from Definition 2.12. In the above example, this case corresponds to the single letter word $\mathbf{inv}_c^L = (12)(13)(23)$ with (13) identified with (12) and (23) identified with (13). In particular, the order of K_+ in this case is one and we have the two single element orbits ε and (12). This yields the following definition for $m = 1$.

Definition 2.16. The *positive Kreweras map* $\mathsf{K}_+^{(1)} : NC_+^{(1)}(W,c) \xrightarrow{\sim} NC_+^{(1)}(W,c)$ is given by sending (cw^{-1}, w) with $w = w^L = t_1 \cdots t_k$ to the element obtained by sending every reflection to its first conjugate inside $\mathbf{inv}_c^L$ — that is, t_i is sent to $c^j t_i c^{-j} \in \mathbf{inv}_c^L$ with $j \in \{1, 2, \dots\}$ minimal with this property — and then rearranging the reflections according to the reflection ordering.

In our example, we see that there is only one reflection in $\mathbf{inv}_c^L = (12)$. Conjugation by the Coxeter element yields $(12) \mapsto (23) \mapsto (13) \mapsto (12) \in \mathbf{inv}_c^L$. In this example, $j = 3$ in the definition, and the positive Kreweras map is given by extending this operation to words.

Theorem 2.17 ([STW25, Thm. 7.3.4])**.** *Let $m \geq 1$. The order of $\mathsf{K}_\circ^{(m)}$ on positive elements in $NC_\nabla^{(m)}(W,c)$ and in $NC_\Delta^{(m-1)}(W,c)$ are both given by*

$$\operatorname{ord}(\mathsf{K}_\circ^{(m)}) = \begin{cases} mh/2 - 1, & \text{if } \psi(s) = s \text{ for all } s \in \mathcal{S}, \\ mh - 2, & \text{otherwise,} \end{cases}$$

with the only exception that the order in type $I_2(h)$ for $\mathsf{K}_\circ^{(2)}$ on positive elements in $NC_\Delta^{(1)}(W,c)$ equals $mh-2$ even for odd h. In particular, in all cases we have $\left(\mathsf{K}_\circ^{(m)}\right)^{mh-2} = \mathbf{1}$.

Indeed, the case $m = 1$ for the action of $\mathsf{K}_\circ$ on positive elements in $NC_\nabla^{(1)}(W,c)$ was not treated in [STW25, Thm. 7.3.4] but the given proof covers that case as well. We also make this order concrete in the classical types.

Corollary 2.18. *The order of* $\mathsf{K}_\circ^{(m)}$ *on positive elements in* $NC_\nabla^{(m)}(W,c)$ *is given by*

$$\operatorname{ord}(\mathsf{K}_\circ^{(m)}) = \begin{cases} mn-2, & \text{in type } A_{n-1}, \\ mn-1, & \text{in type } B_n, \\ m(n-1)-1, & \text{in type } D_n \text{ with } n \text{ even}, \\ 2m(n-1)-2, & \text{in type } D_n \text{ with } n \text{ odd}. \end{cases}$$

2.5. **Cyclic sieving.** We conclude this section with stating the cyclic sieving phenomena concerning the positive Kreweras maps. The notion of cyclic sieving was introduced by Reiner, Stanton and White [RSW04]. Given a finite set S, an action on S of a cyclic group C, and a polynomial $P(q)$ in q with non-negative integer coefficients, we say that the triple $\big(S, P(q), C\big)$ *exhibits the cyclic sieving phenomenon*, if for all ρ dividing $\#C$ and all $g \in C$ of order ρ, we have[3]

$$\#\{s \in S \mid g(s) = s\} = P(e^{2\pi i/\rho}). \tag{2.6}$$

Recall that the usual q-extension of the positive Fuß–Catalan numbers is given by

$$\operatorname{Cat}_+^{(m)}(W;q) = \prod_{i=1}^{n} \frac{[mh+d_i-2]_q}{[d_i]_q},$$

where we use the standard q-notation $[\alpha]_q := (1-q^\alpha)/(1-q) = 1+q+\cdots+q^{\alpha-1}$.

Theorem 2.19. *For a given positive integer* m*, let* W *be a finite irreducible Coxeter group* W*, let* C *be the cyclic group generated by the Kreweras map* $\mathsf{K}_+^{(m+1)}$*, viewed as a group of order* $(m+1)h-2$*, and let* $\widetilde{C}$ *be the cyclic group generated by the Kreweras map* $\widetilde{\mathsf{K}}_+^{(m)}$*, viewed again as a group of order* $(m+1)h-2$*. Then the triples*

$$\Big(NC_+^{(m+1)}(W,c),\ \operatorname{Cat}_+^{(m+1)}(W;q),\ C\Big) \text{ and } \Big(NC_+^{(m)}(W,c),\ \operatorname{Cat}_+^{(m)}(W;q),\ \widetilde{C}\Big)$$

both exhibit the cyclic sieving phenomenon.

We will prove this theorem for the classical types by a case-by-case analysis in Sections 3–5. Furthermore, in Section 6 we develop a tool to count (positive) m-divisible non-crossing partitions that are invariant under powers of the positive Kreweras maps for a given type in a finite computation. We then apply this approach to each exceptional type individually to derive the proposed cyclic sieving phenomenon; see Theorem 6.15 and Appendix E.

3. Positive non-crossing set partitions in type A

In this section, we study positive m-divisible non-crossing partitions and the positive Kreweras maps in the situation of the symmetric group $A_{n-1} = \mathfrak{S}_n$. This situation corresponds to the Coxeter system

$$(W,\mathcal{S}) = \big(A_{n-1}, \{s_i = (i,i+1) \mid 1 \le i < n\}\big) \tag{3.1}$$

[3]This is not the original definition [RSW04, Def.-Prop.(i)], but is easily seen to be equivalent to it.

together with the Coxeter element $c = s_1 \cdots s_{n-1} \in A_{n-1}$ being the long cycle $(1, 2, \ldots, n) \in \mathfrak{S}_n$.

We start with Subsection 3.1, in which we first make the positivity condition for m-divisible non-crossing partitions under the choice (3.1) of Coxeter system explicit; see Proposition 3.1. We then use this to describe *positive* m-divisible non-crossing partitions within the combinatorial model of m-divisible non-crossing partitions due to Edelman [Ede80a], via the translation due to Armstrong [Arm06, Thm. 4.3.8]; see Theorem 3.4.

In Subsection 3.2, we show that, under Armstrong's translation, for $m \geq 2$ the positive Kreweras map $\mathsf{K}_+^{(m)}$ acts as a pseudo-rotation; see Theorem 3.6. This pseudo-rotation is denoted by κ_A.

In Subsection 3.3, we introduce a natural extension of κ_A which also makes sense for $m = 1$; see Definition 3.8. We argue in Theorem 3.9 that, under Armstrong's translation, for $m = 1$ this extension corresponds exactly to the map $\mathsf{K}_+^{(1)}$ from Definition 2.16. The order of the (extended) map κ_A is of course given by Theorem 2.17 for type A; see Theorem 3.10. For the sake of completeness, we also provide an independent, *combinatorial*, proof for that order; see Appendix A.1.

Subsection 3.4 is devoted to the enumeration of positive m-divisible non-crossing partitions. The most refined and most general result is Theorem 3.11 which provides a formula for the number of multichains in the poset of these non-crossing partitions in which the ranks of the elements in the multichain as well as the block structure of the bottom element are prescribed. All other results in that subsection, Corollaries 3.12–3.15, are obtained as consequences.

Subsection 3.5 presents the characterisation of type A positive m-divisible non-crossing partitions that are invariant under powers of the pseudo-rotation κ_A; see Theorem 3.17.

In Subsection 3.6, we then exploit this characterisation to provide a formula for the number of type A positive m-divisible non-crossing partitions that are invariant under powers of the pseudo-rotation κ_A and have a given block structure; see Theorem 3.18. Once again, several less refined enumeration results may be obtained by summation; see Corollaries 3.19 and 3.20.

In the final subsection, Subsection 3.7, we state, and prove, several cyclic sieving phenomena refining Theorem 2.19 in type A; see Theorem 3.23.

3.1. **Combinatorial realisation of the positive non-crossing partitions.** With the choice (3.1) of Coxeter system, we have the following simple combinatorial description of positive m-divisible non-crossing partitions.

Proposition 3.1. *Let m and n be positive integers with $m \geq 2$. The tuple*

$$(w_0, w_1, \ldots, w_m) \in NC^{(m)}(A_{n-1}, c)$$

is positive if and only if $w_m(n) = n$, or, equivalently, if $(w_0 w_1 \cdots w_{m-1})(n) = 1$.

Proof. According to Definition 2.3, the tuple $(w_0, w_1, \ldots, w_m)$ is positive if and only if $cw_m^{-1} = w_0 w_1 \cdots w_{m-1}$ has full support in our generators $s_1, \ldots, s_{n-1}$ given in (3.1). The cycles in the disjoint cycle decomposition of cw_m^{-1} define a non-crossing partition. Thus,

cw_m^{-1} will not have full support if and only if $(cw_m^{-1})(n) \neq 1$. By acting on the left on both sides of this relation by the inverse of the Coxeter element $c = (1, 2, \dots, n)$, we see that this is equivalent to $w_m^{-1}(n) \neq n$. We are interested in the contrapositive: cw_m^{-1} has full support if and only if $w_m(n) = n$. □

Next we recall Armstrong's bijection between $NC^{(m)}(A_{n-1}, c)$ and Edelman's m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ (with the classical definition of "non-crossing" as recalled in Footnote 1), which we denote here by $\widetilde{NC}^{(m)}(A_{n-1})$. We write $\widetilde{NC}(N)$ for the set of *all* non-crossing partitions of $\{1, 2, \dots, N\}$ and observe that

$$\widetilde{NC}^{(m)}(A_{n-1}) = \{\pi \in \widetilde{NC}(mn) \mid \text{all block sizes of } \pi \text{ are divisible by } m\}.$$

Given an element $(w_0, w_1, \dots, w_m) \in NC^{(m)}(A_{n-1}, c)$, the bijection, $\bigtriangledown_{A_{n-1}}$ say, from [Arm06, Thm. 4.3.8] works as follows: for $i = 1, 2, \dots, m$, let $\tau_{m,i}$ be the transformation which maps a permutation $w \in A_{n-1}$ to a permutation $\tau_{m,i}(w) \in \mathfrak{S}_{mn}$ by letting

$$(\tau_{m,i}(w))(mk + i - m) = mw(k) + i - m, \quad k = 1, 2, \dots, n,$$

and $(\tau_{m,i}(w))(l) = l$ for all $l \not\equiv i \pmod m$. With the choice of Coxeter element $c = (1, 2, \dots, n)$, the announced bijection maps $(w_0, w_1, \dots, w_m) \in NC^{(m)}(A_{n-1}, c)$ to

$$\bigtriangledown_{A_{n-1}} (w_0, w_1, \dots, w_m) = (1, 2, \dots, mn)\,(\tau_{m,1}(w_1))^{-1}\,(\tau_{m,2}(w_2))^{-1} \cdots (\tau_{m,m}(w_m))^{-1}, \quad (3.2)$$

where the cycles in the disjoint cycle decomposition correspond to the blocks in the non-crossing partition in $\widetilde{NC}^{(m)}(A_{n-1})$. We refer the reader to [Arm06, Sec. 4.3.2] for the details. For example, let $n = 7$, $m = 3$, $w_0 = (4, 5, 6)$, $w_1 = (3, 6)$, $w_2 = (1, 7)$, and $w_3 = (1, 2, 6)$. Then (w_0, w_1, w_2, w_3) is mapped to

$$\begin{aligned}\bigtriangledown_{A_6}(w_0, w_1, w_2, w_3) &= (1, 2, \dots, 21)\,(7, 16)\,(2, 20)\,(18, 6, 3)\\ &= (1, 2, 21)\,(3, 19, 20)\,(4, 5, 6)\,(7, 17, 18)\,(8, 9, \dots, 16).\end{aligned}$$

Armstrong [Arm06, Thm. 4.3.13] has shown that the cycle structure of the first component of an m-divisible non-crossing partition determines the block structure of its image under his bijection.

Proposition 3.2. *Let $(w_0, w_1, \dots, w_m) \in NC^{(m)}(A_{n-1}, c)$. The non-crossing partition $\bigtriangledown_{A_{n-1}}(w_0, w_1, \dots, w_m) \in \widetilde{NC}^{(m)}(A_{n-1})$ has b_i blocks of size mi if and only if w_0 has b_i cycles of length i in its disjoint cycle decomposition.*

In the earlier example, we have $w_0 = (4, 5, 6) = (1)(2)(3)(7)(4, 5, 6)$ which has one cycle of length 3 and four cycles of length 1. Indeed, the image of (w_0, w_1, w_2, w_3) in the example has one block of size $3 \cdot 3 = 9$ and four blocks of size $3 \cdot 1 = 3$.

Remark 3.3. A simple consequence of Proposition 3.2 is that the image of the set of all non-crossing partitions in $NC^{(m)}(A_{n-1}, c)$ of the form $(\varepsilon, w_1, \dots, w_m)$ under the map $\bigtriangledown_{A_{n-1}}$ is the set of all non-crossing partitions in $\widetilde{NC}^{(m)}(A_{n-1})$ in which *all* blocks have size m.

Theorem 3.4. *Let m and n be positive integers with $m \geq 2$. The image under $\nabla_{A_{n-1}}$ of the positive m-divisible non-crossing partitions in $NC^{(m)}(A_{n-1}, c)$ are those m-divisible non-crossing partitions in $\widetilde{NC}^{(m)}(A_{n-1})$ where 1 and mn are in the same block.*

Proof. First, let $(w_0, w_1, \ldots, w_m) \in NC^{(m)}(A_{n-1}, c)$. By definition, we know that $w_m(n) = n$, and hence $\tau_{m,m}(w_m))^{-1}(mn) = mn$. Consequently,

$$\begin{aligned}\Big(\nabla_{A_{n-1}}(w_0, w_1, \ldots, w_m)\Big)(mn) \\ &= \big((1, 2, \ldots, mn)\,(\tau_{m,1}(w_1))^{-1}\,(\tau_{m,2}(w_2))^{-1}\,\cdots\,(\tau_{m,m}(w_m))^{-1}\big)(mn) \\ &= \big((1, 2, \ldots, mn)\big)(mn) = 1.\end{aligned}$$

Translated to (combinatorial) non-crossing partitions, this means that mn and 1 belong to the same block, which is what we claimed.

Conversely, let $\pi \in \widetilde{NC}^{(m)}(A_{n-1})$ be an m-divisible non-crossing partition in which 1 and mn are in the same block. In other words, if π is interpreted as a permutation, $\pi(mn) = 1$. Let $(w_0, w_1, \ldots, w_m)$ be the element of $NC^{(m)}(A_{n-1}, c)$ such that

$$\nabla_{A_{n-1}}(w_0, w_1, \ldots, w_m) = \pi.$$

We must have

$$\Big(\nabla_{A_{n-1}}(w_0, w_1, \ldots, w_m)\Big)(mn) = 1.$$

The definition (3.2) of $\nabla_{A_{n-1}}$ and the fact that $(\tau_{m,i}(w_i))^{-1}$ leaves multiples of m fixed as long as $1 \leq i \leq m-1$ together imply that $(\tau_{m,m}(w_m))^{-1}(mn) = mn$, or, equivalently, that $w_m(n) = n$. By Proposition 3.1, this means that $(w_0, w_1, \ldots, w_m)$ is positive. □

We let $\widetilde{NC}_+^{(m)}(A_{n-1})$ denote the "positive" elements of $\widetilde{NC}^{(m)}(A_{n-1})$, that is, those for which 1 and mn are contained in the same block. In accordance with our previous conventions, we write $\widetilde{NC}_+(N)$ for the set of *all* non-crossing partitions of $\{1, 2, \ldots, N\}$ for which 1 and N are in the same block. We observe that

$$\widetilde{NC}_+^{(m)}(A_{n-1}) = \{\pi \in \widetilde{NC}_+(mn) \mid \text{all block sizes of } \pi \text{ are divisible by } m\}.$$

We furthermore call the block containing both 1 and mn the *special block* of the partition under consideration.

3.2. **Combinatorial realisation of the positive Kreweras maps.** Next we translate the positive Kreweras map K_+ into a "rotation action" on the positive elements of $\widetilde{NC}^{(m)}(A_{n-1})$. In order to do so, we need to explicitly describe the decomposition (2.5) in type A.

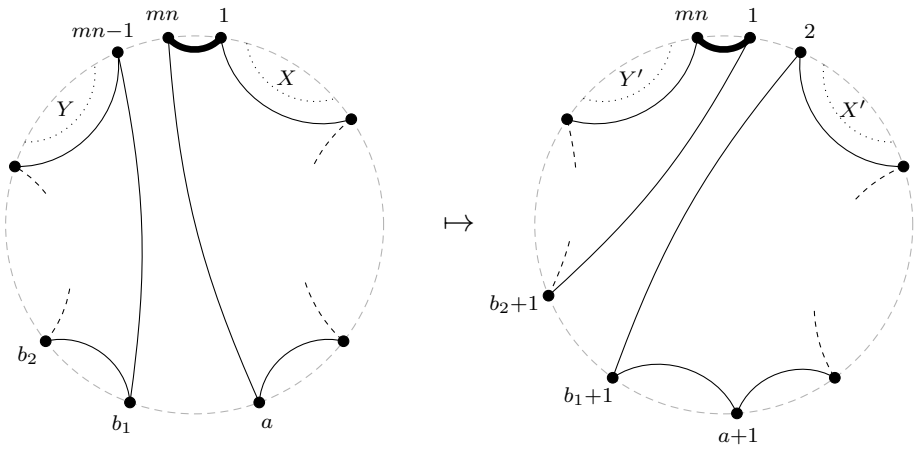

FIGURE 4. The action of the pseudo-rotation κ_A

Lemma 3.5. *Let w be an element of $NC(A_{n-1}, c)$, and let $w^{-1}(n) = a$. Then*

$$w = w^L w^R, \qquad \text{where } w^R = (a, n).$$

In particular, we have $w^L(n) = n$.

Proof. The last $n-1$ reflections in $\mathcal{R}$ given in (2.2) are (i, n), $i = 1, 2, \ldots, n-1$, in this order. Thus, the factorisation $w = w' \circ (a, n)$ for $a = w^{-1}(n)$ is reduced. This implies that $w^L = w'$ and $w^R = (a, n)$, as desired.

The final claim of the lemma is obvious. □

Next, we translate the positive Kreweras map K_+ from Definition 2.12 for type A_{n-1} into combinatorial language.

Theorem 3.6. *Let m and n be positive integers with $m \geq 2$. Under the bijection $\nabla_{A_{n-1}}$, the map $\mathsf{K}_+^{(m)}$ translates into the following map κ_A on $\widetilde{NC}_+^{(m)}(A_{n-1})$: if $mn-1$ is in the same block as 1 and mn of an element π of $\widetilde{NC}_+^{(m)}(A_{n-1})$, then κ_A rotates all blocks of π by one unit in clockwise direction. On the other hand, if $mn-1$ is in a different block, say $\{b_1, b_2, \ldots, mn-1\}$ and $\{1, \ldots, a, mn\}$ are two different blocks, where*

$$1 < \cdots < a < b_1 < b_2 \cdots < mn - 1 < mn,$$

then the image of π under κ_A is the partition which contains the blocks $\{1, b_2+1, b_3+, \ldots, mn\}$ and $\{2, \ldots, a+1, b_1+1\}$ and the blocks which arise from the remaining blocks of π by rotating them by one unit in clockwise direction.

Remark 3.7. Figure 4 provides a schematic illustration of the construction in the statement of the above theorem. Figure 5 then provides a concrete example for the parameters $m = 3$, $n = 8$, $a = 14$, $b_1 = 15$, and $b_2 = 19$. Since this map κ_A essentially acts as rotation, possibly except for the "neighbourhood" of 1, we call it frequently a *pseudo-rotation*.

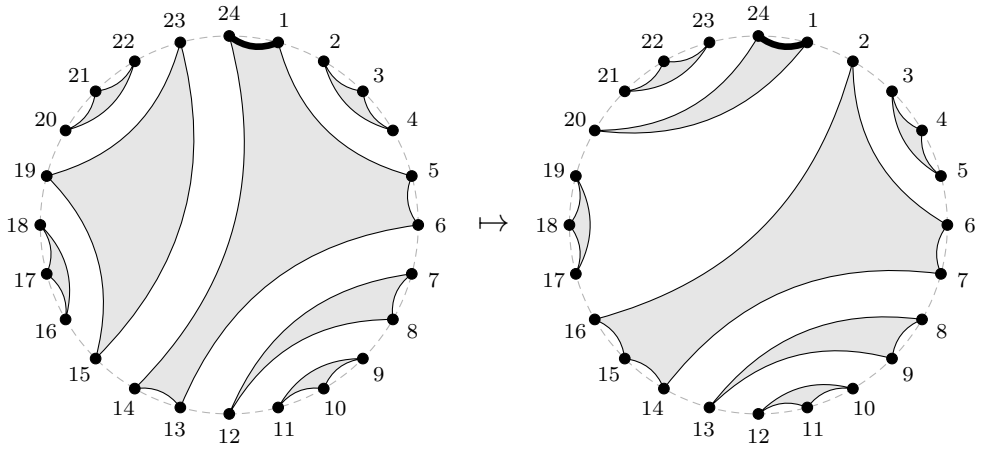

FIGURE 5. The action of the pseudo-rotation κ_A in a concrete example

Proof of Theorem 3.6. The only claim which needs proof is the one on the image of the blocks $\{b_1, b_2, \ldots, mn-1\}$ and $\{1, \ldots, a, mn\}$ in the case where $mn-1$ is not in the same block as 1 and mn. By the structure of m-divisible non-crossing partitions, we have

$$b_1 \equiv b_2 - 1 \equiv a - (m-1) \equiv 0 \pmod{m}.$$

Let us write $b_1 = mB_1$, $b_2 = mB_2 + 1$, and $a = mA - 1$, for suitable integers B_1, B_2, A. If $(w_0, w_1, \ldots, w_m)$ denotes the element of $NC_+^{(m)}(A_{n-1}, c)$ which corresponds to π under the bijection $\nabla_{A_{n-1}}$, then we have

$$\begin{aligned} w_{m-1}(n) &= A, \\ w_{m-1}(B_1) &= n, \\ w_m(B_2) &= B_1, \\ w_m(n) &= n. \end{aligned}$$

Consequently, we have $w_{m-1}^R = (B_1, n)$ and

$$\begin{aligned} cw_{m-1}^R w_m c^{-1}(1) &= B_1 + 1, \\ cw_{m-1}^R w_m c^{-1}(B_2 + 1) &= 1, \\ w_{m-1}^L(B_1) &= A, \\ w_{m-1}^L(n) &= n. \end{aligned}$$

In view of the definition, it follows that $\nabla_{A_{n-1}}\left(\mathsf{K}_+(w_0, w_1, \ldots, w_m)\right)$ maps $a+1 = mA$ to $mB_1 + 1 = b_1 + 1$, $b_1 + 1 = mB_1 + 1$ to $m \cdot 1 - (m-1) + 1 = 2$, $1 = m \cdot 1 - (m-1)$ to $m(B_2+1) - (m-1) + 1 = b_2 + 1$, as we claimed. □

3.3. **The combinatorial positive Kreweras maps.** The definition of κ_A from Theorem 3.6 to make sense on an *arbitrary* non-crossing partition needed that blocks should

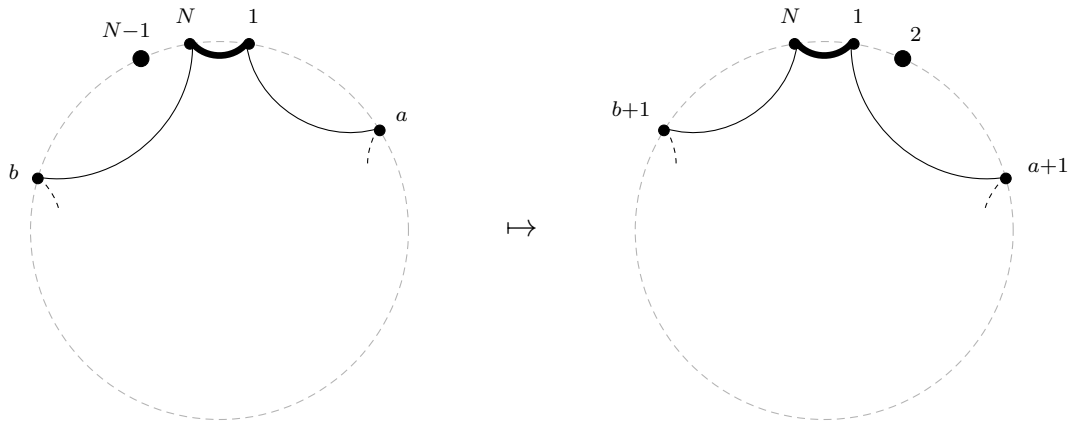

FIGURE 6. Definition of the pseudo-rotation under presence of singleton blocks

be of size at least 2. It is a simple matter to extend this so that it makes sense for non-crossing partitions without any restrictions. See Figure 6 for an illustration of the following definition.

Definition 3.8. We define the action of κ_A on a positive non-crossing partition π of $\{1, 2, \dots, N\}$ to be given by Theorem 3.6 with mn replaced by N (cf. Figures 4 and 5), except when the block containing $N-1$ is a singleton block (in which case Theorem 3.6 would not make sense). If $N-1$ forms its own singleton block, then the image of κ_A is the non-crossing partition which arises from π by replacing the singleton block $\{N-1\}$ by the singleton block $\{2\}$, and by adding 1 to each element of the remaining blocks of π, except to N and 1 which remain untouched; see Figure 6 for an illustration.

As it turns out, this definition is also in agreement with the image under $\nabla_{A_{n-1}}$ of the map $\mathsf{K}_+^{(1)}$ in Definition 2.16 for the case where $m = 1$.

Theorem 3.9. *Let n be a positive integer. Under the bijection $\nabla_{A_{n-1}}$, the map $\mathsf{K}_+^{(1)}$ from Definition 2.16 translates to the combinatorial map κ_A described in Definition 3.8 with $N = n$.*

Sketch of proof. Recall that the bijection $\nabla_{A_{n-1}}$ in the case $m = 1$ reduces to the simple map $w \mapsto cw^{-1}$, where due to our choice (3.1) of Coxeter system, $c = (1, 2, \dots, n)$. This map is usually called the "Kreweras complement". It has an elegant combinatorial realisation (cf. [Arm06, Fig. 4.3 in Sec. 4.2]) on which our arguments will rely.

Let $w = w^L = t_1 \cdots t_k$ be an element of $NC_+^{(1)}(A_{n-1}, c)$, and let π be the non-crossing partition in $\widetilde{NC}_+(n)$ corresponding to the Kreweras complement $(1, 2, \dots, n)w^{-1}$. According to Definition 2.16, the Kreweras map $\mathsf{K}_+^{(1)}$ acts by sending the transposition t_i to $c^j t_i c^{-j}$ with j minimal such that $c^j t_i c^{-j} \in \mathbf{inv}_c^L$. Now, since $c = (1, 2, \dots, n)$, this action will always be $(a, b) \mapsto (a+1, b+1)$, except if $b = n-1$. In the latter case, the action will be

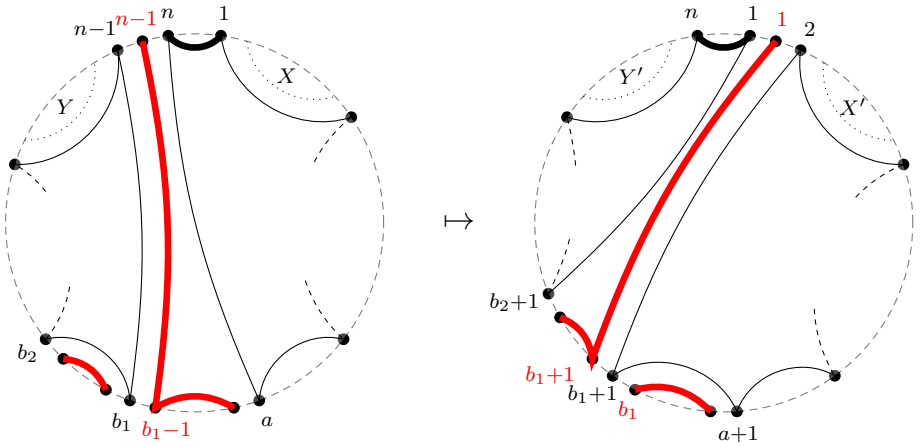

FIGURE 7. The action of the pseudo-rotation κ_A for $m = 1$

$(a, n-1) \mapsto (1, a+2)$ (with $j = 2$) if $a < n-2$, and it will be $(n-2, n-1) \mapsto (1, 2)$ (with $j = 3$). Hence, given the combinatorial realisation of the Kreweras complement, the map $\mathsf{K}_+^{(1)}$ becomes rotation by one unit in clockwise direction under the bijection $\nabla_{A_{n-1}}$ as long as no t_i is of the form $(a, n-1)$. In particular, a singleton block $\{k\}$ in π — which under the Kreweras complement corresponds to an adjacent transposition $(k-1, k)$ among the t_i's — is mapped to $\{k+1\}$ if $k \leq n-1$.

On the other hand, let us assume that there is a t_i with $t_i = (b_1-1, n-1)$ for some $b_1 < n$. This situation is schematically illustrated in Figure 7. The red arcs there correspond to transpositions t_i. In particular, among the t_i's there may occur (a_0, b_1-1), for some a_0 with $a_0 \leq b_1 - 1$, and (b_1, b_2-1), for some b_2 with $b_1 \leq b_2 - 1$ (in both cases, equality corresponds to the degenerate case where the "doubled" number does not appear in the t_i's). If $b_1 < n-1$, then, as discussed just above, $\mathsf{K}_+^{(1)}$ maps $(b_1-1, n-1)$ to $(1, b_1+1)$; see Figure 7. In the Kreweras complement π, this means that, before the action of $\mathsf{K}_+^{(1)}$, there are two blocks $\{b_1, b_2, \ldots, n-1\}$ and $\{1, \ldots, a, n\}$ with

$$1 < \cdots < a < b_1 < b_2 \cdots < n-1 < n;$$

after the action of $\mathsf{K}_+^{(1)}$, there are the two blocks $(1, b_2+1, \ldots, n)$ and $(2, \ldots, a+1, b_1+1)$. It should be noted that this transformation of the blocks is caused by the fact that the overlapping (red) arcs (a_0, b_1-1) and $(b_1-1, n-1)$ become the non-overlapping arcs (a_0+1, b_1) and $(b_1+1, 1)$ in the image after application of $\mathsf{K}_+^{(1)}$, while the non-overlapping arcs $(b_1-1, n-1)$ and (b_1, b_2-1) become the overlapping arcs $(b_1+1, 1)$ and (b_1+1, b_2) in the image. All other blocks in the combinatorial realisation π of $w = w^L = t_1 \cdots t_k$ are rotated by one unit in clockwise direction.

Finally, if one of the t_i's should equal $(n-2, n-1)$, then this corresponds to the singleton block $\{n-1\}$ in π. As discussed above, the action of $\mathsf{K}_+^{(1)}$ maps $t_i = (n-2, n-1)$ to $(1, 2)$, corresponding to the singleton block $\{2\}$ in the Kreweras complement.

All this is exactly in agreement with Definition 3.8. □

We conclude with the following theorem describing the order of the combinatorial map κ_A. In view of our previous discussion, it follows from Corollary 2.18 for the type A_{N-1} with $m = 1$. However, since we believe it is interesting for its own sake, we provide a direct, combinatorial, proof in Appendix A.1.

Theorem 3.10. *Let N be a positive integer with $N \geq 2$. The order of the map κ_A acting on the set of positive non-crossing partitions π of $\{1, 2, \dots, N\}$ is $N - 2$.*

3.4. **Enumeration of positive non-crossing partitions.** Here we present our enumeration results for positive non-crossing partitions in $\widetilde{NC}_+^{(m)}(A_{n-1})$. Our proofs are based on a generating function approach and are largely deferred to Appendix B.

The most refined and most general result is Theorem 3.11 which provides a formula for the number of multichains in the poset of these non-crossing partitions in which the ranks of the elements of the multichain as well as the block structure of the bottom element are prescribed. By specialisation and summation, we obtain, among others, results on the number of such partitions with prescribed block structure and with prescribed rank, and the total number of multichains of given length in $\widetilde{NC}_+^{(m)}(A_{n-1})$.

Theorem 3.11. *Let m, n, l be positive integers, and let $s_1, s_2, \dots, s_l, b_1, b_2, \dots, b_n$ be non-negative integers with $s_1 + s_2 + \cdots + s_l = n - 1$. The number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{l-1}$ in the poset of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ with the property that $\mathrm{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \dots, l - 1$, and that the number of blocks of size mi of π_1 is b_i, $i = 1, 2, \dots, n$, is given by*

$$\frac{mn - b_1 - b_2 - \cdots - b_n}{(mn-1)(b_1 + b_2 + \cdots + b_n)} \binom{b_1 + b_2 + \cdots + b_n}{b_1, b_2, \dots, b_n} \binom{mn-1}{s_2} \cdots \binom{mn-1}{s_l} \tag{3.3}$$

if $b_1 + 2b_2 + \cdots + nb_n = n$ and $s_1 + b_1 + b_2 + \cdots + b_n = n$, and 0 otherwise.

For the proof of the theorem, see the end of Appendix B.2.

The $l = 2$ special case of the previous theorem yields an enumeration formula for positive m-divisible non-crossing partitions with a given block structure.

Corollary 3.12. *Let m and n be positive integers, For non-negative integers $b_1, b_2, \dots, b_n$, the number of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ which have exactly b_i blocks of size mi, $i = 1, 2, \dots, n$, is given by*

$$\frac{1}{mn-1} \binom{b_1 + b_2 + \cdots + b_n}{b_1, b_2, \dots, b_n} \binom{mn-1}{b_1 + b_2 + \cdots + b_n} \tag{3.4}$$

if $b_1 + 2b_2 + \cdots + nb_n = n$, and 0 otherwise.

On the other hand, if we sum Expression (3.3) over all $s_2, s_3, \dots, s_l$ with $s_2 + s_3 + \cdots + s_l = n - s_1 - 1$ using iterated Chu–Vandermonde summation (see e.g. [GKP89, Sec. 5.1, Eq. (5.27)]), then we obtain the following generalisation.

Corollary 3.13. *Let m, n, l be positive integers, For non-negative integers $b_1, b_2, \dots, b_n$, the number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{l-1}$ in the poset of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ for which the number of blocks of size mi of π_1 is b_i,*

$i = 1, 2, \dots, n$, is given by

$$\frac{mn - b_1 - b_2 - \cdots - b_n}{(mn-1)(b_1 + b_2 + \cdots + b_n)} \binom{b_1 + b_2 + \cdots + b_n}{b_1, b_2, \dots, b_n} \binom{(l-1)(mn-1)}{b_1 + b_2 + \cdots + b_n - 1} \tag{3.5}$$

if $b_1 + 2b_2 + \cdots + nb_n = n$, and 0 otherwise.

Summing the result in Theorem 3.11 over all possible block structures, we arrive at the following result.

Corollary 3.14. *Let m, n, l be positive integers, and let $s_1, s_2, \dots, s_l$ be non-negative integers with $s_1 + s_2 + \cdots + s_l = n - 1$. The number of multi-chains $\pi_1 \le \pi_2 \le \cdots \le \pi_{l-1}$ in the poset of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ with the property that $\mathrm{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \dots, l-1$, is given by*

$$\frac{mn - s_2 - s_3 - \cdots - s_l - 1}{(mn-1)n} \binom{n}{s_1} \binom{mn-1}{s_2} \cdots \binom{mn-1}{s_l}. \tag{3.6}$$

Proof. As indicated above the statement of the corollary, the expression (3.6) can be obtained from (3.3) by summing over all possible b_i's, $i = 1, 2, \dots, n$, with $b_1 + 2b_2 + \cdots + nb_n = n$ and $s_1 + b_1 + b_2 + \cdots + b_n = n$. A simpler way to arrive at the same result is to go to (B.17), set all x_i's equal to 1, to see that the number which we want to compute is given by

$$\begin{aligned}
&\frac{mn - s_2 - s_3 - \cdots - s_l - 1}{(mn-1)(s_2 + s_3 + \cdots + s_l + 1)} \binom{mn-1}{s_2} \binom{mn-1}{s_3} \cdots \binom{mn-1}{s_l} \\
&\qquad\qquad \times \langle z^{mn} \rangle \left(\frac{z^m}{1 - z^{mi}} \right)^{s_2 + s_3 + \cdots + s_l + 1} \\
&= \frac{mn - s_2 - s_3 - \cdots - s_l - 1}{(mn-1)(s_2 + s_3 + \cdots + s_l + 1)} \binom{mn-1}{s_2} \binom{mn-1}{s_3} \cdots \binom{mn-1}{s_l} \\
&\qquad\qquad \times \binom{n-1}{n - s_2 - s_3 - \cdots - s_l - 1},
\end{aligned}$$

which, because of $s_1 + s_2 + \cdots + s_l = n - 1$, is easily seen to be equivalent to (3.6). □

Summing the expression in (3.6) over all possible s_i's with $s_1 + s_2 + \cdots + s_l = n - 1$, we obtain the number of all multichains of a given length, and, thus, the *zeta polynomial* of the poset of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$. As a special case, the *total number* of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ falls out.

Corollary 3.15. *Let m, n, l be positive integers, The number of multichains $\pi_1 \le \pi_2 \le \cdots \le \pi_{l-1}$ in the poset of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ is given by*

$$\frac{1 + (l-1)(m-1)}{n-1} \binom{n - 1 + (l-1)(mn-1)}{n-2}. \tag{3.7}$$

In particular, the total number of elements in this poset equals

$$\frac{1}{n}\binom{(m+1)n-2}{n-1}. \tag{3.8}$$

Proof. We compute

$$\begin{aligned}
&\sum_{s_1+\cdots+s_l=n-1} \frac{mn-s_2-s_3-\cdots-s_l-1}{(mn-1)n}\binom{n}{s_1}\binom{mn-1}{s_2}\cdots\binom{mn-1}{s_l}\\
&\quad=\sum_{s_1=0}^{n-1}\frac{mn-(n-s_1-1)-1}{(mn-1)n}\binom{n}{s_1}\binom{(l-1)(mn-1)}{n-s_1-1}\\
&\quad=\sum_{s_1=0}^{n-1}\left(\frac{1}{n}\binom{n}{s_1}\binom{(l-1)(mn-1)}{n-s_1-1}\right.\\
&\qquad\qquad\left.-\frac{(l-1)(mn-1)}{(mn-1)n}\binom{n}{s_1}\binom{(l-1)(mn-1)-1}{n-s_1-2}\right)\\
&\quad=\frac{1}{n}\binom{n+(l-1)(mn-1)}{n-1}-\frac{l-1}{n}\binom{(n-1)+(l-1)(mn-1)}{n-2},
\end{aligned}$$

where, to obtain the second line and the last line, we used again the Chu–Vandermonde summation formula. Upon little manipulation, the last expression can be converted into (3.7). Formula (3.8) results from (3.7) by setting $l=2$. □

The formula in Corollary 3.15 counts multichains of $l-1$ elements. For $m=1$, this formula thus equals the zeta polynomial of the poset of positive non-crossing partitions of $\{1,2,\dots,n\}$ According to [Ede80b, Lem. 2.2(iii)], evaluating the zeta polynomial at $l=-1$ yields the Möbius function. We record this in the following corollary.

Corollary 3.16. *The Möbius function between bottom and top element of the poset of positive non-crossing partitions of $\{1,2,\dots,n\}$ is given by*

$$\frac{(-1)^{n-2}}{n-1}\binom{2n-4}{n-2}. \tag{3.9}$$

3.5. **Characterisation of pseudo-rotationally invariant elements.** In this subsection, we describe the elements of $\widetilde{NC}_+(N)$ that are invariant under a power of κ_A. To be specific, we call such an element *ρ-pseudo-rotationally invariant* if it is invariant under $\kappa_A^{(N-2)/\rho}$ for some positive integer ρ that divides $N-2$. We achieve the description by providing two constructions of such elements together with Theorem 3.17 which then tells us how to obtain all such invariant elements from these two constructions.

In order to be able to describe the two constructions, we need to introduce a few notions. Given a sequence $a_1,a_2,\dots,a_k$, we shall frequently consider the cyclic rotation of these elements and invariably denote it by R, so that $R(a_i)=a_{i+1}$ and the indices are taken modulo k. (Although R depends on the sequence $a_1,a_2,\dots,a_k$, we do not indicate this in the notation since the sequence will always be clear from the context.) We call a block B of

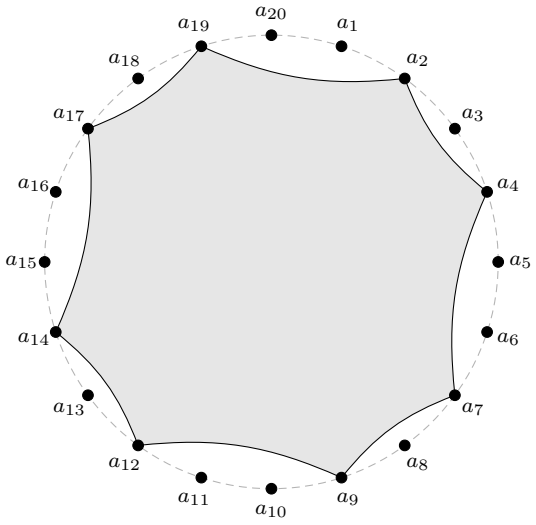

FIGURE 8. A central block

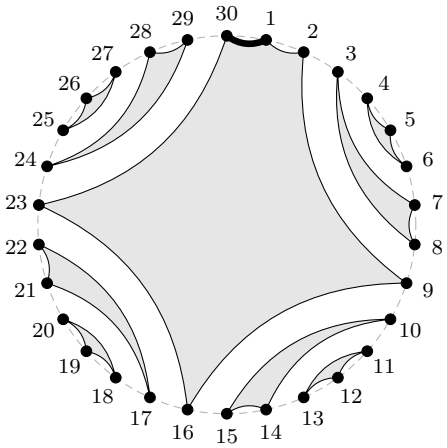

FIGURE 9. A 4-pseudo-rotationally invariant non-crossing partition in $\widetilde{NC}_+(30)$ (and also in $\widetilde{NC}_+^{(3)}(A_9)$)

a non-crossing partition of $\{a_1, a_2, \ldots, a_k\}$ *central* if $B = R^l(B)$ for some l with $1 \leq l < k$. See Figure 8 for an example with $k = 20$ and $l = 5$.

We now present two constructions of (pseudo-rotationally invariant, as it will turn out in Theorem 3.17) positive non-crossing partitions. In the sequel, ρ will always be a positive integer dividing $N - 2$ and different from 1.

CONSTRUCTION 1. We start with a non-crossing partition of $\{2, 3, \ldots, N-1\}$ that is invariant under $R^{(N-2)/\rho}$ (with R acting cyclically on $2 < 3 < \cdots < N-1$) and that contains 2 in the central block. We also allow the degenerate case that the central block

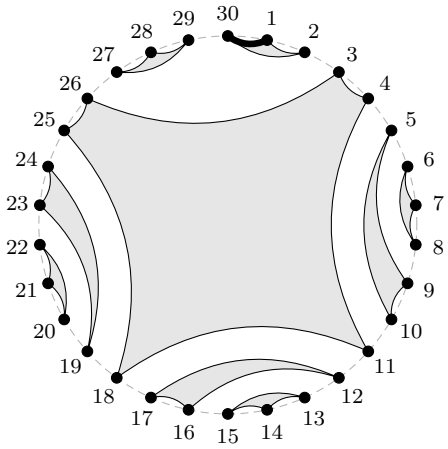

FIGURE 10. Another 4-pseudo-rotationally invariant non-crossing partition in $\widetilde{NC}_+(30)$ (and also in $\widetilde{NC}_+^{(3)}(A_9)$)

is empty. Then we add 1 and N to the central block, thereby creating a positive non-crossing partition. If we are in the degenerate case where we started with an empty central block, the construction has to be understood in the sense that a block $\{1, N\}$ is added to the ρ-pseudo-rotationally invariant non-crossing partition of $\{2, 3, \dots, N-1\}$ in which $\{2, 3, \dots, \frac{N-2}{\rho} + 1\}$ is a union of blocks. Figures 9–11 illustrate this construction. In the first two figures, we chose $\rho = 4$ and $N = 30$. In order to obtain the non-crossing partition in Figure 9, one starts with the non-crossing partition

$$\{\{2, 9, 16, 23\}, \{3, 7, 8\}, \{4, 5, 6\}, \{10, 14, 15\}, \{11, 12, 13\}, \\ \{17, 21, 22\}, \{18, 19, 20\}, \{24, 28, 29\}, \{25, 26, 27\}\}$$

of $\{2, 3, \dots, 29\}$, which is indeed invariant under $R^{(30-2)/4} = R^7$. Then one adds 1 and 30 to the central block $\{2, 9, 16, 23\}$ to obtain the partition shown in Figure 9. If one applies κ_A twice to this partition, then the result is the partition in Figure 10.

An example of the degenerate case where one starts with a rotationally symmetric non-crossing partition of $\{2, 3, \dots, N-1\}$ with empty central block is given in Figure 11. The parameter values in the figure are $N = 32$ and $\rho = 3$. In this example we start with the 3-rotationally invariant non-crossing partition

$$\{\{2, 5, 8\}, \{3, 4\}, \{6, 7\}, \{9, 10, 11\}, \{12, 15, 18\}, \{13, 14\}, \{16, 17\}, \{19, 20, 21\}, \\ \{22, 25, 28\}, \{23, 24\}, \{26, 27\}, \{29, 30, 31\}\}$$

of $\{2, 3, \dots, 31\}$ with empty central block. Adding $\{32, 1\}$ as central block, we obtain the 3-pseudo-rotationally invariant non-crossing partition on the left of Figure 11. The right part of the figure shows the result of one application of the pseudo-rotation κ_A.

CONSTRUCTION 2. This construction requires $\rho = 2$. We start with a non-crossing partition $\bar{\pi}$ of $\{N, 1, 2, \dots, (N-2)/2\}$ in which N and 1 are in the same block. Out of

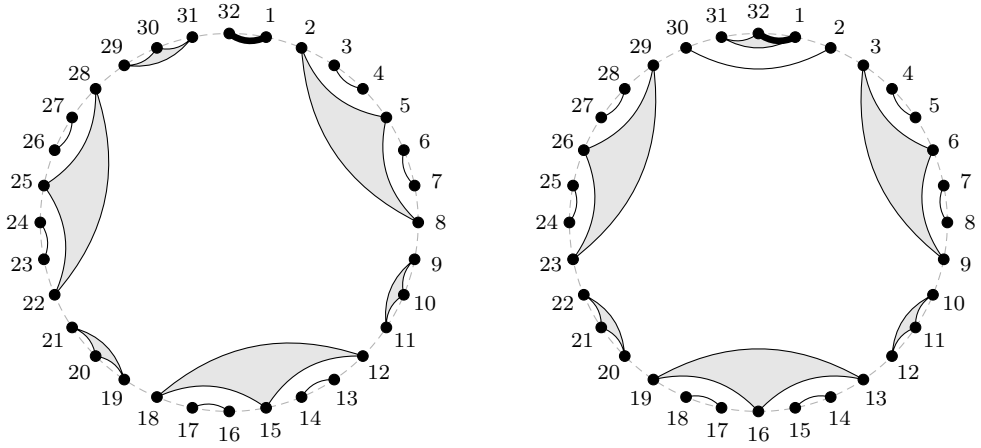

FIGURE 11. Two 3-pseudo-rotationally invariant non-crossing partitions in $\widetilde{NC}_+(32)$ (and also in $\widetilde{NC}_+^{(1)}(A_{31})$)

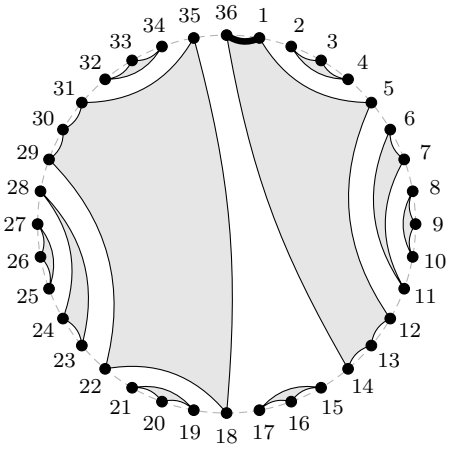

FIGURE 12. A 2-pseudo-rotationally invariant non-crossing partition in $\widetilde{NC}_+(36)$ (and also in $\widetilde{NC}_+^{(3)}(A_{11})$)

these data, we form a positive non-crossing partition π of $\{1, 2, \ldots, N\}$ in the following way: each block of $\bar{\pi}$ is also a block of π. Furthermore, for each block B of $\bar{\pi}$ not containing N and 1, its "twin block" $R^{(N-2)/2}(B)$ is also a block of π. Finally, let B_0 denote the block containing N and 1 of $\bar{\pi}$, say $B_0 = \{N, 1, v_1, v_2, \ldots, v_{a-2}\}$. Then

$$R^{(N-2)/2}\big(\{v_1, v_2, \ldots, v_{a-2}\}\big) \cup \{N/2, N-1\}$$

is also a block of π. Figures 12 and 13 illustrate this construction. In these figures, we chose $N = 36$. In order to obtain the non-crossing partition in Figure 12, one starts with the non-crossing partition

$$\{\{1, 5, 12, 13, 14, 36\}, \{2, 3, 4\}, \{6, 7, 11\}, \{8, 9, 10\}, \{15, 16, 17\}\}$$

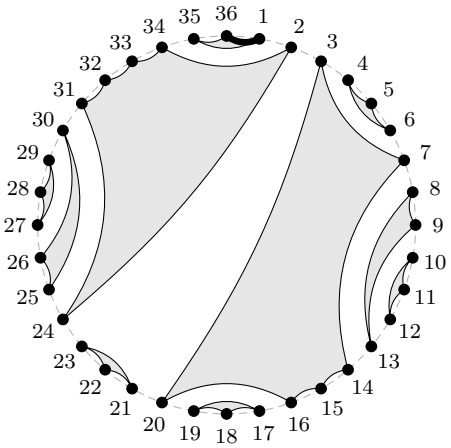

FIGURE 13. Another 2-pseudo-rotationally invariant non-crossing partition in $\widetilde{NC}_+(36)$ (and also in $\widetilde{NC}_+^{(3)}(A_{11})$)

of $\{36, 1, 2, \dots, 17\}$. Then the blocks not containing 1 and 36 are rotated, that is, $R^{(36-2)/2} = R^{17}$ is applied to them, yielding the blocks

$$\{19, 20, 21\},\ \{23, 24, 28\},\ \{25, 26, 27\},\ \{32, 33, 34\}.$$

All of these become blocks of the partition to be constructed; see Figure 12. Finally, we take the block B_0 containing 1 and 36, $B_0 = \{1, 5, 12, 13, 14, 36\}$, remove 1 and 36, apply R^{17} to the remaining set,

$$R^{17}(B_0 \setminus \{1, 36\}) = \{22, 29, 30, 31\},$$

and finally add $36/2 = 18$ and $36 - 1 = 35$ to this to obtain the last block of the partition to be constructed; see Figure 12. If one applies κ_A twice to this partition, then the result is the partition in Figure 13.

Theorem 3.17. *Let N and ρ be positive integers with $N, \rho \geq 2$ and $\rho \mid (N-2)$.*

(1) *If $\rho \geq 3$, all ρ-pseudo-rotationally invariant elements of $\widetilde{NC}_+(N)$ are obtained by starting with a non-crossing partition from Construction 1 and applying the pseudo-rotation κ_A repeatedly to it.*

(2) *All 2-pseudo-rotationally invariant elements of $\widetilde{NC}_+(N)$ are obtained by starting with a non-crossing partition from Construction 1 or Construction 2 and applying the pseudo-rotation κ_A repeatedly to it.*

The proof of this theorem is given in Appendix C.1.

3.6. **Enumeration of pseudo-rotationally invariant elements.** With the characterisation of pseudo-rotationally invariant elements of $\widetilde{NC}_+(N)$ at hand, we can now embark on the enumeration of such elements. Since the map κ_A does in general not preserve the partial order on non-crossing partitions, and the repeated application of κ_A is essential in the aforementioned characterisation (see Theorem 3.17), we are not able to provide results

on chain enumeration (which however still may exist). Rather, the most refined and most general result in this subsection is Theorem 3.18 which provides a formula for the number of these non-crossing partitions in which the block structure is prescribed. In the statement of the theorem, and also later, we use an extended version of the earlier introduced notion of "central block": for a ρ-pseudo-rotationally invariant element π of $\widetilde{NC}_+(N)$ (that is, π is invariant with respect to $\kappa_A^{(N-2)/\rho}$), a block B of π is called *κ_A-central block* (or simply *central block* if κ_A is clear from the context) if $B = \kappa_A^{(N-2)/\rho}(B)$. (How κ_A acts on a block B should be intuitively clear from the definition of κ_A, cf. Figures 4–6.) For example, both non-crossing partitions in Figures 9 and 10 have central blocks: in the partition of Figure 9 the central block is $\{1, 2, 9, 16, 23, 30\}$, while in the partition of Figure 10 the central block is $\{3, 4, 11, 18, 25, 26\}$. On the other hand, neither of the non-crossing partitions in Figures 12 and 13 has a central block. While a singleton block can never be a central block, it may happen that — slightly counter-intuitively — $\{N, 1\}$ is a central block. An example with $N = 32$ and $\rho = 3$ is shown in Figure 11, where indeed $\kappa_A^{(32-2)/3}((1, 32)) = \kappa_A^{10}((1, 32)) = (1, 32)$.

Theorem 3.18. *Let m, n, ρ be positive integers with $\rho \geq 2$ and $\rho \mid (mn-2)$. Furthermore, let $b_1, b_2, \ldots, b_n$ be non-negative integers. The number of positive m-divisible non-crossing partitions of $\{1, 2, \ldots, mn\}$ which are invariant under the ρ-pseudo-rotation $\kappa_A^{(mn-2)/\rho}$, the number of non-central blocks of size mi being ρb_i, $i = 1, 2, \ldots, n$, the central block having size $ma = mn - m\rho \sum_{j=1}^{n} j b_j$, is given by*

$$\binom{b_1 + b_2 + \cdots + b_n}{b_1, b_2, \ldots, b_n} \binom{(mn-2)/\rho}{b_1 + b_2 + \cdots + b_n} \tag{3.10}$$

if $b_1 + 2b_2 + \cdots + nb_n < n/\rho$, or if $\rho = 2$ and $b_1 + 2b_2 + \cdots + nb_n = n/2$, and 0 otherwise.

The proof of this theorem is found in Appendix B.3.

Corollary 3.19. *Let m, n, ρ be positive integers with $\rho \geq 2$ and $\rho \mid (mn-2)$. Furthermore, let b be a non-negative integer. The number of positive m-divisible non-crossing partitions of $\{1, 2, \ldots, mn\}$ which are invariant under the ρ-pseudo-rotation $\kappa_A^{(mn-2)/\rho}$, have ρb non-central blocks and a central block of size ma, is given by*

$$\binom{(n-a-\rho)/\rho}{b-1} \binom{(mn-2)/\rho}{b} \tag{3.11}$$

if $a \equiv n \pmod{\rho}$ and 0 otherwise, while the number of these partitions without restricting the size (or existence) of the central block is

$$\binom{\lfloor n/\rho \rfloor}{b} \binom{(mn-2)/\rho}{b}. \tag{3.12}$$

The proof of this corollary is also found in Appendix B.3.

Corollary 3.20. *Let m, n, a, ρ be positive integers with $\rho \geq 2$ and $\rho \mid (mn-2)$. The number of positive m-divisible non-crossing partitions of $\{1, 2, \ldots, mn\}$ which are invariant under*

the ρ-pseudo-rotation $\kappa_A^{(mn-2)/\rho}$ and have a central block of size ma is given by

$$\binom{((m+1)n-a-\rho-2)/\rho}{(n-a)/\rho} \tag{3.13}$$

if $a \equiv n \pmod{\rho}$ and 0 otherwise, while the number of these partitions without restricting the size (or existence) of the central block is

$$\binom{\lfloor((m+1)n-2)/\rho\rfloor}{\lfloor n/\rho\rfloor}. \tag{3.14}$$

Proof. One sums Expressions (3.11) and (3.12) over all b. The resulting sums can be evaluated by means of the Chu–Vandermonde summation formula. □

3.7. **Cyclic sieving.** With the enumeration results in Subsections 3.4 and 3.6, we are now ready to derive our cyclic sieving results in type A; see Theorem 3.23 below. The definition of cyclic sieving (given in (2.6)) requires certain properties of the "cyclic sieving polynomial" $P(q)$. Namely, one has to show that it is indeed a polynomial and has non-negative coefficients, and one has to compute the evaluation of this polynomial at certain roots of unity. We verify these properties separately in Lemmas 3.21 and 3.22.

We use the standard q-notations $[\alpha]_q := (1-q^\alpha)/(1-q)$, $[n]_q! := [n]_q\,[n-1]_q \cdots [1]_q$, where $[0]_q! := 1$, the q-binomial coefficient

$$\begin{bmatrix} M \\ m \end{bmatrix}_q := \begin{cases} \frac{[M]_q\,[M-1]_q \cdots [M-m+1]_q}{[m]_q!}, & \text{if } k \geq 1, \\ 1, & \text{if } k = 0, \end{cases}$$

and the q-multinomial coefficient

$$\begin{bmatrix} M \\ m_1, m_2, \ldots, m_k \end{bmatrix}_q := \begin{cases} \frac{[M]_q\,[M-1]_q \cdots [M-m_1-m_2-\cdots-m_k+1]_q}{[m_1]_q!\,[m_2]_q! \cdots [m_k]_q!}, & \text{if } m_1+\cdots+m_k \geq 1, \\ 1, & \text{if } m_1+\cdots+m_k = 0. \end{cases}$$

Lemma 3.21. *Let m, n be positive integers and s and $b_1, b_2, \ldots, b_n$ non-negative integers such that $0 \leq s \leq n-1$ and*

$$b_1 + 2b_2 + \cdots + nb_n = n. \tag{3.15}$$

Then the expressions

$$\frac{1}{[n]_q} \begin{bmatrix} mn-2 \\ n-1 \end{bmatrix}_q, \tag{3.16}$$

$$\frac{1}{[n]_q} \begin{bmatrix} n \\ s \end{bmatrix}_q \begin{bmatrix} mn-2 \\ n-s-1 \end{bmatrix}_q, \tag{3.17}$$

$$\frac{1}{[b_1+b_2+\cdots+b_n]_q} \begin{bmatrix} b_1+b_2+\cdots+b_n \\ b_1, b_2, \ldots, b_n \end{bmatrix}_q \begin{bmatrix} mn-2 \\ b_1+b_2+\cdots+b_n-1 \end{bmatrix}_q \tag{3.18}$$

are polynomials in q with non-negative integer coefficients.

The proof of this lemma is found in Appendix D.2.

Lemma 3.22. *Let $m, n, s, b_1, b_2, \ldots, b_n$ be integers satisfying the hypotheses in Lemma 3.21. Furthermore, let σ be a positive integer such that $\sigma \mid (mn-2)$ and $\sigma < mn-2$. Writing $\rho = (mn-2)/\sigma$ and $\omega_\rho = e^{2\pi i\sigma/(mn-2)} = e^{2\pi i/\rho}$, we have*

$$\frac{1}{[n]_q}\begin{bmatrix}(m+1)n-2\\ n-1\end{bmatrix}_q\Bigg|_{q=\omega_\rho} = \binom{\lfloor((m+1)n-2)/\rho\rfloor}{\lfloor n/\rho\rfloor}, \tag{3.19}$$

$$\frac{1}{[n]_q}\begin{bmatrix}mn-2\\ n-1\end{bmatrix}_q\Bigg|_{q=\omega_\rho} = \begin{cases}\binom{(mn-2)/2}{n/2}, & \text{if } \rho=2 \text{ and } n\equiv 0 \pmod 2,\\ \binom{(mn-2)/\rho}{\lfloor n/\rho\rfloor}, & \text{if } n\equiv 1 \pmod\rho,\\ 0, & \text{otherwise,}\end{cases} \tag{3.20}$$

$$\frac{1}{[n]_q}\begin{bmatrix}n\\ s\end{bmatrix}_q\begin{bmatrix}mn-2\\ n-s-1\end{bmatrix}_q\Bigg|_{q=\omega_\rho} = \begin{cases}\binom{\lfloor(n-1)/\rho\rfloor}{\lfloor s/\rho\rfloor}\binom{(mn-2)/\rho}{(n-s-1)/\rho}, & \text{if } n-s-1\equiv 0 \pmod\rho,\\ \binom{(n-2)/2}{s/2}\binom{(mn-2)/2}{(n-s)/2}, & \text{if } \rho=2 \text{ and } n\equiv s\equiv 0 \pmod 2,\\ 0, & \text{otherwise,}\end{cases} \tag{3.21}$$

$$\frac{1}{[b_1+b_2+\cdots+b_n]_q}\begin{bmatrix}b_1+b_2+\cdots+b_n\\ b_1,b_2,\ldots,b_n\end{bmatrix}_q\begin{bmatrix}mn-2\\ b_1+b_2+\cdots+b_n-1\end{bmatrix}_q\Bigg|_{q=\omega_\rho}$$
$$= \begin{cases}\binom{(b_1+b_2+\cdots+b_n-1)/\rho}{\lfloor b_1/\rho\rfloor,\lfloor b_2/\rho\rfloor,\ldots,\lfloor b_n/\rho\rfloor}\binom{(mn-2)/\rho}{(b_1+b_2+\cdots+b_n-1)/\rho}, & \text{if } b_k\equiv 0 \pmod\rho \text{ for } k\neq j\\ & \text{and } b_j\equiv 1 \pmod\rho, \text{ for some } j,\\ \binom{(b_1+b_2+\cdots+b_n)/2}{b_1/2,b_2/2,\ldots,b_n/2}\binom{(mn-2)/2}{\lfloor(b_1+b_2+\cdots+b_n)/2\rfloor}, & \\ & \text{if } \rho=2 \text{ and } n\equiv b_1\equiv\cdots\equiv b_n\equiv 0 \pmod 2,\\ 0, & \text{otherwise.}\end{cases} \tag{3.22}$$

The proof of this lemma is found in Appendix D.3.

Now we are ready to state, and prove, our cyclic sieving results in type A.

Theorem 3.23. *Let $m, n, s, b_1, b_2, \ldots, b_n$ be integers satisfying the hypotheses in Lemma 3.21. Furthermore, let C be the cyclic group of pseudo-rotations of an mn-gon generated by κ_A. Then the triple (M, P, C) exhibits the cyclic sieving phenomenon for the following choices of sets M and polynomials P:*

(1) *$M = \widetilde{NC}_+^{(m)}(A_{n-1})$, and*

$$P(q) = \frac{1}{[n]_q}\begin{bmatrix}(m+1)n-2\\ n-1\end{bmatrix}_q;$$

(2) *M consists of the elements of $\widetilde{NC}_+^{(m)}(A_{n-1})$ all of whose blocks have size m, and*

$$P(q) = \frac{1}{[n]_q} \begin{bmatrix} mn-2 \\ n-1 \end{bmatrix}_q ;$$

(3) *M consists of all elements of $\widetilde{NC}_+^{(m)}(A_{n-1})$ which have rank s (or, equivalently, their number of blocks is $n-s$), and*

$$P(q) = \frac{1}{[n]_q} \begin{bmatrix} n \\ s \end{bmatrix}_q \begin{bmatrix} mn-2 \\ n-s-1 \end{bmatrix}_q ;$$

(4) *M consists of all elements of $\widetilde{NC}_+^{(m)}(A_{n-1})$ whose number of blocks of size mi is b_i, $i = 1, 2, \dots, n$, and*

$$P(q) = \frac{1}{[b_1 + b_2 + \cdots + b_n]_q} \begin{bmatrix} b_1 + b_2 + \cdots + b_n \\ b_1, b_2, \dots, b_n \end{bmatrix}_q \begin{bmatrix} mn-2 \\ b_1 + b_2 + \cdots + b_n - 1 \end{bmatrix}_q .$$

Proof. The polynomials $P(q)$ in the assertion of the theorem are indeed polynomials with non-negative coefficients due to Lemma 3.21.

Now, given a positive integer ρ with $\rho \mid (mn-2)$, we must show

$$P(q)\big|_{q=\omega_\rho} = (\text{number of elements of } \widetilde{NC}_+^{(m)}(A_{n-1}) \text{ invariant under } \kappa_A^{(mn-2)/\rho}),$$

where $\omega_\rho = e^{2\pi i/\rho}$.

For Item (1), this follows by combining (3.14) and (3.19). Item (2) is the special case of Item (4) where $b_1 = n$ (and, hence, $b_2 = b_3 = \cdots = b_n = 0$), and therefore does not need to be treated separately.

We now turn to Item (3). We distinguish two cases, depending on ρ.

Case 1: $\rho \geq 3$. Then, according to Proposition C.5, any positive m-divisible non-crossing partition which is ρ-pseudo-rotationally invariant must have a central block. Moreover, Theorem 3.17(1) says that all of these partitions arise by Construction 1 and subsequent applications of the pseudo-rotation κ_A. As a consequence, the number of non-central blocks must be divisible by ρ. Consequently, for any such non-crossing partition π, we have

$$\#(\text{blocks of } \pi) = 1 + \rho B = n - s,$$

where B is some non-negative integer. In particular, we must have $n - s - 1 \equiv 0 \pmod{\rho}$. The number of these non-crossing partitions is given by Formula (3.12) with b replaced by $B = (n-s-1)/\rho$. It is not difficult to see that this agrees with the expression in the first alternative on the right-hand side of (3.21).

Case 2: $\rho = 2$. In that case, Theorem 3.17(2) says that we obtain all relevant positive m-divisible non-crossing partitions by either Construction 1 or 2 and subsequent applications of κ_A. In the first case, there is a central block and the previous arguments apply again. In the second case, there is no central block. Moreover, if b_i denotes the number of blocks of size mi, then all b_i's are even. In particular, the number of blocks, $n - s$, must

be even. Since $n = b_1 + 2b_2 + \cdots + nb_n$, also n, and thus also s, must be even. The number of relevant non-crossing partitions is then given by (3.11) with $a = 0$ and b replaced by $(n-s)/2$. The resulting expression is identical with the expression in the second alternative on the right-hand side of (3.21).

In all other cases, there are no positive m-divisible non-crossing partitions which are ρ-pseudo-rotationally invariant.

Finally, we address Item (4). Again, we distinguish two cases depending on ρ.

Case 1: $\rho \geq 3$. Let again b_i denote the number of blocks of size mi. Repeating the arguments for Item (3), we see that there is one b_j (namely the one for which mj is the size of the central block) which is congruent to 1 modulo ρ, while all other b_k's, $k \neq j$, are divisible by ρ. The number of relevant non-crossing partitions is then given by (3.10) with b_j replaced by $(b_j - 1)/\rho$ and b_k replaced by b_k/ρ, $k \neq j$. This agrees with the expression in the first alternative on the right-hand side of (3.22).

Case 2: $\rho = 2$. In that case, Theorem 3.17(2) says that we obtain all relevant positive m-divisible non-crossing partitions by either Construction 1 or 2 and subsequent application of κ_A. In the first case, there is a central block and the previous arguments apply again. In the second case, by repeating the arguments for Item (3), we see that all b_i's, and thus also n, must be even. The number of relevant non-crossing partitions is then given by (3.10) with b_k replaced by $b_k/2$, $k = 1, 2, \ldots, n$. The resulting expression is identical with the expression in second alternative on the right-hand side of (3.22).

In all other cases, there are no positive m-divisible non-crossing partitions which are ρ-pseudo-rotationally invariant.

This completes the proof of the theorem. □

Corollary 3.24. *The conclusion of Theorem 2.19 holds for type A_{n-1}.*

Proof. The cyclic sieving phenomenon for $\left(NC_+^{(m+1)}(A_{n-1}, c),\ \mathrm{Cat}_+^{(m+1)}(A_{n-1}; q),\ C\right)$ follows directly from Theorem 3.23(1) with m replaced by $m + 1$.

The image of the embedding of $NC_+^{(m)}(A_{n-1}, c)$ in $NC_+^{(m+1)}(A_{n-1}, c)$ as described in Proposition 2.14 is given by all tuples $(w_0, w_1, \ldots, w_{m+1})$ with $w_0 = \varepsilon$. By Remark 3.3, these tuples correspond to non-crossing set partitions in $\widetilde{NC}^{(m+1)}(A_{n-1})$ all of whose blocks have size $m+1$. The cyclic sieving phenomenon for $\left(NC_+^{(m)}(A_{n-1}, c),\ \mathrm{Cat}_+^{(m)}(A_{n-1}; q),\ \widetilde{C}\right)$ thus follows from Theorem 3.23(2) with m replaced by $m + 1$. □

4. Positive non-crossing set partitions in type B

In this section, we study positive m-divisible non-crossing partitions and the positive Kreweras maps in the situation of the hyperoctahedral group B_n. Recall that B_n can be combinatorially realised as the group of permutations π of $\{1, 2, \ldots, n, -1, -2, \ldots, -n\}$ satisfying $\pi(-i) = -\pi(i)$ for $i = 1, 2, \ldots, n$. In the sequel, we shall most often write

$\bar{i}$ instead of $-i$. We adopt the type B cycle notation from [BW02] consisting of the conventions

$$((i_1, i_2, \dots, i_k)) := (i_1, i_2, \dots, i_k)(\overline{i_1}, \overline{i_2}, \dots, \overline{i_k}),$$
$$[i_1, i_2, \dots, i_k] := (i_1, i_2, \dots, i_k, \overline{i_1}, \overline{i_2}, \dots, \overline{i_k}).$$

The Coxeter system corresponding to the hyperoctahedral group B_n is

$$(W, \mathcal{S}) = \big(B_n, \{s_i = ((i, i+1)) \mid 1 \le i < n\} \cup \{s_n = (n, \overline{n}) = [n]\}\big) \tag{4.1}$$

together with the Coxeter element $c = s_1 \cdots s_n = [1, 2, \dots, n]$.

The structure of this section is the same as the one of Section 3. Namely, in Subsection 4.1 we first make the positivity condition for type B m-divisible non-crossing partitions under the choice (4.1) of Coxeter system explicit; see Proposition 4.1. We then use this to describe *positive* m-divisible non-crossing partitions within Armstrong's combinatorial model of type B m-divisible non-crossing partitions [Arm06, Def. 4.5.5 and Thm. 4.5.6]; see Theorem 4.4.

In Subsection 4.2, we show that also here, under Armstrong's translation, for $m \ge 2$ the positive Kreweras map acts as a pseudo-rotation; see Theorem 4.6. This pseudo-rotation is denoted by κ_B.

In Subsection 4.3, we introduce a natural extension of κ_B which also makes sense for $m = 1$; see Definition 4.8. Also here it turns out that, under Armstrong's translation, for $m = 1$ this extension corresponds exactly to the map $\mathsf{K}_+^{(1)}$ from Definition 2.16; see Theorem 4.9. The order of the (extended) map κ_B is of course given by Theorem 2.17 for type B; see Theorem 4.10. For the sake of completeness, we also provide an independent, *combinatorial*, proof for that order; see Appendix A.2.

Subsection 4.4 is devoted to the enumeration of type B positive m-divisible non-crossing partitions. Here, we avoid attempting to find results on chain enumeration since this seems to require more refined techniques as the ones we have developed here. Thus, our most refined and most general result is Theorem 4.11 which provides a formula for the number of these non-crossing partitions in which the block structure is prescribed. All other results in that subsection, Corollaries 4.12 and 4.13, are obtained as consequences.

Subsection 4.5 is then devoted to the characterisation of type B positive m-divisible non-crossing partitions that are invariant under powers of the pseudo-rotation κ_B; see Theorem 4.14.

In Subsection 4.6, we then exploit this characterisation to provide a formula for the number of type B positive m-divisible non-crossing partitions that are invariant under powers of the pseudo-rotation κ_B and have a given block structure; see Theorem 4.15. Once again, several less refined enumeration results may be obtained by summation; see Corollaries 4.16 and 4.17.

In the final subsection, Subsection 4.7, we establish several cyclic sieving phenomena refining Theorem 2.19 in type B; see Theorem 4.19.

4.1. **Combinatorial realisation of the positive non-crossing partitions.** With the choice (4.1) of Coxeter system, we have the following simple combinatorial description of positive m-divisible non-crossing partitions.

Proposition 4.1. *Let m and n be positive integers with $m \geq 2$. $(w_0, w_1, \ldots, w_m) \in NC^{(m)}(B_n, c)$ is positive if and only if $w_m(n) > 0$.*

Proof. According to Definition 2.3, the tuple $(w_0, w_1, \ldots, w_m)$ is positive if and only if $cw_m^{-1} = w_0 w_1 \cdots w_{m-1}$ has full support in our generators $\{s_1, \ldots, s_n\}$ given in (4.1). The cycles in the disjoint cycle decomposition of cw_m^{-1} define a non-crossing partition. Thus, cw_m^{-1} will not have full support if and only if the preimage of 1 under cw_m^{-1} is a positive number. In other words, there is an $l > 0$ such that $(cw_m^{-1})(l) = 1$. By acting on the left on both sides of this relation by the inverse of the Coxeter element $c = [1, 2, \ldots, n]$, we see that this is equivalent to $w_m^{-1}(l) = \overline{n}$, or, again equivalently, to $w_m(n) = \overline{l}$, for some $l > 0$. We are interested in the contrapositive: cw_m^{-1} has full support if and only if $w_m(n) > 0$. □

Next we recall Armstrong's bijection between $NC^{(m)}(B_n, c)$ and m-divisible non-crossing partitions of $\{1, 2, \ldots, mn, \overline{1}, \overline{2}, \ldots, \overline{mn}\}$ that are invariant under substitution of i by $-i$, for all i, which we denote here by $\widetilde{NC}^{(m)}(B_n)$. We write $\widetilde{NC}(\pm N)$ for the set of *all* non-crossing partitions of $\{1, 2, \ldots, N, \overline{1}, \overline{2}, \ldots, \overline{N}\}$ that are invariant under substitution of i by $-i$, for all i, and observe that

$$\widetilde{NC}^{(m)}(B_n) = \{\pi \in \widetilde{NC}(\pm mn) \mid \text{all block sizes of } \pi \text{ are divisible by } m\}.$$

Following [Rei97], we agree here to call a block that is itself invariant under substitution of i by $-i$, for all i, a *zero block*, a notion that will be used frequently in the sequel. It should be noted that a non-crossing partition can have at most one zero block and that, for parity reasons, the size of a zero block is always divisible by $2m$.

Given an element $(w_0, w_1, \ldots, w_m) \in NC^{(m)}(B_n, c)$, the bijection, $\bigtriangledown_{B_n}$ say, from [Arm06, Thm. 4.5.6] works essentially in the same way as in type A: namely $(w_0, w_1, \ldots, w_m) \in NC^{(m)}(B_n, c)$ is mapped to

$$\bigtriangledown_{B_n}(w_0, w_1, \ldots, w_m) = [1, 2, \ldots, mn]\,(\bar{\tau}_{m,1}(w_1))^{-1}\,(\bar{\tau}_{m,2}(w_2))^{-1} \cdots (\bar{\tau}_{m,m}(w_m))^{-1}, \tag{4.2}$$

where $\bar{\tau}_{m,i}$ is the obvious extension of the transformation $\tau_{m,i}$ from Section 3. More precisely, we let

$$(\bar{\tau}_{m,i}(w))(mk + i - m) = mw(k) + i - m, \quad k = 1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n},$$

and $(\bar{\tau}_{m,i}(w))(l) = l$ and $(\bar{\tau}_{m,i}(w))(\bar{l}) = \bar{l}$ for all $l \not\equiv i \pmod m$, where $m\overline{k}+i-m$ is identified with $\overline{mk + i - m}$ for all k and i. Again, the cycles in the disjoint cycle decomposition correspond to the blocks in the non-crossing partition in $\widetilde{NC}^{(m)}(B_n)$. We refer the reader to [Arm06, Sec. 4.5] for the details. For example, let $n = 5$, $m = 3$, $w_0 = ((2, 4))$,

$w_1 = [1] = (1, \overline{1})$, $w_2 = ((1, 4))$, and $w_3 = ((2, 3))\,((4, 5))$. Then $(w_0; w_1, w_2, w_3)$ is mapped to

$$\begin{aligned}\bigtriangledown_{B_5}(w_0; w_1, w_2, w_3) &= [1, 2, \dots, 15]\,[1]\,((2, 11))\,((6, 9))\,((12, 15)) \\ &= ((1, \overline{2}, \overline{12}))\,((3, 4, 5, 6, 10, 11))\,((7, 8, 9))\,((13, 14, 15)). \qquad (4.3)\end{aligned}$$

Similarly as in type A_{n-1}, the cycle structure of the first component of an m-divisible non-crossing partition determines the block structure of its image under Armstrong's bijection $\bigtriangledown_{B_n}$; see [Arm06, Proof of Thm. 4.3.13].

Proposition 4.2. *Let $(w_0, w_1, \dots, w_m) \in NC^{(m)}(B_n, c)$. The non-crossing partition $\pi = \bigtriangledown_{B_n}(w_0, w_1, \dots, w_m) \in \widetilde{NC}^{(m)}(B_n)$ has a zero block consisting of $2km$ elements if and only if w_0 contains a cycle of the form $[i_1, i_2, \dots, i_k]$ in its disjoint cycle decomposition. Furthermore, π has $2b_k$ non-zero blocks of size mk if and only if w_0 has b_k cycles of the form $((i_1, i_2, \dots, i_k))$ in its disjoint cycle decomposition.*[4]

In the earlier example, we have $w_0 = ((2, 4)) = ((1))((3))((4))((5))((2, 4))$. Indeed, the image of (w_0, w_1, w_2, w_3) in the example has no zero block, two non-zero blocks of size $3 \cdot 2 = 6$ and six non-zero blocks of size $3 \cdot 1 = 3$.

Remark 4.3. A simple consequence of Proposition 4.2 is that the image of the set of all non-crossing partitions in $NC^{(m)}(B_n, c)$ of the form $(\varepsilon, w_1, \dots, w_m)$ under the map $\bigtriangledown_{B_n}$ is the set of all non-crossing partitions in $\widetilde{NC}^{(m)}(B_n)$ in which *all* blocks have size m.

Theorem 4.4. *Let m and n be positive integers with $m \geq 2$. The image under $\bigtriangledown_{B_n}$ of the positive m-divisible non-crossing partitions in $NC^{(m)}(B_n, c)$ are those m-divisible non-crossing partitions in $\widetilde{NC}^{(m)}(B_n)$ where the block containing 1 also contains a negative number.*

Proof. First, let $(w_0, w_1, \dots, w_m) \in NC^{(m)}(B_n, c)$. By definition, we know that $w_m(n) > 0$. Let us write l for $w_m(n)$. We then have

$$\begin{aligned}&\Big(\bigtriangledown_{B_n}(w_0, w_1, \dots, w_m)\Big)(\overline{ml}) \\ &\quad = \big([1, 2, \dots, mn]\,(\tau_{m,1}(w_1))^{-1}\,(\tau_{m,2}(w_2))^{-1} \cdots (\tau_{m,m}(w_m))^{-1}\big)(\overline{ml}) \\ &\quad = \big([1, 2, \dots, mn]\big)(\overline{mn}) = 1.\end{aligned}$$

Translated to non-crossing partitions, this means that $\overline{ml}$ and 1 belong to the same block, which implies our claim.

Conversely, let $\pi \in \widetilde{NC}^{(m)}(B_n)$ be an m-divisible non-crossing partition in which the block containing 1 also contains a negative number. Let $\overline{L}$ be the negative number that

[4]The reader must recall that a "cycle" of the form $((i_1, i_2, \dots, i_k))$ consists actually of *two* disjoint cycles of length k, while a cycle of the form $[i_1, i_2, \dots, i_k]$ is indeed a single cycle, of length $2k$.

is connected with 1 in the geometric representation of π. Since in between $\overline{L}$ and 1 there "sit" blocks all of whose sizes are divisible by m, we must necessarily have $\overline{L} = \overline{ml}$, for some l between 1 and n. In other words, if π is interpreted as a permutation, $\pi(\overline{ml}) = 1$. Let $(w_0, w_1, \dots, w_m)$ be the element of $NC^{(m)}(B_n, c)$ such that

$$\bigtriangledown_{B_n}(w_0, w_1, \dots, w_m) = \pi.$$

We must have

$$\Big(\bigtriangledown_{B_n}(w_0, w_1, \dots, w_m)\Big)(\overline{ml}) = 1.$$

The definition (4.2) of $\bigtriangledown_{B_n}$ and the fact that $(\tau_{m,i}(w_i))^{-1}$ leaves multiples of m fixed as long as $1 \leq i \leq m-1$ together imply that $(\tau_{m,m}(w_m))^{-1}(\overline{ml}) = \overline{mn}$. Phrased differently, we have $(\tau_{m,m}(w_m))^{-1}(ml) = mn$, or, equivalently, $w_m(n) = l > 0$. By definition, this means that $(w_0, w_1, \dots, w_m)$ is positive. □

4.2. **Combinatorial realisation of the positive Kreweras maps.** The next step consists in translating the positive Kreweras map K_+ into a "rotation action" on the positive elements of $NC^{(m)}(B_n, c)$. In order to do so, we need to explicitly describe the decomposition (2.5) in type B.

Lemma 4.5. *Let w be an element of $NC(B_n, c)$, and consider the cycle z in the (type B) disjoint cycle decomposition of w that contains n.*

1. *If $z = ((i_1, i_2, \dots, i_k, n))$ with $i_1 > 0$ (and consequently $0 < i_1 < i_2 < \cdots < i_k < n$), we put $a = n$.*
2. *If $z = ((i_1, \dots, i_s, i_{s+1}, \dots, i_k, n))$ with $i_s < 0$ and $i_{s+1} > 0$ (and consequently $i_1, i_2, \dots, i_s < 0$, $i_{s+1}, \dots, i_{k-1}, i_k > 0$, and $0 < |i_1| < |i_2| < \cdots < |i_k| < n$; the sequence $i_{s+1}, \dots, i_k$ has to be interpreted as the empty sequence if $s = k$), we put $a = i_s$.*
3. *If $z = [i_1, i_2, \dots, i_k, n]$ (and consequently $0 < i_1 < i_2 < \cdots < i_k < n$), we put $a = \overline{n}$.*

Then we have

$$w = w^L w^R, \qquad \text{where } w^R = ((a, n)).$$

(By slight abuse of notation, here we identify $((\overline{n}, n))$ with $[n]$.) This covers all possible cases.

Proof. The last n reflections in $\mathcal{R}$ given in (2.2) are

$$((i, n)),\ i = \overline{1}, \overline{2}, \dots, \overline{n-1},\ \text{and } [n] = (\overline{n}, n), \tag{4.4}$$

in this order. Thus, by definition, we have $w^L = w \circ (w^R)^{-1}$, where the factor w^R consists of a product of these n reflections (we shall see, however, that in our case it will be always at most one reflection), and $\ell_{\mathcal{R}}(w^L) + \ell_{\mathcal{R}}(w^R) = \ell_{\mathcal{R}}(w)$. Furthermore, using Proposition 4.1 with $m = 1$, we see that w^L must have the property that $w^L(n) > 0$. The arguments below are heavily based on the uniqueness of the decomposition $w = w^l \circ w^R$ in the sense of Lemma 2.9.

In Case (1), we have $w(n) = i_1 > 0$. Hence, we may choose $w^L = w$ and $w^R = \varepsilon = ((n, n))$, and uniqueness of decomposition guarantees that this is the correct choice.

In Case (2), the choice $w^R = ((i_s, n))$ implies that w^L would contain the cycles $((i_1, \dots, i_s))((i_{s+1}, \dots, i_k, n))$. Hence, we would have $w^L(n) = i_{s+1} > 0$ if $s < k$, and $w^L(n) = n$ if $s = k$. By uniqueness of decomposition, this must be the correct choice.

Finally we address Case (3). If we choose $w^R = [n]$, then w^L would contain the cycle $((i_1, i_2, \dots, i_k, n))$. Hence, we would have $w^L(n) = i_1 > 0$. Again, by uniqueness of decomposition, this must be the correct choice. □

Next, we translate the positive Kreweras map K_+ from Definition 2.12 for type B_n into combinatorial language.

Theorem 4.6. *Let m and n be positive integers with $m \geq 2$. Under the bijection ∇_{B_n}, the map $\mathsf{K}_+^{(m)}$ translates to the following map κ_B on $\widetilde{NC}_+^{(m)}(B_n)$: if the block of $\overline{mn}$ contains another negative element, then κ_B rotates all blocks of π by one unit in clockwise direction.*

On the other hand, if the block of $\overline{mn}$ of some element $\pi \in \widetilde{NC}_+^{(m)}(B_n)$ contains no other negative element, then, first of all, 1 is also contained in this block. In other words, $\overline{mn}$ and 1 are successive elements in a block (and consequently also mn and $\overline{1}$). For the description of the operation, we distinguish between two cases:

(1) *If there is a block not containing $1, \overline{1}, mn, \overline{mn}$ in which there are positive* and *negative elements, then let b be minimal such that $\overline{d}$ and b are successive elements in a block, with $\overline{d}$ negative and b positive*[5]*. Furthermore, let b be followed by e in this block, and let a and $\overline{mn}$ be successive elements in a block. (The element a must be positive by the assumption defining the subcase in which we are, while e can be positive or negative. See the left half of Figure 14 for an illustration of the various definitions.) Then the image of π under κ_B is the partition in which $\overline{d+1}$, 1, and $e+1$ are three successive elements in a block, and $a+1$, $b+1$, and 2 are successive elements in another block. All other succession relations in π are rotated by one unit in clockwise direction. See Figure 14 for a schematic illustration of this operation.*

(2) *If there is* no *block other than the ones containing $1, \overline{1}, mn, \overline{mn}$ with positive* and *negative elements, then let a, $\overline{mn}$, and 1 be successive elements in a block. (Again, the element a must be positive.) The image of π under κ_B is the partition in which $\overline{a+1}$, 1, and $\overline{2}$ are successive elements in a block, and all other succession relations in π are rotated by one unit in clockwise direction. See Figure 16 for a schematic illustration of this operation.*

Remarks 4.7. (i) Figure 14 provides a schematic illustration of the construction in Case (1) of the statement of the above theorem, while Figure 15 provides a concrete example. In this example, $m = 3$, $n = 4$, $a = 2$, $b = 6$, $d = 11$, and $e = 10$. Case (2) of the

[5]We have $b \neq d$. For, if $b = d$, then $\{b, \overline{d}\} = \{b, \overline{b}\}$ is a block by itself. It is a zero block of size 2. Thus, necessarily $m = 2$. However, for parity reasons, such non-crossing partitions of type B do not exist.

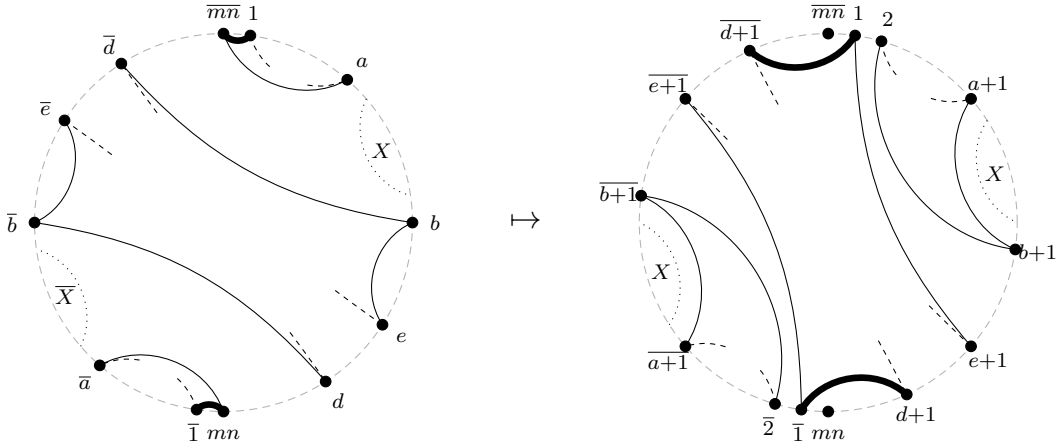

FIGURE 14. The action of the pseudo-rotation κ_B, Case (1)

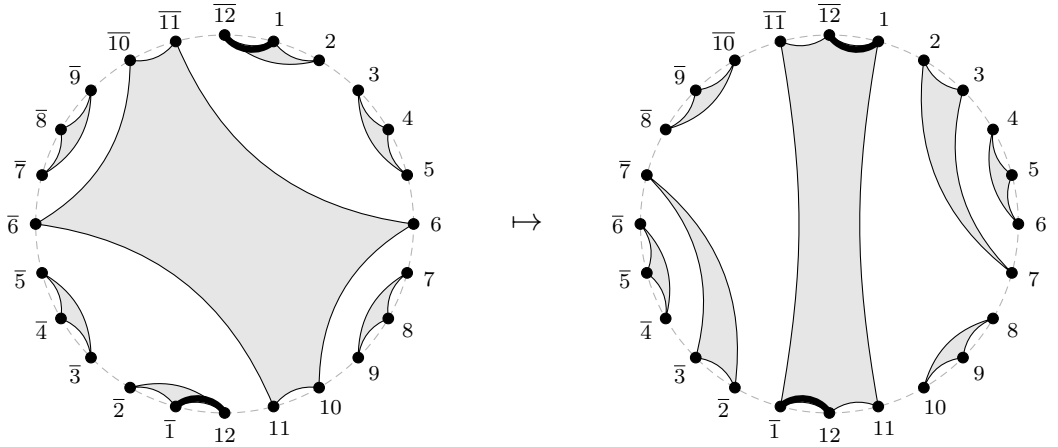

FIGURE 15. The action of the pseudo-rotation κ_B, Case (1)

statement of Theorem 4.6 is illustrated in Figure 16, while Figure 17 provides a concrete example, in which $m = 3$, $n = 4$, and $a = 8$.

(ii) Case (2) can be seen as a degenerate special case of Case (1). To see this, one chooses $b = mn$ and $d = a$ in Case (1). We shall occasionally refer to this point of view in order to simplify arguments in proofs.

Proof of Theorem 4.6. Let us first assume that, in the positive m-divisible type B non-crossing partition π, the block of $\overline{mn}$ contains another negative element. Among those, let $\overline{mi_1 - 1}$ be the one which is the predecessor of $\overline{mn}$ in the block. Obviously, we have $i_1 > 0$. Figure 18 provides sketches of such situations. Let $(w_0, w_1, \ldots, w_m)$ denote the element of $NC_+^{(m)}(B_n, c)$ which corresponds to π under the bijection ∇_{B_n}. By the definition of ∇_{B_n}, we see that $w_{m-1}(n) = i_1$. From Lemma 4.5(1), we infer $w_{m-1}^L = w_{m-1}$ and $w_{m-1}^R = \varepsilon$.

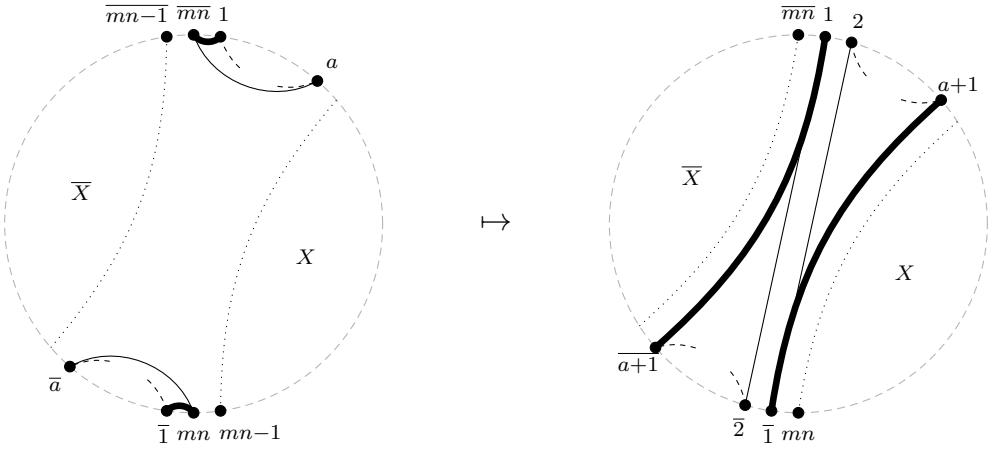

FIGURE 16. The action of the pseudo-rotation κ_B, Case (2)

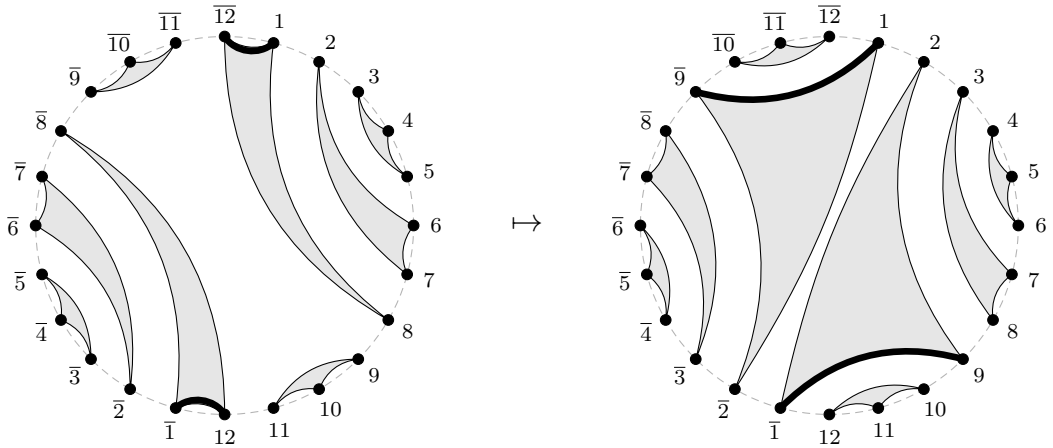

FIGURE 17. The action of the pseudo-rotation κ_B, Case (2)

This means that the pseudo-rotation κ_B reduces to ordinary rotation in this special case, as we claimed.

For the remainder of the proof, we assume that the block of $\overline{mn}$ contains no other negative element. In particular, it must contain a positive number (since $m \geq 2$). By Theorem 4.4, we know that the block of 1 contains a negative number. Hence, if the partition wants to be non-crossing, necessarily 1 must be in the same block as $\overline{mn}$.

Let again $(w_0, w_1, \ldots, w_m)$ denote the element of $NC_+^{(m)}(B_n, c)$ which corresponds to π under the bijection ∇_{B_n}.

CASE (1). Let $a_2, a_3, \ldots, a_{s-1}$ be positive integers with $a < a_2 < a_3 < \cdots < a_{s-1}$ such that a_2 and $a+1$ (sic!) are successive elements in a block of π, a_3 and a_2+1 are successive

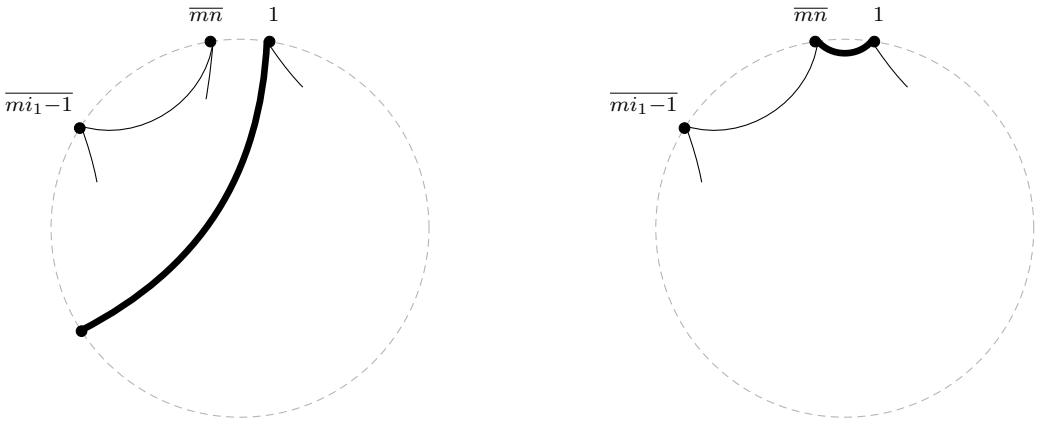

FIGURE 18. The block of $\overline{mn}$ contains another negative element

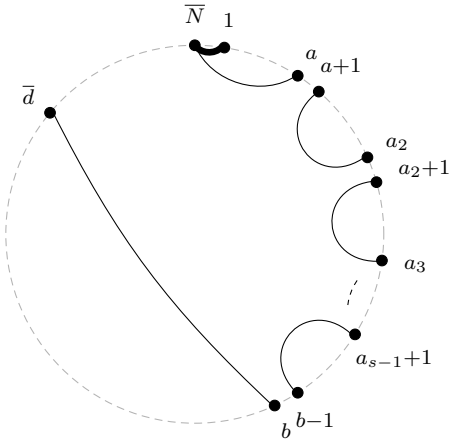

FIGURE 19. The block structure in Case (1)

elements in a block, ..., and $b-1$ and $a_{s-1}+1$ are successive elements in a block. Figure 19 provides a sketch of this situation.

The congruence classes of $a, b, d, e, a_2, \ldots, a_{s-1}$ modulo m are forced: indeed, we have $a \equiv -1 \pmod{m}$ since a is the predecessor of $\overline{mn}$ in their block, we have $a_2 \equiv -1 \pmod{m}$ since a_2 is the predecessor of $a+1$ in their block, we have $a_3 \equiv -1 \pmod{m}$ since a_3 is the predecessor of a_2+1 in their block, ..., we have $b \equiv 0 \pmod{m}$ since $b-1$ is the predecessor of $a_{s-1}+1$ in their block, and we have $d \equiv -1 \pmod{m}$ and $e \equiv 1 \pmod{m}$ since $\overline{d}$, b, e are successive elements in their block. We write $a = mA - 1$, $b = mB$, $d = mD - 1$, $e = mE + 1$, and $a_i = mA_i - 1$, $i = 2, 3, \ldots, s-1$, for suitable integers

$A, B, D, E, A_2, A_3, \dots, A_{s-1}$. By the definition of ∇_{B_n}, we have

$$\begin{aligned} w_{m-1}(\overline{n}) &= A, \\ w_{m-1}(A) &= A_2, \\ w_{m-1}(A_2) &= A_3, \\ w_{m-1}(A_{s-1}) &= B, \\ w_{m-1}(B) &= \overline{D}, \\ w_m(E) &= B, \qquad (4.5) \\ w_m(n) &= n. \qquad (4.6) \end{aligned}$$

Consequently, the cycle of w_{m-1} containing n is of the form

$$((\overline{A}, \overline{A}_2, \dots, \overline{A}_{s-1}, \overline{B}, D, \dots, n)).$$

By Lemma 4.5(2), we have

$$w^R_{m-1} = ((\overline{B}, n)), \quad \text{and } w^L_{m-1} \text{ contains the cycles } ((A, A_2, \dots, A_{s-1}, B))((D, \dots, n)).$$

Together with (4.5) and (4.6), this implies the relations

$$\begin{aligned} w^L_{m-1}(n) &= D, \\ w^L_{m-1}(A) &= A_2, \\ w^L_{m-1}(A_2) &= A_3, \\ w^L_{m-1}(A_{s-1}) &= B, \\ w^L_{m-1}(B) &= A, \\ (cw^R_{m-1}w_m c^{-1})(1) &= B + 1, \\ (cw^R_{m-1}w_m c^{-1})(E + 1) &= 1. \end{aligned}$$

By the definition of $\mathsf{K}^{(m)}_+$ and of ∇_{B_n}, this means that $a + 1$, $b + 1$, and 2 are successive elements in their block in the image of π under $\mathsf{K}^{(m)}_+$ (conjugated by ∇_{B_n}), that $\overline{d+1}$, 1, and $e+1$ are successive elements in a block, while all other "block connections" are rotated by one unit in clockwise direction. This is in accordance with the corresponding assertion in the proposition.

Case (2). Let $a_2, a_3, \dots, a_k$ be positive integers with $a < a_2 < a_3 < \cdots < a_k$ such that a_2 and $a + 1$ are successive elements in a block, a_{i+1} and $a_i + 1$ are successive elements in a block, $i = 2, 3, \dots, s - 2$, and $mn - 1$ and $a_{s-1} + 1$ are successive elements in a block. Figure 20 provides a sketch of this situation.

As in Case (1), we must have $a \equiv -1 \pmod m$, and $a_i \equiv -1 \pmod m$, for $i = 2, 3, \dots, s - 1$. We write again $a = mA - 1$ and $a_i = mA_i - 1$ for suitable integers A and

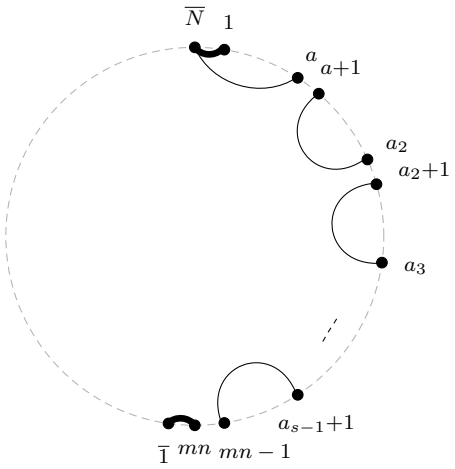

FIGURE 20. The block structure in Case (2)

A_i, for $i = 2, 3, \ldots, s-1$. By the definition of ∇_{B_n}, we have

$$\begin{aligned} w_{m-1}(\overline{n}) &= A, \\ w_{m-1}(A) &= A_2, \\ w_{m-1}(A_2) &= A_3, \\ w_{m-1}(A_{s-2}) &= A_{s-1}, \\ w_{m-1}(A_{s-1}) &= n, \\ w_m(n) &= n. \end{aligned} \tag{4.7}$$

Consequently, the cycle of w_{m-1} containing n is of the form

$$[A, A_2, \ldots, \ldots, A_{s-1}, n].$$

By Lemma 4.5(3), we have

$$w_{m-1}^R = [n], \quad \text{and } w_{m-1}^L \text{ contains the cycle } ((A, A_2, \ldots, A_{s-1}, n)).$$

Together with (4.7), this implies the relations

$$\begin{aligned} w_{m-1}^L(n) &= A, \\ w_{m-1}^L(A) &= A_2, \\ w_{m-1}^L(A_2) &= A_3, \\ w_{m-1}^L(A_{s-2}) &= A_{s-1}, \\ w_{m-1}^L(A_{s-1}) &= n, \\ (cw_{m-1}^R w_m c^{-1})(1) &= \overline{1}. \end{aligned}$$

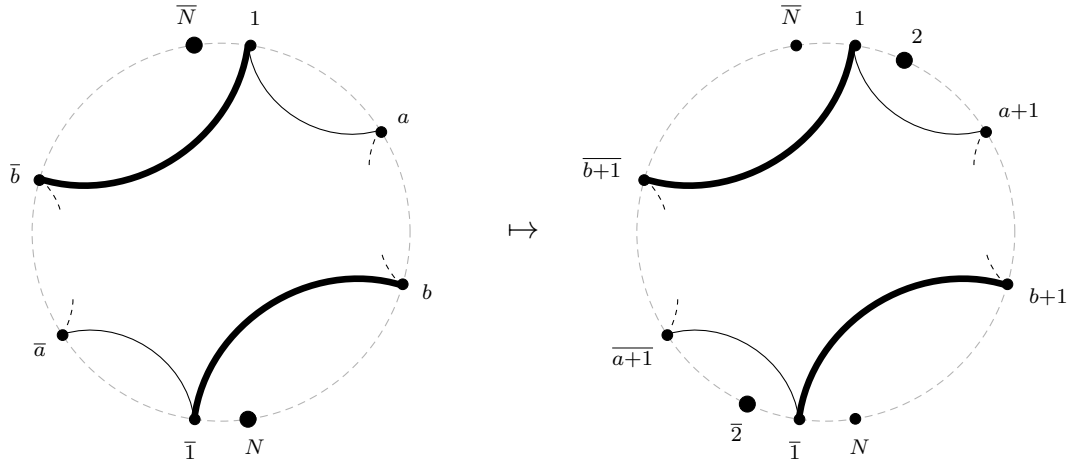

FIGURE 21. Definition of the pseudo-rotation under presence of singleton blocks

By the definition of ∇_{B_n}, this means that $\overline{a+1}$, 1, and $\overline{2}$ are successive elements in their block in the image of π under $\mathsf{K}_+^{(m)}$ (conjugated by ∇_{B_n}), while all other "block connections" are rotated by one unit in clockwise direction. This is in accordance with the corresponding assertion in the proposition. Before finishing the argument, we would like to offer, to the first three readers who made it thus far, a one-dollar bill hand-signed by Christian. The bill is claimed by sending us a *non-trivial* positive non-crossing partition in $\widetilde{NC}_+^{(2)}(B_4)$ and its image under the positive Kreweras map κ_B. This completes the proof of the proposition. □

4.3. **The combinatorial positive Kreweras maps.** The definition of κ_B from Theorem 4.6 to make sense on an *arbitrary* non-crossing partition needed that blocks should be of size at least 2. It is a simple matter to extend this so that it makes sense for non-crossing partitions without any restrictions.

Definition 4.8. We define the action of κ_B on a positive type B non-crossing partition π of $\{1, 2, \dots, N, \overline{1}, \overline{2}, \dots, \overline{N}\}$ to be given by Theorem 4.6 with mn replaced by N (cf. Figures 14 and 16), except when the block containing N is a singleton block (in which case also $\{\overline{N}\}$ is a singleton block; here, Theorem 4.6 would not make sense). If $\{N\}$ is a singleton block, then the image of κ_B is the non-crossing partition which arises from π by replacing the singleton block $\{N\}$ by the singleton block $\{\overline{2}\}$, by replacing the singleton block $\{\overline{N}\}$ by the singleton block $\{2\}$, and i by $i+1$ and $\bar{i}$ by $\overline{i+1}$ for $i = 2, 3, \dots, N-1$ in each of the remaining blocks of π, leaving 1 and $\overline{1}$ untouched; see Figure 21 for an illustration. In the degenerate case of Case (1) in Theorem 4.6 where $b = d$, the partition π contains $\{b, \bar{b}\} = \{d, \bar{d}\}$ as zero block[6] (see the left half of Figure 22 for a schematic illustration of that situation, which can only occur for $m = 1$; cf. Footnote 5). The image of π under κ_B

[6]See Subsection 4.1 for the definition of zero block.

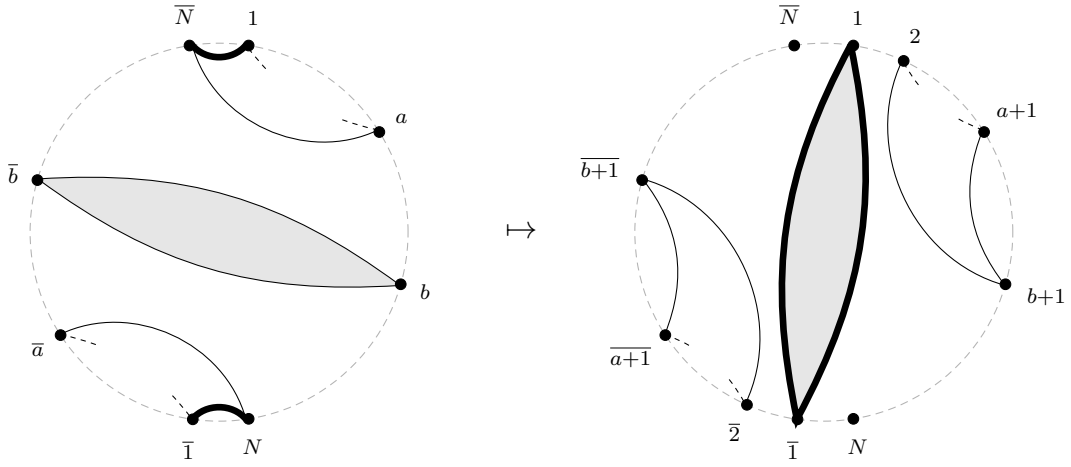

FIGURE 22. The action of the pseudo-rotation κ_B, Case (1) with $b = d$

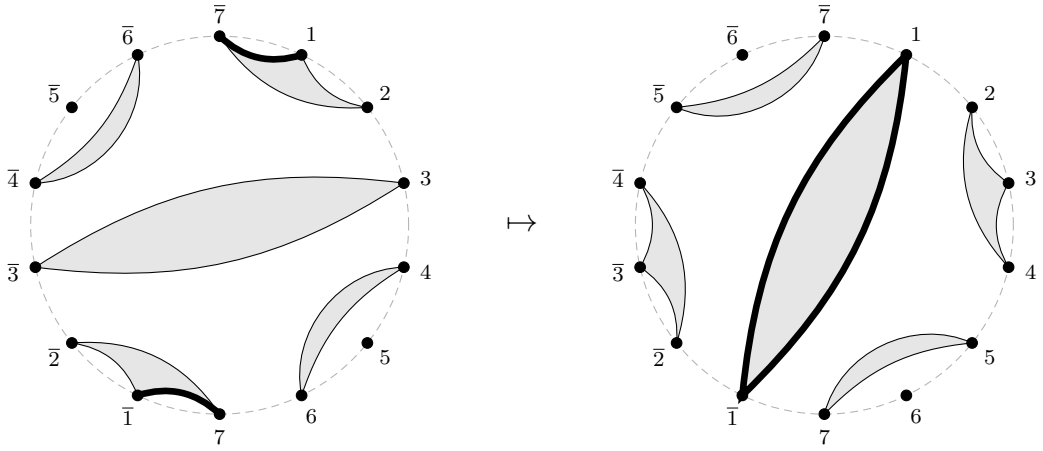

FIGURE 23. The action of the pseudo-rotation κ_B, Case (1) with $b = d$

is then defined as the partition in which 1 and $\overline{1}$ form a block, and $a+1$, $b+1$, and 2 are successive elements in another block. All other succession relations in π are rotated by one unit in clockwise direction. See Figure 22 for a schematic illustration of this operation and Figure 23 for a concrete example in which $N = 7$, $a = 2$, and $b = d = 3$.

As it turns out, this definition is also in agreement with the image under ∇_{B_n} of the map $\mathsf{K}_+^{(1)}$ in Definition 2.16 for the case where $m = 1$.

Theorem 4.9. *Let n be a positive integer. Under the bijection ∇_{B_n}, the map $\mathsf{K}_+^{(1)}$ from Definition 2.16 translates to the combinatorial map κ_B described in Definition 4.8 with $N = n$.*

Since the proof of the theorem is completely analogous to the proof of Theorem 3.9, we omit it here.

We conclude with the following theorem describing the order of the combinatorial map κ_B. In view of our previous discussion, it follows from Corollary 2.18 for the type B_N with $m = 1$. However, since we believe it is interesting for its own sake, we provide a direct, combinatorial, proof in Appendix A.2.

Theorem 4.10. *Let N be a positive integer. The order of the map κ_B acting on the set of positive type B non-crossing partitions π of $\{1, 2, \dots, N, \overline{1}, \overline{2}, \dots, \overline{N}\}$ is $N - 1$.*

4.4. **Enumeration of positive non-crossing partitions.** Here we present our enumeration results for positive non-crossing partitions in $\widetilde{NC}_+^{(m)}(B_n)$. The most refined and most general result is Theorem 4.11 which provides a formula for the number of these non-crossing partitions in which the block structure is prescribed. By specialisation and summation, we obtain results on the number of such partitions with prescribed rank or size of zero block, including the total number of elements in $\widetilde{NC}_+^{(m)}(B_n)$.

Theorem 4.11. *Let m and n be positive integers, and let a and $b_1, b_2, \dots, b_n$ be non-negative integers. The number of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn, \overline{1}, \overline{2}, \dots, \overline{mn}\}$ of type B, where the number of non-zero blocks of size mi is $2b_i$, $i = 1, 2, \dots, n$, the zero block having size $2ma = 2mn - 2m\sum_{j=1}^{n} jb_j$ (if $a = 0$ this is interpreted as the absence of a zero block), is given by*

$$\binom{b_1 + b_2 + \cdots + b_n}{b_1, b_2, \dots, b_n} \binom{mn - 1}{b_1 + b_2 + \cdots + b_n}. \tag{4.8}$$

The proof of this theorem is found in Appendix B.5.

Corollary 4.12. *Let m and n be positive integers, and let a and b be non-negative integers. The number of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn, \overline{1}, \overline{2}, \dots, \overline{mn}\}$ of type B which have $2b$ non-zero blocks and a zero block of size $2ma$ (if $a = 0$ this is interpreted as the absence of a zero block), is given by*

$$\binom{n - a - 1}{b - 1} \binom{mn - 1}{b}, \tag{4.9}$$

while the number of these partitions without restricting the size (or existence) of the zero block is

$$\binom{n}{b} \binom{mn - 1}{b}. \tag{4.10}$$

Via the relation $s = n - b$, the last expression also equals the number of elements of $\widetilde{NC}_+^{(m)}(B_n)$ of fixed rank s.

The proof of this corollary is also found in Appendix B.5.

Corollary 4.13. *Let m and n be positive integers, and let a be a non-negative integer. The number of positive m-divisible non-crossing partitions of $\{1,2,\dots,mn,\overline{1},\overline{2},\dots,\overline{mn}\}$ of type B which have a zero block of size $2ma$ (if $a=0$ this is interpreted as the absence of a zero block) is given by*

$$\binom{(m+1)n-a-2}{n-a}, \tag{4.11}$$

while the number of these partitions without restricting the size (or existence) of the zero block is

$$\binom{(m+1)n-1}{n}. \tag{4.12}$$

Proof. One sums Expressions (4.9) and (4.10) over all b. The resulting sums can be evaluated by means of the Chu–Vandermonde summation formula. □

4.5. **Characterisation of pseudo-rotationally invariant elements.** In this subsection, we describe the elements of $\widetilde{NC}(\pm N)$ that are invariant under a power of κ_B. We call an element of $\widetilde{NC}(\pm N)$ *2ν-pseudo-rotationally invariant* if it is invariant under the action of $\kappa_B^{2(N-1)/(2\nu)}=\kappa_B^{(N-1)/\nu}$. This — perhaps seemingly unintuitive — definition has its background in the cyclic sieving result(s) that we aim to eventually state and prove in Subsection 4.7. While it is shown in Theorem 4.10 that the order of κ_B is $N-1$, in the statement of the cyclic sieving phenomena in Theorem 4.19 we need to consider the cyclic group generated by κ_B as a group of order $2(N-1)$ in order to fit with the uniform statement of cyclic sieving in Theorem 2.19.

We achieve the description of pseudo-rotationally invariant elements in $\widetilde{NC}(\pm N)$ by providing a construction of such elements (see below) together with Theorem 4.14 which then tells us how to obtain all such invariant elements from the construction. To this end, we adopt the notions of rotation R of a sequence and of a central block from Subsection 3.5.

Construction. We start with an (ordinary) 2ν-rotationally invariant non-crossing partition of $\{2,3,\dots,N,\overline{2},\overline{3},\dots,\overline{N}\}$, 2 being contained in the zero block. We also allow the degenerate case that the zero block is empty. Then we add 1 and $\overline{1}$ to the zero block, thereby creating a positive non-crossing partition. We illustrate this construction in the case where $\nu=2$ and $N=15$. We start with the non-crossing partition

$$\{\{2,9,\overline{2},\overline{9}\},\{3,7,8\},\{4,5,6\},\{10,14,15\},\{11,12,13\},\\ \{\overline{3},\overline{7},\overline{8}\},\{\overline{4},\overline{5},\overline{6}\},\{\overline{10},\overline{14},\overline{15}\},\{\overline{11},\overline{12},\overline{13}\}\}$$

of $\{2,3,\dots,\overline{15}\}$, which is indeed 4-rotationally invariant in the above sense. Then one adds 1 and $\overline{1}$ to the zero block $\{2,9,\overline{2},\overline{9}\}$ to obtain the partition shown in Figure 24. If one applies κ_B five times to this partition, then the result is the partition in Figure 25.

Theorem 4.14. *Let N and ν be positive integers with $\nu\geq 2$. Then all 2ν-pseudo-rotationally invariant positive non-crossing partitions of $\{1,2,\dots,N,\overline{1},\overline{2},\dots,\overline{N}\}$ of type B are obtained by starting with a non-crossing partition obtained by the above Construction and applying the pseudo-rotation κ_B repeatedly to it.*

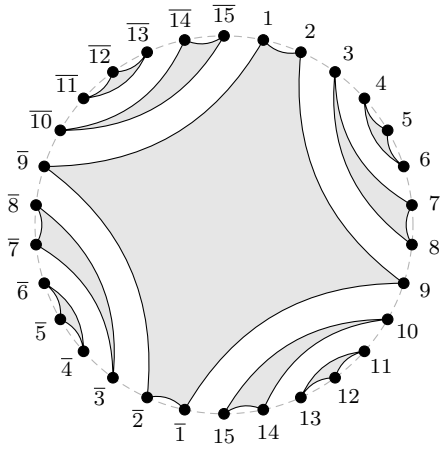

FIGURE 24. A 4-pseudo-rotationally invariant non-crossing partition in $\widetilde{NC}(\pm 15)$ (and also in $\widetilde{NC}_+^{(3)}(B_5)$)

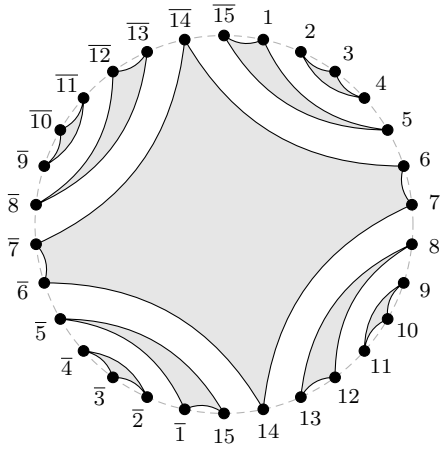

FIGURE 25. Another 4-pseudo-rotationally invariant non-crossing partition in $\widetilde{NC}(\pm 15)$ (and also in $\widetilde{NC}_+^{(3)}(B_5)$)

The proof of this theorem is given in Appendix C.2.

4.6. **Enumeration of pseudo-rotationally invariant elements.** With the characterisation of pseudo-rotationally invariant elements of $\widetilde{NC}(\pm N)$ at hand, we can now embark on the enumeration of such elements. The most refined and most general result in this subsection is Theorem 4.15 which provides a formula for the number of these non-crossing partitions in which the block structure is prescribed.

Theorem 4.15. *Let m, n, a, ν be positive integers with $\nu \mid (mn-1)$. Furthermore, let $b_1, b_2, \dots, b_n$ be non-negative integers. The number of positive m-divisible non-crossing*

partitions of $\{1,2,\dots,mn,\overline{1},\overline{2},\dots,\overline{mn}\}$ *of type* B *which are invariant under the* 2ν*-pseudo-rotation* $\kappa_B^{(mn-1)/\nu}$*, where the number of non-zero blocks of size* mi *is* $2\nu b_i$*,* $i=1,2,\dots,n$*, the zero block having size* $2ma = 2mn - 2m\nu\sum_{j=1}^{n} jb_j > 0$*, is given by*

$$\binom{b_1+b_2+\cdots+b_n}{b_1,b_2,\dots,b_n}\binom{(mn-1)/\nu}{b_1+b_2+\cdots+b_n}. \tag{4.13}$$

Corollary 4.16. *Let* m,n,a,ν *be positive integers with* $\nu\geq 2$ *and* $\nu \mid (mn-1)$*. Furthermore, let* b *be a non-negative integer. The number of positive* m*-divisible non-crossing partitions of* $\{1,2,\dots,mn,\overline{1},\overline{2},\dots,\overline{mn}\}$ *of type* B *which are invariant under the* 2ν*-pseudo-rotation* $\kappa_B^{(mn-1)/\nu}$*, have* $2\nu b$ *non-zero blocks and a zero block of size* $2ma$*, is given by*

$$\binom{(n-a-\nu)/\nu}{b-1}\binom{(mn-1)/\nu}{b} \tag{4.14}$$

if $a\equiv n \pmod{\nu}$ *and* 0 *otherwise, while the number of these partitions without restricting the size of the zero block is*

$$\binom{\lfloor n/\nu\rfloor}{b}\binom{(mn-1)/\nu}{b}. \tag{4.15}$$

Corollary 4.17. *Let* m,n,a,ν *be positive integers with* $\nu\geq 2$ *and* $\nu \mid (mn-1)$*. The number of positive* m*-divisible non-crossing partitions of* $\{1,2,\dots,mn,\overline{1},\overline{2},\dots,\overline{mn}\}$ *of type* B *which are invariant under* $\kappa_B^{(mn-1)/\nu}$ *and have a zero block of size* $2ma$ *is given by*

$$\binom{((m+1)n-a-\nu-1)/\nu}{(n-a)/\nu} \tag{4.16}$$

if $a\equiv n \pmod{\nu}$ *and* 0 *otherwise, while the number of these partitions without restricting the size of the zero block is*

$$\binom{\lfloor((m+1)n-1)/\nu\rfloor}{\lfloor n/\nu\rfloor}. \tag{4.17}$$

Proof. One sums Expressions (4.14) and (4.15) over all b. The resulting sums can be evaluated by means of the Chu–Vandermonde summation formula. □

4.7. **Cyclic sieving.** With the enumeration results in Subsections 4.4 and 4.6, we are now ready to derive our cyclic sieving results in type B; see Theorem 4.19 below. Lemma 4.18 prepares for the proof of this theorem.

Lemma 4.18. *Let* $m,n,s,b_1,b_2,\dots,b_n$ *be integers such that* $0\leq s\leq n$ *and*

$$b_1+2b_2+\cdots+nb_n\leq n. \tag{4.18}$$

Furthermore, let σ be a positive integer such that $\sigma \mid (2mn-2)$. Writing $\rho = 2(mn-1)/\sigma$ and $\omega_\rho = e^{2\pi i\sigma/(2mn-2)} = e^{2\pi i/\rho}$, we have

$$\left.\begin{bmatrix}(m+1)n-1\\ n\end{bmatrix}_{q^2}\right|_{q=\omega_\rho} = \begin{cases}\binom{\lfloor 2((m+1)n-1)/\rho\rfloor}{\lfloor 2n/\rho\rfloor}, & \text{if } \rho \text{ is even},\\ \binom{\lfloor ((m+1)n-1)/\rho\rfloor}{\lfloor n/\rho\rfloor}, & \text{if } \rho \text{ is odd},\end{cases} \tag{4.19}$$

$$\left.\begin{bmatrix}mn-1\\ n\end{bmatrix}_{q^2}\right|_{q=\omega_\rho} = \begin{cases}\binom{mn-1}{n}, & \text{if } \rho = 1 \text{ or } \rho = 2,\\ 0, & \text{otherwise},\end{cases} \tag{4.20}$$

$$\left.\begin{bmatrix}n\\ s\end{bmatrix}_{q^2}\begin{bmatrix}mn-1\\ n-s\end{bmatrix}_{q^2}\right|_{q=\omega_\rho} = \begin{cases}\binom{\lfloor 2n/\rho\rfloor}{\lfloor 2s/\rho\rfloor}\binom{2(mn-1)/\rho}{2(n-s)/\rho}, & \text{if } \rho \text{ is even and } n\equiv s \pmod{\rho/2},\\ \binom{\lfloor n/\rho\rfloor}{\lfloor s/\rho\rfloor}\binom{(mn-1)/\rho}{(n-s)/\rho}, & \text{if } \rho \text{ is odd and } n\equiv s \pmod{\rho},\\ 0, & \text{otherwise},\end{cases} \tag{4.21}$$

$$\left.\begin{bmatrix}b_1+b_2+\cdots+b_n\\ b_1,b_2,\dots,b_n\end{bmatrix}_{q^2}\begin{bmatrix}mn-1\\ b_1+b_2+\cdots+b_n\end{bmatrix}_{q^2}\right|_{q=\omega_\rho}$$
$$= \begin{cases}\binom{2(b_1+b_2+\cdots+b_n)/\rho}{\lfloor 2b_1/\rho\rfloor,\lfloor 2b_2/\rho\rfloor,\dots,\lfloor 2b_n/\rho\rfloor}\binom{2(mn-1)/\rho}{2(b_1+b_2+\cdots+b_n)/\rho}, & \text{if } \rho \text{ is even and } b_k\equiv 0 \pmod{\rho/2} \text{ for all } k,\\ \binom{(b_1+b_2+\cdots+b_n)/\rho}{\lfloor b_1/\rho\rfloor,\lfloor b_2/\rho\rfloor,\dots,\lfloor b_n/\rho\rfloor}\binom{(mn-1)/\rho}{(b_1+b_2+\cdots+b_n)/\rho}, & \text{if } \rho \text{ is odd and } b_k\equiv 0 \pmod{\rho} \text{ for all } k,\\ 0, & \text{otherwise.}\end{cases} \tag{4.22}$$

The proof of this lemma is found in Appendix D.3.

Now we are ready to state, and prove, our cyclic sieving results in type B.

Theorem 4.19. *Let $m, n, s, b_1, b_2, \dots, b_n$ be integers satisfying the hypotheses in Lemma 4.18. Furthermore, let C be the cyclic group of pseudo-rotations of the $2mn$-gon consisting of the elements $\{1, 2, \dots, mn, \overline{1}, \overline{2}, \dots, \overline{mn}\}$ generated by κ_B, viewed as a group of order $2mn-2$. Then the triple (M, P, C) exhibits the cyclic sieving phenomenon for the following choices of sets M and polynomials P:*

(1) *$M = \widetilde{NC}_+^{(m)}(B_n)$, and*

$$P(q) = \begin{bmatrix}(m+1)n-1\\ n\end{bmatrix}_{q^2};$$

(2) *M consists of the elements of $\widetilde{NC}_+^{(m)}(B_n)$ all of whose blocks have size m, and*

$$P(q) = \begin{bmatrix}mn-1\\ n\end{bmatrix}_{q^2};$$

(3) *M consists of all elements of $\widetilde{NC}_+^{(m)}(B_n)$ which have rank s (or, equivalently, their number of non-zero blocks is $2(n-s)$), and*

$$P(q) = \begin{bmatrix} n \\ s \end{bmatrix}_{q^2} \begin{bmatrix} mn-1 \\ n-s \end{bmatrix}_{q^2};$$

(4) *M consists of all elements of $\widetilde{NC}_+^{(m)}(B_n)$ whose number of non-zero blocks of size mi is $2b_i$, $i = 1, 2, \ldots, n$, and*

$$P(q) = \begin{bmatrix} b_1+b_2+\cdots+b_n \\ b_1, b_2, \ldots, b_n \end{bmatrix}_{q^2} \begin{bmatrix} mn-1 \\ b_1+b_2+\cdots+b_n \end{bmatrix}_{q^2}.$$

Proof. The polynomials $P(q)$ in the assertion of the theorem are indeed polynomials with non-negative coefficients since q-binomial and q-multinomial coefficients have this property.

Now, given a positive integer ρ with $\rho \mid (2mn-2)$, we must show

$$P(q)\big|_{q=\omega_\rho} = \text{number of elements of } \widetilde{NC}_+^{(m)}(B_n) \text{ which are invariant under } \kappa_B^{(2mn-2)/\rho},$$

where $\omega_\rho = e^{2\pi i/\rho}$.

We begin with Item (1). If ρ is even, this follows by combining (4.17) with $\nu = \rho/2$ and the first alternative on the right-hand side of (4.19). If ρ is odd, then we claim that an element of $\widetilde{NC}_+^{(m)}(B_n)$ that is invariant under the action of $\kappa_B^{(2mn-2)/\rho}$ is automatically also invariant under $\kappa_B^{(2mn-2)/2\rho} = \kappa_B^{(mn-1)/\rho}$. Indeed, let λ be the multiplicative inverse of 2 modulo ρ. Then there exists a positive integer κ such that $2\lambda = 1 + \rho\kappa$. Thus, we have

$$\lambda \cdot \frac{2mn-2}{\rho} = \frac{2\lambda}{\rho}(mn-1) = \frac{1}{\rho}(mn-1) + \kappa(mn-1). \tag{4.23}$$

According to Theorem 4.10. the order of κ_B is $mn-1$, proving our claim. In this case, the assertion (1) follows by combining (4.17) with $\nu = \rho$ and the second alternative on the right-hand side of (4.19).

Item (2) is the special case of Item (4) where $b_1 = n$ (and, hence, $b_2 = b_3 = \cdots = b_n = 0$), and therefore does not need to be treated separately.

We now turn to Item (3). Again, let first ρ be even. If $\rho = 2$, then the assertion follows immediately from Theorem 4.10 and (4.10). If $\rho > 2$, then, according to Theorem 4.14, any positive m-divisible non-crossing partition which is ρ-pseudo-rotationally invariant arises from the Construction in Subsection 4.5 and subsequent applications of the pseudo-rotation κ_B. As a consequence, the number of non-zero blocks must be divisible by ρ. Hence, for any such non-crossing partition π, we have

$$\#(\text{non-zero blocks of } \pi) = \rho B = 2(n-s),$$

where B is some non-negative integer. In particular, we must have $n \equiv s \pmod{\rho/2}$. The number of these non-crossing partitions is given by Formula (4.15) with $\nu = \rho/2$ and b replaced by $B = 2(n-s)/\rho$. It is not difficult to see that this agrees with the expression in the first alternative on the right-hand side of (4.21).

On the other hand, if ρ is odd, then we reuse the earlier argument that any element of $\widetilde{NC}_+^{(m)}(B_n)$ that is invariant under the action of $\kappa_B^{(2mn-2)/\rho}$ is also invariant under $\kappa_B^{(2mn-2)/2\rho} = \kappa_B^{(mn-1)/\rho}$. Hence, Theorem 4.10, Equation (4.10), Theorem 4.14, and Equation (4.15) with $\nu = \rho$ and b replaced by $(n-s)/\rho$ together confirm that the ρ-pseudo-rotationally invariant elements of $\widetilde{NC}_+^{(m)}(B_n)$ that are relevant for the current case are given by the second alternative on the right-hand side of (4.21).

In all other cases, there are no positive m-divisible non-crossing partitions which are ρ-pseudo-rotationally invariant.

Finally, we address Item (4). Again, we distinguish two cases depending on whether ρ is even or odd.

Let first ρ be even. Repeating the arguments from Item (3), we see that all numbers b_k are divisible by $\rho/2$. If $\rho = 2$, the number of relevant non-crossing partitions is then given by Theorem 4.11. If $\rho > 2$, the number of relevant non-crossing partitions is given by Theorem 4.15 with the appropriate replacements. The resulting expression agrees with the expression in the first alternative on the right-hand side of (4.22).

If ρ is odd, then, by repeating the earlier argument again that any element of $\widetilde{NC}_+^{(m)}(B_n)$ that is invariant under the action of $\kappa_B^{(2mn-2)/\rho}$ is also invariant under $\kappa_B^{(2mn-2)/2\rho} = \kappa_B^{(mn-1)/\rho}$, we infer that all numbers b_k are divisible by ρ. Also here, the number of relevant non-crossing partitions is given by Theorem 4.15 with the appropriate replacements. The resulting expression agrees with the expression in the second alternative on the right-hand side of (4.22).

In all other cases, there are no positive m-divisible non-crossing partitions which are ρ-pseudo-rotationally invariant.

This completes the proof of the theorem. □

Corollary 4.20. *The conclusion of Theorem 2.19 holds for type B_n.*

Proof. The cyclic sieving phenomenon for $\bigl(NC_+^{(m+1)}(B_n, c),\ \mathrm{Cat}_+^{(m+1)}(B_n; q),\ C\bigr)$ follows directly from Theorem 4.19(1) with m replaced by $m+1$.

The image of the embedding of $NC_+^{(m)}(B_n, c)$ in $NC_+^{(m+1)}(B_n, c)$ as described in Proposition 2.14 is given by all tuples $(w_0, w_1, \dots, w_{m+1})$ with $w_0 = \varepsilon$. By Remark 4.3, these tuples correspond to non-crossing set partitions in $\widetilde{NC}^{(m+1)}(B_n)$ all of whose non-zero blocks have size $m+1$. The cyclic sieving phenomenon for $\bigl(NC_+^{(m)}(B_n, c),\ \mathrm{Cat}_+^{(m)}(B_n; q),\ \widetilde{C}\bigr)$ thus follows from Theorem 4.19(2) with m replaced by $m+1$. □

5. Positive non-crossing set partitions in type D

In this section, we study positive m-divisible non-crossing partitions and the positive Kreweras maps in the situation of the even hyperoctahedral group D_n. Recall that D_n can be combinatorially realised as the group of permutations π of $\{1, 2, \dots, n, -1, -2, \dots, -n\}$ satisfying $\pi(-i) = -\pi(i)$ for $i = 1, 2, \dots, n$ which maps an even number of the numbers

$\{1, 2, \dots, n\}$ to a negative number. Again, we shall most often write $\bar{i}$ instead of $-i$. Moreover, the type B cycle notation explained in Section 4 is also used here.

The Coxeter system corresponding to the group D_n is

$$(W, \mathcal{S}) = \big(D_n, \{s_i = ((i, i+1)) \mid 1 \le i \le n-1\} \cup \{s_n = ((n-1, \overline{n}))\}\big) \tag{5.1}$$

together with the Coxeter element $c = s_1 \cdots s_n = [1, 2, \dots, n-1][n]$.

The structure of this section is again the same as the one of Section 3. Namely, in Subsection 5.1 we first make the positivity condition for type D m-divisible non-crossing partitions under the choice (5.1) of Coxeter system explicit; see Proposition 5.1. We then use this to describe *positive* m-divisible non-crossing partitions within the combinatorial model of type D m-divisible non-crossing partitions on an annulus from [KM10] (with the correction reported in [Kim11]); see Theorem 5.4.

In Subsection 5.2, we show that, under the translation described in [KM10, Sec. 7], for $m \ge 2$ the positive Kreweras map acts as a pseudo-rotation on an annulus; see Theorem 5.6. This pseudo-rotation is denoted by κ_D.

In Subsection 5.3, we then show how this realisation naturally gives rise to an extension which also makes sense for $m = 1$; see Definition 5.8. Again it turns out that, under the translation from [KM10], for $m = 1$ this extension corresponds exactly to the map $\mathsf{K}_+^{(1)}$ from Definition 2.16; see Theorem 5.9. The order of the (extended) map κ_D is of course given by Theorem 2.17 for type D. However, for enumerative purposes, we need more refined order properties of κ_D that also take into account the combinatorial structure of the type D positive m-divisible non-crossing partitions. These are the subject of Theorem 5.10. They require *combinatorial* proofs independent of Theorem 2.17. These proofs are found in Appendix A.3.

Subsection 5.4 is devoted to the enumeration of type D positive m-divisible non-crossing partitions. For the sake of simplicity, we content ourselves with the enumeration of all such non-crossing partitions, and those all of whose block sizes are equal to m; see Theorem 5.11.

Subsection 5.5 is then devoted to the characterisation of type D positive m-divisible non-crossing partitions that are invariant under powers of the pseudo-rotation κ_D; see Theorem 5.13. Here, the invariant non-crossing partitions arise from three different constructions.

In Subsection 5.6, we then use this characterisation and, for each of the aforementioned three constructions, provide a formula for the number of corresponding type D positive m-divisible non-crossing partitions that are invariant under powers of the pseudo-rotation κ_D; see Propositions 5.14–5.16.

In the final subsection, Subsection 5.7, we establish two cyclic sieving phenomena which prove Theorem 2.19 in type D; see Theorem 5.20.

5.1. **Combinatorial realisation of the positive non-crossing partitions.** With the choice (5.1) of Coxeter system, we have the following simple combinatorial description of positive m-divisible non-crossing partitions.

Proposition 5.1. *Let m and n be positive integers with $m \geq 2$. The tuple $(w_0, w_1, \dots, w_m) \in NC^{(m)}(D_n, c)$ is positive if and only if $0 < w_m(n-1) \leq n-1$.*

Proof. According to Definition 2.3, the tuple $(w_0, w_1, \dots, w_m)$ is positive if and only if $cw_m^{-1} = w_0 w_1 \cdots w_{m-1}$ has full support in our generators $\{s_1, \dots, s_n\}$ given in (5.1). The cycles in the disjoint cycle decomposition of cw_m^{-1} define a non-crossing partition. Thus, cw_m^{-1} will not have full support if and only if the preimage of 1 under cw_m^{-1} is a positive number or $\overline{n}$. In other words, there is either an $l > 0$ such that $(cw_m^{-1})(l) = 1$, or $(cw_m^{-1})(\overline{n}) = 1$. By acting on the left on both sides of this relation by the inverse of the Coxeter element $c = [1, 2, \dots, n-1][n]$, we see that this is equivalent to either $w_m^{-1}(l) = \overline{n-1}$, or $w_m^{-1}(\overline{n}) = \overline{n-1}$. Again an equivalent statement is that we have either $w_m(n-1) = \bar{l}$, for some $l > 0$, or $w_m(n-1) = n$. We are interested in the contrapositive: cw_m^{-1} has full support if and only if $0 < w_m(n-1) \leq n-1$. □

Next we recall the bijection from [KM10, Sec. 7] between $NC^{(m)}(D_n, c)$ and m-divisible non-crossing partitions on an annulus, with the numbers $\{1, 2, \dots, mn-m, \overline{1}, \overline{2}, \dots, \overline{mn-m}\}$ placed clockwise around the outer circle, the numbers $\{mn-m+1, mn-m+2, \dots, mn, \overline{mn-m+1}, \overline{mn-m+2}, \dots, \overline{mn}\}$ placed *counter-clockwise* around the inner circle, that are invariant under substitution of i by $-i$ for all i, and which satisfy the following additional restrictions:

(D1) Successive elements in a block must be successive numbers when taken modulo m.
(D2) A zero block (see right below for the definition) must contain *all* elements of the inner circle and at least two elements of the outer circle.
(D3) In the case that there is no block that contains elements from the outer *and* the inner circle, the blocks on the inner circle are completely determined by the blocks on the outer circle in the following way: consider a block, B say, on the outer circle that is "visible" from the inner circle. (Here, "visible" means that it is possible to travel from a point of the inner circle to an element of that block without crossing any block connection.) Let b be the "right-most" element in B, which, by definition, is determined by considering the elements of $B \cup (-B)$ in clockwise order and, among these, choosing the element of B that is followed by an element of $-B$. There is a unique $a \equiv b \pmod m$ between $mn-m+1$ and mn. Then one block on the inner circle consists of the m consecutive elements in counter-clockwise order on the inner circle ending in a, and the other block consists of the negatives of these elements.

(Restriction (D3) was missing in [KM10], but is reported in [Kim11, Sec. 7].) We denote these latter m-divisible non-crossing partitions by $\widetilde{NC}^{(m)}(D_n)$.

Again, following [Rei97], we agree here to call a block that is itself invariant under substitution of i by $-i$, for all i, a *zero block*. Furthermore, a block is called *bridging* if it contains elements of the outer *and* the inner circle.

Given an element $(w_0, w_1, \dots, w_m) \in NC^{(m)}(D_n, c)$, the bijection, ∇_{D_n} say, from [KM10, Sec. 7] works essentially in the same way as in type B: namely $(w_0, w_1, \dots, w_m) \in$

$NC^{(m)}(D_n, c)$ is mapped to

$$\nabla_{D_n}(w_0, w_1, \ldots, w_m) = [1, 2, \ldots, m(n-1)]\,[mn-m+1, \ldots, mn-1, mn] \circ (\bar{\tau}_{m,1}(w_1))^{-1}\,(\bar{\tau}_{m,2}(w_2))^{-1} \cdots (\bar{\tau}_{m,m}(w_m))^{-1}, \quad (5.2)$$

where $\bar{\tau}_{m,i}$ is defined as in Section 4 for type B. Again, the cycles in the disjoint cycle decomposition correspond to the blocks in the non-crossing partition in $\widetilde{NC}^{(m)}(D_n)$. We refer the reader to [KM10, Sec. 7] for the details. For example, let $n = 6$, $m = 3$, $w_0 = ((2, \overline{4}))$, $w_1 = ((2, \overline{6}))\,((4, 5))$, $w_2 = ((1, \overline{5}))\,((2, 3))$, and $w_3 = ((3, 6))$. Then $(w_0; w_1, w_2, w_3)$ is mapped to

$$\begin{aligned}\nabla_{D_6}&(w_0; w_1, w_2, w_3)\\ &= [1, 2, \ldots, 15]\,[16, 17, 18]((4, \overline{16}))\,((10, 13))\,((2, \overline{14}))\,((5, 8))\,((9, 18))\\ &= ((1, 2, \overline{15}))\,((3, 4, \overline{17}, \overline{18}, \overline{10}, \overline{14}))\,((5, 9, \overline{16}))\,((6, 7, 8))\,((11, 12, 13)). \end{aligned} \quad (5.3)$$

Similarly as in types A_{n-1} and B_n, the cycle structure of the first component of an m-divisible non-crossing partition determines the block structure of its image under the above bijection ∇_{D_n}.

Proposition 5.2. *Let $(w_0, w_1, \ldots, w_m) \in NC^{(m)}(D_n, c)$. The non-crossing partition $\pi = \nabla_{D_n}(w_0, w_1, \ldots, w_m) \in \widetilde{NC}^{(m)}(D_n)$ has a zero block consisting of $2km+2m$ elements if and only if w_0 contains $[i_1, i_2, \ldots, i_k][n]$ for some $i_1, i_2, \ldots, i_k$ in its disjoint cycle decomposition. Furthermore, π has $2b_k$ non-zero blocks of size mk if and only if w_0 has b_k cycles of the form $((i_1, i_2, \ldots, i_k))$ in its disjoint cycle decomposition.*[7]

In the earlier example, we have $w_0 = ((2, \overline{4})) = ((1))((3))((5))((6))((2, \overline{4}))$. Indeed, the image of (w_0, w_1, w_2, w_3) in the example has no zero block, two non-zero blocks of size $3 \cdot 2 = 6$ and eight non-zero blocks of size $3 \cdot 1 = 3$.

Remark 5.3. A simple consequence of Proposition 5.2 is that the image of the set of all non-crossing partitions in $NC^{(m)}(D_n, c)$ of the form $(\varepsilon, w_1, \ldots, w_m)$ under the map ∇_{D_n} is the set of all non-crossing partitions in $\widetilde{NC}^{(m)}(D_n)$ in which *all* blocks have size m.

Theorem 5.4. *Let m and n be positive integers with $m \geq 2$. The image under ∇_{D_n} of the positive m-divisible non-crossing partitions in $NC^{(m)}(D_n, c)$ are those m-divisible non-crossing partitions in $\widetilde{NC}^{(m)}(D_n)$ where the predecessor of 1 in its block is a negative number on the outer circle of the annulus.*

[7] The reader must recall that a "cycle" of the form $((i_1, i_2, \ldots, i_k))$ consists actually of *two* disjoint cycles of length k, while a cycle of the form $[i_1, i_2, \ldots, i_k]$ is indeed a single cycle, of length $2k$.

Proof. First, let $(w_0, w_1, \dots, w_m) \in NC_+^{(m)}(D_n, c)$. By definition, we know that $0 < w_m(n-1) \le n-1$. Let us write l for $w_m(n-1)$. We then have

$$\begin{aligned}\Big(\bigtriangledown_{D_n}&(w_0, w_1, \dots, w_m)\Big)(\overline{ml}) \\ &= \big([1, 2, \dots, mn-m][mn-m+1, \dots, mn] \\ &\qquad (\tau_{m,1}(w_1))^{-1} (\tau_{m,2}(w_2))^{-1} \cdots (\tau_{m,m}(w_m))^{-1}\big)(\overline{ml}) \\ &= \big([1, 2, \dots, mn-m][mn-m+1, \dots, mn]\big)(\overline{m(n-1)}) = 1.\end{aligned}$$

Translated to non-crossing partitions, this means that $\overline{ml}$ and 1 belong to the same block, which implies our claim.

Conversely, let $\pi \in \widetilde{NC}^{(m)}(D_n)$ be an m-divisible non-crossing partition in which 1 is preceded in its block by a negative number, $\overline{L}$ say, on the outer circle. Since in between $\overline{L}$ and 1 there "sit" blocks all of whose sizes are divisible by m, we must necessarily have $\overline{L} = \overline{ml}$, for some l between 1 and $n-1$. In other words, if π is interpreted as a permutation, $\pi(\overline{ml}) = 1$. Let $(w_0, w_1, \dots, w_m)$ be the element of $NC^{(m)}(D_n, c)$ such that

$$\bigtriangledown_{D_n}(w_0, w_1, \dots, w_m) = \pi.$$

We must have

$$\Big(\bigtriangledown_{D_n}(w_0, w_1, \dots, w_m)\Big)(\overline{ml}) = 1.$$

The definition (5.2) of $\bigtriangledown_{D_n}$ and the fact that $(\tau_{m,i}(w_i))^{-1}$ leaves multiples of m fixed as long as $1 \le i \le m-1$ together imply that $(\tau_{m,m}(w_m))^{-1}(\overline{ml}) = \overline{m(n-1)}$. Phrased differently, we have $(\tau_{m,m}(w_m))^{-1}(ml) = m(n-1)$, or, equivalently, $w_m(n-1) = l > 0$. By definition, this means that $(w_0, w_1, \dots, w_m)$ is positive. □

5.2. **Combinatorial realisation of the positive Kreweras maps.** The next step consists in translating the positive Kreweras map K_+ into a "rotation action" on the elements of $NC_+^{(m)}(D_n, c)$. In order to do so, we need to explicitly describe the decomposition (2.5) in type D.

Lemma 5.5. *Let w be an element of $NC(D_n, c)$, and consider the cycle z in the (type D) disjoint cycle decomposition of w that contains $n-1$.*

1. *If $z = ((i_1, i_2, \dots, i_k, n-1))$ with $0 < i_1 < i_2 < \cdots < i_k < n-1$, then $w^R = \varepsilon$.*
2. *If $z = ((i_1, \dots, i_s, i_{s+1}, \dots, i_k, n-1))$ with $i_1, i_2, \dots, i_s < 0$, $i_{s+1}, \dots, i_{k-1}, i_k > 0$, and $|i_1| < |i_2| < \cdots < |i_k| < n-1$ (as before, the sequence $i_{s+1}, \dots, i_k$ has to be interpreted as the empty sequence if $s = k$), then $w^R = ((i_s, n-1))$.*
3. *If $z = ((i_1, \dots, i_s, n-1, j_1, \dots, j_t, \pm n))$ with $0 < i_1 < i_2 < \cdots < i_s < n-1$ and $0 > j_1 > j_2 > \cdots > j_t > -i_1$, then $w^R = ((n-1, \pm n))$.*
4. *If $z = [i_1, i_2, \dots, i_k, n-1][n]$ (and consequently $0 < i_1 < i_2 < \cdots < i_k < n-1$), then $w^R = ((n-1, n))((n-1, \overline{n})) = [n-1][n]$.*

Here, in Item (3), $\pm n$ *means either* n *or* $\overline{n}$*, and the two alternatives must be read consistently. This covers all possible cases.*

Proof. The last n reflections in $\mathcal{R}$ given in (2.2) are

$$((i, n-1)),\ i = \overline{1}, \overline{2}, \ldots, \overline{n-2},\ \text{and}\ ((n-1, \overline{n})),\ ((n-1, n)), \tag{5.4}$$

in this order. Thus, by definition, we have $w^L = w \circ (w^R)^{-1}$, where the factor w^R consists of a product of these n reflections, and $\ell_{\mathcal{R}}(w^L) + \ell_{\mathcal{R}}(w^R) = \ell_{\mathcal{R}}(w)$. Furthermore, using Proposition 5.1 with $m = 1$, we see that w^L must have the property that $0 < w^L(n-1) \leq n-1$. The arguments below are heavily based on the uniqueness of the decomposition $w = w^l \circ w^R$ in the sense of Lemma 2.9.

In Case (1), we have $w(n-1) = i_1 > 0$. Hence, we may choose $w^L = w$ and $w^R = \varepsilon$, and uniqueness of decomposition guarantees that this is the correct choice.

In Case (2), the choice $w^R = ((i_s, n-1))$ implies that w^L would contain the cycles $((i_1, \ldots, i_s))((i_{s+1}, \ldots, i_k, n-1))$. Hence, we would have $w^L(n-1) = i_{s+1} > 0$ if $s < k$, and $w^L(n-1) = n-1$ if $s = k$. By uniqueness of decomposition, this must be the correct choice.

Case (3) is similar. The choice $w^R = ((n-1, \pm n))$ implies that w^L would contain the cycles $((i_1, i_2, \ldots, i_s, n-1))((j_1, j_2, \ldots, j_t, \pm n))$. Hence, we would have $w^L(n-1) = i_1 > 0$, which, by uniqueness of decomposition, must be the correct choice.

Finally we address Case (4). If we choose $w^R = [n-1][n]$, then w^L would contain the cycle $((i_1, i_2, \ldots, i_k, n-1))$. Hence, we would have $w^L(n-1) = i_1 > 0$. Again, by uniqueness of decomposition, this must be the correct choice. □

Next, we translate the positive Kreweras map K_+ from Definition 2.12 for type D_n into combinatorial language.

Theorem 5.6. *Let* m *and* n *be positive integers with* $m \geq 2$*. Under the bijection* ∇_{D_n}*, the map* $\mathsf{K}_+^{(m)}$ *translates to the following map* κ_D *on* $\widetilde{NC}_+^{(m)}(D_n)$*: if the block of* $\overline{m(n-1)}$ *contains another negative element which is different from* $\overline{mn-1}$*, then* κ_D *rotates all blocks of* π *by one unit (in clockwise direction on the outer circle of the annulus, and in counterclockwise direction on the inner circle).*

On the other hand, if the block of $\overline{m(n-1)}$ *of some element* $\pi \in \widetilde{NC}_+^{(m)}(D_n)$ *contains no other negative element which is different from* $\overline{mn-1}$*, then, first of all,* 1 *is also contained in this block. In other words,* $\overline{m(n-1)}$ *and* 1 *are successive elements in a block (and consequently also* $m(n-1)$ *and* $\overline{1}$*). For the description of the operation, we distinguish between three cases:*

(1) *If there is a block not containing* $1, \overline{1}, m(n-1), \overline{m(n-1)}$ *in which there are positive* and *negative elements from the outer circle, then let* b *be minimal such that* $\overline{d}$ *and* b *are successive elements in such a block, with* $\overline{d}$ *negative and* b *positive. Furthermore, let* b *be followed by* e *in this block, and let* a *and* $\overline{m(n-1)}$ *be successive elements in a block. (The element* a *must be positive and on the outer circle by the assumption*

defining the subcase in which we are[8]*, while* e *can be positive or negative, and on the outer or inner circle. See the left half of Figure 26 for an illustration of the various definitions.) Then the image of* π *under* κ_D *is the partition in which* $\overline{d+1}$, 1, *and* $e+1$ *are three successive elements in a block, and* $a+1$, $b+1$, *and* 2 *are successive elements in another block. All other succession relations in* π *are rotated by one unit in clockwise direction on the outer circle and in counter-clockwise direction on the inner circle. See Figure 26 for a schematic illustration of this operation.*

(2) *If there is no block not containing* $1, \overline{1}, m(n-1), \overline{m(n-1)}$ *in which there are positive* and *negative elements from the outer circle, and if the predecessor of* $\mp mn$*, denoted by* $\overline{d}$ *(with* d *positive), is on the outer circle, then let* $\overline{b}$ *be the successor of* $\mp mn$ *and* a *be the predecessor of* $\overline{m(n-1)}$*. (The element* a *must be positive and on the outer circle or equal to* $\mp(mn-1)$ *by the assumption defining the subcase in which we are. Furthermore, the element* $\overline{b}$ *is either a negative element on the outer circle or equal to* $\pm(mn-m+1)$*. See the left half of Figure 29 for an illustration of the described situation.) Then the image of* π *under* κ_D *is the partition in which* $\overline{d+1}$, 1, *and* $\overline{b+1}$ *are three successive elements in a block, and* $a+1$, $\pm(mn-m+1)$, *and* 2 *are successive elements in another block. All other succession relations in* π *are rotated by one unit in clockwise direction on the outer circle and in counter-clockwise direction on the inner circle. See Figure 29 for a schematic illustration of this operation.*

(3) *If there is no block containing elements from the outer* and *the inner circle, then let* a, $\overline{m(n-1)}$, *and* 1 *be successive elements in a block. (Again, the element* a *must be positive.) By Condition* (D3) *in the definition of* $\widetilde{NC}^{(m)}(D_n)$*, the blocks on the inner circle are* $\{\overline{mn}, mn-m+1, \dots, mn-1\}$ *and its negative. The image of* π *under* κ_D *is the partition in which* $\overline{a+1}$, 1, *and* $\overline{2}$ *are successive elements, other succession relations on the outer circle are rotated by one unit in clockwise direction, while on the inner circle the blocks in the image partition are* $\{mn-m+2, \dots, mn-1, mn, \overline{mn-m+1}\}$ *and its negative. (These are the original blocks rotated by* two *units in counter-clockwise direction) See Figure 33 for a schematic illustration of this operation.*

Remarks 5.7. (i) Figure 26 provides a schematic illustration of the construction in Case (1) of the statement of the above theorem, while Figures 27 and 28 provide two concrete examples. In both examples, $m=3$, $n=5$, $a=2$, $b=6$, $d=11$, and $e=10$. However, they differ in the existence of a zero block: in Figure 27 there is a zero block, while in Figure 28 there is none. Case (2) of the statement of Theorem 5.6 is illustrated in Figure 29, while Figures 30–32 provide three concrete examples. In these examples, $m=4$, $n=6$, $a=\overline{23}$, $b=17$, and $d=19$, respectively $m=3$, $n=6$, $a=\overline{17}$, $b=13$, and $d=14$, respectively $m=3$, $n=6$, $a=23$, $b=21$, and $d=19$. Finally, Figure 33 provides a schematic illustration of the construction in Case (3) of the statement of the

[8]Indeed, it is that block not containing $1, \overline{1}, m(n-1), \overline{m(n-1)}$ in which there are positive *and* negative elements on the outer circle which constitutes a "barrier" making it possible for a to lie on the inner circle.

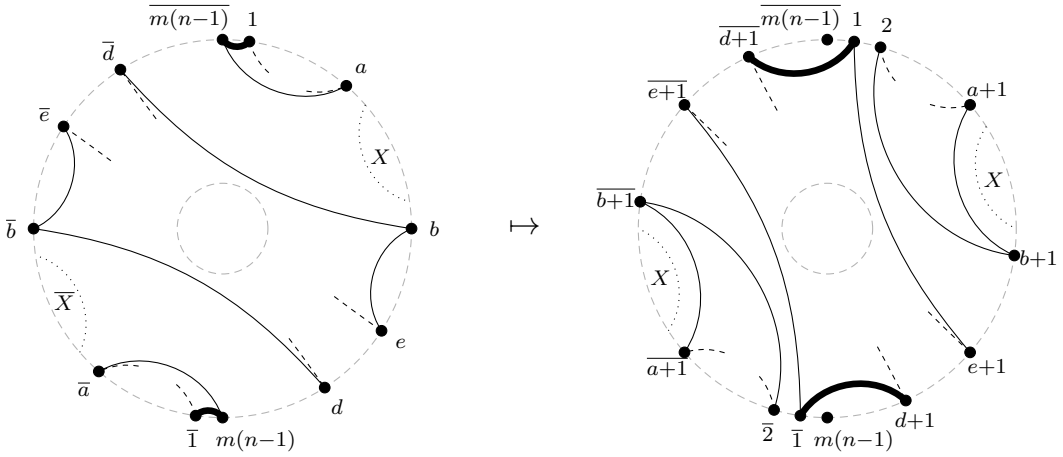

FIGURE 26. The action of the pseudo-rotation κ_D, Case (1)

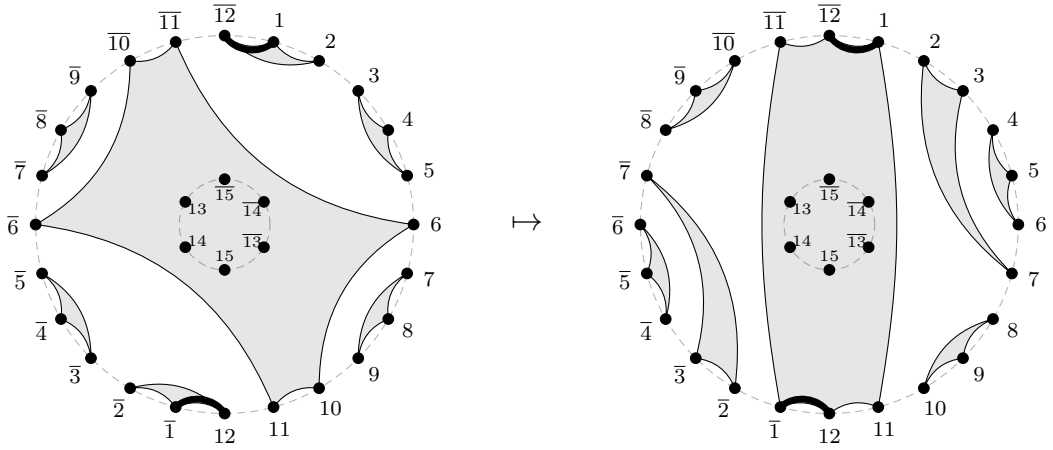

FIGURE 27. The action of the pseudo-rotation κ_D, Case (1)

above theorem, while Figure 34 contains a concrete example, in which $m = 3$, $n = 5$, and $a = 8$.

(ii) Case (2) can be seen as a degenerate special case of Case (1). To see this, one chooses $b = \mp mn$ in Case (1).

Proof of Theorem 5.6. Let us first assume that, in the positive m-divisible type D non-crossing partition π, the block of $\overline{m(n-1)}$ contains another negative element which is different from $\overline{mn-1}$. Among those, let $\overline{mi_1-1}$ be the one which is the predecessor of $\overline{m(n-1)}$ in the block. Obviously, we have $i_1 > 0$. Figure 35 provides sketches of such situations. Let $(w_0, w_1, \ldots, w_m)$ denote the element of $NC_+^{(m)}(D_n, c)$ which corresponds to π under the bijection ∇_{D_n}. By the definition of ∇_{D_n}, we see that $w_{m-1}(n-1) = i_1$.

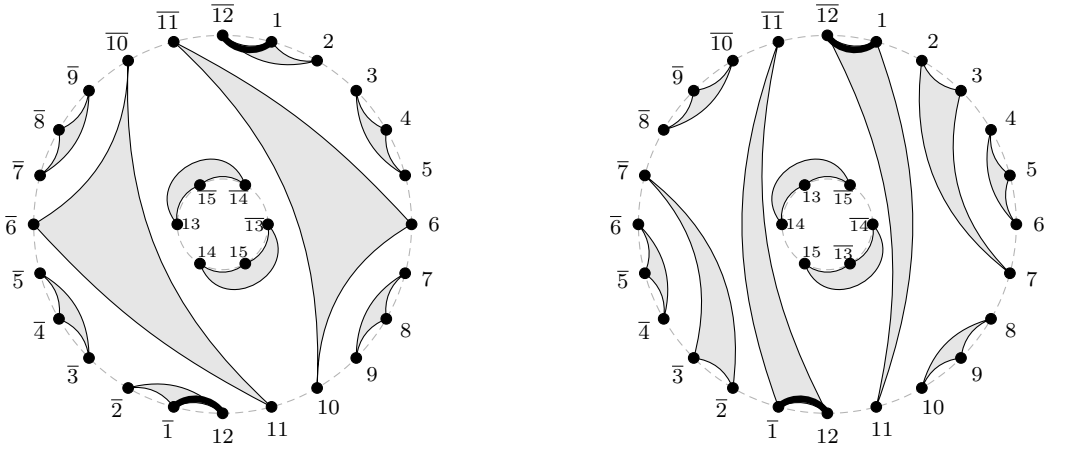

FIGURE 28. The action of the pseudo-rotation κ_D, Case (1)

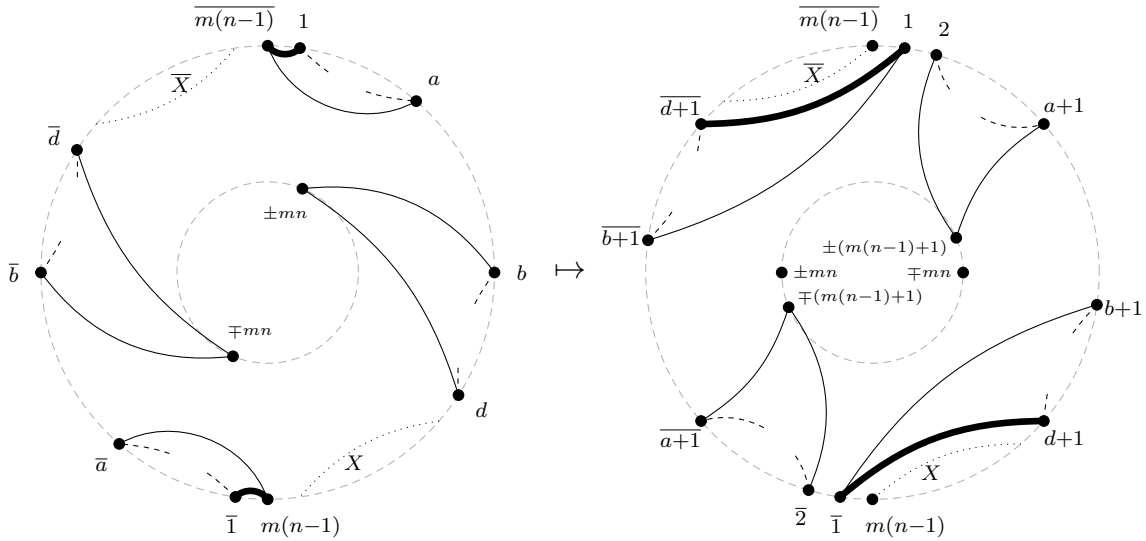

FIGURE 29. The action of the pseudo-rotation κ_D, Case (2)

From Lemma 5.5(1), we infer $w^L_{m-1} = w_{m-1}$ and $w^R_{m-1} = \varepsilon$. This means that the pseudo-rotation κ_D reduces to ordinary rotation in this special case, as we claimed.

For the remainder of the proof, we assume that the block of $\overline{m(n-1)}$ contains no other negative element which is different from $\overline{mn-1}$. By arguments that are completely analogous to the corresponding ones in the proof of Theorem 4.6, one shows that 1 is an element of that block.

Let again $(w_0, w_1, \dots, w_m)$ denote the element of $NC^{(m)}_+(D_n, c)$ which corresponds to π under the bijection ∇_{D_n}.

CASE (1). This is proved in the same way as Case (1) in the proof of Theorem 4.6.

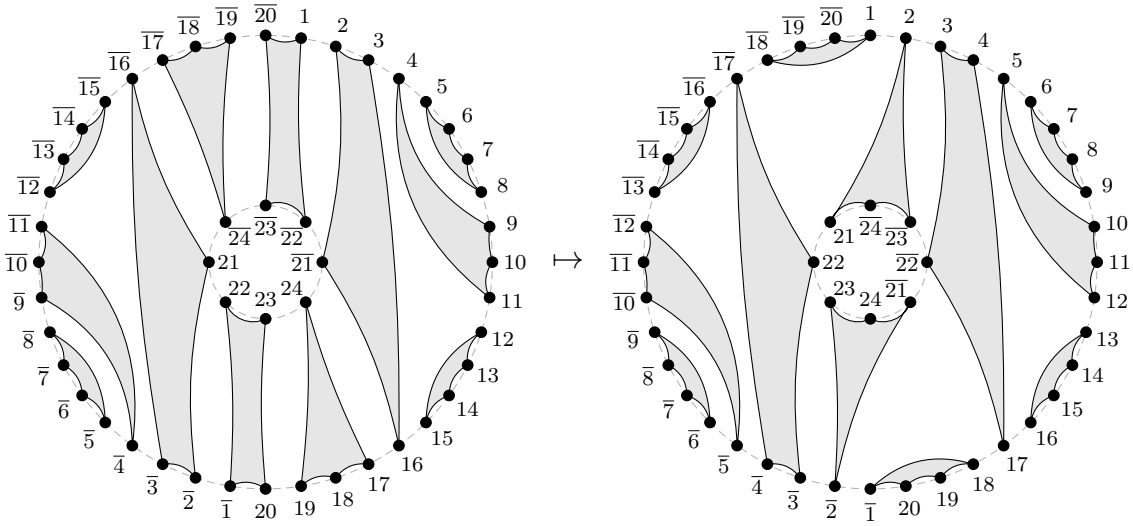

FIGURE 30. The action of the pseudo-rotation κ_D, Case (2)

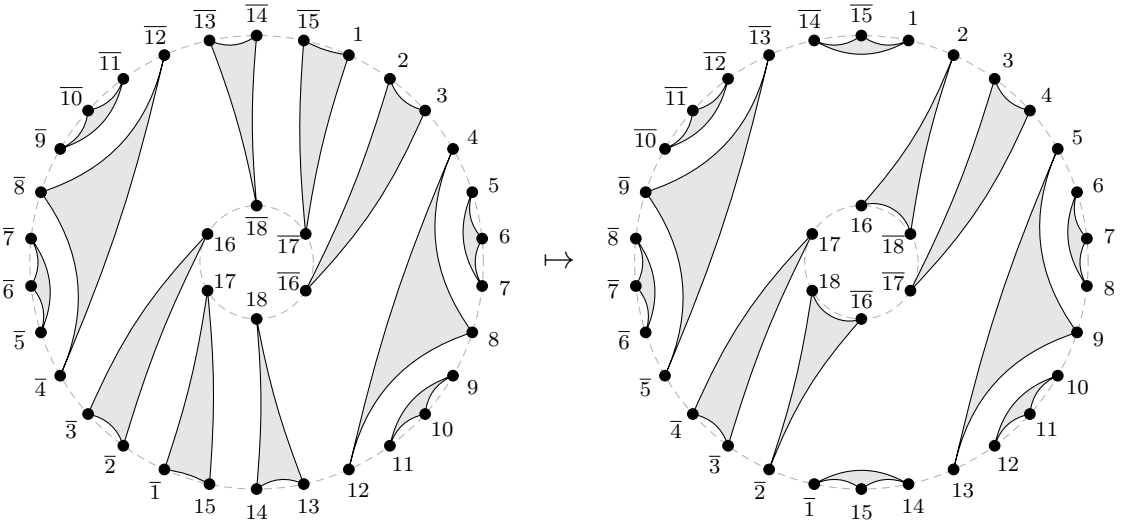

FIGURE 31. The action of the pseudo-rotation κ_D, Case (2)

CASE (2). This case is also similar to Case (1) in the proof of Theorem 4.6. We are at times sketchy here, leaving some details to the reader.

By the assumptions of this case, and by what we already showed, we know that a, $\overline{m(n-1)}$, and 1 are successive elements in a block, and that $\overline{d}$, $\mp mn$ and $\overline{b}$ are successive elements in a block.

We write $a = mA - 1$, $b = mB + 1$, and $d = mD - 1$, for suitable integers A, B, and D. Here, D is positive. The integer A can be between 1 and $n-1$ or equal to $\mp n$, and B can be between A and $D-1$ or equal to $\mp(n-1)$. In view of the above observations, by the

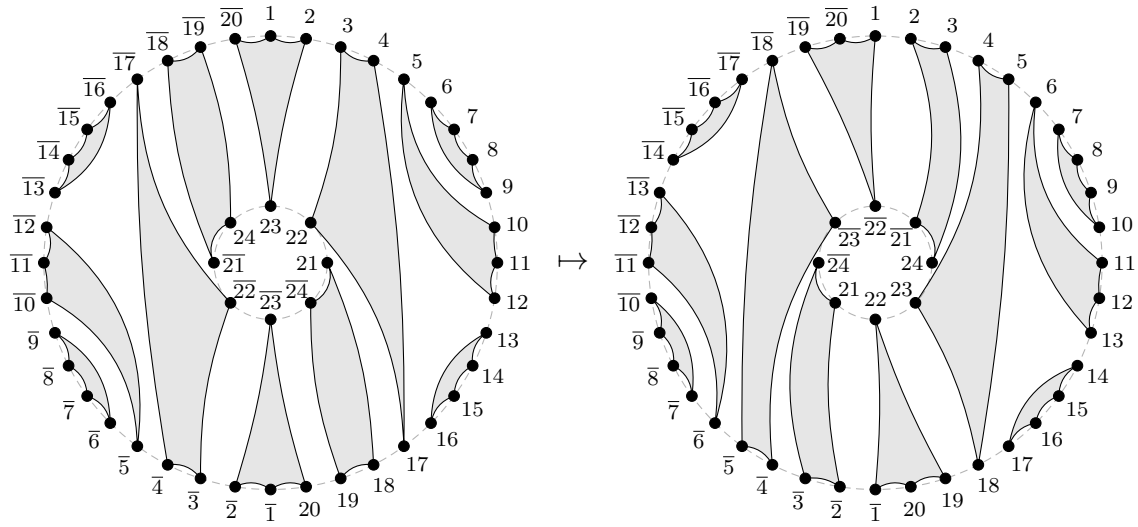

FIGURE 32. The action of the pseudo-rotation κ_D, Case (2)

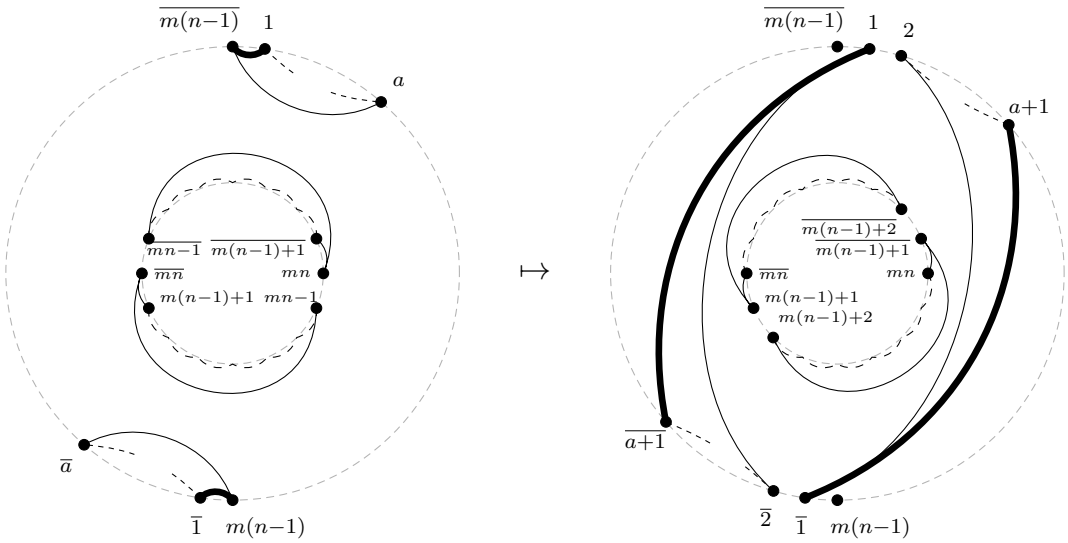

FIGURE 33. The action of the pseudo-rotation κ_D, Case (3)

definition of ∇_{D_n}, we have

$$w_{m-1}(\overline{n-1}) = A,$$
$$w_{m-1}(\pm n) = D,$$
$$w_m(n-1) = n-1, \tag{5.5}$$
$$w_m(B) = \pm n. \tag{5.6}$$

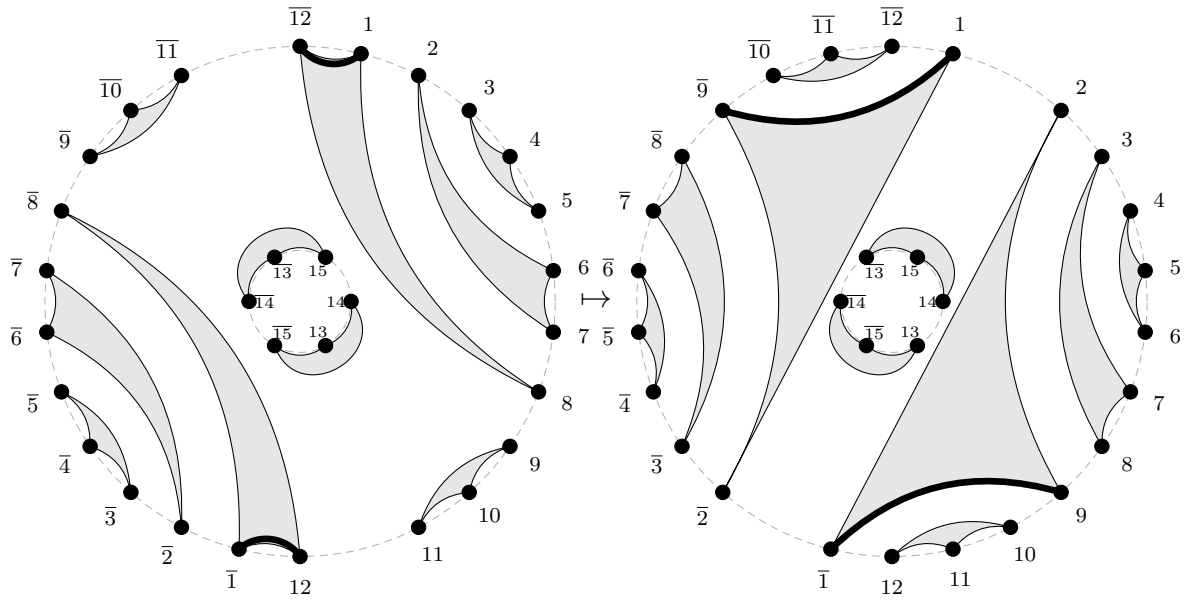

FIGURE 34. The action of the pseudo-rotation κ_D, Case (3)

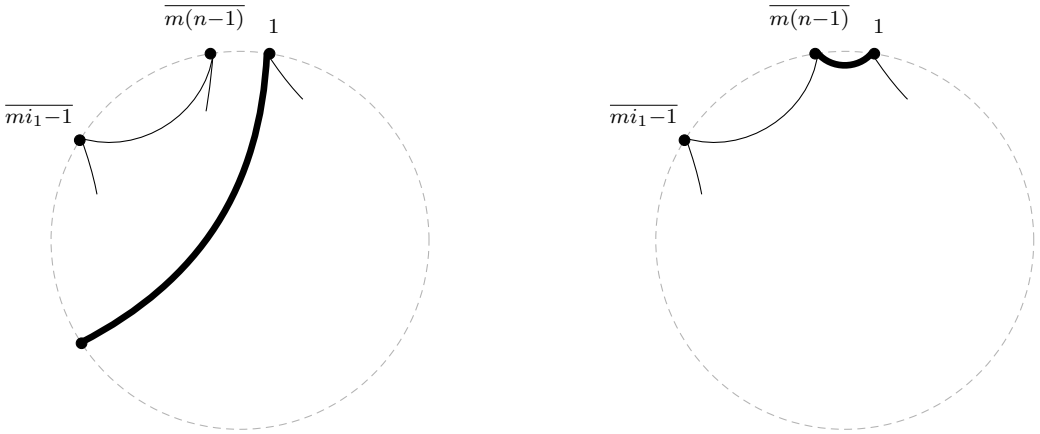

FIGURE 35. The block of $\overline{m(n-1)}$ contains another negative element

Consequently, the cycle of w_{m-1} containing $n-1$ is of the form

$$((D,\dots,n-1,\overline{A},\dots,\pm n)).$$

(At this point, one has to argue that $n-1$ and $\pm n$ are indeed in the same cycle of w_{m-1}.) By Lemma 5.5(3), we have

$$w_{m-1}^R = ((n-1,\pm n)), \quad \text{and} \quad w_{m-1}^L \text{ contains the cycles } ((D,\dots,n-1))((\overline{A},\dots,\pm n)).$$

Together with (5.5) and (5.6), this implies the relations

$$\begin{aligned} w^L_{m-1}(n-1) &= D,\\ w^L_{m-1}(\pm n) &= \overline{A},\\ (cw^R_{m-1}w_m c^{-1})(1) &= \pm n,\\ (cw^R_{m-1}w_m c^{-1})(B+1) &= \overline{1}. \end{aligned}$$

By the definition of ∇_{D_n}, this means that $\overline{d+1}$, 1, and $\overline{b+1}$ are successive elements in their block in the image of π under $\mathsf{K}^{(m)}_+$ (conjugated by ∇_{D_n}), that $a+1$, $\pm(mn-m+1)$, and 2 are successive elements in their block, while all other "block connections" are rotated by one unit in clockwise direction on the outer circle and by one unit in counter-clockwise direction on the inner circle. This is in accordance with the corresponding assertion in the proposition.

Case (3). We are again somewhat sketchy here. We write again $a = mA - 1$ for a suitable positive integer A. By the definition of ∇_{D_n}, we have

$$\begin{aligned} w_{m-1}(n-1) &= \overline{A},\\ w_{m-1}(n) &= \overline{n},\\ w_m(n-1) &= n-1, \qquad (5.7)\\ w_m(n) &= n. \qquad (5.8) \end{aligned}$$

Consequently, the cycle of w_{m-1} containing $n-1$ is of the form

$$[A,\dots,\dots,n-1][n].$$

By Lemma 5.5(4), we have

$$w^R_{m-1} = [n-1][n], \quad \text{and} \quad w^L_{m-1} \text{ contains the cycles } ((A,\dots,n-1))((n)).$$

Together with (5.7) and (5.8), this implies the relations

$$\begin{aligned} w^L_{m-1}(n-1) &= A,\\ w^L_{m-1}(n) &= n,\\ (cw^R_{m-1}w_m c^{-1})(1) &= \overline{1},\\ (cw^R_{m-1}w_m c^{-1})(n) &= \overline{n}. \end{aligned}$$

By the definition of ∇_{D_n}, this means that $\overline{a+1}$, 1, and $\overline{2}$ are successive elements in their block in the image of π under $\mathsf{K}^{(m)}_+$ (conjugated by ∇_{D_n}), that mn, $\overline{mn-m+1}$, and $mn-m+2$ are successive elements in their block, while the other "block connections" are rotated by one unit in clockwise direction on the outer circle and by *two* units in counter-clockwise direction on the inner circle. This is in accordance with the corresponding assertion in the proposition.

This completes the proof of the proposition. □

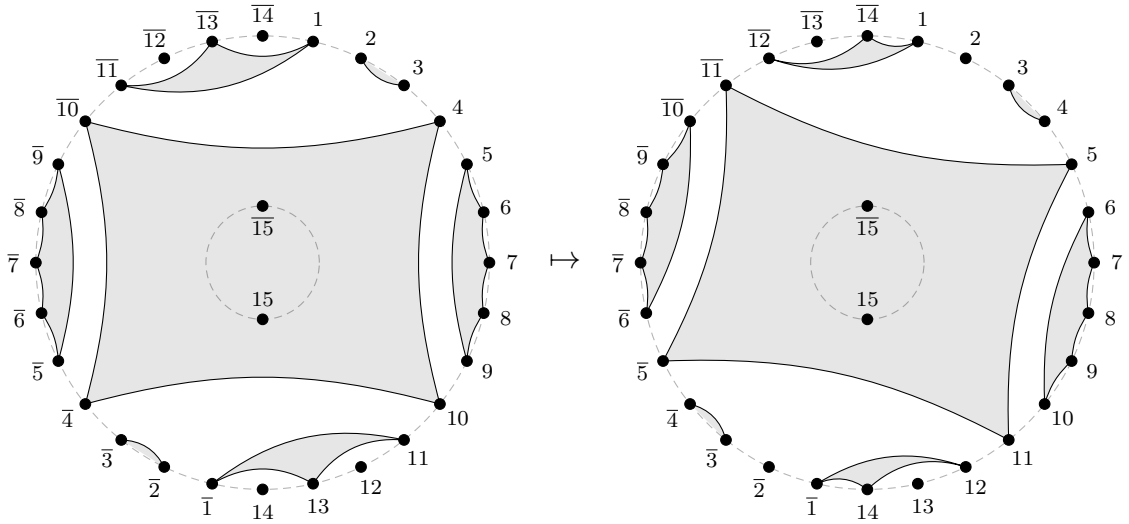

FIGURE 36. Example of the action of the pseudo-rotation κ_D under presence of singleton blocks

5.3. **The combinatorial positive Kreweras maps.** The definition of κ_D from Theorem 5.6 to make sense on an *arbitrary* non-crossing partition needed that blocks should be of size at least 2. It is a simple matter to extend this so that it makes sense for non-crossing partitions without any restrictions.

Definition 5.8. We define the action of κ_D on a positive type D non-crossing partition π of $\{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$ as in Theorem 5.6 with $m = 1$ (cf. Figures 26, 29 and 33), except when the block containing $n-1$ is a singleton block (in which case also $\{\overline{n-1}\}$ is a singleton block; here, Theorem 5.6 would not make sense). If $\{n-1\}$ is a singleton block, then the image of κ_D is the non-crossing partition which arises from π by replacing the singleton block $\{n-1\}$ by the singleton block $\{\overline{2}\}$, by replacing the singleton block $\{\overline{n-1}\}$ by the singleton block $\{2\}$, and i by $i+1$ and $\overline{i}$ by $\overline{i+1}$ for $i = 2, 3, \ldots, n-2$ in each of the remaining blocks of π, leaving 1, $\overline{1}$, n, and $\overline{n}$ untouched; see Figure 36 for an example with $n = 15$.

As it turns out, this definition is also in agreement with the image under ∇_{D_n} of the map $\mathsf{K}_+^{(1)}$ in Definition 2.16 for the case where $m = 1$.

Theorem 5.9. *Let n be a positive integer. Under the bijection ∇_{D_n}, the map $\mathsf{K}_+^{(1)}$ from Definition 2.16 translates to the combinatorial map κ_D described in Definition 5.8.*

Since the proof of the theorem is completely analogous to the proof of Theorem 3.9, we omit it here.

We conclude with the following theorem describing the order of the combinatorial map κ_D. In view of our previous discussion, the first part of the theorem follows from Corollary 2.18 for the type D_N with $m = 1$. We provide a direct, combinatorial, proof

which also allows us to show the more refined assertions in the second part of the theorem. See Appendix A.3.

Theorem 5.10. *Let m and n be positive integers. The order of the map κ_D acting on $\widetilde{NC}_+^{(m)}(D_n)$ is $m(n-1)-1$ if n is even, and it is equal to $2m(n-1)-2$ if n is odd. More precisely, given an element π of $\widetilde{NC}_+^{(m)}(D_n)$ without bridging block, we have $\kappa_D^{m(n-1)-1}(\pi) = \pi$. On the other hand, if $\pi \in \widetilde{NC}_+^{(m)}(D_n)$ contains a bridging block, then $\kappa_D^{m(n-1)-1}(\pi) = \pi$ if n is even, while only $\kappa_D^{2m(n-1)-2}(\pi) = \pi$ if n is odd.*

5.4. **Enumeration of positive non-crossing partitions.** Here we present our enumeration results for positive non-crossing partitions in $\widetilde{NC}_+^{(m)}(D_n)$. In the theorem below we give the total number of these non-crossing partitions, and as well the number of those of them in which all block sizes are equal to m.

Theorem 5.11. *Let m and n be positive integers. The number of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn, \overline{1}, \overline{2}, \dots, \overline{mn}\}$ of type D equals*

$$\frac{2m(n-1)+n-2}{n}\binom{(m+1)(n-1)-1}{n-1}, \tag{5.9}$$

while the number of these partitions of which all blocks have size m equals

$$\frac{2m(n-1)-n}{n}\binom{m(n-1)-1}{n-1}. \tag{5.10}$$

Proof. Elements of $\widetilde{NC}_+^{(m)}(D_n)$ may or may not have a bridging block. The case of those without bridging block may be reduced to a type B situation. For the other case, we refer to the special case of an enumeration result from Subsection 5.6.

To be precise, as we argue in the proof of Theorem 5.10 given in Appendix A.3, partitions in $\widetilde{NC}_+^{(m)}(D_n)$ without a bridging block are in bijection with partitions in $\widetilde{NC}_+^{(m)}(B_{n-1})$. By (4.12) with n replaced by $n-1$, the number of these partitions is

$$\binom{(m+1)(n-1)-1}{n-1}.$$

On the other hand, a partition in $\widetilde{NC}_+^{(m)}(D_n)$ with a bridging block must arise from Construction 3 in Subsection 5.5 below with $\nu = 1$. By Proposition 5.15 below with $\nu = 1$, the number of these partitions is

$$2\binom{(m+1)(n-1)-1}{n}.$$

It is easy to verify that the sum of these two numbers equals (5.9).

Similarly, by (4.10) with $b = n$ and subsequent replacement of n by $n-1$, the number of partitions in $\widetilde{NC}_+^{(m)}(D_n)$ without a bridging block, all of whose blocks have size m, is

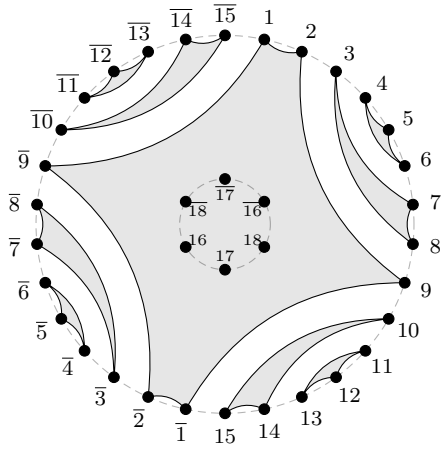

FIGURE 37. A 4-pseudo-rotationally invariant non-crossing partition in $\widetilde{NC}_+^{(3)}(D_6)$

given by

$$\binom{m(n-1)-1}{n-1}.$$

Moreover, by Proposition 5.17 below with $\nu = 1$, the number of partitions in $\widetilde{NC}_+^{(m)}(D_n)$ that contain a bridging block and all of whose blocks have size m, is given by

$$2\binom{m(n-1)-1}{n}.$$

It is easy to verify that the sum of these two numbers equals (5.10). □

5.5. **Characterisation of pseudo-rotationally invariant elements.** In this subsection, we describe the elements of $\widetilde{NC}_+^{(m)}(D_n)$ that are invariant under a power of κ_D. We call an element of $\widetilde{NC}_+^{(m)}(D_n)$ *ρ-pseudo-rotationally invariant* if it is invariant under the action of $\kappa_D^{2(m(n-1)-1)/\rho}$.

We achieve the description of pseudo-rotationally invariant elements in $\widetilde{NC}_+^{(m)}(D_n)$ by providing three constructions of such elements (see below) together with Theorem 5.13 which then tells us how to obtain all such invariant elements from the constructions.

CONSTRUCTION 1. We start with a 2ν-pseudo-rotationally invariant positive non-crossing partition of type B of $\{1, 2, \ldots, m(n-1), \overline{1}, \overline{2}, \ldots, \overline{m(n-1)}\}$, all of whose blocks have size divisible by m. By Theorem 4.14, this partition contains a zero block. We add the elements $\{m(n-1)+1, \ldots, mn, \overline{m(n-1)+1}, \ldots, \overline{mn}\}$ of the inner circle to this zero block, thereby creating a 2ν-pseudo-rotationally invariant positive m-divisible non-crossing partition of type D. We illustrate this construction in the case where $\nu = 2$, $m = 3$, and $n = 6$. We

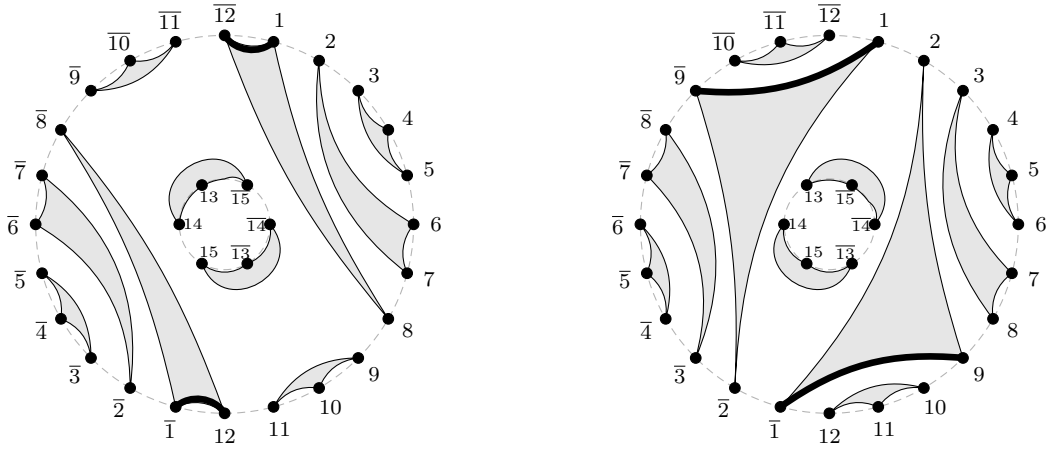

FIGURE 38. Two 2-pseudo-rotationally invariant non-crossing partitions in $\widetilde{NC}_{+}^{(3)}(D_5)$

start with the non-crossing partition

$$\{\{1,2,9,\overline{1},\overline{2},\overline{9}\},\{3,7,8\},\{4,5,6\},\{10,14,15\},\{11,12,13\},\\ \{\overline{3},\overline{7},\overline{8}\},\{\overline{4},\overline{5},\overline{6}\},\{\overline{10},\overline{14},\overline{15}\},\{\overline{11},\overline{12},\overline{13}\}\}$$

of $\{1,2,\dots,\overline{15}\}$, which is indeed 4-pseudo-rotationally invariant in the above sense (see Figure 24). Then one adds the elements $\{16,17,18,\overline{16},\overline{17},\overline{18}\}$ of the inner circle to the zero block; see Figure 37.

CONSTRUCTION 2. We start with a positive non-crossing partition of type B of $\{1,2,\dots,m(n-1),\overline{1},\overline{2},\dots,\overline{m(n-1)}\}$ without zero block, all of whose blocks have size divisible by m. We remark that this partition has order $m(n-1)-1$ as type B non-crossing partition; see Theorem 4.10. We then add the elements $\{m(n-1)+1,\dots,mn,\overline{m(n-1)+1},\dots,\overline{mn}\}$ of the inner circle, which are partitioned into two blocks of size m according to the rule (D3) in Subsection 5.1. Thereby we create a positive m-divisible non-crossing partition of type D which has order $m(n-1)-1$ (see Theorem 5.10). Thus, this partition is 2-pseudo-rotationally invariant. We illustrate this construction in the case where $m=3$ and $n=5$. We start with the type B positive non-crossing partition

$$\{\{1,8,\overline{12}\},\{2,6,7\},\{3,4,5\},\{9,10,11\},\{\overline{1},\overline{8},12\},\{\overline{2},\overline{6},\overline{7}\},\{\overline{3},\overline{4},\overline{5}\},\{\overline{9},\overline{10},\overline{11}\}\}$$

of $\{1,2,\dots,\overline{12}\}$ (see Figure 17). Then we add the elements $\{13,14,15,\overline{13},\overline{14},\overline{15}\}$ on the inner circle, partitioned into the two blocks $\{\overline{15},13,14\}$ and $\{15,\overline{13},\overline{14}\}$ according to Condition (D3) in the definition of $\widetilde{NC}^{(m)}(D_n)$. See the left part of Figure 38. On the right, the result of application of κ_D (according to Case (3) of Theorem 5.6) is shown.

CONSTRUCTION 3. According to Lemma 5.12 below, this construction only applies if $\nu \mid n$ and $\nu \mid (m+1)$.

We start with a block B of size divisible by m that contains $\overline{m(n-1)}$ as only negative element on the outer circle, and that contains 1 and possibly further elements; if it contains

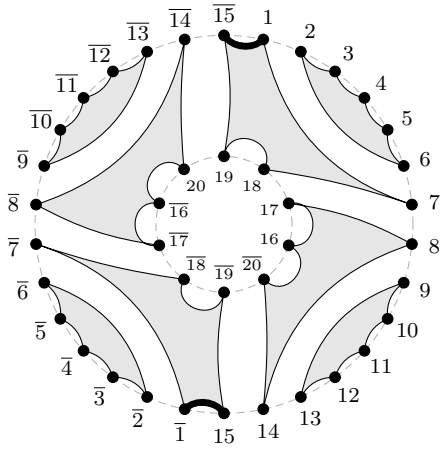

FIGURE 39. A 4-pseudo-rotationally invariant non-crossing partition in $\widetilde{NC}_+^{(5)}(D_4)$

elements from the inner circle then, according to rule (D1) in Subsection 5.1, among these there must be $mn-1$ or $\overline{mn-1}$). To this block, we add more blocks of size divisible by m in a non-crossing fashion such that, together, they form a non-crossing partition of successive elements on the outer circle starting with $\overline{m(n-1)}, 1, \dots$, and of x successive elements on the inner circle. (According to rule (D1), the latter must end in $mn-1$ or $\overline{mn-1}$ when read counter-clockwise.) We call this (partial) non-crossing partition the *fundamental seed*. We place $\nu - 1$ *modified* copies along the annulus, which, together with the fundamental seed, cover the elements $\{\overline{m(n-1)}, 1, 2, \dots, m(n-1)-1\}$ on the outer circle and the elements $\{\mp mn, \pm(m(n-1)+1), \pm(m(n-1)+2), \dots, \pm(mn-1)\}$ on the inner circle. The modification consists in deleting the "first" element from B on the outer circle (that is, $\overline{m(n-1)}$) and replacing it by an element on the inner circle. A copy of the obtained partition is then formed by replacing i by $-i$, for all i. Together, a positive m-divisible non-crossing partition of type D which is ν-pseudo-rotationally invariant is obtained. We illustrate this construction in the case where $\nu = 2$, $m = 5$, and $n = 4$; cf. Figure 39. We start with the block $\{\overline{15}, 1, 7, 18, 19\}$ and augment it to the partial non-crossing partition

$$\{\{\overline{15}, 1, 7, 18, 19\}, \{2, 3, 4, 5, 6\}\}.$$

A modified copy of this fundamental seed is formed,

$$\{\{8, 14, \overline{20}, 16, 17\}, \{9, 10, 11, 12, 13\}\},$$

and added to it. Finally, a copy of the obtained partition is formed by replacing i by $-i$, for all i. Together, a positive m-divisible non-crossing partition of type D is obtained; see again Figure 39. This non-crossing partition is indeed 4-pseudo-rotationally invariant, that is, it is invariant under the action of κ_D^7.

Lemma 5.12. *Let $\nu \mid \big(m(n-1)-1\big)$. Construction 3 is only applicable if $\nu \mid n$ and $\nu \mid (m+1)$. Moreover, the number of elements in the fundamental seed equals mn/ν.*

Proof. The two conditions are in fact equivalent. Indeed, if $\nu \mid n$, then $\nu \mid (m+1)$ follows from $\nu \mid \big(m(n-1)-1\big)$. Conversely, if $\nu \mid (m+1)$, then, again due to $\nu \mid \big(m(n-1)-1\big)$, we infer $\nu \mid mn$. Since ν and m must be mutually prime because of $\nu \mid (m+1)$, it follows that $\nu \mid n$.

In order to show that $\nu \mid (m+1)$, we write x for the number of elements on the inner circle that are contained in the fundamental seed according to Construction 3. Since the $\nu-1$ modified copies of the fundamental seed that are placed around the annulus in Construction 3 have one more element on the inner circle than the fundamental seed itself, and since the fundamental seed together with its $\nu-1$ copies covers one half of the inner circle, we have

$$x + (\nu-1)(x+1) = m,$$

or, equivalently,

$$\nu(x+1) = m+1. \tag{5.11}$$

This implies that $\nu \mid (m+1)$.

The second part of the assertion of the lemma is obvious since the fundamental seed together with its $\nu-1$ copies covers half of the annulus, that is, mn elements. □

Theorem 5.13. *Let m, n, ρ be positive integers with $\rho \geq 2$. Then all ρ-pseudo-rotationally invariant positive non-crossing partitions of $\{1, 2, \dots, mn, \overline{1}, \overline{2}, \dots, \overline{mn}\}$ of type D are obtained by starting with a non-crossing partition obtained by Construction 1, 2, or 3, and applying the pseudo-rotation κ_D repeatedly to it.*

More precisely, a partition arising from Construction 1 is 2ν-pseudo-rotationally invariant. Furthermore, a partition arising from Construction 2 is 2-pseudo-rotationally invariant. Finally, a partition arising from Construction 3 is 2ν-pseudo-rotationally invariant if n/ν is even, and it is ν-pseudo-rotationally invariant if n/ν is odd.

The proof of this theorem is given in Appendix C.3.

5.6. **Enumeration of pseudo-rotationally invariant elements.** With the characterisation of pseudo-rotationally invariant elements of $\widetilde{NC}_+^{(m)}(D_n)$ at hand, we can now embark on the enumeration of such elements. Proposition 5.14 gives the total number of elements of $\widetilde{NC}_+^{(m)}(D_n)$ that arise from Constructions 1 and 2, while Proposition 5.15 gives the total number of elements of $\widetilde{NC}_+^{(m)}(D_n)$ that arise from Constructions 3. Proposition 5.16 and Proposition 5.17 provide the cardinalities of the subsets of those elements all of whose block sizes are m.

Proposition 5.14. *The total number of elements of $\widetilde{NC}_+^{(m)}(D_n)$ that arise from Constructions 1 and 2 in Subsection 5.5 equals*

$$\binom{(m+1)(n-1)-1}{n-1}. \tag{5.12}$$

If ρ is even, then the number of elements of $\widetilde{NC}_+^{(m)}(D_n)$ that are ρ-pseudo-rotationally invariant and arise from Constructions 1 and 2 in Subsection 5.5 equals

$$\binom{\lfloor 2((m+1)(n-1)-1)/\rho \rfloor}{\lfloor 2(n-1)/\rho \rfloor}. \tag{5.13}$$

Proof. As we already argued in Theorem 5.10 and Theorem 5.13, the elements of $\widetilde{NC}_+^{(m)}(D_n)$ that are ρ-pseudo-rotationally invariant and arise from Constructions 1 and 2 (the latter is only possible for $\rho = 1$ and $\rho = 2$) are in bijection with elements of $\widetilde{NC}_+^{(m)}(B_{n-1})$ that are ρ-rotationally invariant. This is seen by starting with an element of $\widetilde{NC}_+^{(m)}(D_n)$ and deleting the inner circle, thereby obtaining an element of $\widetilde{NC}_+^{(m)}(B_{n-1})$. Clearly, this map is reversible. If $\rho = 1$, then, by (4.12) with n replaced by $n-1$, this number equals (5.12). If ρ is even, then, by (4.17) with n replaced by $n-1$ and $\nu = \rho/2$, this number equals (5.13). □

Proposition 5.15. *The total number of elements of $\widetilde{NC}_+^{(m)}(D_n)$ that arise from Construction 3 in Subsection 5.5 equals*

$$2\binom{((m+1)(n-1)-\nu)/\nu}{n/\nu}. \tag{5.14}$$

The proof of this proposition is postponed to Appendix B.6.

Proposition 5.16. *The total number of elements of $\widetilde{NC}_+^{(m)}(D_n)$ that arise from Constructions 1 and 2 in Subsection 5.5 with all block sizes equal to m is*

$$\binom{m(n-1)-1}{n-1}. \tag{5.15}$$

If ρ is even, then the number of elements of $\widetilde{NC}_+^{(m)}(D_n)$ that are ρ-pseudo-rotationally invariant and arise from Constructions 1 and 2 in Subsection 5.5, and all of whose blocks have size m, equals

$$\binom{\lfloor 2(m(n-1)-1)/\rho \rfloor}{\lfloor 2(n-1)/\rho \rfloor}. \tag{5.16}$$

Proof. As we argued before, the elements of $\widetilde{NC}_+^{(m)}(D_n)$ that are ρ-pseudo-rotationally invariant and arise from Constructions 1 and 2 (the latter is only possible for $\rho = 1$ and $\rho = 2$) are in bijection with elements of $\widetilde{NC}_+^{(m)}(B_{n-1})$ that are ρ-rotationally invariant. If $\rho = 1$, then, by (4.10) with $b = n$ and subsequent replacement of n by $n-1$, this number equals (5.15). If ρ is even, then, by (4.15) with $b = \lfloor n/\nu \rfloor$ and subsequent replacement of n by $n-1$ and setting $\nu = \rho/2$, this number equals (5.16). □

Proposition 5.17. *The total number of elements of $\widetilde{NC}_+^{(m)}(D_n)$ that arise from Construction 3 in Subsection 5.5 and all of whose blocks have size m equals*

$$2\binom{(m(n-1)-1)/\nu}{n/\nu}. \tag{5.17}$$

The proof of this proposition is also postponed to Appendix B.6.

5.7. **Cyclic sieving.** With the enumeration results in Subsections 5.4 and 5.6, we are now ready to derive our cyclic sieving results in type D; see Theorem 5.20 below. Lemmas 5.18 and 5.19 prepare for the proof of this theorem.

Lemma 5.18. *Let m and n be positive integers. Then the expression*

$$\frac{[2m(n-1)+n-2]_q}{[n]_q}\begin{bmatrix}(m+1)(n-1)-1\\ n-1\end{bmatrix}_{q^2} \tag{5.18}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[2m(n-1)+n-2]_q}{[n]_q}\begin{bmatrix}(m+1)(n-1)-1\\ n-1\end{bmatrix}_{q^2} = \begin{bmatrix}(m+1)(n-1)-1\\ n-1\end{bmatrix}_{q^2} + q^n\,(1+q^n)\begin{bmatrix}(m+1)(n-1)-1\\ n\end{bmatrix}_{q^2}.$$

Since q-binomial coefficients are polynomials in q with non-negative coefficients, the claim is obviously true. □

Lemma 5.19. *Let m and n be integers. Furthermore, let σ be a positive integer such that $\sigma \mid (2m(n-1)-2)$. Writing $\rho = 2(m(n-1)-1)/\sigma$ and $\omega_\rho = e^{2\pi i\sigma/(2m(n-1)-2)} = e^{2\pi i/\rho}$,*

we have

$$\frac{[2m(n-1)+n-2]_q}{[n]_q}\begin{bmatrix}(m+1)(n-1)-1\\ n-1\end{bmatrix}_{q^2}\Bigg|_{q=\omega_\rho}$$
$$=\begin{cases}\frac{2m(n-1)+n-2}{n}\binom{\lfloor 2((m+1)(n-1)-1)/\rho\rfloor}{\lfloor 2(n-1)/\rho\rfloor}, & \text{if } \rho \text{ is even and } \rho \mid n,\\ \binom{\lfloor 2((m+1)(n-1)-1)/\rho\rfloor}{\lfloor 2(n-1)/\rho\rfloor}, & \text{if } \rho \text{ is even and } \rho \nmid n,\\ \frac{2m(n-1)+n-2}{n}\binom{\lfloor ((m+1)(n-1)-1)/\rho\rfloor}{\lfloor (n-1)/\rho\rfloor}, & \text{if } \rho \text{ is odd and } \rho \mid n,\\ \binom{\lfloor ((m+1)(n-1)-1)/\rho\rfloor}{\lfloor (n-1)/\rho\rfloor}, & \text{if } \rho \text{ is odd and } \rho \nmid n,\end{cases} \tag{5.19}$$

$$\frac{[2m(n-1)-n]_q}{[n]_q}\begin{bmatrix}m(n-1)-1\\ n-1\end{bmatrix}_{q^2}\Bigg|_{q=\omega_\rho}$$
$$=\begin{cases}\frac{(2m(n-1)-n)}{n}\binom{m(n-1)-1}{n-1}, & \text{if } \rho=2 \text{ and } n \text{ is even},\\ \binom{m(n-1)-1}{n-1}, & \text{if } \rho=2 \text{ and } n \text{ is odd},\\ 2\binom{2(m(n-1)-1)/\rho}{2n/\rho}, & \text{if } \rho \text{ is even, } \rho\geq 4, \text{ and } \rho \mid n,\\ \frac{2m(n-1)-n}{n}\binom{m(n-1)-1}{n-1}, & \text{if } \rho=1,\\ 2\binom{(m(n-1)-1)/\rho}{n/\rho}, & \text{if } \rho \text{ is odd, } \rho\geq 3, \text{ and } \rho \mid n,\\ 0, & \text{otherwise.}\end{cases} \tag{5.20}$$

Due to its technical nature, the proof of this lemma is deferred to the end of Appendix D.

Theorem 5.20. *Let m and n be positive integers. Furthermore, let C be the cyclic group of pseudo-rotations of the annulus with $\{1,2,\dots,m(n-1),\overline{1},\overline{2},\dots,\overline{m(n-1)}\}$ on the outer circle and $\{m(n-1)+1,\dots,mn,\overline{m(n-1)+1},\dots,\overline{mn}\}$ on the inner circle generated by κ_D, viewed as a group of order $2m(n-1)-2$. Then the triple (M,P,C) exhibits the cyclic sieving phenomenon for the following choices of sets M and polynomials P:*

(1) *$M=\widetilde{NC}_+^{(m)}(D_n)$, and*
$$P(q)=\frac{[2m(n-1)+n-2]_q}{[n]_q}\begin{bmatrix}(m+1)(n-1)-1\\ n-1\end{bmatrix}_{q^2};$$

(2) *M consists of the elements of $\widetilde{NC}_+^{(m)}(D_n)$ all of whose blocks have size m, and*
$$P(q)=\frac{[2m(n-1)-n]_q}{[n]_q}\begin{bmatrix}m(n-1)-1\\ n-1\end{bmatrix}_{q^2}.$$

Proof. The polynomials $P(q)$ in the assertion of the theorem are indeed polynomials with non-negative coefficients due to Lemma 5.18.

Now, given a positive integer ρ with $\rho \mid (2m(n-1)-2)$, we must show

$$P(q)\big|_{q=\omega_\rho}=\text{number of elements of } \widetilde{NC}_+^{(m)}(D_n) \text{ which are invariant under } \kappa_D^{(2m(n-1)-2)/\rho}, \tag{5.21}$$

where $\omega_\rho=e^{2\pi i/\rho}$.

We begin with Item (1). Let first ρ be even. By (5.13), there are

$$\binom{\lfloor 2((m+1)(n-1)-1)/\rho \rfloor}{\lfloor 2(n-1)/\rho \rfloor} \tag{5.22}$$

elements of $\widetilde{NC}_+^{(m)}(D_n)$ that are ρ-pseudo-rotationally invariant and do not contain a bridging block (and hence arise from Constructions 1 and 2 in Subsection 5.5).

Moreover, by Theorem 5.13, there exist elements of $\widetilde{NC}_+^{(m)}(D_n)$ that are ρ-pseudo-rotationally invariant and *do* contain a bridging block (and hence arise from Construction 3 in Subsection 5.5) only if $n/(\rho/2)$ is even or if n/ρ is odd. In both cases, it follows that $\rho \mid n$, and, due to Proposition 5.15 with $\nu = \rho/2$, there are then

$$2\binom{(2(m+1)(n-1)-\rho)/\rho}{2n/\rho} \tag{5.23}$$

such elements of $\widetilde{NC}_+^{(m)}(D_n)$ that arise from Construction 3. The sum of (5.22) and (5.23) equals the first alternative on the right-hand side of (5.19), which confirms (5.21) in this case.

On the other hand, if $\rho \nmid n$, then there are no ρ-pseudo-rotationally invariant elements of $\widetilde{NC}_+^{(m)}(D_n)$ that arise from Construction 3. Consequently, the expression in (5.22) is already equal to the total number of ρ-pseudo-rotationally invariant elements, which agrees with the second alternative on the right-hand side of (5.19). This confirms (5.21) in this other case.

Now let ρ be odd. By an argument that is analogous to the one around (4.23), we infer that a ρ-pseudo-rotationally invariant element of $\widetilde{NC}_+^{(m)}(D_n)$ without a bridging block (and thus arising from Construction 1 or 2 in Subsection 5.5) is automatically also 2ρ-pseudo-rotationally invariant. By (5.13) with ρ replaced by 2ρ, there are

$$\binom{\lfloor ((m+1)(n-1)-1)/\rho \rfloor}{\lfloor (n-1)/\rho \rfloor} \tag{5.24}$$

elements of $\widetilde{NC}_+^{(m)}(D_n)$ that are 2ρ-pseudo-rotationally invariant (and also ρ-pseudo-rotationally invariant) and do not contain a bridging block.

Moreover, by Theorem 5.13, there exist elements of $\widetilde{NC}_+^{(m)}(D_n)$ that are ρ-pseudo-rotationally invariant and *do* contain a bridging block (and hence arise from Construction 3 in Subsection 5.5) only if $\rho \mid n$, and, due to Proposition 5.15 with $\nu = \rho$, there are then

$$2\binom{((m+1)(n-1)-\rho)/\rho}{n/\rho} \tag{5.25}$$

such elements of $\widetilde{NC}_+^{(m)}(D_n)$ that arise from Construction 3. The sum of (5.24) and (5.25) equals the third alternative on the right-hand side of (5.19), which confirms (5.21) in this case.

On the other hand, if $\rho \nmid n$, then there are no ρ-pseudo-rotationally invariant elements of $\widetilde{NC}_+^{(m)}(D_n)$ that arise from Construction 3. Consequently, the expression in (5.24) is already equal to the total number of ρ-pseudo-rotationally invariant elements, which agrees with the fourth alternative on the right-hand side of (5.19). This confirms (5.21) in this other case.

We turn to Item (2). We will be brief here since the arguments are very similar to those above addressing Item (1). Again, let first ρ be even.

If $\rho = 2$ and n is even, then the assertion follows immediately from (5.16) with $\rho = 2$, and from Theorem 5.13 and Proposition 5.17 with $\nu = 1$ and comparison with the first alternative on the right-hand side of (5.20).

If $\rho = 2$ and n is odd, then, according to Theorem 5.13 with $\nu = 1$, elements of $\widetilde{NC}_+^{(m)}(D_n)$ that are 2-pseudo-rotationally invariant and all of whose blocks have size m cannot arise from Construction 3. The assertion follows from (5.16) with $\rho = 2$ and comparison with the second alternative on the right-hand side of (5.20).

If ρ is even and $\rho \geq 4$, then an element of $\widetilde{NC}_+^{(m)}(D_n)$ which is ρ-pseudo-rotationally invariant and of which all blocks have size m cannot arise from Construction 1 since a zero block must contain all elements of the inner circle and therefore has more than $2m$ elements. It also cannot arise from Construction 2 since, according to Theorem 5.13, these elements are 2-pseudo-rotationally invariant (but not pseudo-rotationally invariant of higher order). Consequently, such an element must arise from Construction 3, either with $\nu = \rho/2$ and where $n/(\rho/2)$ is even, or with $\nu = \rho$ and where n/ρ is odd. In both cases, it follows that $\rho \mid n$. The number of these elements is then given by Proposition 5.17 with $\nu = \rho/2$ which agrees with the third alternative on the right-hand side of (5.20).

Now let ρ be odd. If $\rho = 1$, comparison of (5.15) and the fourth alternative on the right-hand side of (5.20) confirms (5.21) in this case.

On the other hand, if $\rho \geq 3$, then, as above in the case of even $\rho \geq 4$, ρ-pseudo-rotationally invariant elements of $\widetilde{NC}_+^{(m)}(D_n)$ all of whose blocks have size m cannot arise from Construction 1 or 2. Consequently, such elements must arise from Construction 3 with $\nu = \rho$. The number of these elements is given by Proposition 5.17 with $\nu = \rho$, which agrees with the fifth alternative on the right-hand side of (5.20).

In all other cases, there are no positive m-divisible non-crossing partitions which are ρ-pseudo-rotationally invariant.

This completes the proof of the theorem. □

Corollary 5.21. *The conclusion of Theorem 2.19 holds for type D_n.*

Proof. The cyclic sieving phenomenon for $\left(NC_+^{(m+1)}(D_n, c),\ \mathrm{Cat}_+^{(m+1)}(D_n; q),\ C\right)$ follows directly from Theorem 5.20(1) with m replaced by $m+1$.

The image of the embedding of $NC_+^{(m)}(D_n, c)$ in $NC_+^{(m+1)}(D_n, c)$ as described in Proposition 2.14 is given by all tuples $(w_0, w_1, \ldots, w_{m+1})$ with $w_0 = \varepsilon$. By Remark 5.3, these tuples

correspond to non-crossing set partitions in $\widetilde{NC}^{(m+1)}(D_n)$ all of whose non-zero blocks have size $m+1$. The cyclic sieving phenomenon for $\left(NC_+^{(m)}(D_n,c),\ \mathrm{Cat}_+^{(m)}(D_n;q),\ \widetilde{C}\right)$ thus follows from Theorem 5.20(2) with m replaced by $m+1$. □

6. Positive m-divisible non-crossing partitions in dihedral and exceptional types

In this section, we study the action of the two positive Kreweras maps in the dihedral and the exceptional types. The main tool will be a general approach for counting (positive) m-divisible non-crossing partitions that are invariant under powers of the positive Kreweras maps. We then apply this approach to each type individually to derive the proposed cyclic sieving phenomena.

It is well known that any two Coxeter elements c and c' are conjugate, and that this conjugation induces a bijection between $NC_+^{(m)}(W,c)$ and $NC_+^{(m)}(W,c')$. This bijection is moreover $\mathsf{K}_\circ$-equivariant. (Recall that $\mathsf{K}_\circ = \mathsf{K}_\circ^{(m)}$ has been defined in Subsection 2.4.) It therefore suffices to prove the proposed cyclic sieving phenomena in Theorem 2.19 for the *bipartite Coxeter element* $c = c_A c_B$ given by the decomposition $\mathcal{S} = A \sqcup B$ such that all simple generators in A pairwise commute, all simple generators in B pairwise commute, and where we write c_A and c_B for the products of the generators in A and in B. This has the computational advantage that we have

$$\mathrm{w}_\circ(\mathrm{c}) = \underbrace{\mathrm{c}_A\ \mathrm{c}_B\ \mathrm{c}_A\ \cdots}_{h}. \tag{6.1}$$

That is, $\mathrm{w}_\circ(\mathrm{c})$ is formed, as a word, of h alternating copies of c_A and c_B where we recall that h is the Coxeter number. We thus do not need to consider the situation "up to commutations". The word $\mathbf{inv}_c^m\mathbf{inv}_c^L$ then consists of $m+1$ concatenated copies of $\mathbf{inv}_c$ with the last n letters removed, and this word equals

$$\mathbf{inv}_c^m\mathbf{inv}_c^L = \underbrace{\mathrm{c}_A\ \mathrm{c}_B\ \mathrm{c}_A\ \cdots}_{(m+1)h-2}. \tag{6.2}$$

We refer to Example 6.6 for several examples and note that $\mathbf{inv}_c^{m-1}\mathbf{inv}_c^L$ consists of $mh-2$ alternating copies of c_A and c_B.

This additional regularity (that is in general missing for arbitrary Coxeter elements) allows us to describe subwords that are invariant under powers of the positive Kreweras maps.

Let $(m+1)h-2 = k\cdot\rho$, where we assume ρ to be even if $\psi \equiv \varepsilon$. (The situation $mh-2 = k\cdot\rho$ is obtained by simply replacing m by $m-1$.) We always consider the application of $(\mathsf{K}_\circ)^k$ which is a $(\rho/2)$-fold symmetry if $\psi \equiv \varepsilon$ and an ρ-fold symmetry if $\psi \not\equiv \varepsilon$ (except in dimension 2 with $m=1$, compare Theorem 2.17). It follows that the orbit of a letter r inside $\mathbf{inv}_c^m\mathbf{inv}_c^L$ contains $\rho/2$ letters if and only if either $\psi \equiv \varepsilon$ or ρ is even with $\psi(r) = r$. This yields the following first important restriction concerning the symmetries we need to consider.

Lemma 6.1. *Let* $\mathrm{r}_1 \cdots \mathrm{r}_{n'}$ *be a subword of* $\mathbf{inv}_c^m \mathbf{inv}_c^L$ *which is invariant under the application of* $(\mathsf{K}_\circ)^k$*. Then* ρ *divides* $2n'$*.*

Proof. As the subword is invariant when applying $(\mathsf{K}_\circ)^k$, it must be the union of orbits of letters of this operation. The statement follows as we have observed that every orbit contains either $\rho/2$ or ρ many letters. □

When counting positive m-divisible non-crossing partitions that are invariant under powers of either of the two positive Kreweras maps, we only consider subwords of $\mathbf{inv}_c^m \mathbf{inv}_c^L$ that are reduced factorisations of non-crossing partitions. We therefore only consider subwords of length at most n. In other words, we only need to consider parameters ρ dividing $2n'$ for some $n' = \ell_{\mathcal{R}}(w) \leq n$ with $w \in NC(W, c)$.

After we have seen that we need to understand only a small number of symmetries, we next discuss how to find the appropriate numerical parameters for the situation that one considers for a ρ- or $(\rho/2)$-fold symmetry, respectively. To this end, we make use of the following lemma, explaining for which parameters m we have that ρ divides $(m+1)h - 2$.

Lemma 6.2. *Let* α, β, γ *be coprime natural numbers, that is,* $\gcd(\alpha, \beta, \gamma) = 1$*, and let* $\mathcal{Y}$ *be the set of integers* Y *such that* α *divides* $\beta \cdot Y + \gamma$*. Then*

(1) $\mathcal{Y} = \emptyset$ *if* $\gcd(\alpha, \beta) > 1$*, and*
(2) $\mathcal{Y} = \{Y \mid Y \equiv Y_0 \pmod{\alpha}\}$ *for a fixed* Y_0 *if* $\gcd(\alpha, \beta) = 1$*.*

Proof. Suppose Y is an integer such that α divides $\beta \cdot Y + \gamma$, or, phrased differently, $\beta \cdot Y + \gamma \equiv 0 \pmod{\alpha}$.

If $\gcd(\alpha, \beta) > 1$, then that common divisor of α and β does not divide γ (by coprimeness of all three), so no solution exists. Otherwise β has a multiplicative inverse modulo α, say β^{-1} with $\beta^{-1}\beta \equiv 1 \pmod{\alpha}$. Multiplication of both sides of the congruence by β^{-1} gives $Y \equiv -\beta^{-1}\gamma \pmod{\alpha}$, and hence all solutions form the single residue class $\{Y_0 + k\alpha \mid k \in \mathbb{Z}\}$, with $Y_0 \equiv -\beta^{-1}\gamma \pmod{\alpha}$. □

One now applies this lemma to the situation $mh + (h-2) = k \cdot \rho$, where we want to determine all possible parameters m for a given parameter ρ. First, we set

$$y = \begin{cases} 2, & \text{if } \rho \text{ and } h \text{ are even,} \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 6.2 then says that there exists an m such that ρ divides $hm + (h-2)$ if and only if ρ/y and h/y are coprime. In this case, m is given by the fixed residue class m' modulo ρ/y, $0 \leq m' < \rho/y$, given by

$$m' = \tfrac{2-h}{y}(h/y)^{-1} \pmod{\rho/y}, \tag{6.3}$$

and we moreover obtain

$$\tfrac{y}{\rho}\big((m+1)h - 2\big) = ah + \tfrac{y}{\rho}\big((m'+1)h - 2\big) \tag{6.4}$$

for the parameter a given by

$$m = a\tfrac{\rho}{y} + m',$$

and integral $\frac{y}{\rho}\big((m'+1)h-2\big)$. We thus define in this case the reflection $\mathrm{r}_{\mathrm{fix}}$ inside $\mathbf{inv}_c$ to be the letter in position

$$i_{\mathrm{fix}} = \tfrac{n}{2} \cdot \tfrac{y}{\rho}\big((m'+1)h-2\big) + 1, \tag{6.5}$$

and $\mathbf{inv}_c^{L'}$ to be the subword of $\mathbf{inv}_c$ consisting of the first $i_{\mathrm{fix}}-1$ letters (so that $\mathrm{r}_{\mathrm{fix}}$ is the first letter in $\mathbf{inv}_c$ not contained in $\mathbf{inv}_c^{L'}$). This yields the following lemma.

Lemma 6.3. *We have:*

(i) *if h is even, the length of $\mathbf{inv}_c^{L'}$ is a multiple of n;*
(ii) *if h is odd, the length of $\mathbf{inv}_c^{L'}$ is a multiple of $n/2$.*

In the second case in the lemma, we recall in Lemma 6.5 that n is indeed even.

Proof of Lemma 6.3. If h is even, we consider the two situations that ρ is even or odd. If ρ is even then $y=2$ and

$$\tfrac{n}{2} \cdot \tfrac{y}{\rho}\big((m'+1)h-2\big) = n \cdot \tfrac{1}{\rho}\big((m'+1)h-2\big),$$

and ρ divides $\big((m'+1)h-2\big)$. If ρ is odd then $y=1$ and

$$\tfrac{n}{2} \cdot \tfrac{y}{\rho}\big((m'+1)h-2\big) = n \cdot \tfrac{1}{\rho}\big((m'+1)h/2-1\big),$$

and ρ divides $\big((m'+1)h/2-1\big)$.

If h is odd, then $y=1$ and

$$\tfrac{n}{2} \cdot \tfrac{y}{\rho}\big((m'+1)h-2\big) = \tfrac{n}{2} \cdot \tfrac{1}{\rho}\big((m'+1)h-2\big)$$

and ρ divides $\big((m'+1)h-2\big)$. □

We refer the reader to Example 6.6 were we give several examples of the above computations. Next, we use this description of m as a residue class modulo ρ/y to obtain another important restriction. Let r be a letter in $\mathbf{inv}_c^m\mathbf{inv}_c^L$ and let $\mathrm{r}_1\cdots\mathrm{r}_{\rho'}$ be the subword of $\mathbf{inv}_c^m\mathbf{inv}_c^L$ obtained from r under the application of $(\mathsf{K}_\circ)^k$ (in particular, $\rho' \in \{\rho/2, \rho\}$). We then have the following lemma.

Lemma 6.4. *We have that*

(1) *the sequence $r_1, r_2, \dots, r_{\rho'}$ of reflections in the subword $\mathrm{r}_1\cdots\mathrm{r}_{\rho'}$, and*
(2) *the order in which the letters $\mathrm{r}_1, \mathrm{r}_2, \dots, \mathrm{r}_{\rho'}$ are visited when applying $(\mathsf{K}_\circ)^k$*

are independent of the parameter m.

Proof. The above considerations imply that both the sequence $r_1, r_2, \dots, r_{\rho'}$ of reflections and the order in which the letters are visited do not change when passing between the parameters $m = a\frac{\rho}{y} + m'$ and $m + \frac{\rho}{y} = m + (a+1)\frac{\rho}{y} + m'$. □

From now on, we distinguish several disjoint cases which in total cover all possible situations:

(1) The Coxeter number h is odd.
(2) The Coxeter number h is even and
  (a) $\psi \equiv \varepsilon$, that is, $w_\circ s w_\circ = s$ for all $s \in \mathcal{S}$,

(b) $\psi \not\equiv \varepsilon$ and ρ is even,
(c) $\psi \not\equiv \varepsilon$ and ρ is odd.

Here, we observe that it is well-known that $\psi \equiv \varepsilon$ implies that h is even, as seen in the following auxiliary lemma.

Lemma 6.5. *We have h is odd if and only if $c_B = \psi(c_A)$, and h is even if and only if $\psi(c_A) = c_A, \psi(c_B) = c_B$. In particular, n is even if h is odd.*

Proof. This follows from Equation (6.1) together with the observation that c_A, c_B, and $w_\circ$ are involutions. As $\psi(c_A) = c_B$ implies that $|A| = |B|$ and $\mathcal{S} = A \sqcup B$, we deduce (the well-known property) that $n = |\mathcal{S}|$ is even in this case. □

This lemma implies that

$$(\mathrm{c}_A\mathrm{c}_B)^h = \mathrm{w}_\circ(\mathrm{c})\psi(\mathrm{w}_\circ(\mathrm{c})), \tag{6.6}$$

without commutations in Case (2a), respectively in Case (1) with the additional assumptions that $\mathrm{c}_B = \psi(\mathrm{c}_A)$ and $\mathrm{c} = \mathrm{c}_A\mathrm{c}_B$. On the other hand, (6.6) cannot hold in Cases (2b) and (2c) as in both we have $c_A = \psi(c_A)$ inside W but $\mathrm{c}_A \neq \psi(\mathrm{c}_A)$ as words.

Before continuing, we look at explicit examples of all these four cases.

Example 6.6. To visualise the previous analysis, we consider all four situations.

(i) In type A_4, we are in Case (1). In this situation with $m = 3$, we have

$$\mathbf{inv}_c^3\mathbf{inv}_c^L \longleftrightarrow \underset{\mathbf{1}}{\mathrm{c}_A}\ \underset{12}{\mathrm{c}_B}\ \underset{5}{\mathrm{c}_A}\ \underset{\mathbf{16}}{\mathrm{c}_B}\ \underset{9}{\mathrm{c}_A} \mid \underset{2}{\mathrm{c}_B}\ \underset{\mathbf{13}}{\mathrm{c}_A}\ \underset{6}{\mathrm{c}_B}\ \underset{17}{\mathrm{c}_A}\ \underset{\mathbf{10}}{\mathrm{c}_B} \mid \underset{3}{\mathrm{c}_A}\ \underset{14}{\mathrm{c}_B}\ \underset{\mathbf{7}}{\mathrm{c}_A}\ \underset{18}{\mathrm{c}_B}\ \underset{11}{\mathrm{c}_A} \mid \underset{\mathbf{4}}{\mathrm{c}_B}\ \underset{15}{\mathrm{c}_A}\ \underset{8}{\mathrm{c}_B}$$

where the numbers below all copies of c_A and of c_B indicate the order in which the orbit of one letter in the first copy of c_A is visited by applying $\mathsf{K}_\circ$. One can thus consider $k \in \{1, 2, 3, 6, 9, 18\}$ as $(m + 1)h - 2 = 18$. For example, the numbers $1, 16, 13, 10, 7, 4$ (highlighted above) sit in every k^{th} alternating copy of c_A and c_B, here with $k = 3$.

(ii) In type B_3, we are in Case (2a). In this situation with $m = 2$, we have

$$\mathbf{inv}_c^3\mathbf{inv}_c^L \longleftrightarrow \underset{\mathbf{1}}{\mathrm{c}_A}\ \underset{4}{\mathrm{c}_B}\ \underset{\mathbf{7}}{\mathrm{c}_A}\ \underset{8}{\mathrm{c}_B}\ \underset{\mathbf{5}}{\mathrm{c}_A}\ \underset{2}{\mathrm{c}_B} \mid \underset{\mathbf{3}}{\mathrm{c}_A}\ \underset{6}{\mathrm{c}_B} \mid \underset{2}{\mathrm{c}_A}\ \underset{\mathbf{5}}{\mathrm{c}_B}\ \underset{8}{\mathrm{c}_A}\ \underset{\mathbf{3}}{\mathrm{c}_B}\ \underset{6}{\mathrm{c}_A}\ \underset{\mathbf{1}}{\mathrm{c}_B} \mid \underset{4}{\mathrm{c}_A}\ \underset{\mathbf{7}}{\mathrm{c}_B}$$

where the numbers below all copies of c_A indicate the order in which the orbit of one letter in the first copy of c_A is visited by applying $\mathsf{K}_\circ$. One can thus consider $k \in \{1, 2, 4\}$ as $(m + 1)h - 2 = 8$. For example, the numbers $1, 7, 5, 3$ sit in every $(2k)^{\text{th}}$ alternating copy of c_A and c_B, here with $k = 2$.

(iii) In type A_3, we are in Case (2b) or in Case (2c). In this situation with $m = 4$, we have

$$\begin{aligned}\mathbf{inv}_c^3\mathbf{inv}_c^L &\longleftrightarrow \underset{\substack{\mathbf{1}\\\mathbf{10}}}{\mathrm{c}_A}\ \underset{\substack{6\\15}}{\mathrm{c}_B}\ \underset{\substack{2\\11}}{\mathrm{c}_A}\ \underset{\substack{\mathbf{7}\\\mathbf{16}}}{\mathrm{c}_B} \mid \underset{\substack{3\\12}}{\mathrm{c}_A}\ \underset{\substack{8\\17}}{\mathrm{c}_B}\ \underset{\substack{\mathbf{4}\\\mathbf{13}}}{\mathrm{c}_A}\ \underset{\substack{9\\18}}{\mathrm{c}_B} \mid \underset{\substack{5\\14}}{\mathrm{c}_A}\ \mathrm{c}_B \\ &\longleftrightarrow \underset{\substack{\mathbf{1}\\10}}{\mathrm{c}_A}\ \underset{\substack{6\\\mathbf{15}}}{\mathrm{c}_B}\ \underset{\substack{2\\\mathbf{11}}}{\mathrm{c}_A}\ \underset{\substack{\mathbf{7}\\16}}{\mathrm{c}_B} \mid \underset{\substack{\mathbf{3}\\12}}{\mathrm{c}_A}\ \underset{\substack{8\\\mathbf{17}}}{\mathrm{c}_B}\ \underset{\substack{4\\\mathbf{13}}}{\mathrm{c}_A}\ \underset{\substack{\mathbf{9}\\18}}{\mathrm{c}_B} \mid \underset{\substack{\mathbf{5}\\14}}{\mathrm{c}_A}\ \mathrm{c}_B\end{aligned}$$

where the numbers below all copies of c_A indicate the order in which the orbit of one letter in the first copy of c_A is visited by applying $\mathsf{K}_\circ$, with the top letter number

indicating this letter, and the bottom number indicating the letter after applying the involution ψ. One can thus consider $k \in \{1,2,3,6,9,18\}$ as $(m+1)h-2 = 18$. Thus, we are in Case (2b) for $k \in \{1,3,9\}$ and in Case (2c) for $k \in \{2,6,18\}$. For example, the numbers $1,10,7,16,4,13$ come in pairs $\big(\mathrm{r}, \psi(\mathrm{r})\big)$, and the pairs sit in every $(2k)^{\text{th}}$ alternating copy of c_A and c_B, here with $k=3$. Similarly, the numbers $1,15,11,7,3,17,13,9,5$ sit in every k^{th} alternating copy of c_A and c_B, here with $k=2$. Moreover, this word is travelled by $(\mathsf{K}_\circ)^k$ in the order

$$1,8,6,4,2,9,7,5,3,$$

so the letters in positions $1,4,5,8,9$ are copies of r and the letters in positions $2,3,6,7$ are copies of $\psi(\mathrm{r})$, as desired. The other possible k's are analogous.

Given the various cases, we can directly further describe the orbits in Lemma 6.4.

Lemma 6.7. *In the situation of Lemma 6.4, the order in which letters are visited by* $\mathsf{K}_\circ$ *is given by cyclically visiting*

CASE (1): *every* h^{th} *letter,*
CASE (2a): *every* $(h/2)^{th}$ *letter,*
CASE (2b): *every* $(h/2)^{th}$ *letter if* $\psi(r) = r$ *or consecutive pair of letters if* $\psi(r) \neq r$,
CASE (2c): *every* h^{th} *letter.*

Moreover, in

CASE (2b) WITH $\psi(r) \neq r$: *first, the* $h/2$ *letters* $\mathrm{r}_1, \mathrm{r}_3, \dots, \mathrm{r}_{h-1}$ *are visited, and then the* $h/2$ *letters* $\mathrm{r}_2, \mathrm{r}_4, \dots, \mathrm{r}_h$ *are visited, in this order, where* $\psi(r_{2k-1}) = r_{2k}$ *for* $1 \le k \le h/2$,
CASE (2c) WITH $\psi(r) = r$: *the* $h/2$ *letters* $\mathrm{r}_1, \mathrm{r}_2, \dots, \mathrm{r}_{h/2}$ *are visited, in this order.*

Proof. This is a direct consequence of the way the letters in $\mathbf{inv}_c^m \mathbf{inv}_c^L$ are visited under the application of $\mathsf{K}_\circ$. In particular observe that we have seen in the discussion after Lemma 6.2 that h respectively $h/2$ are indeed coprime with ρ'. This can be easily rechecked in all cases in Example 6.6. Observe in type A_3 — which corresponds to Case (2b) — that the word is $1,10,7,16,4,13$, and the involution ψ combines $1,10$, $7,16$, and $4,13$, respectively. Since $h/2 = 2$, the order is $1,4,7$ followed by $10,13,16$, as expected. □

The following lemma describes the orbits of a letter inside $\mathbf{inv}_c^m \mathbf{inv}_c^L$ as subwords of $(m+1)h-2$ alternating copies of c_A and c_B in the four cases.

Lemma 6.8. *Let* $\mathrm{r}_1, \dots, \mathrm{r}_{\rho'}$ *be a subword of* $\mathbf{inv}_c^m \mathbf{inv}_c^L$. *Then this subword is an orbit of the action of* $(\mathsf{K}_\circ)^k$ *if and only if*

CASE (1): *its length* ρ' *equals* ρ *and it is invariant under the cyclic shift sending a letter* s *in the* j^{th} *alternating copy of* c_A *or* c_B *to the letter* s (*if* k *is even*) *or* $\psi(\mathrm{s})$ (*if* k *is odd*) *in the* $(j+k)^{th}$ *alternating copy of* c_A *or* c_B;
CASE (2a): *its length* ρ' *equals* $\rho/2$ *and it is invariant under the cyclic shift sending a letter* s *in the* j^{th} *alternating copy of* c_A *or* c_B *to the letter* s *in the* $(j+2k)^{th}$ *alternating copy of* c_A *or* c_B;

CASE (2b): *its length ρ' equals ρ if $\psi(\mathrm{r}_1) \neq \mathrm{r}_1$ or $\rho' = \rho/2$ if $\psi(\mathrm{r}_1) = \mathrm{r}_1$, and it is invariant under the cyclic shift sending a letter* s *in the j^{th} alternating copy of* c_A *or* c_B *to the letter* s *in the $(j+2k)^{th}$ alternating copy of* c_A *or* c_B *and under the involution sending this letter* s *to the letter $\psi(\mathrm{s})$ in the same copy of* c_A *or of* c_B*;*

CASE (2c): *its length ρ' equals ρ and it is invariant under the cyclic shift sending a letter* s *in the j^{th} alternating copy of* c_A *or* c_B *to the letter* s *in the $(j+k)^{th}$ alternating copy of* c_A *or* c_B *after the following replacement: cyclically visit every h^{th} letter starting with the leftmost. Leave the first $(\rho+1)/2$ unchanged while replace the later $(\rho-1)/2$ by $\psi(\mathrm{r})$ in the same alternating copy of* c_A *or* c_B*. Finally replace every letter smaller* r *with* $\mathrm{r} <_c r_1$ *by $\psi(\mathrm{r})$ in the same alternating copy of* c_A *or* c_B*.*

Proof. Recall that $\mathbf{inv}_c^m\mathbf{inv}_c^L$ is given by $(m+1)h-2$ alternating copies of c_A and c_B. Let now r be a letter of $\mathbf{inv}_c^m\mathbf{inv}_c^L$ corresponding to some letter s inside some copy of c_A. (The case that s is inside some copy of c_B is completely analogous.) As we have seen in Subsection 2.4, the orbit of r under $\mathsf{K}_\circ$ is given in this case by all letters r′ corresponding to all letters s and to all letters $\psi(\mathrm{s})$ inside all copies of c_A and of c_B. The four described cases are then obtained by a direct analysis of the order in which the letters in this orbit of r under $\mathsf{K}_\circ$ are visited as given in Lemma 6.4.

The only difficult situation is the replacement at the end of Item (iv): we have seen in Lemma 6.4 that the letters are visited in the cyclic order $\mathrm{r}_1, \mathrm{r}_{h+1}, \ldots, r_{(\rho'-1)h+1}$. As $\rho' = \rho$ is odd, we have that ρ and h are coprime by Lemma 6.2. One can therefore iterate through the letters $\mathrm{r}_1, \ldots, \mathrm{r}_{\rho'}$ each once in this order. If r_1 is the very first letter inside $\mathbf{inv}_c$, then this iteration first visits the $(\rho+1)/2$ letters corresponding again to the letter r_1 and then visits the $(\rho-1)/2$ letters corresponding to $\psi(\mathrm{r}_1)$. This is captured by the first replacement above. If r_1 is instead any letter inside $\mathbf{inv}_c$, then the resulting word is correct exactly in the positions starting at r_1, while the letters left of r_1 inside $\mathbf{inv}_c$ have again to be replaced. □

To make it explicit, we record from Lemma 6.8 the possible orbit sizes in the various cases.

Corollary 6.9. *In the situation of Lemma 6.8, the orbit sizes are given by*

(i) *ρ if we are in Case (1), in Case (2b) with $\psi(\mathrm{r}_1) \neq \mathrm{r}_1$, or in Case (2c), and*
(ii) *$\rho/2$ if we are in Case (2a), or in Case (2b) with $\psi(\mathrm{r}_1) = \mathrm{r}_1$.*

We next describe how to decompose a subword of $\mathbf{inv}_c^m\mathbf{inv}_c^L$ which is invariant under $(\mathsf{K}_\circ)^k$ into orbits, sharpening in particular Lemma 6.1.

Lemma 6.10. *Let $\mathrm{r}_1, \ldots, \mathrm{r}_{n'}$ be a subword of $\mathbf{inv}_c^m\mathbf{inv}_c^L$ which is invariant under the action of $(\mathsf{K}_\circ)^k$. The orbit decomposition of $\mathrm{r}_1, \ldots, \mathrm{r}_{n'}$ is then given as follows:*

CASE (1): *ρ divides n' and the orbits consist of those letters that are multiples of k' positions apart, where k' is given by $n' = \rho \cdot k'$.*

CASE (2a): *$\rho/2$ divides n' and the orbits consist of those letters that are multiples of k' positions apart, where k' is given by $n' = \rho/2 \cdot k'$.*

CASE (2b): *divide the word $\mathrm{r}_1 \ldots \mathrm{r}_{n'}$ into subwords $\mathrm{r}'_1, \mathrm{r}'_2, \ldots, \mathrm{r}'_{n''}$ and $\mathrm{r}''_1, \mathrm{r}''_2, \ldots, \mathrm{r}''_{n'''}$ such that the former have the property that $\psi(\mathrm{r}'_i) \neq \mathrm{r}'_i$ and the latter have the property*

$\psi(\mathrm{r}''_i) = \mathrm{r}''_i$. *Then* ρ *divides* n'' *and the orbits of size* ρ *consist of those letters in* $\mathrm{r}'_1, \mathrm{r}'_2, \dots, \mathrm{r}'_{n''}$ *that are multiples of* k' *positions apart, where* k' *is given by* $n'' = \rho \cdot k'$, *and* $\rho/2$ *divides* n''' *and the orbits of size* $\rho/2$ *consist of those letters in* $\mathrm{r}''_1, \mathrm{r}''_2, \dots, \mathrm{r}''_{n'''}$ *that are multiples of* k'' *positions apart, where* k'' *is given by* $n''' = \rho/2 \cdot k''$.

CASE (2c): ρ *divides* n' *and the orbits consist of those letters that are multiples of* k' *positions apart, where* k' *is given by* $n' = \rho \cdot k'$.

Proof. This is a direct consequence of the description of the orbits in Lemma 6.8 as there we have seen that the letters in an orbit are equally far apart inside $\mathbf{inv}_c^m \mathbf{inv}_c^L$. □

Given Lemmas 6.8 and 6.10, we obtain a purely numerical (and computationally tractable) way to give a necessary and sufficient condition on a reduced factorisation $\mathrm{r}_1 \cdots \mathrm{r}_{n'}$ of a non-crossing partition $w = r_1 \cdots r_{n'} \in NC(W, c)$ to correspond to an invariant positive m-divisible non-crossing partition for some m. We then later explicitly give, as a polynomial in the parameter m (or rather its appropriate residue class), the number of subwords of $\mathbf{inv}_c^m \mathbf{inv}_c^L$ for $\mathrm{c} = \mathrm{c}_A \mathrm{c}_B$ that are invariant under the positive Kreweras map that spell out this word. To this end, we assume

- in Case (1) that $\mathrm{c}_B = \psi(\mathrm{c}_A)$ and
- in Case (2) that the letters s and $\psi(\mathrm{s})$ inside c_A and inside c_B are equal or next to each other.

The situation is almost the same in all cases, but with the slight modifications already present in the previous lemmas. We need one more auxiliary lemma to nicely describe Case (2b).

Lemma 6.11. *Let* h *be even, let* r *be a letter in* $\mathbf{inv}_c$, *and let* s *be the corresponding letter in the* h *alternating copies of* c_A *and* c_B. *Let moreover* r′ *be the letter in* $\mathbf{inv}_c$ *given by* $\psi(\mathrm{r})$ *with corresponding letter* s′. *Then* $\mathrm{s}' = \psi(\mathrm{s})$ *in the same copy of* c_A *or of* c_B *as* s.

Proof. This is a well-known fact that holds for bipartite Coxeter elements with h even: write $w_\circ = (c_A c_B)^{h/2} = s_1 \cdots s_N$, and let $r = s_1 \cdots s_i s_{i+1} s_i \cdots s_1$. As $\psi(r) = w_\circ r w_\circ$, we obtain $\psi(r) = \psi(s_1) \cdots \psi(s_i)\psi(s_{i+1})\psi(s_i) \cdots \psi(s_1)$, implying the statement because $\psi(c_A) = c_A$ and $\psi(c_B) = c_B$ from Lemma 6.5. □

This lemma enables us in Case (2b) in Lemma 6.8 to not consider all letters r and $\psi(\mathrm{r})$, but to restrict attention to a consistent choice of one of the two. To this end, let $\mathrm{r}_1, \dots, \mathrm{r}_{n'}$ be a subword of $\mathbf{inv}_c^m \mathbf{inv}_c^L$ in Case (2b) with $\psi(\mathrm{r}_1) \neq \mathrm{r}_1$. Then we have seen in Lemma 6.7 that this is an orbit of the action of $(\mathsf{K}_\circ)^k$ if and only if $n' = \rho$, $\psi(\mathrm{r}_{2i}) = \mathrm{r}_{2i+1}$, and the subword $\mathrm{r}_1, \mathrm{r}_3, \dots, \mathrm{r}_{n'-1}$ is invariant under the cyclic shift sending a letter s in the j^{th} alternating copy of c_A or c_B to the letter s in the $(j+2k)^{\text{th}}$ alternating copy of c_A or c_B. We refer to Example 6.6(iii) for comparison.

For the following proposition, recall the discussion after Lemma 6.2 and in particular the definition of y and of m'.

Proposition 6.12. *Let* $\mathrm{r}_1 \cdots \mathrm{r}_{n'}$ *be a reduced* $\mathcal{R}$*-word for* $w = r_1 \cdots r_{n'} \in NC(W, c)$, *and let* $i_1, \dots, i_{n'}$ *be the sequence of indices of the letters* $\mathrm{r}_1, \dots, \mathrm{r}_{n'}$ *inside* $\mathbf{inv}_c$. *Let* ρ *be such*

that ρ/y and h/y are coprime and let $m \equiv m' \pmod{\rho/y}$, where we assume in Case (2a) that h is even. Then there exists a subword of $\mathbf{inv}_c^m\mathbf{inv}_c^L$ which is invariant under the ρ-fold application of $\mathsf{K}_\circ$ and which spells out $\mathrm{r}_1 \cdots \mathrm{r}_{n'}$ if and only if the cyclic distances

$$\begin{array}{rcll} a_1 & \equiv & i_{k'+1} - i_1 & \pmod{N} \\ a_2 & \equiv & i_{k'+2} - i_2 & \pmod{N} \\ & \vdots & & \\ a_{n'-k'} & \equiv & i_{n'} - i_{n'-k'} & \pmod{N} \\ a_{n'-k'+1} & \equiv & i_1 - i_{n'-k'+1} + n & \pmod{N} \\ & \vdots & & \\ a_{n'} & \equiv & i_{k'} - i_{n'} + n & \pmod{N} \end{array}$$

inside $\mathbf{inv}_c$ satisfy

$$a_1 \equiv a_2 \equiv \cdots \equiv a_{n'} \equiv 0 \quad \begin{array}{ll} \pmod{\rho} & \text{in Case (1),} \\ \pmod{\rho/2} & \text{in Case (2),} \end{array}$$

where in

CASE (1): *ρ divides n', with k' defined by $n' = \rho \cdot k'$;*

CASE (2a): *$\rho/2$ divides n', with k' defined by $n' = \rho/2 \cdot k'$;*

CASE (2c): *ρ divides n', with k' defined by $n' = \rho \cdot k'$, and after one has taken into account the modification described in Lemma 6.8;*

CASE (2b): *$\rho/2$ divides n'', with k'' defined by $n'' = \rho/2 \cdot k''$, and where first, for every letter r_i, also $\psi(\mathrm{r}_i)$ must be in $\mathrm{r}_1 \cdots \mathrm{r}_{n'}$, and one uses the subword of $\mathrm{r}_1 \cdots \mathrm{r}_{n'}$ given by $\mathrm{r}'_1 \cdots \mathrm{r}'_{n''}$ after Lemma 6.11. In this case, let $i'_1, \ldots, i'_{n''}$ be the sequence of indices of the letters $\mathrm{r}'_1, \ldots, \mathrm{r}'_{n''}$ inside $\mathbf{inv}_c$.*

Proof. We prove Case (2a), the others are completely analogous, with the appropriate modifications in the last two cases.

First assume that there is a $(\rho/2)$-fold invariant subword of $\mathbf{inv}_c^m\mathbf{inv}_c^L$ spelling out $\mathrm{r}_1 \cdots \mathrm{r}_{n'}$. This means that this subword is invariant under the action of $(\mathsf{K}_\circ)^k$.

We have seen in Lemma 6.10 that orbits of the reflections $r_1, \ldots, r_{n'}$ in this word under $(\mathsf{K}_\circ)^k$ must be all letters that are k' positions apart for k' given by $n' = \rho/2 \cdot k'$. The letters that are k' positions apart must thus be spread equally inside the word $\mathbf{inv}_c^m\mathbf{inv}_c^L$, implying the numerical condition

$$a_1 = a_2 = \cdots = a_{n'},$$

where the shift by n in $a_{n'-k'+1}, \ldots, a_{n'}$ comes from the fact that these distances have passed the end of the word $\mathbf{inv}_c^m\mathbf{inv}_c^L$ and there are exactly n letters missing in the final $\mathbf{inv}_c^L$. Moreover, the congruence condition

$$a_1 \equiv a_2 \equiv \cdots \equiv a_{n'} \equiv 0 \pmod{\rho/2}$$

means that all letters that are k' positions apart are indeed in the same orbit of $\mathsf{K}_\circ$, as desired.

On the other hand, if these numerical conditions are satisfied for a reduced $\mathcal{R}$-word $\mathrm{r}_1 \cdots \mathrm{r}_{n'}$ of some $w \in NC(W,c)$, then $\mathrm{r}_1 \cdots \mathrm{r}_{n'}$ fits properly into $\mathbf{inv}_c^m \mathbf{inv}_c^L$ as described in Lemma 6.8. □

Using this proposition, we now have a numerical algorithm to determine which $w \in NC(W,c)$ and which reduced $\mathcal{R}$-words of w possibly contribute to the counting formula of ρ- respectively $(\rho/2)$-fold symmetric positive m-divisible non-crossing partitions. It is now only left to determine to which m they contribute and how much they contribute.

Recall from the paragraph below Example 2.15 that, for all $m \geq 1$, both versions of the positive Kreweras maps $\mathsf{K}_+^{(m)}$ and $\widetilde{\mathsf{K}}_+^{(m)}$ can be considered as the action of $\mathsf{K}_\circ$ on the positive elements in $NC_\nabla^{(m)}(W,c)$, respectively in $NC_\Delta^{(m)}(W,c)$.

Theorem 6.13. *Let $(W,\mathcal{S})$ be a finite Coxeter system, let $\mathrm{c} = \mathrm{c}_A\mathrm{c}_B$ be a fixed word for the bipartite Coxeter element $c = c_A c_B \in W$ such that $\mathrm{c}_B = \psi(\mathrm{c}_A)$ if we are in Case (1) and such that the letters s and $\psi(\mathrm{s})$ inside c_A and inside c_B are equal or next to each other in Case (2). Let $(m+1)h - 2 = k \cdot \rho$ such that ρ is even in Case (2a). Let $a = \lfloor m/\frac{\rho}{y} \rfloor$, and let i_{fix} be as defined in (6.5). Then the number of positive $(m+1)$-divisible non-crossing partitions $NC_+^{(m+1)}(W,c)$ that are invariant under the ρ-fold application of K_+ is given by the sum*

$$\sum \binom{a + \mathrm{asc}(\mathrm{r}_1, \ldots, \mathrm{r}_{k'}, \mathrm{r}_{\mathrm{fix}})}{k'}, \tag{6.7}$$

which ranges over all factorisations $\mathrm{r}_1 \cdots \mathrm{r}_{n'}$ of all elements $w \in NC(W,c)$ that satisfy the properties in Proposition 6.12, and where in

Case (1): *ρ divides n' and k' is given by $n' = \rho \cdot k'$;*

Case (2a): *$\rho/2$ divides n' and k' is given by $n' = \rho/2 \cdot k'$;*

Case (2c): *ρ divides n' and k' is given by $n' = \rho \cdot k'$; furthermore, $\mathrm{r}'_1, \ldots, \mathrm{r}'_{k'}$ is the sequence obtained from $\mathrm{r}_1, \ldots, \mathrm{r}_{k'}$ inside $\mathbf{inv}_c$ via the modification described in Lemma 6.8;*

Case (2b): *$\rho/2$ divides n'' where $\mathrm{r}'_1 \cdots \mathrm{r}'_{n''}$ is the subword of $\mathrm{r}_1 \cdots \mathrm{r}_{n'}$ given in Proposition 6.12, and k'' is given by $n'' = \rho \cdot k''$.*

Proof. This is a direct consequence of the analysis in Proposition 6.12, together with the same counting argument as used for Corollary 2.10. □

Corollary 6.14. *In the situation of Theorem 6.13, the number of positive m-divisible non-crossing partitions $NC_+^{(m)}(W,c)$ that are invariant under the k-fold application of $\widetilde{\mathsf{K}}_+$ is given by the same counting formula where one only considers factorisations $\mathrm{r}_1 \cdots \mathrm{r}_n$ of the Coxeter element $c = c_L c_R \in W$.*

To finish these counting considerations, we recall the steps to count positive m-divisible non-crossing partitions that are invariant under a ρ- respectively $(\rho/2)$-fold symmetry, and compare it to the evaluation of the q-extension of the positive Fuß–Catalan numbers at the corresponding primitive root of unity:

(1) Compute all reduced $\mathcal{R}$-words of the Coxeter element $c = c_A c_B$ using the transitivity of the Hurwitz action (see [Bes03, Prop. 1.6.1]).
   (a) In the case of K_+, one then computes as well all prefixes of these reduced $\mathcal{R}$-words to obtain all reduced factorisations of elements in $NC(W, c)$.
(2) Represent all reduced words computed in Item 1 as indices of positions of subwords of $\mathbf{inv}_c$.
(3) Use Proposition 6.12 to determine which factorisations contribute to the enumeration. Disregard all factorisations not satisfying these properties.
(4) Use Theorem 6.13 to determine the contribution of each factorisation as a polynomial in the parameter a.
(5) Evaluate $\mathrm{Cat}_+^{(m)}(W; q)$ or $\mathrm{Cat}_+^{(m+1)}(W; q)$ (depending on whether one considers $\widetilde{\mathsf{K}}_+$ or K_+) at the root ζ^k where ζ is a primitive $\big((m+1)h-2\big)^{\text{th}}$ root of unity and express the result in terms of the parameter a.
(6) Compare the two results in the previous two steps to verify the cyclic sieving phenomenon in this situation.

We use the described machinery to prove Theorem 2.19 for dihedral and exceptional types and record this here for reference.

Theorem 6.15. *The conclusion of Theorem 2.19 holds for dihedral and exceptional types.*

The concrete calculations are postponed to Appendix E.

7. Further open problems

We use this final section to make explicit some more open problems raised by our work.

7.1. **Chain enumeration.** The attentive reader may have observed that our enumerative treatment for types B_n and D_n does not include any results on chain enumeration, as opposed to type A_{n-1} (cf. Subsection 3.4). It may entirely be possible that formulae analogous to the one in Theorem 3.11 exist also for types B_n and D_n. However, we have not been able to push our approach through in these cases.

Open Problem 7.1. Find formulae for the number of multichains in $\widetilde{NC}_+^{(m)}(B_n)$ and/or $\widetilde{NC}_+^{(m)}(D_n)$ where the ranks of the elements of the chain are prescribed, and/or even the block structure of the bottom element of the chain.

More generally, the question of formulae for chain enumeration of pseudo-rotationally invariant elements arises for all types. Here, we have not even been successful in type A_{n-1}. In all (classical) types, the reason is that the positive Kreweras map does not preserve the order relation, but the characterisation of pseudo-rotationally invariant partitions (see Subsections 3.5, 4.5 and 5.5) involves repeated application of the positive Kreweras map in an essential way.

Open Problem 7.2. Find formulae for the number of multichains in the subposets of $\widetilde{NC}_+^{(m)}(A_{n-1})$, $\widetilde{NC}_+^{(m)}(B_n)$, and/or $\widetilde{NC}_+^{(m)}(D_n)$ consisting of ρ-pseudo-rotationally invariant

elements, where the ranks of the elements of the chain are prescribed, and/or even the block structure of the bottom element of the chain.

If there is a positive solution of the above problem in type D, then we may also ask for more general cyclic sieving phenomena in this type.

Open Problem 7.3. Prove a cyclic sieving phenomenon for the subset of elements of $\widetilde{NC}_+^{(m)}(D_n)$ where their rank is prescribed, and, more generally, where their block structure is prescribed.

Closely related to chain enumeration in a poset is its Möbius function; see [Ede80b]. Corollary 3.16 leads us right-away to the next open problem.

Open Problem 7.4. Determine the Möbius function for $\widetilde{NC}_+^{(m)}(B_n)$ and/or $\widetilde{NC}_+^{(m)}(D_n)$. More generally, determine the Möbius function for the subposets of $\widetilde{NC}_+^{(m)}(A_{n-1})$, $\widetilde{NC}_+^{(m)}(B_n)$, and/or $\widetilde{NC}_+^{(m)}(D_n)$ consisting of ρ-pseudo-rotationally invariant elements.

7.2. **Block enumeration.** In [RS18], Reiner and Sommers proved cyclic sieving phenomena for m-divisible non-crossing partitions with prescribed block structure. Their uniform definition of "block structure" is also applicable in exceptional types. This suggests the following problems.

Question 7.5. Does the positive Kreweras map preserve the block structure, as defined in [RS18], of positive m-divisible non-crossing partitions?

Open Problem 7.6. In case of an affirmative answer to the above question, prove a cyclic sieving phenomenon for *positive* m-divisible non-crossing partitions in exceptional types with prescribed block structure.

7.3. **A positive Panyushev map on non-nesting partitions.** Recall that $\Delta \subseteq \Phi^+$ denotes the simple and positive roots for the Coxeter systems $(W, \mathcal{S})$. For crystallographic Coxeter systems, define the *root poset* $\Phi^+(W)$ to be the partial order on Φ^+ induced by the cover relations given by $\alpha \prec \beta$ if $\beta - \alpha \in \Delta$. The *non-nesting partitions* associated with W were defined by Panyushev [Pan09] as the set of all antichains in Φ^+,

$$NN(W) := \big\{\mathcal{A} \subseteq \Phi^+ \mid \mathcal{A} \text{ antichain}\big\}.$$

As for non-crossing partitions, there is also a positive variant of non-nesting partitions given by

$$NN_+(W) := \big\{\mathcal{A} \subseteq \Phi^+ \setminus \Delta \mid \mathcal{A} \text{ antichain}\big\}.$$

These (positive) non-nesting partitions turn out to share many enumerative properties with non-crossing partitions; see [Arm06] for a detailed treatment thereof. In particular, we have

$$\big|NN(W)\big| = \big|NC(W,c)\big| \text{ and } \big|NN_+(W)\big| = \big|NC_+(W,c)\big|.$$

Obviously, these equalities suggest to construct bijections, and indeed multiple bijections were constructed in various types [CDD+07, FG10, RS10], the only uniformly described

bijection was given in [AST13]. One particular avenue to follow — as explained by Armstrong, Thomas, and the second author in [AST13] — are two maps on $NC(W,c)$ and on $NN(W)$ given by the Kreweras map $\widetilde{\mathsf{K}} : NC(W,c) \longrightarrow NC(W,c)$ and the *Panyushev map* $\mathsf{P} : NN(W) \longrightarrow NN(W)$ which is given by mapping an antichain $\mathcal{A}$ to the collection of minimal elements in the root poset among all elements that are not below $\mathcal{A}$. (This map has in fact been earlier defined by Cameron and Fon-Der-Flaass [CFDF95] for general posets.[9]) In [AST13], a uniformly described bijection between non-crossing and non-nesting partitions is constructed sending the Kreweras map to the Panyushev map. This bijection is uniquely determined (but actually strongly overdetermined) by certain inductive properties on parabolic subgroups. The proof of its existence is given by explicitly constructing it in the various types. Observe that this bijection does not send positive non-crossing partitions to positive non-nesting partitions. Moreover, as mentioned above for the Kreweras map, the Panyushev map does not stabilise positive non-nesting partitions either. But again, there is a similarly defined *positive Panyushev map* $\mathsf{P}_+ : NN_+(W) \longrightarrow NN_+(W)$ given by the same rule as for P but considered on the root poset Φ^+ with the simple roots Δ removed.

Before raising two questions on the connections between positive non-crossing and positive non-nesting partitions, we copy two conjectures on the maps P and P_+ by Panyushev [Pan09, Conj. 2.1 and 2.3], the first of which was proven in [AST13] by constructing an equivariant bijection between $NN(W)$ and $NC(W,c)$ as briefly mentioned above.

Theorem 7.7 (Armstrong, Stump, Thomas, conjectured by Panyushev)**.** *Let W be a finite, crystallographic Coxeter group with Coxeter number h, and let P be the Panyushev map on $NN(W)$. The order of P is given by*

$$\operatorname{ord}(\mathsf{P}) = \begin{cases} (m+1)h, & \text{if } \psi \not\equiv \varepsilon, \\ (m+1)h/2, & \text{if } \psi \equiv \varepsilon. \end{cases}$$

Conjecture 7.8 (Panyushev)**.** *Under the same assumptions as in the previous theorem with P_+ being the positive Panyushev map on $NN_+(W)$, the order of P_+ is given by*

$$\operatorname{ord}(\mathsf{P}_+) = \begin{cases} (m+1)h-2, & \text{if } \psi|_{\mathcal{R}\setminus\mathcal{S}} \not\equiv \varepsilon, \\ (m+1)h/2-1, & \text{if } \psi|_{\mathcal{R}\setminus\mathcal{S}} \equiv \varepsilon. \end{cases}$$

(Here, we have corrected a subtle mistake in the original conjecture in that we added the restriction of ψ to non-simple reflections in the above case distinction. The only difference occurs in type A_2, where ψ interchanges the two simple reflections s_1, s_2 while fixing the third reflection $s_1 s_2 s_1 = s_2 s_1 s_2$.) The combinatorial models in types A and B for the bijection in [AST13] immediately give that, in these types, the triple

$$\Big(NN_+(W), \operatorname{Cat}_+(W;q), \langle \mathsf{P}_+ \rangle\Big)$$

[9] Still, we prefer to call this map on non-nesting partitions the "Panyushev map" since it was Panyushev who studied this map for root posets.

exhibits the cyclic sieving phenomenon. This holds as well in the exceptional types, which was checked using a computer, thus leaving only type D open. Together with Theorem 2.17 that now raises the following two questions.

Question 7.9. Is there an equivariant bijection between $NN_+(W)$ and $NC_+(W,c)$ for the Kreweras map respectively the Panyushev map, similar to the bijection between $NN(W)$ and $NC(W,c)$?

We remark that there are as well m-extensions $NN^{(m)}(W)$ and $NN_+^{(m)}(W)$ of (positive) non-nesting partitions, cf. [Ath04, Cor. 1.3]. No extension of the Panyushev map in that direction is known so far. In particular, we completely lack an understanding of uniform connections between m-divisible non-crossing partitions and the m-extension of non-nesting partitions.

Open Problem 7.10. Construct a generalised (positive) Panyushev map on $NN^{(m)}(W)$ and on $NN_+^{(m)}(W)$ which generalises the (positive) Panyushev map and at the same time provides the same orbit structure as the generalised (positive) Kreweras map on $NC^{(m)}(W,c)$ and on $NC_+^{(m)}(W,c)$.

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Appendix A. Proofs: The order of the positive Kreweras map

This appendix is devoted to the proofs of Theorems 3.10, 4.10 and 5.10 describing the order of the positive Kreweras map in classical types.

A.1. **The order of the positive Kreweras map in type A_{n-1}: Proof of Theorem 3.10.** The proof proceeds in several steps. Let π be some non-crossing partition of $\{1, 2, \dots, N\}$.

Step 1. We argue that singleton blocks are not "important", in the sense that it suffices to prove the claim for non-crossing partitions without singleton blocks.
Step 2. We prove that a block of π of the form $\{N, 1, 2, \dots, k\}$ returns to its position after $N-2$ applications of κ_A.
Step 3. We prove that a block of π containing N and 1 but not $N-1$ returns to its position after $N-2$ applications of κ_A.
Step 4. We argue that Steps 1–3 suffice to complete the proof.
We now enter in the details of each step.

Step 1. If we concentrate ourselves on a particular singleton block and what it does under successive applications of the pseudo-rotation κ_A, then we see that κ_A always rotates this block by 1 unit in clockwise direction, except that the singleton block $\{N-1\}$ is mapped to $\{2\}$, that is, it is — so-to-speak — rotated by 3 units in clockwise direction. In particular, after $N-2$ applications of κ_A, a singleton block returns to its original position.

Let us suppose for the moment that the assertion of the lemma is already established for the case of non-crossing partitions without singleton blocks. We furthermore suppose that π contains exactly k singleton blocks. It is straightforward to see that, if we remove the singleton blocks and consider the remainder of π as a non-crossing partition on the remaining elements, the pseudo-rotation κ_A acts as the identity if $\{N-1\}$ is a singleton block, and otherwise in the way illustrated in Figure 4. In particular, if we denote the non-crossing partition arising from π' by ignoring its singleton blocks, then, according to our assumption, after $N-k-2$ applications of κ_A on π' we return back to π'. This implies that $N-2$ applications of κ_A to π produce a non-crossing partition $\hat{\pi}$ in which the singleton blocks are in their original place, and which is identical with π' after removal of the singleton blocks. Since there is a unique way to reinsert the singleton blocks into π' such that they sit in their original position, this shows that, under the above assumption, the map κ_A^{N-2} acts as the identity on π.

From now on, we assume that π does not contain any singleton blocks.

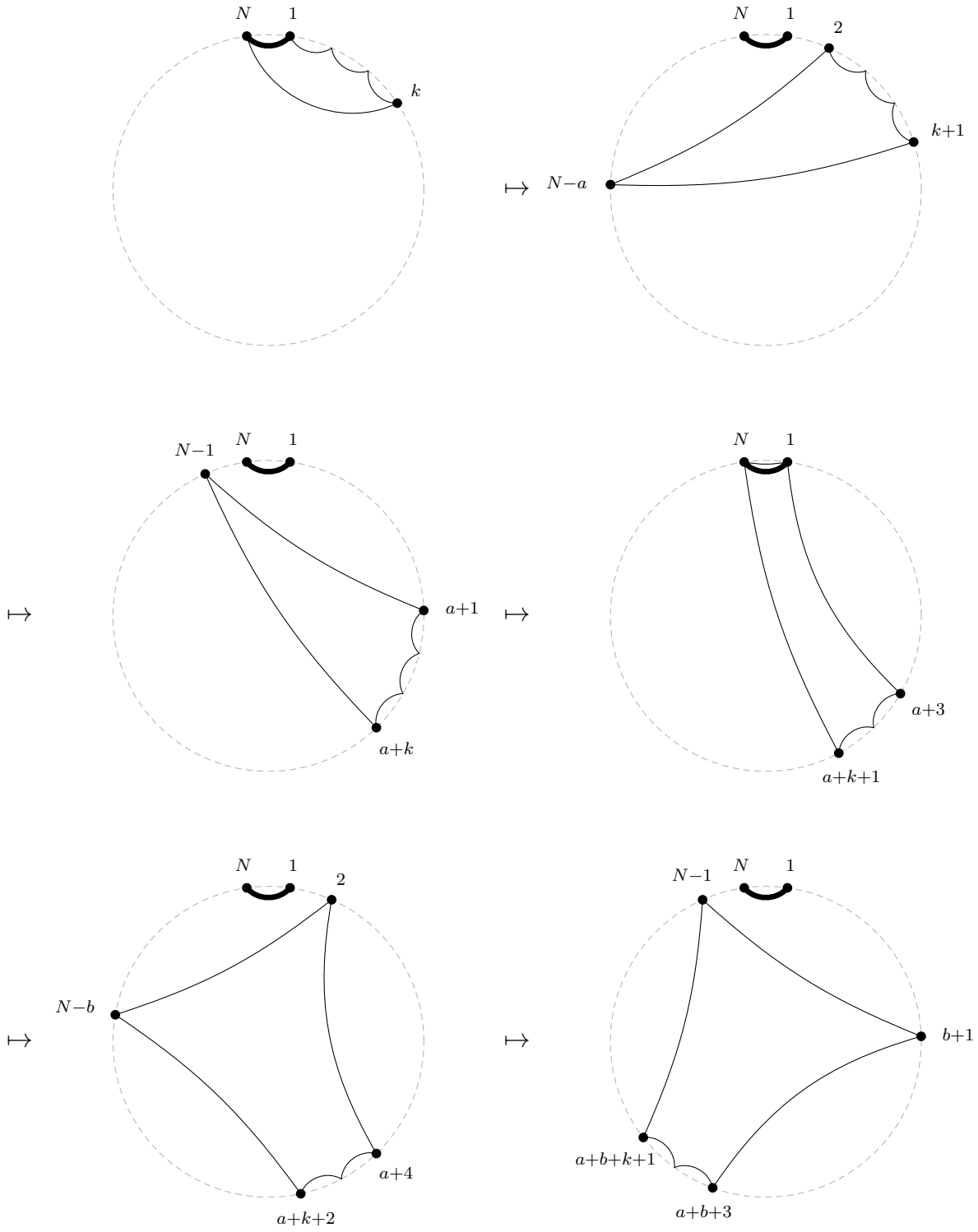

FIGURE 40. Pseudo-rotation of a block of successive elements

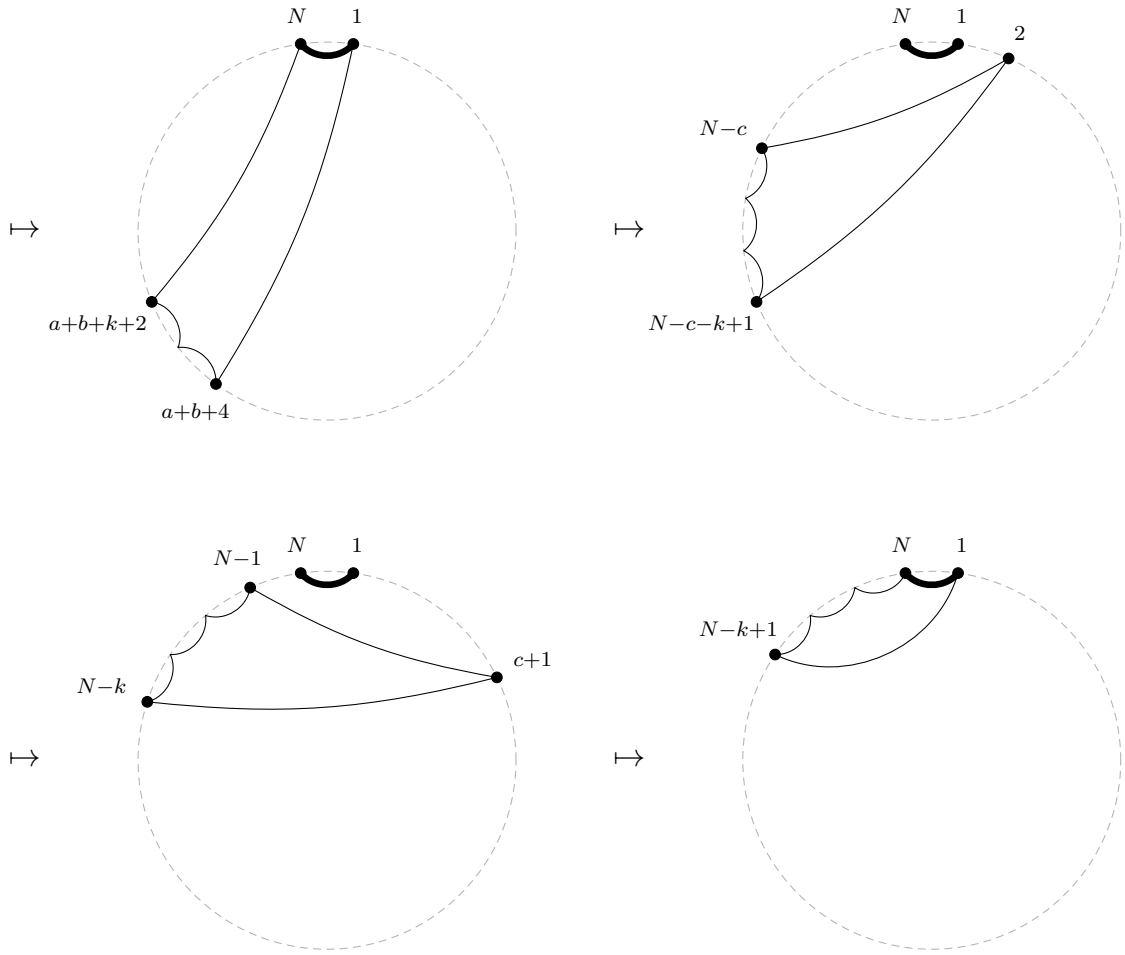

FIGURE 41. Pseudo-rotation of a block of successive elements

STEP 2. Let $\{N, 1, 2, \ldots, k\}$ be a block of π. See Figure 40 for an illustration. The first application of κ_A maps this block to $\{N-a, 2, \ldots, k+1\}$, for a suitable positive integer a; see the second image in Figure 40. Each of the next $a-1$ applications of κ_A rotates this block by one unit in clockwise direction, so that we obtain $\{N-1, a+1, \ldots, a+k\}$. See the third image in Figure 40 for the resulting non-crossing partition. If we we now apply κ_A another time, then the block on which we are concentrating ourselves becomes $\{N, 1, a+3, \ldots, a+k+1\}$; see the fourth image in Figure 40. It should be noted that $a+3$ is not an error: one element in the maximal sequence of successive elements has been "lost" under this operation. Another application of κ_A then turns this block into $\{N-b, 2, a+4, \ldots, a+k+2\}$, for a suitable positive integer b. See the fifth image in Figure 40.

There follow $b-1$ applications of κ_A which act as ordinary rotations on our block of interest. Our block becomes $\{N-1, b+1, a+b+3, \ldots, a+b+k+1\}$; see the sixth image in Figure 40. The next application of κ_A has the effect that it "removes" $b+1$ from this

block. More precisely, the block becomes $\{N, 1, a+b+4, \dots, a+b+k+2\}$; see the first image in Figure 41.

Comparison with the fourth image in Figure 40 shows that, except for the location of the maximal sequence of successive elements, we are in the same situation again. In other words, we may now run through the situations illustrated by the fourth, fifth, sixth image in Figure 40, and the first image in Figure 41 several times, until the "added" element ($N-b$ in the fifth image in Figure 40) eventually "joins" the maximal sequence of successive elements (i.e., $N-b = a+k+3$). Once this happens, we are in the situation illustrated by the second image in Figure 41.

Referring to this image, $c-1$ further applications of κ_A lead to a block of the form $\{N-k, \dots, N-1, c+1\}$; see the third image in Figure 41. One more application of κ_A turns this block into $\{N-k+1, \dots, N, 1\}$; see the last image in Figure 41. (One "loses" $c+1$ under this operation.) Finally, the next $k-1$ applications of κ_A act as ordinary rotations on our block, so that we return to the situation illustrated by the first image in Figure 40.

It can be checked that, in total, we performed $N-2$ applications of κ_A.

STEP 3. Now we consider the case of a block that contains more than one maximal sequence of successive elements. Let the block contain $N, 1, 2, \dots, k$, and let the maximal sequence of successive elements (cyclically) preceding N be $a+1, \dots, a+l$. The first image in Figure 42 provides an illustration.

We now apply κ_A. Thereby, our block turns into $\{N-u, 2, \dots, k+1, *, a+2, \dots, a+l+1\}$; see the second image in Figure 42. Here, the symbol $*$ indicates further possible elements in the block lying between $k+1$ and $a+2$. The next $u-1$ applications of κ_A act as ordinary rotations on our block; see the third image in Figure 42.

By a further application of κ_A, the maximal sequence of k successive elements in the block is shortened by one element. The result is that our block has become $\{N, 1, u+3, \dots, u+k+1, *, u+a+2, \dots, u+a+l+1\}$; see the fourth image in Figure 42.

In a sense, we already know from Step 2 what happens next: the next application of κ_A removes N and 1 from our block, rotates the other elements by one unit, and adds $N-v$ and 2 to it, for a suitable positive integer v; see the fifth image in Figure 42. There follow $v-1$ applications of κ_A which act as ordinary rotations on our block; see the sixth image in Figure 42. The following application of κ_A then turns our block into the form $\{N, 1, u+v+4, \dots, u+v+k+2, *, u+v+a+3, \dots, u+v+a+l+2\}$; see the first image in Figure 43, a form we have already seen before as comparison with the fourth image in Figure 42 reveals. Thus, this procedure may repeat itself several times until, eventually, we reach the situation where our block becomes $\{2, z+2, \dots, z+k, *, z+a+1, \dots, z+a+l+1\}$; see the second image in Figure 43. It should be noted that the maximal sequence of successive elements consisting originally of l elements now has been enlarged to the $l+1$ elements $z+a+1, \dots, z+a+l+1$.

Now follow $N-z-a-l-2$ applications of κ_A, each of them acting as ordinary rotations on our block of interest; see the third image in Figure 43. The next application of κ_A makes us "lose" the "isolated" element $N-z-a-l$, so that we obtain the block

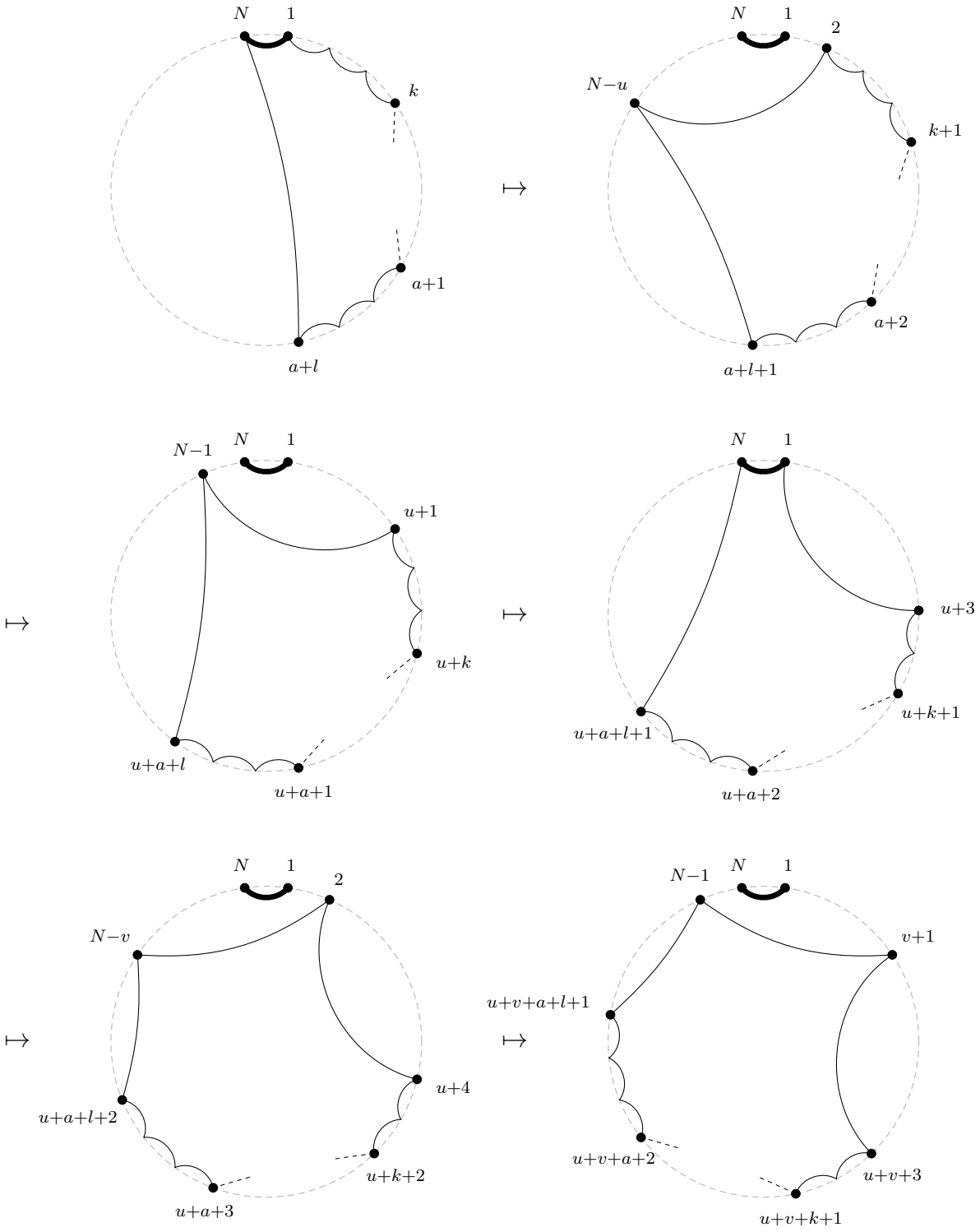

FIGURE 42. Pseudo-rotation of a block

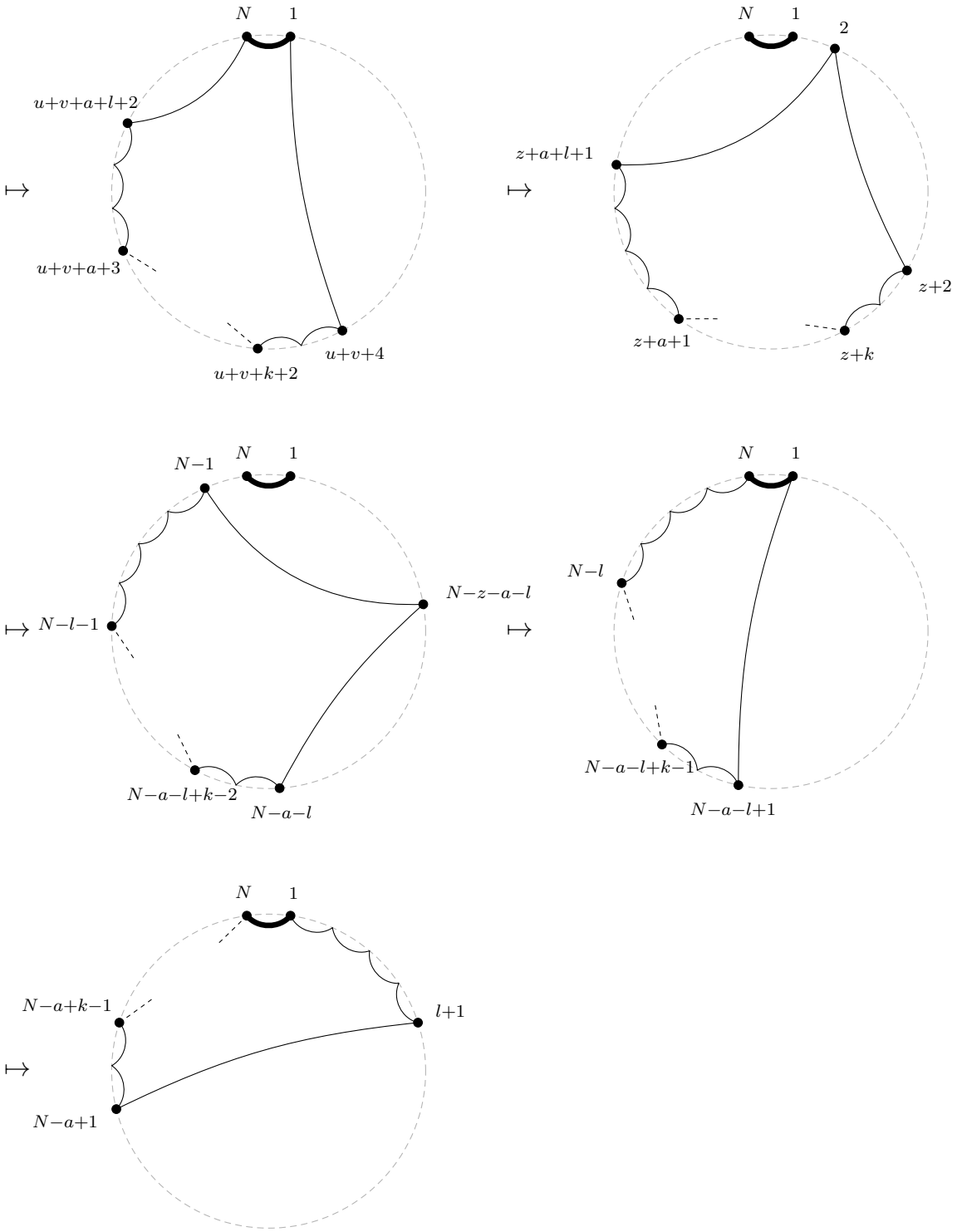

FIGURE 43. Pseudo-rotation of a block

$\{N-a-l+1,\dots,N-a-l+k-1,*,N-l,\dots,N,1\}$; see the fourth image in Figure 43. Finally, l further applications of κ_A let us arrive at $\{N,1,\dots,l+1,N-a+1,\dots,N-a+k-1,*\}$; see the last image in Figure 43.

At this point, let us summarise what happened. Comparison of the first image in Figure 42 and the last image in Figure 43 shows that the (cyclically) maximal sequence of successive elements that originally contained N and 1 at its beginning (namely $N,1,\dots,k$ in the first image in Figure 42) got rotated further but "lost" its first two elements. (It became $N-a+1,\dots,N-a+k-1$; see the last image in Figure 43.) By contrast, the maximal sequence of successive elements that preceded that sequence (namely $a+1,\dots,a+l$ in the first image in Figure 42) now got rotated to the top. It gained two elements so that it now contains N and 1 at its beginning. (It became $N,1,\dots,l+1$; see the last image in Figure 43.)

This process now will repeat itself. Every time a maximal sequence of successive elements is rotated away from $\{N,1\}$, it will "lose" two elements, and every time a maximal sequence of successive elements is moved into $\{N,1\}$ it will gain two elements. In particular, such a maximal sequence is almost always rotated ordinarily. At one point, however, it "loses" its first element, and at a later point it "gains" a new last element. These two events together explain an extra "jump" of 1. A further such "jump" happens when the maximal sequence is moved into $\{N,1\}$. Thus, after $N-2$ applications of κ_A, our block will be back in its original position.

Step 4. We have shown that, for a non-crossing partition π without singleton block, a block containing a maximal sequence of (cyclically) successive elements $N,1,\dots,k$ will be back in its original position after $N-2$ applications of the pseudo-rotation κ_A. If we now consider any other block, then we may apply κ_A as many times as it is necessary to move that block into a position where it contains a (cyclically) maximal sequence of successive elements beginning with N and 1. Then the above arguments apply. This shows that also any other block returns back to its original position after $N-2$ applications of κ_A.

Since, in Step 1, we have shown that it suffices to prove the assertion for non-crossing partitions without singleton blocks, this completes the proof of the theorem. □

A.2. **The order of the positive Kreweras map in type B_n: Proof of Theorem 4.10.** This proof is rather similar in spirit to the one of Theorem 3.10. We shall therefore be somewhat briefer here.

Again, we proceed in several steps.

Step 1. We argue that singleton blocks are not "important", in the sense that it suffices to prove the claim for non-crossing partitions without singleton blocks.

Step 2. We prove that a block of π of the form $\{\overline{N},1,2,\dots,k\}$, with $k\le N$, becomes its negative, $\{N,\overline{1},\overline{2},\dots,\overline{k}\}$, after $N-1$ applications of κ_B.

Step 3. We prove that a block of π containing $\overline{N}$ and 1 but not $\overline{N-1}$ becomes its negative after $N-1$ applications of κ_B.

Step 4. We argue that Steps 1–3 suffice to complete the proof.

We now discuss each step somewhat more in detail.

STEP 1. The arguments here are completely analogous to the ones in Step 1 of the preceding proof of Theorem 3.10. We leave the details to the reader.

STEP 2. Let $\{\overline{N}, 1, 2, \ldots, k\}$ be a block of π. There are two cases depending on whether or not there is a block other than the above block and its "companion" $\{N, \overline{1}, \overline{2}, \ldots, \overline{k}\}$ containing both positive and negative elements. They are illustrated in Figure 44 respectively in Figure 45.

Let us start with the case where there is such an additional block containing both positive and negative elements; see the first image in Figure 44. The first application of κ_B maps our block to $\{2, \ldots, k+1, b+1\}$, for a suitable positive integer b; see the second image in Figure 44. Each of the next $N-b$ applications of κ_B rotates this block by one unit, so that we obtain $\{N-b+2, \ldots, N-b+k+1, \overline{1}\}$. See the third image in Figure 44 for the resulting non-crossing partition.

There will be some further applications of κ_B which rotate this block ordinarily, until a situation is reached which is indicated in the fourth image of Figure 44 (which assumes that $c-1$ applications of κ_B have taken place, with $c \geq 2$): there is a block containing N and otherwise only negative elements, and the connection between $\overline{c}$ and $N+k-b+c$ is the closest to this block which is between a negative and a positive element. The next application of κ_B then turns our block of interest into $\{N-b+c+2, \ldots, N+k-b+c+1, \overline{1}\}$; see the fifth image in Figure 44.

Comparison with the third image in Figure 44 shows that, except for the location of the maximal sequence of (cyclically) successive elements, we are in the same situation again. In other words, we may now run through the situations illustrated by the third, fourth, and fifth image in Figure 44 several times, until our maximal sequence of successive elements eventually reaches N and thereby joins with $\overline{1}$ to form the block $\{N-k+1, \ldots, N, \overline{1}\}$; see the sixth image in Figure 44.

Finally, another $k-1$ applications of κ_B rotate this block to $\{N, \overline{1}, \overline{2}, \ldots, \overline{k}\}$, the negative of the block with which we started. Thus, we are back to the situation illustrated by the first image in Figure 44.

It can be checked that, in total, we performed $N-1$ applications of κ_B.

The second case, namely when there is no additional block containing both positive and negative elements, can be quickly reduced to the previous case; see Figure 45. Namely, the first application of κ_B maps our block to $\{2, 3, \ldots, k+1, \overline{1}\}$. This is the same situation as the one which we encountered in the third image in Figure 44 (namely the one with $b = N$). Thus, the subsequent arguments from the previous case apply again and show that also here $N-1$ applications of κ_B map the pair of blocks $\{\overline{N}, 1, \ldots, k\}$ and $\{N, \overline{1}, \ldots, \overline{k}\}$ to itself.

STEP 3. Now we consider the case of a block that contains more than one maximal sequence of successive elements. Let the block contain $\overline{N}, 1, 2, \ldots, k$ but not $\overline{N-1}$. In principle, we distinguish between several cases depending on whether further maximal sequences of successive elements contain positive or negative elements. We shall see, however, that it suffices to look at the following situation.

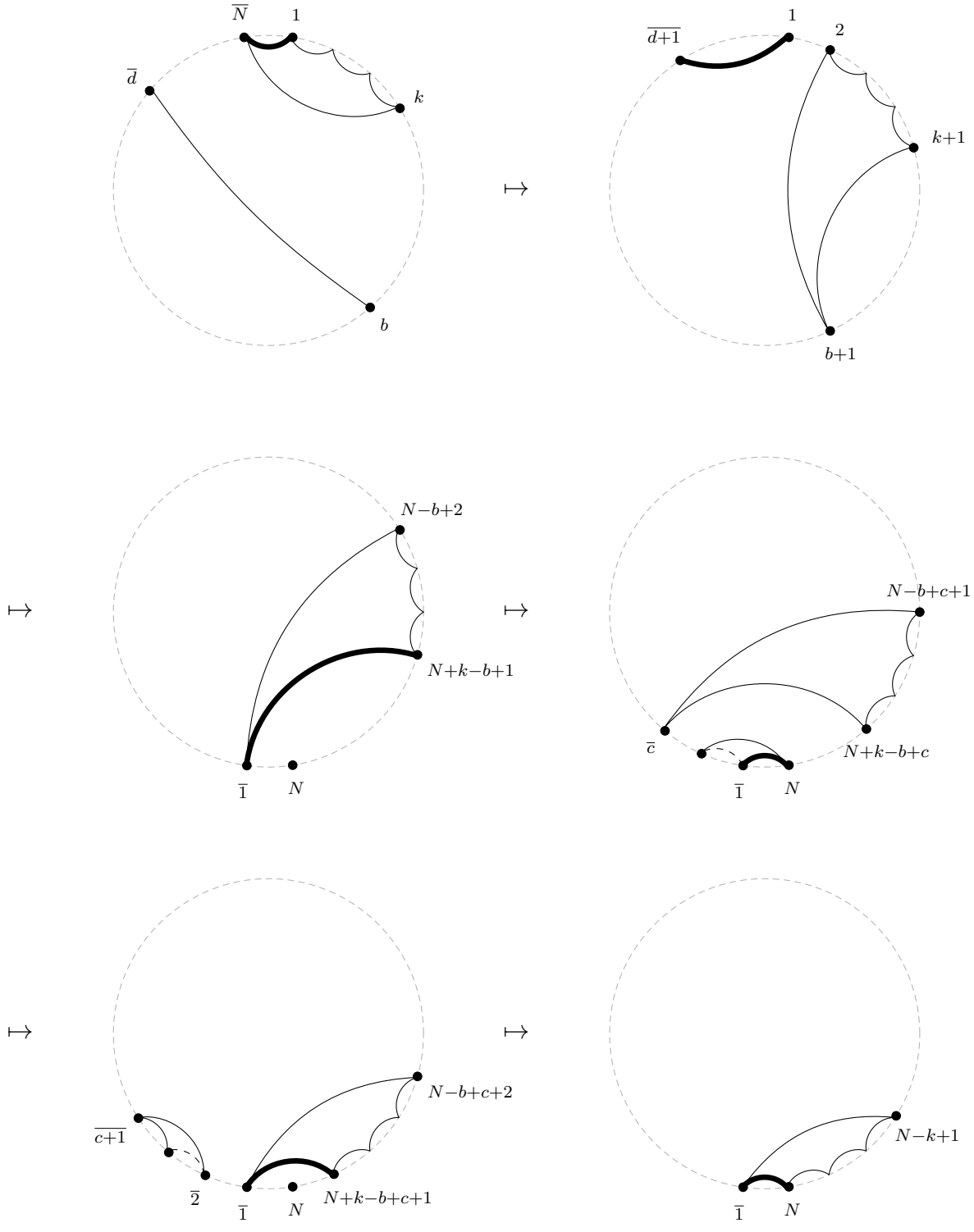

FIGURE 44. Pseudo-rotation of a block of successive elements

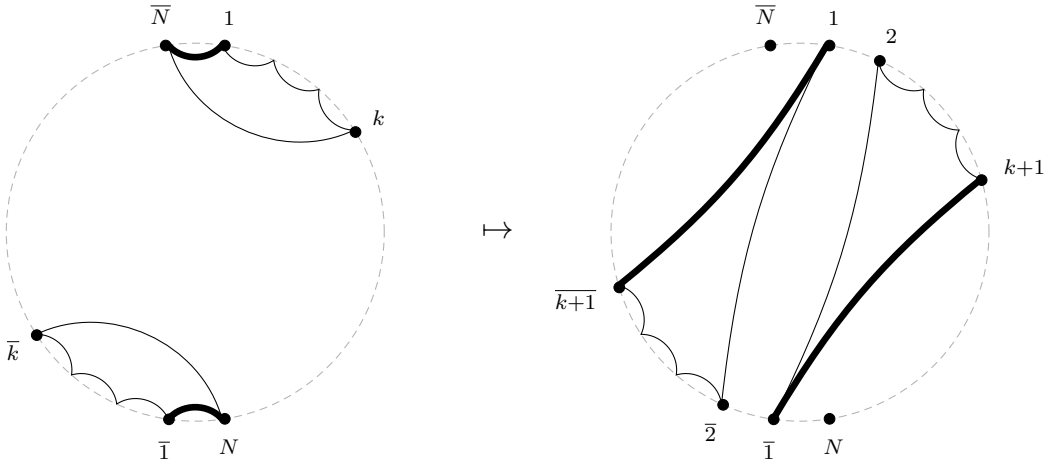

FIGURE 45. Pseudo-rotation of a block of successive elements

Let the maximal sequence of successive elements (cyclically) preceding $\overline{N}$ be $a+1,\dots,a+l$ (consisting of positive numbers). The first image in Figure 46 provides an illustration.

The first application of κ_B maps this block to $\{2,\dots,k+1,*,a+2,\dots,a+l+1,b+1\}$; see the second image in Figure 46. (We tacitly include the Case (2) of the map κ_B by the choice of $b=N$ and the identification $b+1=N+1\equiv\overline{1}$.) Here, as in the proof of Theorem 3.10, the symbol $*$ indicates further possible elements in the block lying between $k+1$ and $a+2$.

The next applications of κ_B act as ordinary rotations on our block. Thereby, the element $b+1$ will be rotated into $\overline{1}$ (cf. the third image in Figure 46) and eventually into the situation illustrated in the fourth image in Figure 46. Here, the connection between $\overline{c}$ and $N-b+a+l+c$ is supposed to be the closest to the block containing N and $\overline{1}$ among the blocks containing a positive and a negative element.

The next application of κ_B brings us into the situation illustrated in the fifth image in Figure 46. Comparison with the third image shows that we shall now (possibly) revisit such situations several times until our maximal sequence of l successive elements reaches N and joins with $\overline{1}$. See the sixth image in Figure 46.

Further applications of κ_B map our block to $\{N-a,\dots,N-a+k-1,*,N,\overline{1},\dots,\overline{l}\}$; see the first image in Figure 47.

At this point, we should pause for a moment and make an intermediate summary of what we achieved so far: we started with a block containing two (or more) maximal sequences of (cyclically) successive numbers, one maximal sequence starting with $\overline{N},1,\dots,$ the one preceding $\overline{N}$ being one consisting entirely of positive numbers; see the first image in Figure 46. If we regard the first image in Figure 47, and take into account that also the "negative" of the considered block is a block in the type B non-crossing partition, then we have now obtained a block with one maximal sequence of successive elements starting with $\overline{N},1,\dots,$ however with the preceding maximal sequence of successive elements consisting

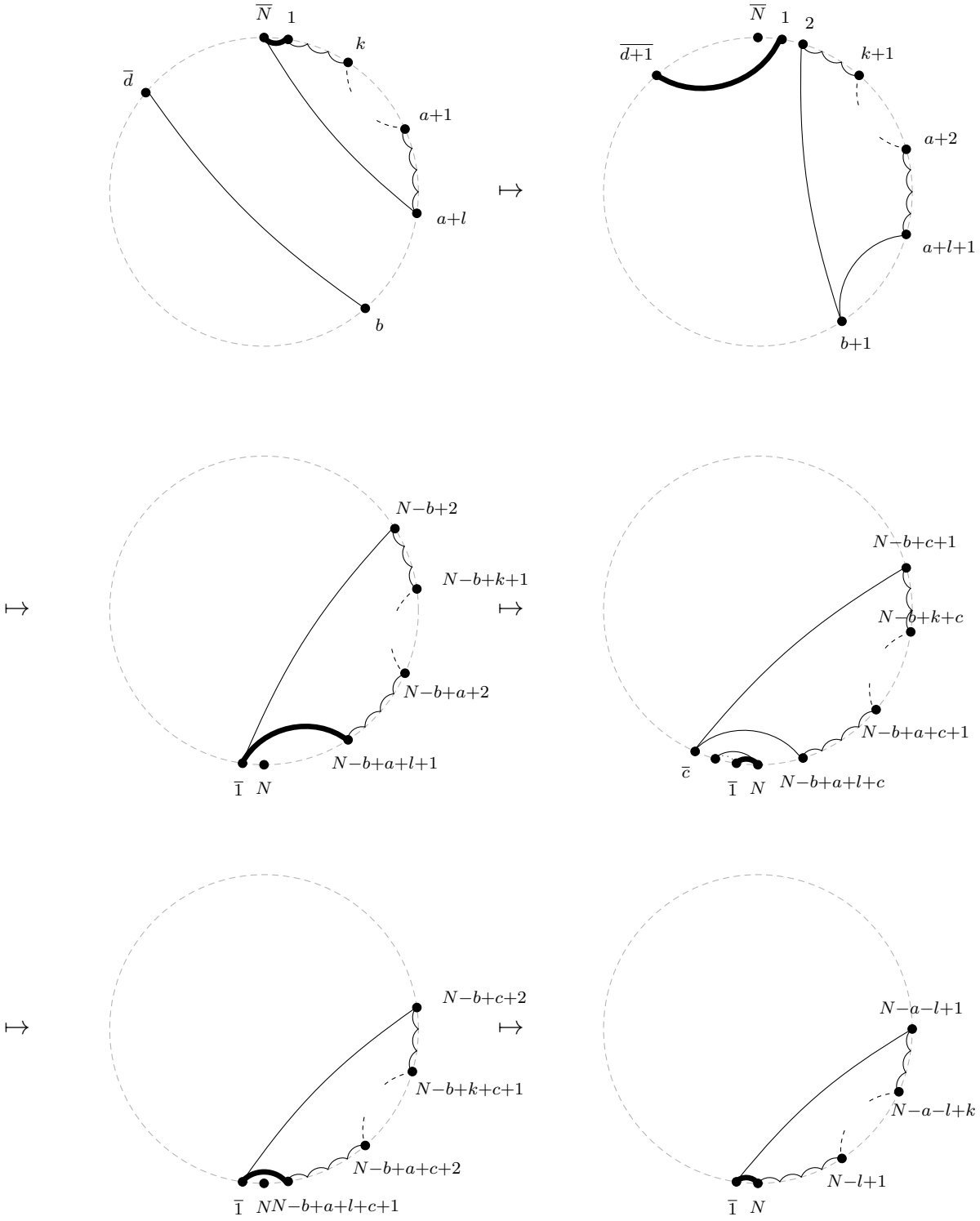

FIGURE 46. Pseudo-rotation of a block

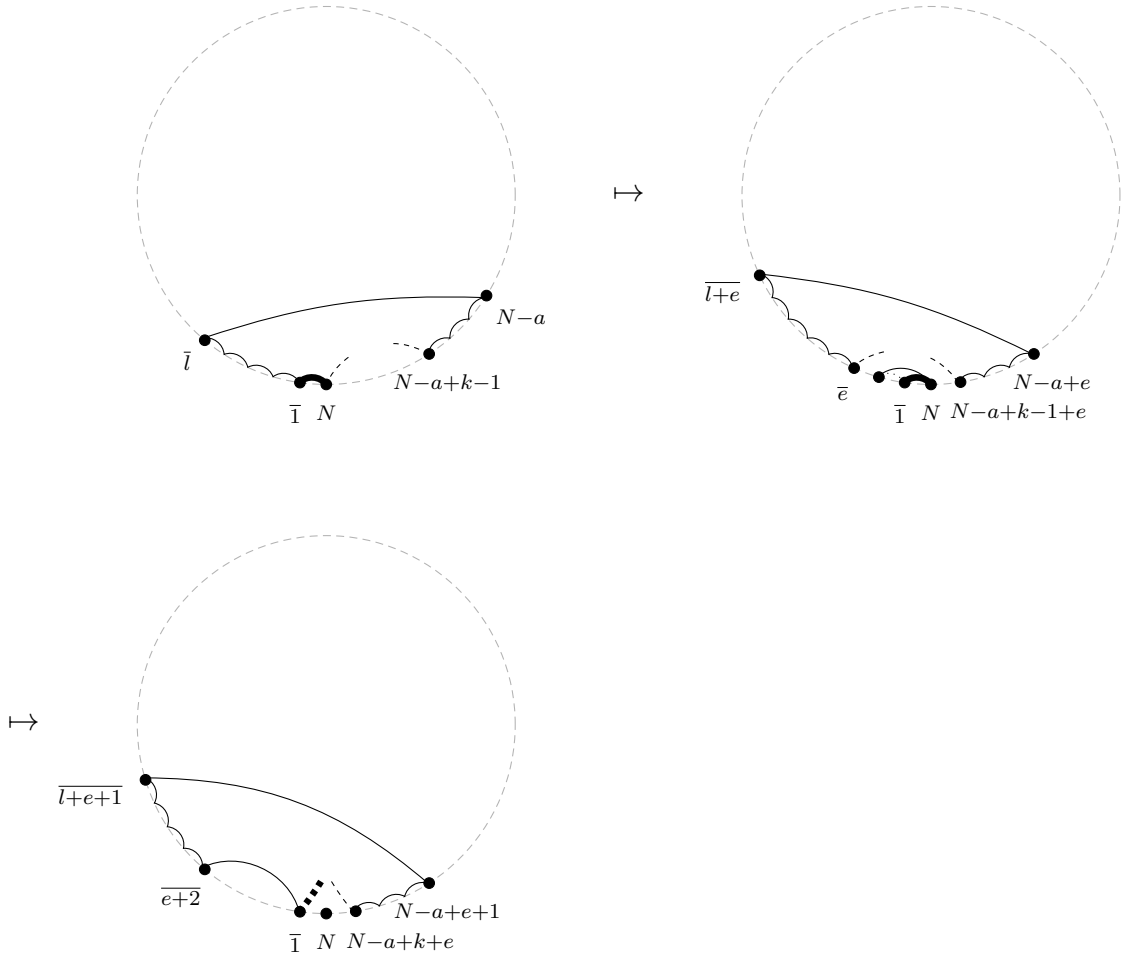

FIGURE 47. Pseudo-rotation of a block

entirely of *negative* elements. Thus, if we continue our analysis, then we will also have covered that case.

If we now continue applying the pseudo-rotation κ_B, then we know what is going to happen: a few applications will act as ordinary rotation by one unit on our block, until we reach the situation where the connection between the maximal sequence of successive negative numbers (the "leading" maximal sequence) and the preceding maximal sequence of successive numbers will be next to a block containing $N, \overline{1}$, and otherwise only negative numbers; see the second image in Figure 47. The next application of κ_B makes the "leading" maximal sequence lose its last element, connects what remains from it to $\overline{1}$, which itself is connected to the "next" maximal sequence; see the third image in Figure 47. Further applications of κ_B let one potentially run through situations illustrated in the second and third images in Figure 47 several times, until the "next" maximal sequence reaches N,

thereby also joining with $\overline{1}$. Thus, we have again reached a situation as in the first image in Figure 47, with now two maximal sequences of successive numbers on the negative side.

We may now summarise: whenever a maximal sequence of (cyclically) successive elements in a block is in the position $\{\overline{N}, 1, \dots, k\}$, then, when κ_B is applied, it gets rotated by one unit, it "loses" its last element, and in turn it "gains" an element in front of it (cf. the first and second images in Figure 46). Subsequent applications of κ_B act as ordinary rotations on this maximal sequence plus "front runner", until the front runner moves beyond $\overline{1}$. Upon further applications of κ_B, the "front runner" is eventually lost and replaced by $\overline{1}$, and this may happen several times (cf. the third, fourth, and fifth images in Figure 46), until our maximal sequence has reached N and thereby joins with $\overline{1}$.

The short (and rough) version of this process is that our maximal sequence of successive elements gets rotated by one unit by each application of κ_B, and, on the way, "loses" its last element and eventually "gains" the element in front of it. This implies that after $N-1$ applications of κ_B a block is mapped to its "mirror image" consisting of all negatives of its elements. Since we started with a type B non-crossing partition, this shows that κ_B^{N-1} acts as the identity on the pair consisting of the block and its "mirror image".

Step 4. So far, we have shown that, for a non-crossing partition π without singleton block, a block containing a maximal sequence of (cyclically) successive elements $\overline{N}, 1, \dots, k$ is mapped to its negative or to itself (if it is a zero block) by κ_B^{N-1}. If we now consider any other block, then we may apply κ_B as many times as it is necessary to move that block into a position where it contains a (cyclically) maximal sequence of successive elements beginning with $\overline{N}$ and 1. Then the above arguments apply. This shows that also any other block is mapped to its negative or to itself by $N-1$ applications of κ_B.

Since, in Step 1, we have shown that it suffices to prove the assertion for non-crossing partitions without singleton blocks, this completes the proof of the theorem. □

A.3. **The order of the positive Kreweras map in type D_n: Proof of Theorem 5.10.** As earlier, it is not difficult to see that it suffices to prove the assertion of the theorem for non-crossing partitions without singleton blocks. So, let π be a partition in $\widetilde{NC}_+^{(m)}(D_n)$ without singleton blocks. We distinguish between the following three cases:

Case 1. π contains a zero block;
Case 2. π contains no zero block and no bridging block, so that the inner circle is split into two blocks;
Case 3. π contains at least one bridging block.

We now discuss each case in more detail.

Case 1. When we apply κ_D repeatedly to a partition π in $\widetilde{NC}_+^{(m)}(D_n)$ with a zero block, then it is always Case (1) of Theorem 5.6 which occurs. If we ignore the elements of π on the inner circle, then π reduces to a partition $\hat{\pi}$ in $\widetilde{NC}_+^{(m)}(B_{n-1})$, and the action of κ_D on π behaves like the action of κ_B on $\hat{\pi}$ as described in Case (1) of Theorem 4.6. Theorem 4.10

with $N = m(n-1)$ says that $\kappa_B^{m(n-1)-1}$ acts on $\hat{\pi}$ as the identity. Consequently, $\kappa_D^{m(n-1)-1}$ acts on π as the identity.

CASE 2. When we apply κ_D repeatedly to a partition π in $\widetilde{NC}_+^{(m)}(D_n)$ without a zero block and without any bridging block, then it is the Cases (1) and (3) of Theorem 5.6 which occur. If we ignore the elements of π on the inner circle, then π reduces to a partition $\hat{\pi}$ in $\widetilde{NC}_+^{(m)}(B_{n-1})$, and the action of κ_D on π behaves like the action of κ_B on $\hat{\pi}$ as described in Theorem 4.6, with Case (1) corresponding to Case (1) of Theorem 5.6, and Case (2) corresponding to Case (3) of Theorem 5.6. Here again, Theorem 4.10 with $N = m(n-1)$ says that $\kappa_B^{m(n-1)-1}$ acts on $\hat{\pi}$ as the identity, and hence $\kappa_D^{m(n-1)-1}$ acts on π as the identity.

CASE 3. Let π be an element of $\widetilde{NC}_+^{(m)}(D_n)$ with a bridging block. By repeated application of κ_D, we may move this block into a position where its "first" element on the outer circle (in clockwise direction) is $\overline{m(n-1)}$, its next element is 1, and its "last" element on the outer circle is f, say. The successor of f in the block then is $\pm(mn-m+f+1)$, an element of the inner circle (to achieve this, $mn-m+f+1$ might have to be reduced modulo m), and the predecessor of $\overline{m(n-1)}$ is $\pm(mn-1)$, also an element of the inner circle. Let us call this block B, that is, $B = \{\overline{m(n-1)}, 1, *, f, \pm(mn-m+f+1), \ldots, \pm(mn-1)\}$. Here, as before, the symbol $*$ indicates further possible elements in the block lying between 1 and f, while the dots $\ldots$ indicate that all possible elements lying between $\pm(mn-m+f+1)$ and $\pm(mn-1)$ are included in the block. See the first image in Figure 48. In a degenerate case, it may also be that B contains no element from the inner circle. Nevertheless, in such a case we consider it still as a bridging block since another application of κ_D will make it a bridging block (via Case (2) of Theorem 5.6).

We claim that, after $m(n-1)-1$ applications of κ_D, the block B is moved to the "opposite side" of the annulus; more precisely, it becomes the block $\tilde{B}$, say, which consists of the elements of B on the outer circle with opposite sign, and of the elements of B on the inner circle if n is odd, and otherwise of the elements of B on the inner circle with opposite sign.

In order to see this, we notice that, for the first application of κ_D, we are in Case (2) of Theorem 5.6, with $a = \pm(mn-1)$. This first application of κ_D then removes $\overline{m(n-1)}$ from the block, adds $\mp(mn-m+1)$ to it, and replaces i by $i+1$ and $\bar{i}$ by $\overline{i+1}$ for $i \in B$ respectively $\bar{i} \in B$. See the second image in Figure 48. From here on, application of κ_D acts as ordinary rotation, that is, elements on the outer circle are shifted by one unit in clockwise direction, and elements on the inner circle are shifted by one unit in counter-clockwise direction, until a situation is reached where Case (2) of Theorem 5.6 applies again and involves our block of interest. More precisely, this situation occurs when our block has been (ordinarily) rotated $d-f-1$ times, for some $d > f+1$, the block containing $\overline{1}$ contains $m(n-1)$ and otherwise only negative elements, and there is no other block containing positive and negative elements that separates the latter block from our block of interest. The third image in Figure 48 provides a schematic illustration of that situation. In particular, our block has been (pseudo-)rotated to $\{d-f+1, *, d, \pm mn, \ldots, \mp(mn-$

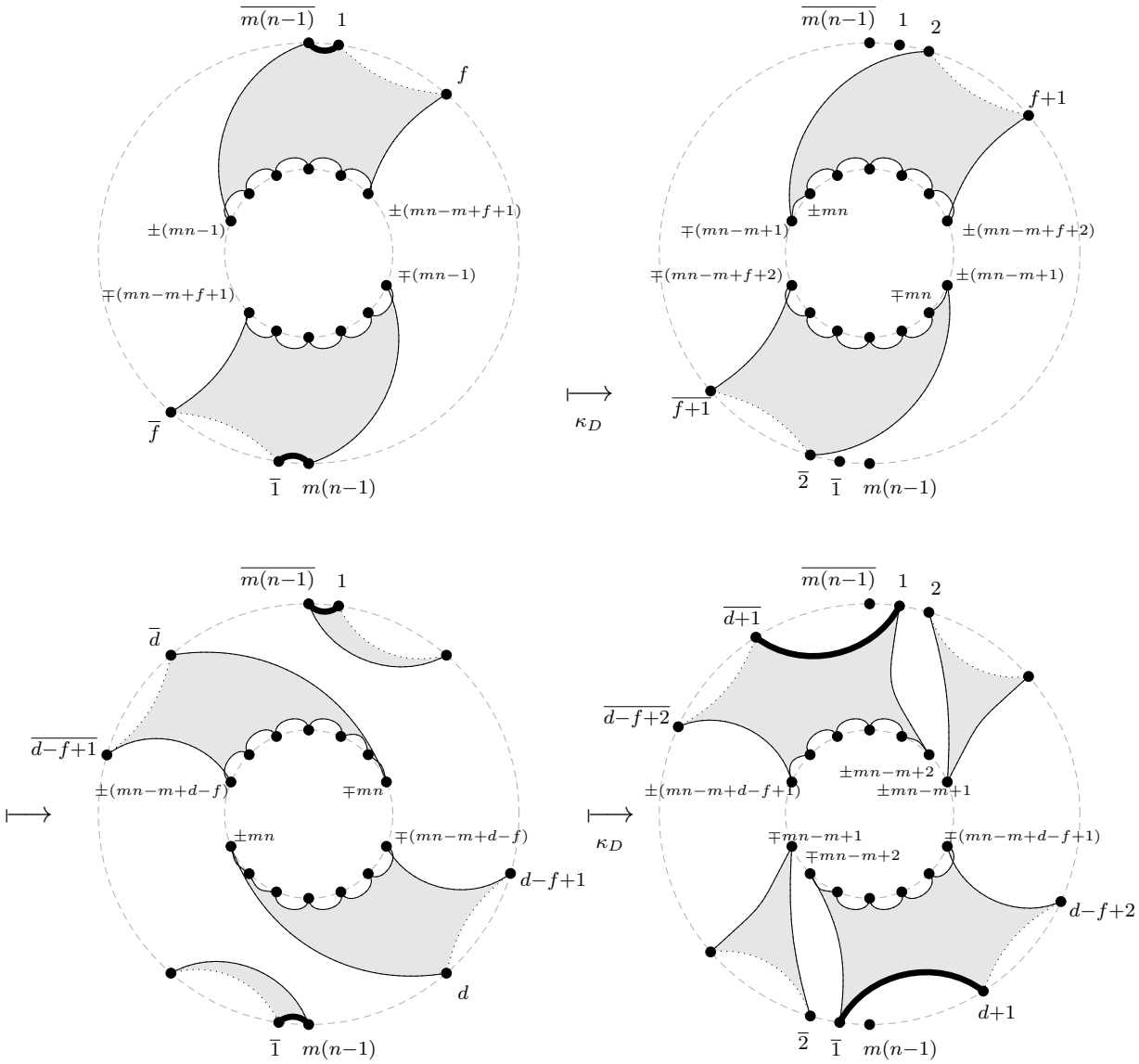

FIGURE 48. Movements of a bridging block under the pseudo-rotation κ_D

$m+d-f)\}$, where, again, $mn-m+d-f$ might have to be reduced modulo m. It should be observed that the fact that there are no blocks separating the block containing $\overline{1}$ and our block of interest implies that the number of elements lying on the outer circle between d and $m(n-1)$ must be divisible by m. Hence, we have $d \equiv -1 \pmod{m}$.

As we already mentioned, for the next application of κ_D we are in Case (2) of Theorem 5.6. It makes our block of interest "lose" $\pm mn$, "gain" $\overline{1}$, and otherwise i is replaced by $i+1$, respectively $\overline{i}$ by $\overline{i+1}$, for elements i respectively $\overline{i}$ in the block. That is, our

block has become $\{d-f+2, *, d+1, \overline{1}, \pm(mn-m+2), \dots, \mp(mn-m+d-f+1)\}$. See the fourth image in Figure 48.

At this point, we should pause for a moment and make an intermediate summary of what happened so far: we started with a block containing $\overline{m(n-1)}$ and 1, some further positive elements on the outer circle with maximal element f, and some successive elements on the inner circle ending in $\pm(mn-1)$ (when read in counter-clockwise direction). In the first application of κ_D we "lost" $\overline{m(n-1)}$ on the outer circle and we "gained" $\mp(mn-m+1)$ on the inner circle, while all other elements were rotated by one unit, in clockwise direction on the outer circle and in counter-clockwise direction on the inner circle. Subsequently, this (new) block was ordinarily rotated (in the same sense concerning the different directions on the outer circle and the inner circle). In the last application of κ_D, we "gained" $\overline{1}$ but "lost" the "first" element on the inner circle (read in counter-clockwise direction), while all other elements were rotated by one unit. This implies in particular that the successive elements on the inner circle made — so-to-speak — a further "jump" by one unit in addition to the "ordinary" rotations.

From here on, further applications of κ_D let the "first" and "last" elements on outer and inner circle of the block "move forward" by one unit; the map κ_D either acts by ordinary rotation or via Case (1) of Theorem 5.6. Both act by ordinary rotation on the inner circle, while in the latter case the action is equivalent to an action of κ_B on the outer circle (ignoring the inner circle). Since we have shown in Theorem 4.10 that the action of κ_B on positive type B non-crossing partitions of $\{1, 2, \dots, m(n-1), \overline{1}, \overline{2}, \dots, \overline{m(n-1)}\}$ has order $m(n-1)-1$, the above arguments show that the elements on the outer circle of block B will be "moved forward" — in clockwise direction — by $m(n-1)$ units, that is, to their negatives, while the elements on the inner circle will be "moved forward" — in counter-clockwise direction — by $m(n-1)$ units. The latter implies that they are mapped to the elements on the inner circle of B if n is odd, and otherwise to their negatives. This establishes the claim.

We are now in the position to conclude the proof. We have shown that $m(n-1)-1$ applications of κ_D map the elements on the outer circle of a bridging block to their negatives, and the elements on the inner circle to themselves if n is odd, while to their negatives if n is even. Moreover, the other blocks (which must lie on the outer circle since the inner circle has only $2m$ elements and all block sizes are divisible by m) are moved by κ_B-operations when κ_D is applied. Consequently, by Theorem 4.10, $m(n-1)-1$ applications of κ_D map them to their negatives. Thus, if n is even, *all* blocks are mapped to their negatives, which means that we obtained the original partition since it was invariant under substitution of i by $-i$, for all i, from the very beginning. On the other hand, if n is odd, then we need another $m(n-1)-1$ applications of κ_D to get back to our original partition. □

Appendix B. Proofs: Enumeration results

This appendix is devoted to the proofs of Theorems 3.11, 4.11 and 5.11 in which we enumerate multichains of positive m-divisible non-crossing set partitions in classical types

according to their block structure, and to the proofs of Theorems 3.18 and 4.15 and Propositions 5.15 and 5.17 in which we enumerate pseudo-rotationally invariant positive m-divisible non-crossing set partitions of classical types.

B.1. **Preparations and auxiliary results.** We use the generating function approach from [Kra13]. We start by recalling the two forms of Lagrange inversion (cf. [Hen74, Thm. 1.9b], [Sta99, Thm. 5.4.2].

Lemma B.1 (Lagrange inversion)**.** *Let $f(z)$ be a formal power series with $f(0) = 0$ and $f'(0) \neq 0$, and let $F(z)$ be its compositional inverse. Then, for all integers a and b,*

$$\left\langle z^0 \right\rangle z^a F^{b-1}(z) = \left\langle z^0 \right\rangle z^b f^{a-1}(z) f'(z) \tag{B.1}$$

and

$$a \left\langle z^0 \right\rangle z^a F^b(z) = -b \left\langle z^0 \right\rangle z^b f^a(z). \tag{B.2}$$

More generally, if $g(z)$ is a Laurent series with only finitely many terms containing negative powers of z, then

$$\left\langle z^{-1} \right\rangle g'(z) F^{-b}(z) = b \left\langle z^b \right\rangle g(f(z)). \tag{B.3}$$

Let x_i, y_i, $i = 1, 2, \dots$, be variables, and $x_0 = y_0 = 1$. For a positive m-divisible non-crossing partition π let the weight $w_{\mathbf{y}}^{(m)}(\pi)$ be defined by

$$w_{\mathbf{y}}^{(m)}(\pi) = \prod_{i=1}^{\infty} y_i^{\#(\text{special blocks of } \pi \text{ of size } mi)} x_i^{\#(\text{non-special blocks of } \pi \text{ of size } mi)}.$$

Recall that the special block of a positive m-divisible non-crossing partition π is the one which contains both 1 and mn, so that, clearly, in $w_{\mathbf{y}}^{(m)}(\pi)$ there appears exactly one variable y_i, and it does so with exponent 1. For example, the weight $w_{\mathbf{y}}^{(3)}(\,.\,)$ of the positive m-divisible non-crossing partition on the left of Figure 5 is $y_2 x_1^6$, while the weight of the non-crossing partition on the right of the same figure is $y_1 x_1^5 x_2$.

Sometimes, we shall consider the special case where y_i is set equal to x_i for all i. We abbreviate this specialised weight by $w^{(m)}$, that is,

$$w^{(m)}(\pi) = \prod_{i=1}^{\infty} x_i^{\#(\text{blocks of } \pi \text{ of size } mi)}.$$

For example, the specialised weight $w^{(m)}(\,\cdot\,)$ of the two positive m-divisible non-crossing partitions in Figure 5 is $x_1^6 x_2$.

We write $|\pi|$ for the *size* of π, that is, for the number of elements of the set that is partitioned by π.

First we quote an auxiliary result from [Kra13], providing a functional equation for the generating function for *all* m-divisible non-crossing partitions with respect to the specialised weight $w^{(m)}$. For the sake of completeness, we also sketch its proof.

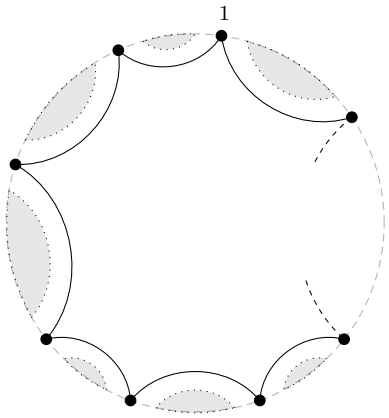

FIGURE 49. Illustration of the decomposition explaining (B.4): between the arcs of the block containing 1 there are (smaller) non-crossing partitions, indicated by shaded "half disks".

Lemma B.2. *Let $C^{(m)}(z)$ be the generating function $\sum_\pi w^{(m)}(\pi) z^{|\pi|}$, where the sum is over all m-divisible non-crossing partitions. Then*

$$C^{(m)}(z) = \sum_{i=0}^{\infty} x_i z^{mi} \left(C^{(m)}(z)\right)^{mi}. \tag{B.4}$$

Proof. The equation is best explained by a picture; see Figure 49. Given an m-divisible non-crossing partition π, let B be the block containing the element 1. This block has a size that is divisible by m, say mi. The (weight) contribution of that block is then $x_i z^{mi}$. Between the "arcs" of that block B we find smaller m-divisible non-crossing partitions, each one contributing $C^{(m)}(z)$ to the term. Since the block B has mi arcs, this explains the summand in (B.4) by standard generating function calculus. The equation follows once one observes that the summand for $i = 0$ corresponds to the (empty) partition of $\emptyset$. □

From now on,

$$F(z) = \frac{z}{\sum_{i=0}^{\infty} x_i z^{mi}}, \tag{B.5}$$

and $f(z)$ denotes its compositional inverse.

Slightly rewriting (B.4), we see that it is equivalent to

$$z = \frac{zC^{(m)}(z)}{\sum_{i=0}^{\infty} x_i (zC^{(m)}(z))^{im}} = F(zC^{(m)}(z)).$$

Thus, we have $zC^{(m)}(z) = f(z)$.

We quote an easy corollary from [Kra13].

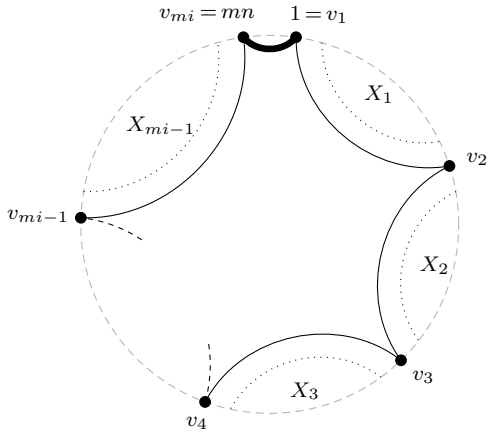

FIGURE 50. The decomposition of a positive m-divisible non-crossing partition

Corollary B.3. *The generating function* $\sum_\pi w^{(m)}(\pi)$*, where the sum is over all* m*-divisible non-crossing partitions of* $\{1, 2, \dots, mn\}$*, equals*

$$\frac{1}{mn+1} \left\langle z^0 \right\rangle z F^{-mn-1}(z), \tag{B.6}$$

where $F(z)$ *is defined by* (B.5)*.*

Proof. By Lemma B.2, the generating function in question is

$$\left\langle z^{mn} \right\rangle C^{(m)}(z) = \left\langle z^0 \right\rangle z^{-mn-1} f(z).$$

The assertion now follows from the application of the Lagrange inversion formula in the form (B.2). □

B.2. **Enumeration of positive m-divisible non-crossing partitions in type A.** We turn to the subset of all *positive* m-divisible non-crossing partitions.

Lemma B.4. *Let* $C_+^{(m)}(z)$ *be the generating function* $\sum_\pi w_{\mathbf{y}}^{(m)}(\pi) z^{|\pi|}$*, where the sum is over all positive* m*-divisible non-crossing partitions. Then*

$$C_+^{(m)}(z) = \sum_{i=1}^{\infty} y_i z^{mi} \left(C^{(m)}(z) \right)^{mi-1}. \tag{B.7}$$

Proof. The vertices 1 and mn must be in one of the blocks, say in a block of size mi. Then, if $v_1 = 1, v_2, \dots, v_{mi} = mn$ are all the vertices in this block in clockwise order, then the vertices between v_j and v_{j+1}, $j = 1, 2, \dots, mi - 1$ are involved in a smaller (arbitrary, not necessarily positive) m-divisible non-crossing partition (see Figure 50). Equation (B.7) then follows by standard generating function calculus, the block containing 1 and mn contributing the term $y_i z^{mi}$, and the "small" m-divisible non-crossing partitions between v_j and v_{j+1} each contributing a term $C^{(m)}(z)$, $j = 1, 2, \dots, mi - 1$. □

The analogue of Corollary B.3 for positive m-divisible non-crossing partitions reads as follows.

Corollary B.5. *The generating function* $\sum_\pi w_{\mathbf{y}}^{(m)}(\pi)$, *where the sum is over all positive* m*-divisible non-crossing partitions of* $\{1,2,\dots,mn\}$, *equals*

$$\sum_{i=1}^{\infty} \frac{mi-1}{mn-1} y_i \left\langle z^0 \right\rangle z^{mi-1} F^{-mn+1}(z), \tag{B.8}$$

where $F(z)$ *is defined by* (B.5).

Proof. Because of $C^{(m)}(z) = f(z)/z$, we know from (B.7) that the desired generating function equals

$$\left\langle z^0 \right\rangle \sum_{i=1}^{\infty} y_i z^{-mn+1} f^{mi-1}(z). \tag{B.9}$$

Hence, the claim follows from (B.2) with $a = mi-1$ and $b = -mn+1$. □

Proposition B.6. *Let* m, n, l *be positive integers with* $l \geq 2$. *Furthermore, let* $s_2', s_3, \dots, s_l$ *be non-negative integers with* $s_2' + s_3 + \cdots + s_l = n-1$. *The generating function*

$$\sum{}' w_{\mathbf{y}}^{(m)}(\pi_1), \tag{B.10}$$

where the sum is over all multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{l-1}$ *of positive* m*-divisible non-crossing partitions of* $\{1,2,\dots,mn\}$, *where the rank of* π_i *is* $s_2' + s_3 + \cdots + s_i$, $i = 2,3,\dots,l-1$, *is given by*

$$\begin{aligned} &\frac{1}{mn-1}\binom{mn-1}{s_3}\cdots\binom{mn-1}{s_l} \\ &\times \sum_{k=0}^{s_3+\cdots+s_l} (-1)^{k+s_3+\cdots+s_l}\binom{s_3+\cdots+s_l}{k} \sum_{a=1}^{\infty} \left\langle z^0 \right\rangle (ma-1) y_a z^{ma+k-1} F^{-mn-k+1}(z). \end{aligned} \tag{B.11}$$

If $l = 2$, *empty sums have to interpreted as* 0, *and empty products have to be interpreted as* 1.

Proof. We prove the assertion by induction on l. If $l = 2$, the assertion is true due to (B.8) and our conventions for $l = 2$. Let us now assume that the assertion is true for multichains consisting of $l-2$ elements and consider the multichain $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{l-1}$. The induction hypothesis implies that the generating function

$$\sum w_{\mathbf{y}}^{(m)}(\pi_2),$$

where the sum is over all multichains $\pi_2 \leq \cdots \leq \pi_{l-1}$ of positive m-divisible non-crossing partitions of $\{1,2,\dots,mn\}$, where the rank of π_i is $(s_2' + s_3) + s_4 + \cdots + s_i$, $i = 3,\dots,l-1$,

is given by

$$\frac{1}{mn-1}\binom{mn-1}{s_4}\cdots\binom{mn-1}{s_l}$$
$$\times \sum_{k=0}^{s_4+\cdots+s_l}(-1)^{k+s_4+\cdots+s_l}\binom{s_4+\cdots+s_l}{k}\sum_{a=1}^{\infty}\left\langle z^0\right\rangle (ma-1)y_a z^{ma+k-1}F^{-mn-k+1}(z). \quad \text{(B.12)}$$

Next we model the relation $\pi_1 \le \pi_2$ by appropriate replacements of the variables x_b and y_a in the generating function (B.12). The partition π_1 arises from π_2 by replacing non-special blocks of π_2 of size mb by m-divisible non-crossing partitions of mb elements, and by replacing the special block, say of size ma, by a *positive* m-divisible non-crossing partition of ma elements. Hence, to model this in the generating function, we will replace in (B.12) the variable x_b by the coefficient of u^{mb} in $tC^{(m)}(u)$, and we will replace the variable y_a by the coefficient of u^{ma} in $tC_+^{(m)}(u)$. (Recall that the series $C^{(m)}(u)$ is the generating function for *all* m-divisible non-crossing partitions, while $C_+^{(m)}(u)$ is the one for *positive* m-divisible non-crossing partitions.) Here, the variable t keeps track of the number of blocks of π_2 and, thus, by the equation

$$n - (\text{number of blocks of } \pi_2) = (\text{rank of } \pi_2) = s_2',$$

respectively equivalently,

$$(\text{number of blocks of } \pi_2) = s_3 + s_4 + \cdots + s_l + 1,$$

of the rank of π_2. From the result we will then extract the coefficient of $t^{s_3+s_4+\cdots+s_l+1}$ to obtain the generating function (B.10).

Next, we compute the effect of substitution of $\left\langle u^{mb}\right\rangle tC^{(m)}(u)$ for x_b, $b = 1, 2, \ldots$, in the series $F(z)$ (recall its definition in Equation (B.5)):

$$\begin{aligned} z\left(1+\sum_{b=1}^{\infty} z^{mb}\left\langle u^{mb}\right\rangle tC^{(m)}(u)\right)^{-1} &= z\left(1+t(C^{(m)}(z)-1)\right)^{-1} \\ &= z\left(1+t\left(\frac{f(z)}{z}-1\right)\right)^{-1}. \end{aligned} \quad \text{(B.13)}$$

On the other hand, we need to compute the effect of substitution of $\langle u^{ma}\rangle\, tC_+^{(m)}(u) = \langle u^{ma}\rangle\, t\sum_{\ell=1}^{\infty} y_\ell u f(u)^{m\ell-1}$ (see (B.8), respectively (B.9)) for y_a in $\sum_{a=1}^{\infty}(ma-1)y_a z^{ma-1}$.

We obtain

$$\begin{aligned}\sum_{a=1}^{\infty}(ma-1)z^{ma-1}&\left(\langle u^{ma}\rangle\, t\sum_{\ell=1}^{\infty}y_\ell u f(u)^{m\ell-1}\right)\\ &=\sum_{a=1}^{\infty}\left(\frac{d}{dz}-\frac{1}{z}\right)(z^{ma})\cdot\left(\langle u^{ma}\rangle\, t\sum_{\ell=1}^{\infty}y_\ell u f(u)^{m\ell-1}\right)\\ &=t\left(\frac{d}{dz}\sum_{\ell=1}^{\infty}y_\ell z f^{m\ell-1}(z)-\frac{1}{z}\sum_{\ell=1}^{\infty}y_\ell z f^{m\ell-1}(z)\right)\\ &=t\sum_{\ell=1}^{\infty}(m\ell-1)y_\ell z f'(z)f^{m\ell-2}(z).\end{aligned}\tag{B.14}$$

Using the substitutions (B.13) and (B.14) in (B.12), subsequent extraction of the coefficient of $t^{s_3+s_4+\cdots+s_l+1}$ leads to

$$\begin{aligned}&\left\langle t^{s_3+s_4+\cdots+s_l+1}\right\rangle\frac{1}{mn-1}\binom{mn-1}{s_4}\cdots\binom{mn-1}{s_l}\sum_{k=0}^{s_4+\cdots+s_l}(-1)^{k+s_4+\cdots+s_l}\binom{s_4+\cdots+s_l}{k}\\ &\qquad\cdot t\sum_{\ell=1}^{\infty}\left\langle z^0\right\rangle(m\ell-1)y_\ell z f'(z)f^{m\ell-2}(z)z^{-mn+1}\left(1+t\left(\frac{f(z)}{z}-1\right)\right)^{mn+k-1}\\ &=\frac{1}{mn-1}\binom{mn-1}{s_4}\cdots\binom{mn-1}{s_l}\sum_{k=0}^{s_4+\cdots+s_l}(-1)^{k+s_4+\cdots+s_l}\binom{s_4+\cdots+s_l}{k}\\ &\quad\cdot\sum_{\ell=1}^{\infty}\left\langle z^0\right\rangle(m\ell-1)y_\ell f'(z)f^{m\ell-2}(z)z^{-mn+2}\\ &\qquad\cdot\binom{mn+k-1}{s_3+s_4+\cdots+s_l}\left(\frac{f(z)}{z}-1\right)^{s_3+s_4+\cdots+s_l}\\ &=\frac{1}{mn-1}\binom{mn-1}{s_4}\cdots\binom{mn-1}{s_l}\sum_{k=0}^{s_4+\cdots+s_l}(-1)^{k}\binom{mn+k-1}{s_3+s_4+\cdots+s_l}\binom{s_4+\cdots+s_l}{k}\\ &\qquad\cdot\sum_{\ell=1}^{\infty}\sum_{j=0}^{s_3+s_4+\cdots+s_l}(-1)^{s_3+j}\binom{s_3+s_4+\cdots+s_l}{j}\\ &\qquad\qquad\cdot\left\langle z^0\right\rangle(m\ell-1)y_\ell f'(z)f^{m\ell+j-2}(z)z^{-mn-j+2}\end{aligned}\tag{B.15}$$

The sums over k and j have become completely independent, and the sum over k can be evaluated by means of the Chu–Vandermonde summation formula (see e.g. [GKP89,

Sec. 5.1, Eq. (5.27)]). Namely, we have

$$\begin{aligned}
&\sum_{k=0}^{s_4+\cdots+s_l}(-1)^k\binom{mn+k-1}{s_3+s_4+\cdots+s_l}\binom{s_4+\cdots+s_l}{k}\\
&\quad=(-1)^{mn-s_3-s_4-\cdots-s_l-1}\sum_{k=0}^{s_4+\cdots+s_l}\binom{-s_3-s_4-\cdots-s_l-1}{mn+k-s_3-s_4-\cdots-s_l-1}\binom{s_4+\cdots+s_l}{s_4+\cdots+s_l-k}\\
&\quad=(-1)^{mn-s_3-s_4-\cdots-s_l-1}\binom{-s_3-1}{mn-s_3-1}\\
&\quad=(-1)^{s_4+\cdots+s_l}\binom{mn-1}{mn-s_3-1}.
\end{aligned}\tag{B.16}$$

If we substitute this in (B.15) and use the Lagrange inversion formula (B.1) from Lemma B.1 with $a=m\ell+j-1$ and $b=-mn-j+2$, we obtain exactly (B.11). □

Proof of Theorem 3.11. We use Proposition B.6 with $s_2'=s_1+s_2$. We have

$$n-(\text{number of blocks of }\pi_1)=(\text{rank of }\pi_1)=s_1,$$

or, equivalently,

$$\text{number of blocks of }\pi_1=s_2+s_3+\cdots+s_l+1.$$

Hence, we must replace both x_b and y_b by tx_b in (B.11) and extract the coefficient of

$$t^{s_2+s_3+\cdots+s_l+1}x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}.$$

(In the sequel, we write $t^{s_2+s_3+\cdots+s_l+1}\mathbf{x}^{\mathbf{b}}$ for short for the above monomial.) To begin with, we perform the substitution $y_b\to x_b$ in the sum over a in (B.11) and simplify:

$$\begin{aligned}
&\sum_{a=1}^{\infty}\langle z^0\rangle(ma-1)x_az^{ma+k-1}F^{-mn-k+1}(z)\\
&\qquad=\langle z^0\rangle\, z^{k+1}F^{-mn-k-1}(z)\frac{\sum_{a=1}^{\infty}(ma-1)x_az^{ma}}{\left(1+\sum_{a=1}^{\infty}x_az^{ma}\right)^2}\\
&\qquad=\langle z^0\rangle\, z^{k+1}F^{-mn-k-1}(z)\left(-F'(z)+z^{-2}F^2(z)\right)\\
&\qquad=\langle z^0\rangle\left(-z^{k+1}F'(z)F^{-mn-k-1}(z)+z^{k-1}F^{-mn-k+1}(z)\right)\\
&\qquad=\langle z^0\rangle\left(z^{k+1}\frac{1}{mn+k}\frac{d}{dz}F^{-mn-k}(z)+z^{k-1}F^{-mn-k+1}(z)\right)\\
&\qquad=\langle z^0\rangle\left(-\frac{k}{mn+k}z^kF^{-mn-k}(z)+z^{k-1}F^{-mn-k+1}(z)\right).
\end{aligned}$$

This is now used in (B.11). Subsequently, as we already said, we replace x_b by tx_b for all b and extract the coefficient of $t^{s_2+s_3+\cdots+s_l+1}\mathbf{x}^{\mathbf{b}}$. This leads to

$$\begin{aligned}
&\frac{1}{mn-1}\binom{mn-1}{s_3}\cdots\binom{mn-1}{s_l}\sum_{k=0}^{s_3+\cdots+s_l}(-1)^{k+s_3+s_4+\cdots+s_l}\binom{s_3+s_4+\cdots+s_l}{k}\\
&\qquad\cdot\left\langle t^{s_2+s_3+\cdots+s_l+1}\mathbf{x}^{\mathbf{b}}z^{mn}\right\rangle\left(-\frac{k}{mn+k}\left(1+t\sum_{i=1}^{\infty}x_iz^{mi}\right)^{mn+k}\right.\\
&\qquad\qquad\left.+\left(1+t\sum_{i=1}^{\infty}x_iz^{mi}\right)^{mn+k-1}\right)\\
&=\frac{1}{mn-1}\binom{mn-1}{s_3}\cdots\binom{mn-1}{s_l}\sum_{k=0}^{s_3+\cdots+s_l}(-1)^{k+s_3+s_4+\cdots+s_l}\binom{s_3+s_4+\cdots+s_l}{k}\\
&\qquad\cdot\left(-\frac{k}{mn+k}\binom{mn+k}{s_2+s_3+\cdots+s_l+1}\left\langle\mathbf{x}^{\mathbf{b}}z^{mn}\right\rangle\left(\sum_{i=1}^{\infty}x_iz^{mi}\right)^{s_2+s_3+\cdots+s_l+1}\right.\\
&\qquad\qquad\left.+\binom{mn+k-1}{s_2+s_3+\cdots+s_l+1}\left\langle\mathbf{x}^{\mathbf{b}}z^{mn}\right\rangle\left(\sum_{i=1}^{\infty}x_iz^{mi}\right)^{s_2+s_3+\cdots+s_l+1}\right)\\
&=\frac{1}{mn-1}\binom{mn-1}{s_3}\cdots\binom{mn-1}{s_l}\sum_{k=0}^{s_3+\cdots+s_l}(-1)^{k+s_3+s_4+\cdots+s_l}\binom{s_3+s_4+\cdots+s_l}{k}\\
&\qquad\cdot\frac{-k+(mn+k-s_2-s_3-\cdots-s_l-1)}{s_2+s_3+\cdots+s_l+1}\binom{mn+k-1}{s_2+s_3+\cdots+s_l}\\
&\qquad\qquad\cdot\left\langle\mathbf{x}^{\mathbf{b}}z^{mn}\right\rangle\left(\sum_{i=1}^{\infty}x_iz^{mi}\right)^{s_2+s_3+\cdots+s_l+1}.
\end{aligned}$$

The sum over k is completely analogous to the sum over k in the proof of Proposition B.6 and, hence, can as well be evaluated by means of the Chu–Vandermonde summation formula. Repeating the arguments from there, we arrive at the expression

$$\begin{aligned}
&\frac{mn-s_2-s_3-\cdots-s_l-1}{(mn-1)(s_2+s_3+\cdots+s_l+1)}\binom{mn-1}{s_2}\binom{mn-1}{s_3}\cdots\binom{mn-1}{s_l}\\
&\qquad\times\left\langle\mathbf{x}^{\mathbf{b}}z^{mn}\right\rangle\left(\sum_{i=1}^{\infty}x_iz^{mi}\right)^{s_2+s_3+\cdots+s_l+1}. \qquad \text{(B.17)}
\end{aligned}$$

Noting that the conditions in the statement of the theorem say that

$$b_1+2b_2+3b_3+\cdots+nb_n=n$$

and

$$s_1+b_1+b_2+\cdots+b_n=n,$$

and, hence,

$$b_1 + b_2 + \cdots + b_n = s_2 + s_3 + \cdots + s_l + 1,$$

we arrive at (3.3) without any difficulty. □

B.3. **Enumeration of pseudo-rotationally invariant positive m-divisible non-crossing partitions in type A.** Again, we use a generating function approach as in Appendix B.2. When we consider ρ-pseudo-rotationally invariant non-crossing partitions, we need to introduce a "simplified" weight, which we denote by $w_\rho^{(m)}$, and which is defined by

$$w_\rho^{(m)}(\pi) = \prod_{i=1}^{\infty} x_i^{\#(\text{non-central blocks of } \pi \text{ of size } mi)/\rho}.$$

For example, the weight $w_4^{(3)}(\,.\,)$ of both non-crossing partitions in Figures 9 and 10 is x_1^2, while the weight $w_2^{(3)}(\,.\,)$ of both partitions in Figures 12 and 13 is $x_1^4 x_2$.

Lemma B.7. *Let ρ, m, n be positive integers with $\rho \geq 2$, $\rho \mid (mn-2)$. The generating function $\sum_\pi w_\rho^{(m)}(\pi)$, where the sum is over all positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ which are invariant under the ρ-pseudo-rotation $\kappa_A^{(mn-2)/\rho}$ and have a central block of size 2 equals*

$$\left\langle z^0 \right\rangle F^{-(mn-2)/\rho}(z), \tag{B.18}$$

where $F(z)$ is defined by (B.5) and $f(z)$ is the compositional inverse of $F(z)$, as before.

Remark B.8. Since m divides the sizes of all blocks, m-divisible non-crossing partitions as described in the lemma only exist for $m = 1$ and $m = 2$. Indeed, the formula (B.18) produces 0 for $m \geq 3$.

Proof of Lemma B.7. We call a subpartition of a non-crossing partition a *connected component* if it is a partition of consecutive elements, of $\{a, a+1, \dots, b\}$ say, and a and b are in the same block. In Construction 1 in Subsection 3.5 with $N = mn$ and a central block of size 2, $\{2, 3, \dots, \frac{mn-2}{\rho}+1\}$ is a union of blocks. Likewise, $\{mn - \frac{mn-2}{\rho}, \dots, mn-2, mn-1\}$ is a union of blocks. These blocks are organised in connected components. Let the connected component next to mn cover the (consecutive) elements $\{a, \dots, mn-2, mn-1\}$. The first application of κ_A moves this connected component "into $\{mn, 1\}$", while a block $\{a+1, 2\}$ is created covering this moved component. The next $mn - a - 2$ applications of κ_A rotate this arc-like block ordinarily, until a further application of κ_A — so-to-speak — maps it back to $\{mn, 1\}$, while the connected component now covers the elements $\{2, 3, \dots, mn - a + 1\}$.

What the above arguments show is that the set of non-crossing partitions that one obtains from a particular partition, $\hat{\pi}$ say, of $\{mn - \frac{mn-2}{\rho}, \dots, mn-2, mn-1\}$ in Construction 1 and subsequent applications of κ_A will be the same as from the partition where one has circularly permuted the connected components of $\hat{\pi}$ and subsequent applications of κ_A. Consequently, using elementary generating function calculus, if $G(z)$ denotes the

generating function $\sum_\gamma w^{(m)}(\gamma)$, where the sum is over all possible connected components, the generating function that we want to compute equals

$$\frac{mn-2}{\rho}\left\langle z^{(mn-2)/\rho}\right\rangle \sum_{k=1}^{\infty}\frac{1}{k}G^k(z) = -\frac{mn-2}{\rho}\left\langle z^{(mn-2)/\rho}\right\rangle \log\big(1-G(z)\big). \tag{B.19}$$

In order to find an expression for $G(z)$, we observe that this can be achieved by following the proof of Lemma B.4 (with the role of the arc covering the connected component being played by the arc $\{1, mn\}$ there). The result is

$$G(z) = \sum_{i=1}^{\infty} x_i z^{mi}\left(C^{(m)}(z)\right)^{mi-1} = 1 - \frac{1}{C^{(m)}(z)} = 1 - \frac{z}{f(z)},$$

where we used (B.4) and the definition of $f(z)$ to find the right-hand side. We substitute this in (B.19) to find that the generating function that we are interested in is

$$\begin{aligned} -\frac{mn-2}{\rho}\left\langle z^{(mn-2)/\rho}\right\rangle \log\frac{z}{f(z)} &= \left\langle z^{(mn-2)/\rho}\right\rangle z\frac{d}{dz}\log\frac{f(z)}{z} \\ &= \left\langle z^{(mn-2)/\rho}\right\rangle \frac{zf'(z)-f(z)}{f(z)} \\ &= \left\langle z^0\right\rangle z^{1-(mn-2)/\rho} f^{-1}(z) f'(z). \end{aligned}$$

By Lagrange inversion in the form (B.1), this expression can be transformed into the one in (B.18). □

Lemma B.9. *Let ρ, m, n, a be positive integers with $\rho \geq 2$, $\rho \mid (mn-2)$, and $a \equiv n \pmod{\rho}$. The generating function $\sum_\pi w_\rho^{(m)}(\pi)$, where the sum is over all positive m-divisible non-crossing partitions of $\{1, 2, \ldots, mn\}$ which are invariant under the ρ-pseudo-rotation $\kappa_A^{(mn-2)/\rho}$ and have a central block of size ma equals*

$$\frac{mn-2}{ma-2}\left\langle z^{(mn-2)/\rho}\right\rangle f^{(ma-2)/\rho}(z) = \left\langle z^0\right\rangle z^{(ma-2)/\rho} F^{-(mn-2)/\rho}(z), \tag{B.20}$$

where $F(z)$ is defined by (B.5) and $f(z)$ is the compositional inverse of $F(z)$, as before. In the special case where $ma = 2$, the left-hand side of (B.20) must be ignored.

Remark B.10. The condition $a \equiv n \pmod{\rho}$ is actually redundant, but is recorded in the statement of the lemma for the convenience of the reader. More precisely, according to Theorem 3.17, the non-crossing partitions in the statement of the lemma arise exclusively from Construction 1. In that construction, the number of the non-central blocks must be divisible by ρ. Consequently, the number of elements contained in non-central blocks is divisible by $m\rho$. Thus, we have $mn = ma + m\rho B$, for some B. Clearly, this implies $a \equiv n \pmod{\rho}$.

Proof of Lemma B.9. We have already treated the case where $ma = 2$ in Lemma B.7 above. So, let us assume that $ma \neq 2$ from now on.

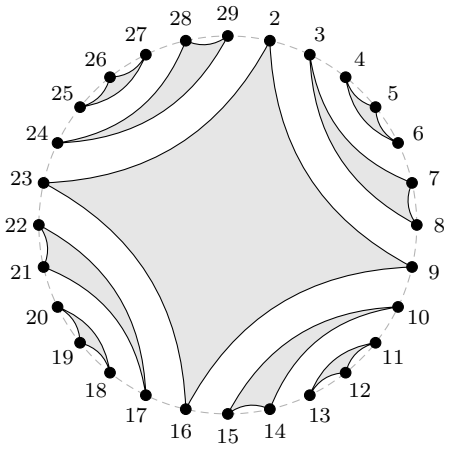

FIGURE 51. A 4-rotationally invariant non-crossing partition of $\{2, 3, \dots, 29\}$

The equality in (B.20) is a direct consequence of the Lagrange inversion formula (B.2) with a replaced by $(ma-2)/\rho$ and $b = -(mn-2)/\rho$. Hence, it suffices to derive one of the two formulae.

Theorem 3.17(1) says that — essentially — enumerating positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ which are invariant under the ρ-pseudo rotation $\kappa_A^{(mn-2)/\rho}$ is equivalent to enumerating (ordinary) non-crossing partitions of $mn-2$ elements, with all block sizes except the one of the central block multiples of m, which are invariant under the ρ-rotation $R^{(mn-2)/\rho}$.

For the following considerations it will be useful to simultaneously look at Figure 51 which shows a non-crossing partition of $\{2, 3, \dots, 29\}$ which is invariant under the 4-pseudo-rotation R^7 (in the sense described in Construction 1, that is, identifying 30 and 2) and which has a central block of size $4 = 3 \cdot 2 - 2$. This is the non-crossing partition which underlies the partitions in Figures 9 and 10 through Construction 1.

We want to count non-crossing partitions of $\{2, 3, \dots, mn-1\}$ with all block sizes except the one of the central block multiples of m, the central block containing the element 2 and having size $ma-2$, which remain invariant under a ρ-rotation. The insertion of 1 and mn and subsequent action of κ_A which is described in Theorem 3.17 to obtain *all* positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ then implies that the result has to be multiplied by $\frac{(mn-2)/\rho}{(ma-2)/\rho} = \frac{mn-2}{ma-2}$. (In particular, we divide by $(ma-2)/\rho$ in order to avoid overcounting: any element of the central block can be moved to the element 2 by repeated application of the rotation R!)

Since the partition is invariant under the ρ-rotation $R^{(mn-2)/\rho}$, it suffices to concentrate on the "fundamental" part of the partition, which we may choose as the restriction of the partition to $\mathcal{F} = \{2, 3, \dots, \frac{mn-2}{\rho} + 1\}$.

Now let $s_1 = 2, s_2, \dots, s_{(ma-2)/\rho}$ be the elements of the central block which are at the same time contained in the fundamental section $\mathcal{F}$. The restriction of our original partition to the vertices in between s_i and s_{i+1}, $i = 1, 2, \dots, \frac{ma-2}{\rho}$ (with the convention that $s_{(ma-2)/\rho+1} =$

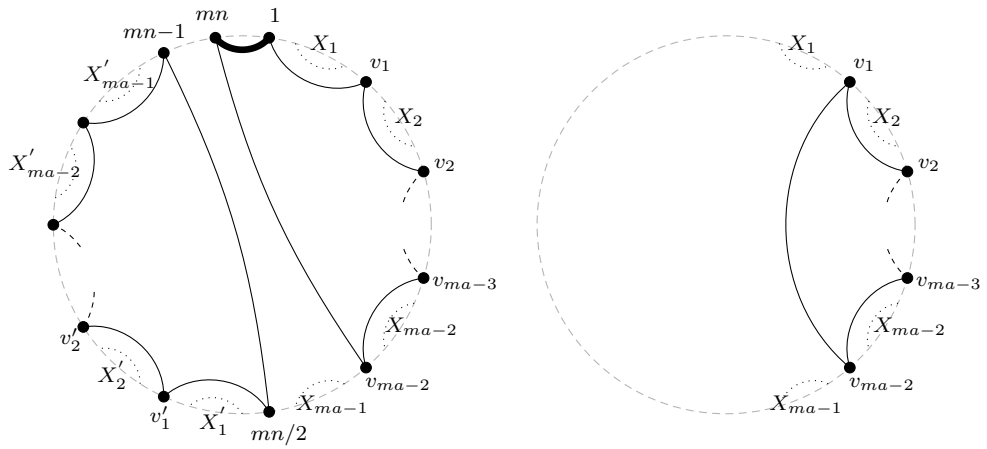

FIGURE 52. The decomposition of a 2-pseudo-rotationally invariant non-crossing partition in $\widetilde{NC}_+^{(m)}(A_{n-1})$ without central block

$s_1 + \frac{mn-2}{\rho} = \frac{mn-2}{\rho} + 2$) is an m-divisible non-crossing partition on $s_{i+1} - s_i - 1$ vertices. That is, the restriction to the fundamental section $\mathcal{F}$ consists of $(ma-2)/\rho$ m-divisible non-crossing partitions "interleaved" by elements of the central block. From Lemma B.2 and the subsequent arguments we know that the generating function for m-divisible non-crossing partitions is given by $f(z)/z$. Then, by elementary generating function calculus, we infer that the generating function $\sum_\pi w_\rho^{(m)}(\pi)$ of the statement in the lemma is equal to

$$\frac{mn-2}{ma-2} \left\langle z^{(mn-2)/\rho} \right\rangle f^{(ma-2)/\rho}(z).$$

This is exactly (B.20). □

Lemma B.11. *Let m and n be positive integers with $2 \mid mn$. The generating function $\sum_\pi w_2^{(m)}(\pi)$, where the sum is over all positive m-divisible non-crossing partitions of $\{1, 2, \ldots, mn\}$ without central block which are invariant under the 2-pseudo-rotation $\kappa_A^{(mn-2)/2}$ equals*

$$\sum_{a=1}^{\infty} \frac{mn-2}{2} x_a \left\langle z^{(mn-2)/2} \right\rangle f^{ma-1}(z) = \sum_{a=1}^{\infty} (ma-1) x_a \left\langle z^{-(ma-1)} \right\rangle F^{-(mn-2)/2}(z), \quad \text{(B.21)}$$

where $F(z)$ is defined by (B.5) and $f(z)$ is the compositional inverse of $F(z)$, as before.

Proof. According to Theorem 3.17(2), all 2-pseudo-rotationally invariant positive m-divisible non-crossing partitions of $\{1, 2, \ldots, mn\}$ are obtained by starting with a non-crossing partition obtained by Construction 2, π say, and applying the pseudo-rotation κ_A repeatedly to it. Since, by Theorem 3.17(1), ρ-pseudo-rotational invariant positive non-crossing partitions necessarily have a central block if $\rho \geq 3$, the fact that π does not contain a central block implies that the partitions $\kappa_A^i(\pi)$, $i = 0, 1, \ldots, \frac{mn-2}{2} - 1$, are all different. Therefore,

what we need to do is to count the number of non-crossing partitions of $\{1,2,\dots,mn\}$ which can be obtained by Construction 2, and then multiply the result by $(mn-2)/2$.

If we fix the size of the block containing mn and 1 in Construction 2 to be ma, then we claim that the generating function $\sum_\pi w_2^{(m)}(\pi)$ for the non-crossing partitions arising from Construction 2 with the size of the block containing mn and 1 being equal to ma is given by

$$x_a z^{ma} \left(C^{(m)}(z)\right)^{ma-1} = x_a z f^{ma-1}(z), \tag{B.22}$$

where $C^{(m)}(z)$ is the generating function for m-divisible non-crossing partitions from Lemma B.2. To see this, the reader must consult Figure 52: the left part of the figure shows a schematic illustration of Construction 2 with $N = mn$ and a replaced by ma, while the right part extracts the relevant portions that explain the above expression. Here, $X_1, X_2, \dots, X_{ma-1}$ symbolise m-divisible non-crossing partitions that may be inserted between $1, v_1, \dots, v_{ma-2}$ and after v_{am-2}, while $X_i' = R^{(mn-2)/2} X_i$ and $v_i' = R^{(mn-2)/2} v_i$. The generating function for each of the X_i's is given by $C^{(m)}(z)$. By elementary generating function calculus, this gives the expression in (B.22).

In view of this expression, the number we want to compute is

$$\frac{mn-2}{2} x_a \left\langle z^{mn/2} \right\rangle z f^{ma-1}(z) = \frac{mn-2}{2} x_a \left\langle z^{(mn-2)/2} \right\rangle f^{ma-1}(z).$$

This number has to be summed over all possible a's. This leads directly to the left-hand side expression in (B.21). The right-hand side expression is then a consequence of the Lagrange inversion formula (B.2). □

Proof of Theorem 3.18. We distinguish two cases depending on whether $a > 0$ or $a = 0$, that is, whether the choice of b_k's implies the existence of a central block or not.

If $a > 0$, that is, $n > \rho \sum_{j=1}^n j b_j$, then, using the right-hand side expression in (B.20) and (B.5), we see that the number which we want to compute is given by

$$\begin{aligned}
\left\langle \mathbf{x}^{\mathbf{b}} z^0 \right\rangle z^{(ma-2)/\rho} F^{-(mn-2)/\rho}(z) &= \left\langle \mathbf{x}^{\mathbf{b}} z^0 \right\rangle z^{(ma-mn)/\rho} \left(1 + \sum_{\ell=1}^\infty x_\ell z^{m\ell}\right)^{(mn-2)/\rho} \\
&= \left\langle \mathbf{x}^{\mathbf{b}} z^{m(n-a)/\rho} \right\rangle \sum_{k=0}^{(mn-2)/\rho} \binom{(mn-2)/\rho}{k} \left(\sum_{\ell=1}^\infty x_\ell z^{m\ell}\right)^k \\
&= \binom{(mn-2)/\rho}{b_1 + b_2 + \cdots + b_n} \binom{b_1 + b_2 + \cdots + b_n}{b_1, b_2, \dots, b_n},
\end{aligned}$$

where the short notation $\mathbf{x}^{\mathbf{b}}$ stands again for $x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$. This is exactly the asserted expression.

On the other hand, if $\rho = 2$ and $n = 2\sum_{j=1}^n j b_j$, then, using the right-hand side expression in (B.21) and (B.5), we see that the number which we want to compute is given

by

$$\begin{aligned}
&\sum_{a=1}^{n}(ma-1)\left\langle x_a^{-1}\mathbf{x}^{\mathbf{b}}z^{-(ma-1)}\right\rangle F^{-(mn-2)/2}(z)\\
&\qquad=\sum_{a=1}^{n}(ma-1)\left\langle x_a^{-1}\mathbf{x}^{\mathbf{b}}z^{\frac{mn}{2}-ma}\right\rangle\left(1+\sum_{\ell=1}^{\infty}x_\ell z^{m\ell}\right)^{(mn-2)/2}\\
&\qquad=\sum_{a=1}^{n}(ma-1)\binom{(mn-2)/2}{b_1+b_2+\cdots+b_n-1}\binom{b_1+b_2+\cdots+b_n-1}{b_1,\dots,b_{a-1},b_a-1,b_{a+1},\dots,b_n}\\
&\qquad=\binom{(mn-2)/2}{b_1+b_2+\cdots+b_n}\binom{b_1+b_2+\cdots+b_n}{b_1,b_2,\dots,b_n}\sum_{a=1}^{n}\frac{(ma-1)b_a}{\frac{mn}{2}-b_1-b_2-\cdots-b_n}.
\end{aligned}$$

Since $\sum_{a=1}^{n}(ma-1)b_a=\frac{mn}{2}-b_1-b_2-\cdots-b_n$, this proves the formula in (3.10) also in this case.

According to Proposition C.5 and Theorem 3.17, in all other cases there are no ρ-pseudo-rotationally invariant positive m-divisible non-crossing partitions. □

Proof of Corollary 3.19. The expression (3.11) could be obtained from (3.10) by summing over all possible b_i's, $i=1,2,\dots,n$, with $a=n-\rho(b_1+2b_2+\cdots+nb_n)$ and $b_1+b_2+\cdots+b_n=b$. It is however simpler to start again with the right-hand side of (B.20) (if $a>0$), respectively the right-hand side of (B.21) (if $a=0$), set all x_j's (except x_0 which, by definition, equals 1) equal to t, and finally extract the coefficient of t^b. Indeed, since in ρ-pseudo-rotationally invariant m-divisible non-crossing partitions the number of non-central blocks of size mi must be a multiple of ρ, implying that the size of the central block must be $ma=mn-m\rho B$, for some positive integer B, we have $a\equiv n \pmod{\rho}$.

Hence, if $a>0$, from (B.20) we obtain

$$\begin{aligned}
&\left\langle t^b z^0\right\rangle z^{(ma-2)/\rho}F^{-(mn-2)/\rho}(z)\Big|_{x_1=x_2=\cdots=t}\\
&\qquad=\left\langle t^b z^0\right\rangle z^{(ma-mn)/\rho}\left(1+t\sum_{\ell=1}^{\infty}z^{m\ell}\right)^{(mn-2)/\rho}\\
&\qquad=\left\langle t^b z^{(mn-ma)/\rho}\right\rangle\sum_{k=0}^{(mn-2)/\rho}\binom{(mn-2)/\rho}{k}t^k\left(\frac{z^m}{1-z^m}\right)^k\\
&\qquad=\left\langle z^{m\frac{n-a}{\rho}-mb}\right\rangle\binom{(mn-2)/\rho}{b}\left(\frac{1}{1-z^m}\right)^b\\
&\qquad=\binom{(mn-2)/\rho}{b}\binom{(n-a-\rho)/\rho}{b-1},
\end{aligned}$$

which is exactly (3.11).

On the other hand, if $a = 0$ (and hence $n \equiv 0 \pmod 2$ and $\rho = 2$), from (B.21) we obtain

$$\begin{aligned}
&\left\langle t^b z^0 \right\rangle \sum_{\alpha=1}^{\infty} (m\alpha - 1) x_\alpha z^{m\alpha-1} F^{-(mn-2)/2}(z)\Big|_{x_1=x_2=\cdots=t} \\
&\quad = \left\langle t^b z^0 \right\rangle \sum_{\alpha=1}^{\infty} (m\alpha - 1) t z^{m\alpha - \frac{mn}{2}} \left(1 + t \sum_{\ell=1}^{\infty} z^{m\ell}\right)^{(mn-2)/2} \\
&\quad = \sum_{\alpha=1}^{\infty} (m\alpha - 1) \left\langle t^{b-1} z^{\frac{mn}{2} - m\alpha} \right\rangle \sum_{k=0}^{(mn-2)/2} \binom{(mn-2)/2}{k} t^k \left(\frac{z^m}{1-z^m}\right)^k \\
&\quad = \sum_{\alpha=1}^{\infty} (m\alpha - 1) \left\langle z^{\frac{mn}{2} - m\alpha - m(b-1)} \right\rangle \binom{(mn-2)/2}{b-1} \left(\frac{1}{1-z^m}\right)^{b-1} \\
&\quad = \sum_{\alpha=1}^{\infty} (m\alpha - 1) \binom{(mn-2)/2}{b-1} \binom{\frac{n}{2} - \alpha - 1}{b-2} \\
&\quad = \binom{(mn-2)/2}{b-1} \sum_{\alpha=1}^{\infty} \left(m\left(\alpha - \tfrac{n}{2}\right) + \left(m\tfrac{n}{2} - 1\right)\right) \binom{\frac{n}{2} - \alpha - 1}{b-2} \\
&\quad = \binom{(mn-2)/2}{b-1} \sum_{\alpha=1}^{\infty} \left(-m(b-1)\binom{\frac{n}{2} - \alpha}{b-1} + \left(m\tfrac{n}{2} - 1\right) \binom{\frac{n}{2} - \alpha - 1}{b-2}\right) \\
&\quad = \binom{(mn-2)/2}{b-1} \left(-m(b-1)\binom{\frac{n}{2}}{b} + \left(m\tfrac{n}{2} - 1\right) \binom{\frac{n}{2} - 1}{b-1}\right) \\
&\quad = \binom{(mn-2)/2}{b-1} \binom{\frac{n}{2} - 1}{b-1} \frac{m\frac{n}{2} - b}{b} \\
&\quad = \binom{(mn-2)/2}{b} \binom{\frac{n}{2} - 1}{b-1},
\end{aligned}$$

which is exactly (3.11) with $a = 0$.

Finally, by summing the expression (3.11) over all $a \equiv n \pmod \rho$, we obtain (3.12). □

B.4. **Enumeration of positive m-divisible non-crossing partitions in type B.** As in Appendix B.2, we use the generating function approach from [Kra13]. Let x_i, $i = 1, 2, \ldots$, be variables, and $x_0 = 1$. For a positive m-divisible non-crossing partition π of type B let the weight $w^{(m)}(\pi)$ be defined by

$$w^{(m)}(\pi) = \prod_{i=1}^{\infty} x_i^{\#(\text{non-zero blocks of } \pi \text{ of size } mi)/2}.$$

For example, the weight $w^{(3)}(\,.\,)$ of the positive m-divisible non-crossing partitions of type B in Figure 15 is x_1^3, while the weight of the non-crossing partitions in Figure 17 is x_1^4.

Lemma B.12. *Let n be a positive integer. The generating function $\sum_\pi w^{(1)}(\pi)$, where the sum is over all positive non-crossing partitions of $\{1,2,\dots,n,\overline{1},\overline{2},\dots,\overline{n}\}$ of type B which have a zero block of size 2 equals*

$$\left\langle z^0 \right\rangle F^{-(n-1)}(z), \tag{B.23}$$

where $F(z)$ is defined by (B.5) and $f(z)$ is the compositional inverse of $F(z)$, as before.

Proof. Let π be a positive non-crossing partition as described in the statement of the lemma, and let the zero block be $\{a,\overline{a}\}$. We may map π to the pair (a,σ), where σ is the non-crossing partition we see between $\overline{a+1}$ and $a-1$, considered as a non-crossing partition of $\{1,2,\dots,n-1\}$, by appropriate relabelling. Thus, we have defined a mapping from the non-crossing partitions of the statement of the lemma and pairs (a,σ), where $1 \le a \le n$ and σ is a non-crossing partition of $\{1,2,\dots,n-1\}$. However, this mapping, while injective, is not surjective due to the condition in Theorem 4.4 that the element 1 must be in a block that also contains negative elements. To be precise, given a non-crossing partition σ, there are $n-\#(\text{blocks of } \sigma)$ numbers a such that there is a corresponding non-crossing partition of type B as in the statement of the lemma under this mapping. This observation can be modelled by generating functions as follows: the generating function for the non-crossing partitions in the statement of the lemma is given by

$$\left\langle z^{n-1} \right\rangle \left(\left(n - \tfrac{d}{dt}\right) C^{(1)}(z)\Big|_{x_i \to t x_i} \right) \Big|_{t=1}.$$

Here, the replacement of the x_i's is restricted to the x_i's with $i \ge 1$ (x_0 being 1 by definition from the outset). We use again that $C^{(1)}(z) = f(z)/z$ and subsequently apply Lagrange inversion in the form (B.2). This leads to

$$\begin{aligned}
n\left\langle z^n \right\rangle f(z) - \left\langle z^n \right\rangle \left(\tfrac{d}{dt} f(z)\Big|_{x_i \to t x_i} \right) \Big|_{t=1} &= n\left\langle z^0 \right\rangle z^{-n} f(z) - \left\langle z^0 \right\rangle z^{-n} \left(\tfrac{d}{dt} f(z)\Big|_{x_i \to t x_i} \right) \Big|_{t=1} \\
&= \left\langle z^0 \right\rangle z F^{-n}(z) - \frac{1}{n}\left\langle z^0 \right\rangle z \left(\tfrac{d}{dt} F^{-n}(z)\Big|_{x_i \to t x_i} \right) \Big|_{t=1} \\
&= \left\langle z^0 \right\rangle z F^{-n}(z) - \frac{1}{n}\left\langle z^0 \right\rangle z^{1-n} \tfrac{d}{dt} \left(1 + t\sum_{i=1}^{\infty} x_i z^i \right)^n \Big|_{t=1} \\
&= \left\langle z^0 \right\rangle z F^{-n}(z) - \left\langle z^0 \right\rangle z^{1-n} \left(\sum_{i=1}^{\infty} x_i z^i \right) \left(1 + \sum_{i=1}^{\infty} x_i z^i \right)^{n-1} \\
&= \left\langle z^0 \right\rangle z F^{-n}(z) - \left\langle z^0 \right\rangle \left(\frac{z}{F(z)} - 1 \right) F^{-(n-1)}(z) \\
&= \left\langle z^0 \right\rangle F^{-(n-1)}(z),
\end{aligned}$$

as desired. □

Lemma B.13. *Let m,n,a be positive integers. The generating function $\sum_\pi w^{(m)}(\pi)$, where the sum is over all positive m-divisible non-crossing partitions of $\{1,2,\dots,mn,\overline{1},\overline{2},\dots,$*

$\overline{mn}\}$ *of type* B *which have a zero block of size* $2ma$ *equals*

$$\frac{mn-1}{ma-1}\left\langle z^{mn-1}\right\rangle f^{ma-1}(z)=\left\langle z^{0}\right\rangle z^{ma-1}F^{-(mn-1)}(z), \tag{B.24}$$

where $F(z)$ *is defined by* (B.5) *and* $f(z)$ *is the compositional inverse of* $F(z)$*, as before. In the special case where* $m=a=1$*, the left-hand side of* (B.24) *must be ignored.*

Proof. We have already treated the case where $m=a=1$ in Lemma B.12 above. So, let us assume that $ma\neq 1$ from now on.

Given an element of $\widetilde{NC}_{+}^{(m)}(B_n)$, by applying the pseudo-rotation κ_B repeatedly to it, we will achieve to obtain a non-crossing partition in which the zero block contains $\overline{mn}$ *and* 1. Another application of κ_B then just normally rotates the zero block, and thus yields a non-crossing partition in which the zero block contains 1 *and* 2. Hence, we may obtain all elements of $\widetilde{NC}_{+}^{(m)}(B_n)$ with a zero block via the Construction in Subsection 4.5 with $\nu=1$. More precisely, we obtain all elements of $\widetilde{NC}_{+}^{(m)}(B_n)$ with a zero block of size $2ma$ by starting with an ordinary m-divisible non-crossing partition of $\{2,3,\dots,mn,\overline{2},\overline{3},\dots,\overline{mn}\}$ containing a zero block of size $2ma-2$ and that remains invariant under the map that exchanges i and $\overline{i}$ for all i, then adding 1 and $\overline{1}$ to the zero block, and subsequently applying the pseudo-rotation κ_B repeatedly.

The remaining arguments are completely analogous to the proof of Lemma B.9 and are therefore left to the reader. □

Lemma B.14. *Let* m *and* n *be positive integers. The generating function* $\sum_{\pi} w^{(m)}(\pi)$*, where the sum is over all positive* m*-divisible non-crossing partitions of* $\{1,2,\dots,mn,\overline{1},\overline{2},\dots,\overline{mn}\}$ *of type* B *without zero block equals*

$$\sum_{a=1}^{\infty}(mn-1)x_a\left\langle z^{mn-1}\right\rangle f^{ma-1}(z)=\sum_{a=1}^{\infty}(ma-1)\,x_a\left\langle z^{-(ma-1)}\right\rangle F^{-(mn-1)}(z), \tag{B.25}$$

where $F(z)$ *is defined by* (B.5) *and* $f(z)$ *is the compositional inverse of* $F(z)$*, as before.*

Proof. We claim that all positive m-divisible non-crossing partitions of $\{1,2,\dots,mn,\overline{1},\overline{2},\dots,\overline{mn}\}$ of type B without zero block can be obtained by starting with a non-crossing partition that has a block containing 1 and $\overline{2}$, and otherwise only negative elements, and then applying the pseudo-rotation κ_B repeatedly to this partition. Indeed, the claim follows immediately by a careful reading of the proof of Lemma C.6.

Consequently, what we need to do is to count the number of non-crossing partitions of $\{1,2,\dots,mn,\overline{1},\overline{2},\dots,\overline{mn}\}$ as described above, and then multiply the result by $mn-1$.

If we fix the size of the block containing 1 and $\overline{2}$ in the above claim to be ma, then, by elementary generating function calculus, we obtain that the generating function $\sum_{\pi} w^{(m)}(\pi)$ for these non-crossing partitions is given by

$$x_a z^{ma}\left(C^{(m)}(z)\right)^{ma-1}=x_a z f^{ma-1}(z),$$

where $C^{(m)}(z)$ is the generating function for m-divisible non-crossing partitions from Lemma B.2. Hence, the number we want to compute is

$$(mn-1)x_a \left\langle z^{mn} \right\rangle z f^{ma-1}(z) = (mn-1)x_a \left\langle z^{mn-1} \right\rangle f^{ma-1}(z).$$

This number has to be summed over all possible a's. This leads directly to the left-hand side expression in (B.25). The right-hand side expression is then a consequence of the Lagrange inversion formula (B.2). □

Proof of Theorem 4.11. We distinguish two cases depending on whether $a > 0$ or $a = 0$, that is, whether the choice of b_k's implies the existence of a zero block or not.

If $a > 0$, that is, $n > \sum_{j=1}^{n} jb_j$, then, using the right-hand side expression in (B.24) and (B.5), we see that the number which we want to compute is given by

$$\begin{aligned}
\left\langle \mathbf{x}^{\mathbf{b}} z^0 \right\rangle z^{ma-1} F^{-(mn-1)}(z) &= \left\langle \mathbf{x}^{\mathbf{b}} z^0 \right\rangle z^{ma-mn} \left(1 + \sum_{\ell=1}^{\infty} x_\ell z^{m\ell}\right)^{mn-1} \\
&= \left\langle \mathbf{x}^{\mathbf{b}} z^{m(n-a)} \right\rangle \sum_{k=0}^{mn-1} \binom{mn-1}{k} \left(\sum_{\ell=1}^{\infty} x_\ell z^{m\ell}\right)^{k} \\
&= \binom{mn-1}{b_1+b_2+\cdots+b_n} \binom{b_1+b_2+\cdots+b_n}{b_1, b_2, \ldots, b_n},
\end{aligned}$$

where the short notation $\mathbf{x}^{\mathbf{b}}$ stands again for $x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$. This is exactly the asserted expression.

On the other hand, if $a = 0$ and hence $n = \sum_{j=1}^{n} jb_j$, then, using the right-hand side expression in (B.25) and (B.5), we see that the number which we want to compute is given by

$$\begin{aligned}
&\sum_{a=1}^{n} (ma-1) \left\langle x_a^{-1} \mathbf{x}^{\mathbf{b}} z^{-(ma-1)} \right\rangle F^{-(mn-1)}(z) \\
&\qquad = \sum_{a=1}^{n} (ma-1) \left\langle x_a^{-1} \mathbf{x}^{\mathbf{b}} z^{mn-ma} \right\rangle \left(1 + \sum_{\ell=1}^{\infty} x_\ell z^{m\ell}\right)^{mn-1} \\
&\qquad = \sum_{a=1}^{n} (ma-1) \binom{mn-1}{b_1+b_2+\cdots+b_n-1} \binom{b_1+b_2+\cdots+b_n-1}{b_1, \ldots, b_{a-1}, b_a-1, b_{a+1}, \ldots, b_n} \\
&\qquad = \binom{mn-1}{b_1+b_2+\cdots+b_n} \binom{b_1+b_2+\cdots+b_n}{b_1, b_2, \ldots, b_n} \sum_{a=1}^{n} \frac{(ma-1)b_a}{mn-b_1-b_2-\cdots-b_n}.
\end{aligned}$$

Since $\sum_{a=1}^{n} (ma-1)b_a = mn - b_1 - b_2 - \cdots - b_n$, this proves the formula in (4.8). □

Proof of Corollary 4.12. The expression (4.9) could be obtained from (4.8) by summing over all possible b_i's, $i = 1, 2, \ldots, n$, with $a = n-(b_1+2b_2+\cdots+nb_n)$ and $b_1+b_2+\cdots+b_n = b$. It is however simpler to start again with the right-hand sides of (B.24) respectively (B.25),

set all x_j's equal to t (except x_0 which, by definition, equals 1), and finally extract the coefficient of t^b.

For the case where $a > 0$, we would start with the right-hand side of (B.24) and then perform the above described steps. A more general calculation is done later in the proof of Corollary 4.16. More precisely, if we put $\nu = 1$ there, then the calculation there reduces to the one which is relevant here. The result is (4.14) with $\nu = 1$, which indeed agrees with (4.9).

On the other hand, if $a = 0$, then from (B.25) we get

$$\begin{aligned}
&\left\langle t^b z^0 \right\rangle \sum_{\alpha=1}^{\infty} (m\alpha - 1)\, x_\alpha \left\langle z^{-(m\alpha-1)} \right\rangle F^{-(mn-1)}(z)\Big|_{x_1=x_2=\cdots=t} \\
&\quad = \left\langle t^b z^0 \right\rangle \sum_{\alpha=1}^{\infty} (m\alpha - 1) t z^{m\alpha-mn} \left(1 + t \sum_{\ell=1}^{\infty} z^{m\ell}\right)^{mn-1} \\
&\quad = \left\langle t^{b-1} z^{m(n-1)} \right\rangle \left(\frac{m}{(1-z^m)^2} - \frac{1}{1-z^m}\right) \sum_{k=0}^{mn-1} \binom{mn-1}{k} t^k \left(\frac{z^m}{1-z^m}\right)^k \\
&\quad = \left\langle z^{m(n-b)} \right\rangle \binom{mn-1}{b-1} \left(m \left(\frac{1}{1-z^m}\right)^{b+1} - \left(\frac{1}{1-z^m}\right)^b\right) \\
&\quad = \binom{mn-1}{b-1} \left(m \binom{n}{b} - \binom{n-1}{b-1}\right) \\
&\quad = \binom{mn-1}{b-1} \frac{(n-1)!}{b!\,(n-b)!} (mn-b),
\end{aligned}$$

which can be simplified to (4.9) with $a = 0$.

By summing the expression (4.9) over all possible a, we obtain (4.10). □

B.5. **Enumeration of pseudo-rotationally invariant positive m-divisible non-crossing partitions in type B.** Again, we use a generating function approach as in Appendix B.2. Abusing notation from Appendix B.3, we introduce the weight function $w_\nu^{(m)}$, defined by

$$w_\nu^{(m)}(\pi) = \prod_{i=1}^{\infty} x_i^{\#(\text{non-zero blocks of } \pi \text{ of size } mi)/2\nu}.$$

For example, the weight $w_2^{(3)}(\,.\,)$ of both non-crossing partitions in Figures 24 and 25 is x_1^2.

Lemma B.15. *Let ν, m, n, a be positive integers with $\nu \geq 1$, $\nu \mid (mn-1)$, and $a \equiv n \pmod{\nu}$. The generating function $\sum_\pi w_\nu^{(m)}(\pi)$, where the sum is over all positive m-divisible non-crossing partitions of $\{1, 2, \ldots, mn, \overline{1}, \overline{2}, \ldots, \overline{mn}\}$ of type B which are invariant under the 2ν-pseudo-rotation $\kappa_B^{(mn-1)/\nu}$ and have a zero block of size ma equals*

$$\frac{mn-1}{ma-1} \left\langle z^{(mn-1)/\nu} \right\rangle f^{(ma-1)/\nu}(z) = \left\langle z^0 \right\rangle z^{(ma-1)/\nu} F^{-(mn-1)/\nu}(z), \tag{B.26}$$

where $F(z)$ is defined by (B.5) and $f(z)$ is the compositional inverse of $F(z)$, as before. In the special case where $m = a = 1$, the left-hand side of (B.26) must be ignored.

Remark B.16. Similarly to Lemma B.9, also here the condition $a \equiv n \pmod{\nu}$ is actually redundant.

Proof of Lemma B.15. Using Theorem 4.14, this is completely analogous to the proofs of Lemmas B.7 and B.9 and is therefore left to the reader. □

Proof of Theorem 4.15. Using the right-hand side expression in (B.26) and (B.5), we see that the number which we want to compute is given by

$$\begin{aligned}\left\langle \mathbf{x}^{\mathbf{b}} z^0 \right\rangle z^{(ma-1)/\nu} F^{-(mn-1)/\nu}(z) &= \left\langle \mathbf{x}^{\mathbf{b}} z^0 \right\rangle z^{(ma-mn)/\nu} \left(1 + \sum_{\ell=1}^{\infty} x_\ell z^{m\ell}\right)^{(mn-1)/\nu} \\ &= \left\langle \mathbf{x}^{\mathbf{b}} z^{m(n-a)/\nu} \right\rangle \sum_{k=0}^{(mn-1)/\nu} \binom{(mn-1)/\nu}{k} \left(\sum_{\ell=1}^{\infty} x_\ell z^{m\ell}\right)^k \\ &= \binom{(mn-1)/\nu}{b_1+b_2+\cdots+b_n} \binom{b_1+b_2+\cdots+b_n}{b_1, b_2, \ldots, b_n},\end{aligned}$$

where the short notation $\mathbf{x}^{\mathbf{b}}$ stands again for $x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$. This is exactly the asserted expression. □

Proof of Corollary 4.16. The expression (4.14) could be obtained from (4.13) by summing over all possible b_i's, $i = 1, 2, \ldots, n$, with $a = n - \nu(b_1 + 2b_2 + \cdots + nb_n)$ and $b_1 + b_2 + \cdots + b_n = b$. It is however simpler to start again with the right-hand side of (B.26), set all x_j's equal to t (except x_0 which, by definition, equals 1), and finally extract the coefficient of t^b. Indeed, since in 2ν-pseudo-rotationally invariant m-divisible non-crossing partitions of type B the number of non-zero blocks of size mi must be a multiple of 2ν, implying that the size of the zero block must be $2ma = 2mn - 2m\nu B$, for some positive integer B, we have $a \equiv n \pmod{\nu}$. Hence, we obtain

$$\begin{aligned}&\left\langle t^b z^0 \right\rangle z^{(ma-1)/\nu} F^{-(mn-1)/\nu}(z)\Big|_{x_1=x_2=\cdots=t} \\ &\quad = \left\langle t^b z^0 \right\rangle z^{(ma-mn)/\nu} \left(1 + t\sum_{\ell=1}^{\infty} z^{m\ell}\right)^{(mn-1)/\nu} \\ &\quad = \left\langle t^b z^{(mn-ma)/\nu} \right\rangle \sum_{k=0}^{(mn-1)/\nu} \binom{(mn-1)/\nu}{k} t^k \left(\frac{z^m}{1-z^m}\right)^k \\ &\quad = \left\langle z^{m\frac{n-a}{\nu} - mb} \right\rangle \binom{(mn-1)/\nu}{b} \left(\frac{1}{1-z^m}\right)^b \\ &\quad = \binom{(mn-1)/\nu}{b} \binom{(n-a-\nu)/\nu}{b-1},\end{aligned}$$

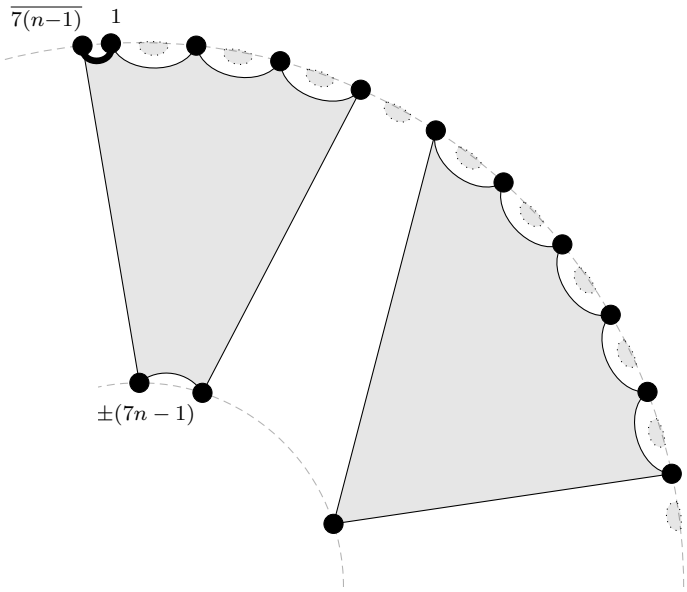

FIGURE 53. A fundamental seed with two bridging blocks

which is exactly (4.14).

By summing the expression (4.14) over all possible a, that is, over all a with $a \equiv n \pmod{\nu}$, we obtain (4.15). □

B.6. **Enumeration of pseudo-rotationally invariant positive m-divisible non-crossing partitions in type D.**

Proof of Proposition 5.15. We assume that the reader recalls the definition of "fundamental seed" given in Construction 3 in Subsection 5.5.

Let B_1 be the block containing $\overline{m(n-1)}$ and 1. We assume that it contains i_1 elements of the inner circle, $i_1 \geq 0$. We ignore the choice of signs of these elements for the moment, but put on record that, in the end, the result must be multiplied by 2 since there are two possible choices for these signs.

Likewise, let $B_2, B_3, \ldots, B_k$ be the further blocks of the fundamental seed that contain elements from the outer *and* the inner circle. For $j = 2, 3, \ldots, k$, we assume that block B_j contains i_j elements of the inner circle, $i_j \geq 1$. For $j = 1, 2, \ldots, k$, we let the block size of B_j equal ma_j.

As in the proof of Lemma 5.12, we write x for the number of elements on the inner circle that are contained in the fundamental seed according to Construction 3, and we write my for the total number of elements in this fundamental seed.

The generating function $\sum z^{|\sigma|_o}$, with the sum running over all m-divisible non-crossing partitions σ defining a fundamental seed with k bridging blocks, is given by

$$\sum_{\substack{i_1+\cdots+i_k=x\\ i_1\geq 0,\, i_2,\ldots,i_k\geq 1\\ a_1,\ldots,a_k\geq 1}} z\,(zC(z))^{ma_1-i_1-1}(zC(z))^{ma_2-i_2}\cdots(zC(z))^{ma_k-i_k}, \tag{B.27}$$

where $C(z)$ is short for our earlier series $C^{(m)}(z)$ in which all x_i's are set equal to 1, and where $|\sigma|_o$ denotes the number of elements on the outer circle that are covered by σ. Indeed, by definition (see Lemma B.2) $C(z) = C^{(m)}(z)$ (with $x_i = 1$ for all i) is the generating function for m-divisible non-crossing partitions. Between any two vertices of a bridging block that lie on the outer circle we may place an m-divisible non-crossing partition, as well as between the last vertex on the outer circle of a bridging block and the first vertex of the next bridging block; the only exception is that $\overline{m(n-1)}$ and 1, the first two elements on the outer circle of the first bridging block, B_1, must be successive, and hence nothing can be placed between these two. Since block B_j has $ma_j - i_j$ elements on the outer circle, this explains the summand in (B.27). Figure 53 gives a schematic illustration of a seed with two bridging blocks, in which $m = 7$, $a_1 = 1$, $i_1 = 2$, $a_2 = 1$, $i_2 = 1$, $x = i_1 + i_2 = 3$, and $y = a_1 + a_2 = 2$. The shaded "half disks" indicate smaller non-crossing partitions that are placed between vertices on the outer circle.

Recalling our earlier notation $f(z) = zC(z)$ and summing over $a_1, a_2, \ldots, a_k$, we obtain

$$\sum_{\substack{i_1+\cdots+i_k=x\\ i_1\geq 0,\, i_2,\ldots,i_k\geq 1}} z\,f^{mk-i_1-\cdots-i_k-1}(z)\big(1-f^m(z)\big)^{-k} = \sum_{\substack{i_1+\cdots+i_k=x\\ i_1\geq 0,\, i_2,\ldots,i_k\geq 1}} z\,f^{mk-x-1}(z)\big(1-f^m(z)\big)^{-k}.$$

It is an easy combinatorial exercise to see that the number of tuples $(i_1, i_2, \ldots, i_k)$ summing to x, where i_1 is non-negative and all other i_j's are positive, equals $\binom{x}{k-1}$. Thus we obtain

$$\binom{x}{k-1} z\,f^{mk-x-1}(z)\big(1-f^m(z)\big)^{-k}$$

for our generating function. However, since we have to apply κ_D multiple times, we have to multiply the result by $(m(n-1)-1)/\nu$, and we have to divide by k since any of the k bridging blocks could have been the "first" block in a fundamental seed. Subsequently, we

must sum the result over all possible k. In other words, we must compute

$$\begin{aligned}\sum_{k\geq 1}\frac{(m(n-1)-1)/\nu}{k}\binom{x}{k-1}z\,f^{mk-x-1}(z)\bigl(1-f^m(z)\bigr)^{-k}\\ &=\sum_{k\geq 1}\frac{my-x-1}{x+1}\binom{x+1}{k}z\,f^{mk-x-1}(z)\bigl(1-f^m(z)\bigr)^{-k}\\ &=\frac{my-x-1}{x+1}\left(z\,f^{-x-1}(z)\left(1+\frac{f^m(z)}{1-f^m(z)}\right)^{x+1}-z\,f^{-x-1}(z)\right)\\ &=\frac{my-x-1}{x+1}z\,f^{-x-1}(z)\left((1-f^m(z))^{-x-1}-1\right).\end{aligned}$$

Here we used (5.11) to obtain the second line.

In the last expression, we want to extract the coefficient of z^{my-x}. Equivalently, we want to compute

$$\frac{my-x-1}{x+1}\left\langle z^{my-x-1}\right\rangle f^{-x-1}(z)\left((1-f^m(z))^{-x-1}-1\right).$$

By using the Lagrange inversion formula in the form (B.3), with $b=my-x-1$ and $g(z)=z^{-x-1}\bigl((1-z^m)^{-x-1}-1\bigr)$, this is the same as

$$\begin{aligned}\frac{my-x-1}{x+1}\frac{1}{my-x-1}\left\langle z^{-1}\right\rangle F^{-my+x+1}(z)\frac{d}{dz}\left(z^{-x-1}\left((1-z^m)^{-x-1}-1\right)\right)\\ &=\left\langle z^{-1}\right\rangle F^{-my+x+1}(z)z^{-x-2}\left(1+((m+1)z^m-1)\,(1-z^m)^{-x-2}\right).\end{aligned}$$

Since we specialised all x_i's to 1, the series $F(z)$ equals $z(1-z^m)$. If we substitute this in the above expression, then we get

$$\begin{aligned}\left\langle z^{-1}\right\rangle z^{-my-1}(1-z^m)^{-my+x+1}\left(1+((m+1)z^m-1)\,(1-z^m)^{-x-2}\right)\\ &=\left\langle z^{my}\right\rangle(1-z^m)^{-my+x+1}+((m+1)z^m-1)\,(1-z^m)^{-my-1}\\ &=\binom{my-x-2+y}{y}+(m+1)\binom{my+y-1}{y-1}-\binom{my+y}{y}\\ &=\binom{((m+1)(n-1)-\nu)/\nu}{n/\nu}.\end{aligned}$$

This is the desired result since, as we mentioned at the beginning of the proof, we must multiply the counting result by 2. □

Proof of Proposition 5.17. We proceed as in the proof of Proposition 5.15. In particular, we first concentrate on fundamental seeds with k bridging blocks. Also here, we ignore the signs of the elements of the inner circle for the moment, so that, in the end, we must multiply the result by 2. The corresponding generating function is given by (B.27), except that there is no summation over a_i's, but rather all a_i's are equal to 1 since here all block

sizes are equal to m. In other words, the expression that we must consider here is

$$\sum_{\substack{i_1+\cdots+i_k=x\\ i_1\geq 0, i_2,\ldots,i_k\geq 1}} z\,(zC(z))^{m-i_1-1}(zC(z))^{m-i_2}\cdots(zC(z))^{m-i_k}.$$

Again using our earlier notation $f(z)=zC(z)$, this can be rewritten as

$$\sum_{\substack{i_1+\cdots+i_k=x\\ i_1\geq 0, i_2,\ldots,i_k\geq 1}} z\,f^{mk-i_1-\cdots-i_k-1}(z) = \sum_{\substack{i_1+\cdots+i_k=x\\ i_1\geq 0, i_2,\ldots,i_k\geq 1}} z\,f^{mk-x-1}(z).$$

As before, the sum over the i_j's is not difficult to carry out, and we obtain

$$\binom{x}{k-1} z\,f^{mk-x-1}(z)$$

for our generating function. This expression must be multiplied by $(m(n-1)-1)/\nu$, divided by k, and then summed over all k. Hence, we must compute

$$\begin{aligned}\sum_{k\geq 1}\frac{(m(n-1)-1)/\nu}{k}\binom{x}{k-1} z\,f^{mk-x-1}(z)&\\ &=\sum_{k\geq 1}\frac{my-x-1}{x+1}\binom{x+1}{k} z\,f^{mk-x-1}(z)\\ &=\frac{my-x-1}{x+1}\left(z\,f^{-x-1}(z)(1+f^m(z))^{x+1}-z\,f^{-x-1}(z)\right)\\ &=\frac{my-x-1}{x+1}z\,f^{-x-1}(z)\left((1+f^m(z))^{x+1}-1\right).\end{aligned}$$

Again we used (5.11) to obtain the second line.

In the last expression, we want to extract the coefficient of z^{my-x}. Equivalently, we want to compute

$$\frac{my-x-1}{x+1}\left\langle z^{my-x-1}\right\rangle f^{-x-1}(z)\left((1+f^m(z))^{x+1}-1\right).$$

By using the Lagrange inversion formula in the form (B.3), with $b=my-x-1$ and $g(z)=z^{-x-1}\big((1+z^m)^{x+1}-1\big)$, this is the same as

$$\begin{aligned}\frac{my-x-1}{x+1}\frac{1}{my-x-1}\left\langle z^{-1}\right\rangle F^{-my+x+1}(z)\frac{d}{dz}\left(z^{-x-1}\left((1+z^m)^{x+1}-1\right)\right)&\\ =\left\langle z^{-1}\right\rangle F^{-my+x+1}(z)z^{-x-2}\left(1+((m-1)z^m-1)\,(1+z^m)^x\right).&\end{aligned}$$

Since here all block sizes are m, we must specialise $x_0=1$ (as always), $x_1=1$, and $x_i=0$ for $i\geq 2$. With these choices, the series $F(z)$ equals $z/(1+z^m)$. If we substitute this in

the above expression, then we get

$$\begin{aligned}\left\langle z^{-1}\right\rangle z^{-my-1}&(1+z^m)^{my-x-1}\left(1+((m-1)z^m-1)\,(1+z^m)^x\right)\\
&=\left\langle z^{my}\right\rangle (1+z^m)^{my-x-1}+((m-1)z^m-1)\,(1+z^m)^{my-1}\\
&=\binom{my-x-1}{y}+(m-1)\binom{my-1}{y-1}-\binom{my-1}{y}\\
&=\binom{(m(n-1)-1)/\nu}{n/\nu}.\end{aligned}$$

This is the desired result since, as we mentioned at the beginning of the proof, we must multiply the counting result by 2. □

Appendix C. Proofs: Characterising pseudo-rotationally invariant positive non-crossing partitions

This appendix is devoted to the proofs of Theorems 3.17, 4.14 and 5.13 in which we characterise pseudo-rotationally invariant positive non-crossing set partitions in classical types.

C.1. **Characterisation of pseudo-rotationally invariant positive non-crossing partitions in type A_{n-1}.** Before we are able to prove Theorem 3.17, we need to first establish some auxiliary results which provide a detailed examination how blocks are moved by successive applications of the pseudo-rotation κ_A. In order to have a convenient language to carry this analysis out, we need some more definitions.

Recall from Subsection 3.6 that, for a ρ-pseudo-rotationally invariant element π of $\widetilde{NC}_+(N)$ (that is, π is invariant with respect to $\kappa_A^{(N-2)/\rho}$), a block B of π is called *κ_A-central block* (or simply *central block* if κ_A is clear from the context) if $B=\kappa_A^{(N-2)/\rho}(B)$. (How κ_A acts on a block B should be intuitively clear from the definition of κ_A, cf. Figures 4–6.) For example, both non-crossing partitions in Figures 9 and 10 have central blocks: in the partition of Figure 9 the central block is $\{1,2,9,16,23,30\}$, while in the partition of Figure 10 the central block is $\{3,4,11,18,25,26\}$. On the other hand, neither of the non-crossing partitions in Figures 12 and 13 has a central block. While a singleton block can never be a central block, it may happen that — slightly counter-intuitively — $\{N,1\}$ is a central block. An example with $N=32$ and $\rho=3$ is shown in Figure 11, where indeed $\kappa_A^{(32-2)/3}((1,32))=\kappa_A^{10}((1,32))=(1,32)$.

Furthermore, we define a *cyclic interval* of $\{1,2,\dots,N\}$ to be an interval of the form $\{e,e+1,\dots,f-1,f\}$, where all elements of the interval are taken modulo N. Thus, for $N=12$, $\{3,4,5,6\}$, $\{9,10,11\}$, as well as $\{10,11,12,1,2\}$ are cyclic intervals. Given an element π of $\widetilde{NC}_+(N)$, the *width* of a block B of π, denoted by $\mathrm{wd}(B)$, is defined as the minimal cardinality of a cyclic interval containing B. Such a minimal-cardinality cyclic interval will also be called a *support* for B. (With this definition, a support of a block does not need to be unique.) A block B in a ρ-pseudo-rotationally invariant element of

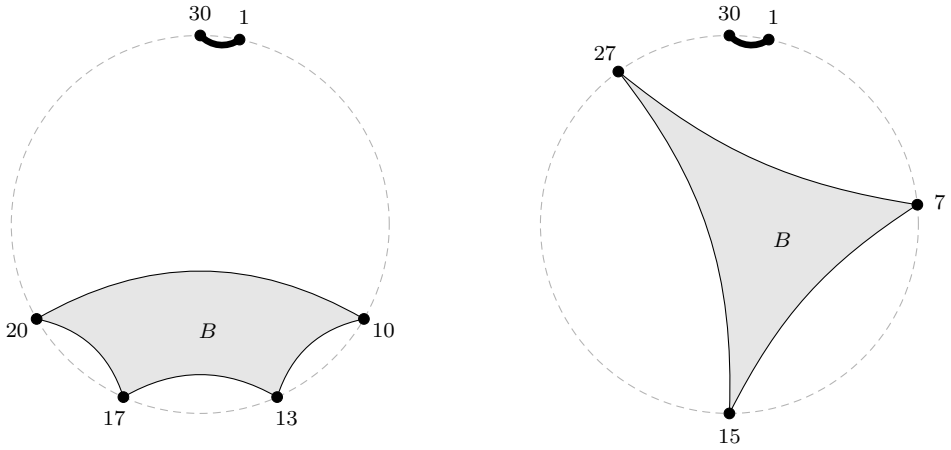

FIGURE 54. Example (on the left) and non-example (on the right) of a block B in Lemma C.1

$\widetilde{NC}_+(N)$ is called *fat* if $\mathrm{wd}(B) > \frac{N-2}{\rho} + 1$. See Figure 10, where $\{3, 4, 11, 18, 25, 26\}$ is a fat block of width $24 > \frac{30-2}{4} + 1 = 8$.

The key for the proof of Theorem 3.17 is to show that, for $\rho \geq 3$, every ρ-pseudo-rotationally invariant element of $\widetilde{NC}_+(N)$ has a central block. We are going to finally achieve this in Proposition C.5. In turn, that proposition needs another fact that we shall prove in Lemma C.4: for every ρ-pseudo-rotationally invariant element of $\widetilde{NC}_+(N)$, a fat block is automatically a central block. In the proof of Lemma C.4, we need certain properties of the pseudo-rotation κ_A which we state and prove separately in the next two lemmas, before we embark on Lemma C.4 and Proposition C.5 themselves and their proofs.

Lemma C.1. *Let B be a block of $\pi \in \widetilde{NC}_+(N)$ which has a support not containing 1 and N. Then $\mathrm{wd}\left(\kappa_A^j(B)\right) \leq \mathrm{wd}(B)$ for any j.*

Remark C.2. On the left of Figure 54, there is an example of a block whose only support is $[10, 11, \ldots, 20]$, which does not contain 1 and $N = 30$. On the other hand, the right of Figure 54 shows a block whose only support is $[27, 28, 29, 30, 1, 2, \ldots, 15]$, which does contain 1 and $N = 30$.

Proof of Lemma C.1. Clearly, the case of $|B| = 1$ being obvious, we may assume $\mathrm{wd}(B) > 1$. Without loss of generality we may restrict our attention to the case where the mentioned support of B has the form $C = [e, e+1, \ldots, N-1]$; see the left image in Figure 4 for a schematic illustration. After application of κ_A, the block $\kappa_A(B)$ is contained in the cyclic interval $\{e+2, \ldots, 1\}$, while $e+1$ and 2 are both contained in a different block "overarching" $\kappa_A(B)$; see the right image in Figure 4. During the next $N-e-2$ applications of κ_A, this overarching block persists, ensuring that $\mathrm{wd}\left(\kappa_A^j(B)\right) \leq \mathrm{wd}(B)$ for all j with $0 \leq j \leq N-e-1$. A further application of κ_A "restores" B, i.e., $\kappa_A^{N-e}(B)$ is a(n ordinarily) rotated version of B. (That we get indeed a rotated version follows from Theorem 3.10

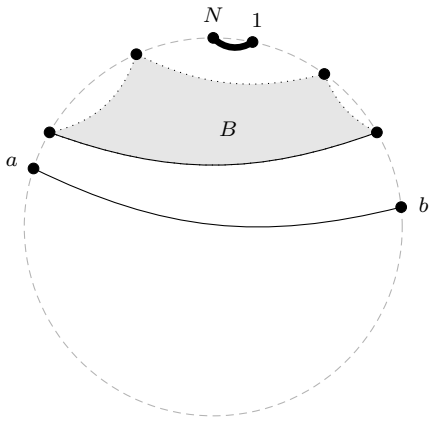

FIGURE 55. Illustration of a block B "covered" as described in Lemma C.3

and the observation that the next applications of κ_A will (ordinarily) rotate our block.) Thus we have $\mathrm{wd}\left(\kappa_A^j(B)\right) = \mathrm{wd}(B)$ for all j with $N-e \le j \le N-2$. This concludes the proof of the lemma. □

Lemma C.3. *Let B be a block in a non-crossing partition $\pi \in \widetilde{NC}_+(N)$ which is "covered" by two elements a and b (that is, a and b belong to a block B' different from B) with $a < N$, $b > 1$, and $b+N-a \le N/2$. (See Figure 55 for a schematic illustration of such a configuration.) Then $\mathrm{wd}\left(\kappa_A^j(B)\right) \le b+N-a-1$ for all j.*

Proof. The argument is similar to the ones in the proof of the previous lemma. Without loss of generality we may assume that a and b are (cyclically) successive elements in B'. Thus, the arc $a-b$ of the block B' is "overarching" B. During the first $N-a-1$ applications of κ_A, this overarching block persists, ensuring that $\mathrm{wd}\left(\kappa_A^j(B)\right) \le b+N-a-1$ for all j with $0 \le j \le N-a-1$. (It is here, where we use the condition $b+N-a \le N/2$, otherwise we could not be sure to draw this conclusion for the width of the pseudo-rotations of B.) A further application of κ_A produces the block $\kappa_A^{N-a}(B)$ which arises from $\kappa_A^{N-a-1}(B)$ by adding one to all elements of the latter block (modulo N), removing 1 and inserting an element e with $e \le b+N-a$ to it. Clearly, for this new block we have also $\mathrm{wd}\left(\kappa_A^{N-a}(B)\right) = e-2+1 \le b+N-a-1$. In order to now complete the proof, we appeal to Lemma C.1. □

We are now in the position for proving that fat blocks are at the same time central.

Lemma C.4. *A fat block B in a ρ-pseudo-rotationally invariant positive non-crossing partition is automatically a central block.*

Proof. Let π be a ρ-pseudo-rotationally invariant element of $\widetilde{NC}_+(N)$, and let B be a fat block of π. By way of contradiction, let us assume that B is not a central block, that is, $\kappa_A^{(N-2)/\rho}(B) \ne B$. Then there must exist another block of π, say B' with $B' \ne B$, for which $\kappa_A^{(N-2)/\rho}(B') = B$.

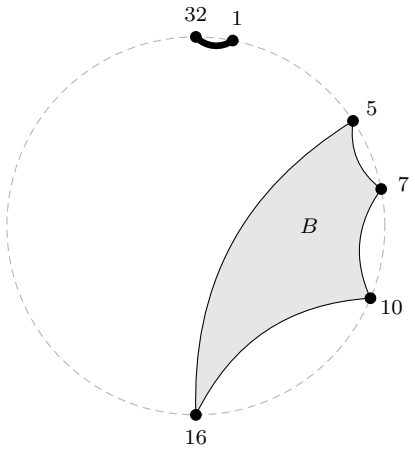

FIGURE 56. A block B with support $[5, \dots, 16]$ not containing 1 and $N = 32$, and with width $12 \leq N/2 = 16$

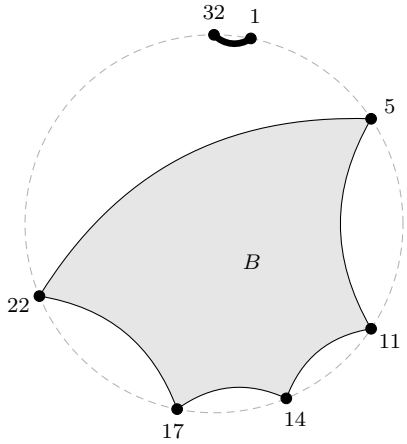

FIGURE 57. A block B with support $[5, \dots, 22]$ not containing 1 and $N = 32$, and with width $18 > N/2 = 16$

We distinguish between three cases (cf. Figures 56–58):

CASE 1. B has a support not containing N and 1, and $\mathrm{wd}(B) \leq N/2$.
CASE 2. B has a support not containing N and 1, and $\mathrm{wd}(B) > N/2$.
CASE 3. All supports of B contain N and 1.

CASE 1. Since π has a central block if and only if $\kappa_A^j(\pi)$ has a central block for any j, we may without loss of generality assume that B is embedded in $C = \{e, e+1, \dots, N-1\}$ with $|C| = \mathrm{wd}(B)$, that is, $N - e = \mathrm{wd}(B)$. It should be noted that e and $N-1$ must be elements of the block B.

We must split our argument again, depending on whether B' is contained in C or not.

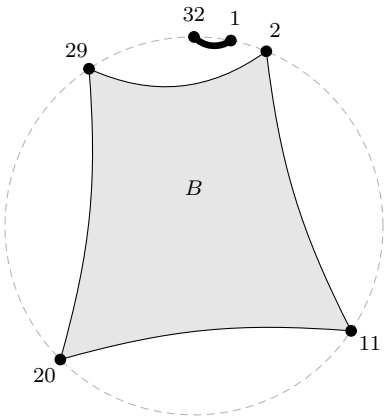

FIGURE 58. A block B with supports $[11,\dots,32,1,2]$, $[20,\dots,32,1,\dots,11]$, and $[29,\dots,32,1,\dots,20]$, all of them containing 1 and $N=32$

If B' is contained in C then $\mathrm{wd}(B') < \mathrm{wd}(B)$. This is absurd since, by Lemma C.1, we have $\mathrm{wd}\left(\kappa_A^j(B')\right) \le \mathrm{wd}(B')$ for all j.

On the other hand, if B' is not contained in C, then it must actually be disjoint from C. In particular, $\max B' < e$. If $N \notin B'$, then $\kappa_A^j(B')$ is a(n ordinarily) rotated version of B' for $1 \le j \le (N-2)/\rho$. Thus,

$$\max \kappa_A^{(N-2)/\rho}(B') = \max B' + \frac{N-2}{\rho} < e + \frac{N-2}{\rho} < N-1,$$

the last inequality following from the assumption that B is fat. If $N \in B'$, then $\max \kappa_A(B') = e+1$. Further $\frac{N-2}{\rho} - 1$ applications of κ_A now (ordinarily) rotate $\kappa_A(B')$. Consequently,

$$\max \kappa_A^{(N-2)/\rho}(B') = e + 1 + \left(\frac{N-2}{\rho} - 1\right) = e + \frac{N-2}{\rho} < N-1.$$

This shows that in both cases $\kappa_A^{(N-2)/\rho}(B')$ cannot contain $N-1$, a contradiction.

CASE 2. Again, we argue differently depending on whether B' is contained in the support of B or not.

If B' is contained in the support of B, then $\mathrm{wd}(B') < \mathrm{wd}(B)$. Again, this is absurd since, by Lemma C.1, we have $\mathrm{wd}\left(\kappa_A^j(B')\right) \le \mathrm{wd}(B')$ for all j.

On the other hand, if B' is not contained in the support of B, then we may apply Lemma C.3 with a the largest element of B — that is, $a = N-1$ — and b the smallest element of B — that is, $b = e$. The conclusion is that $\mathrm{wd}\left(\kappa_A^j(B')\right) < N/2 < \mathrm{wd}(B)$, a contradiction.

CASE 3. If B' is contained in the support of B, then $\mathrm{wd}(B') < \mathrm{wd}(B)$. Lemma C.3 applies with with a the largest element of B and b the smallest element of B. The conclusion is as in the above case and leads to a contradiction.

Finally, let B' be not contained in the support of B. If B' is not fat, then we know that $\mathrm{wd}(B') < \mathrm{wd}(B)$. By Lemma C.1, this is absurd. On the other hand, if B' is fat, then B' fits into Case 1 or Case 2, depending on its width. In both cases, by what we have already established we see that B' must be a central block. But this means that $\kappa_A^{(N-2)/\rho}(B') = B' \neq B$, which is again absurd.

This completes the proof of the lemma. □

Proposition C.5. *If $\rho \geq 3$, there is no ρ-pseudo-rotationally invariant positive non-crossing partition without central block.*

Proof. Let π be a ρ-pseudo-rotationally invariant element of $\widetilde{NC}_+(N)$. Arguing by contradiction, we assume that π contains no central block. Then, according to Lemma C.4, it is at the same time without a fat block, and the same is true for any pseudo-rotation of π, that is, $\kappa_A^j(\pi)$ does not contain a fat block for any j.

From now on we write M for $(N-2)/\rho$. We claim that, in the orbit of π under the action of the pseudo-rotation κ_A, we find a non-crossing partition in which the block containing N and 1 has a maximal element k with $k \leq M$, and the block containing $N-1$ has a support of the form $[N-a, N-a+1, \ldots, N-1]$ with $a \leq M+1$. See the first image of Figure 60 for a schematic illustration of this block structure.

To see this, we proceed as follows. It is obvious that by sufficiently many applications of κ_A to π we may obtain a non-crossing partition in which the block containing N and 1 has the claimed form. (Here we also use that we cannot have any fat blocks.) However, at this point it is not at all obvious that the block containing $N-1$, B' say, has the claimed form; it may rather be a block with a support containing N and 1; see Figure 59 for an illustration (at this point, e, f, and $\widehat{b}$ should be ignored). Moreover, B' may be "covered" by another block B'' with a support containing N and 1. Let $\widehat{B}$ be the block with maximal width with a support containing N and 1. We let the support of $\widehat{B}$ be $[e, e+1, \ldots, f]$. See again Figure 59 for an illustration of that situation. (In the figure, we have $\widehat{B} \neq B'$. However, it might well be that $\widehat{B} = B'$.)

According to our assumptions, the block $\widehat{B}$ cannot be fat, hence $\mathrm{wd}(\widehat{B}) \leq M+1$. Furthermore, all blocks which are disjoint from $\widehat{B}$ cannot be fat either.

Now we apply κ_A $N-e$ times. From the definition of the pseudo-rotation κ_A applied to our particular situation, we may see that $\widehat{B}$ will be "moved into N", in the sense that $\kappa_A^{N-e}(\widehat{B})$ will have a support of the form $[N, 1, \ldots, l]$ with $l \leq f+N-e$, while the blocks originally outside the support of $\widehat{B}$ will just be (ordinarily) rotated $N-e$ times. Thus, we have indeed reached the claimed situation, illustrated in the first image of Figure 60. Here we announce the award of a published version of this article, autographed by C. To receive the award, a 5-pseudo-rotationally invariant positive non-crossing partition in $\widetilde{NC}_+^{(3)}(A_8)$ different from the one-block-partition has to be sent to the authors.

In view of the fact that π contains a central block if and only if $\kappa_A^j(\pi)$ has a central block for any j, we now assume without loss of generality that π contains a block B which has a support of the form $[N, 1, \ldots, k]$, with $k \leq M$, and the block B' containing $N-1$ has

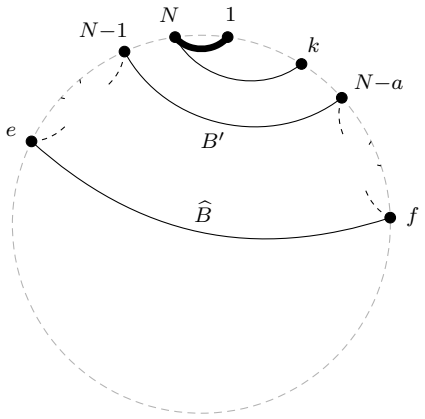

FIGURE 59. The situation at the beginning of the proof of Proposition C.5

a support of the form $[N-a, N-a+1, \dots, N-1]$ with $a \le M+1$. The first image in Figure 60 provides a schematic sketch of the combinatorial structure of π. In the figure, the block B' containing $N-1$ is indicated as well. It is indeed disjoint from B since

$$k \le M < N - M - 1 \le N - a.$$

If $B = \{N, 1\}$ then it is a central block as is easy to see directly. Hence, we may restrict ourselves to the case where $k \ge 2$, or, in other words, to the case where B contains at least three elements.

We now investigate what happens when we apply κ_A repeatedly to π. In $\kappa_A(\pi)$, we find the block $\kappa_A(B)$ that contains $2, k+1, N-a+1$, and we find the block $\kappa_A(B')$ that contains N and 1; see the second image in Figure 60.

The next $a-2$ applications of κ_A (ordinarily) rotate $\kappa_A(B')$, until we arrive at $\kappa_A^{a-1}(\pi)$, in which $\kappa_A^{a-1}(B)$ is a block containing $a, k+a-1, N-1$; see the third image in Figure 60. At no point do we arrive at a structure that would be in agreement with the structure at the very beginning; compare the second and third image in Figure 60 with the first image.

If $a = M+1$, then we have already obtained $\kappa_A^{a-1}(\pi) = \kappa_A^M(\pi) = \kappa_A^{(N-2)/\rho}(\pi)$, which should be the same as π. However, comparison of the first and third image in Figure 60 shows that we should have $(N-a) \in \kappa_A^M(B)$, which is impossible since

$$N - a \ge N - M - 1 > 2M \ge k + a - 1 \tag{C.1}$$

for $\rho \ge 3$. Thus, we have reached a contradiction.

If $a \le M$, then we have to apply κ_A another time. We arrive at $\kappa_A^a(\pi)$, in which the block $\kappa_A^a(B)$ contains $N, 1, k+a$; see the fourth image in Figure 60. The structure of the obtained non-crossing partition is the same as in the beginning — cf. the first image in Figure 60 — except that the maximum element of the block containing $N, 1$ (namely $k+a$) is larger than at the beginning (where it was k). Since this block cannot be a fat block according to our assumptions, we have $k+a \le M$. We now run through the arguments of

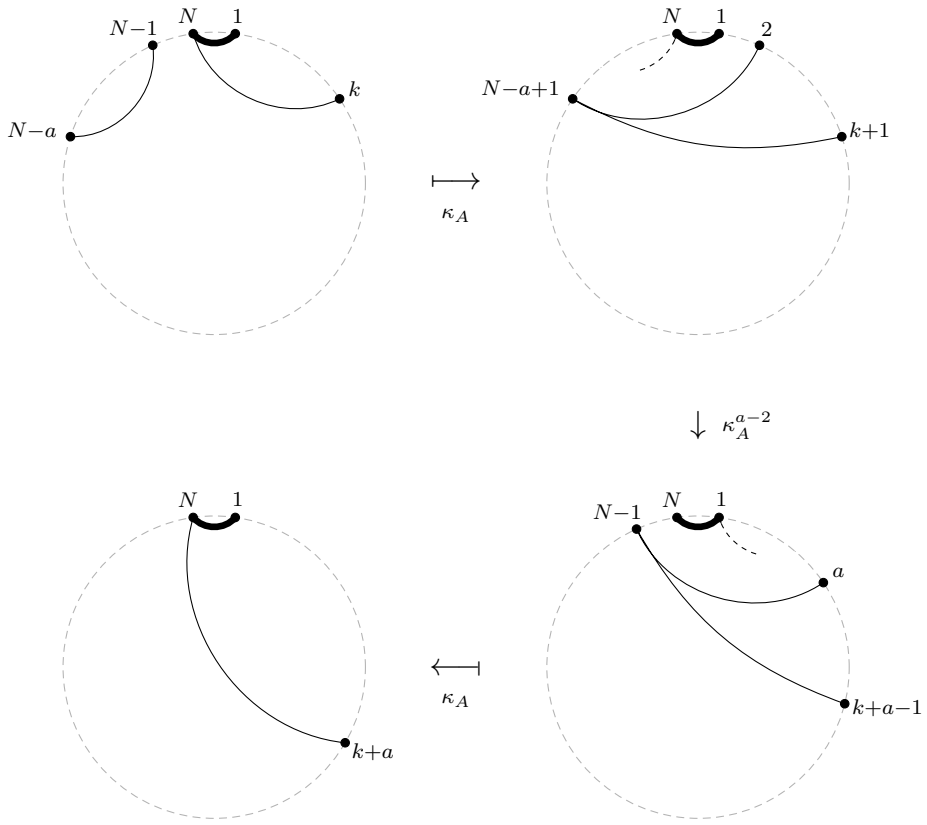

FIGURE 60. A schematic illustration of the construction in Proposition C.5

this proof another time, starting with $\kappa_A^a(\pi)$ instead of π. After finitely many rounds we arrive at a contradiction.

This completes the proof of the lemma. □

Proof of Theorem 3.17. It is not difficult to see from the analysis of what happens to a block under repeated application of the pseudo-rotation κ_A in Step 3 of the proof of Theorem 3.10 (given in Appendix A.1) that Constructions 1 and 2 from Subsection 3.5 do indeed yield ρ-pseudo-rotationally invariant positive non-crossing partitions.

For the converse, that is, to show that ρ-pseudo-rotationally invariant positive non-crossing partitions cannot arise in a different way, we discuss the two cases separately.

(1) Let $\rho \geq 3$ and π be a ρ-pseudo-rotationally invariant non-crossing partition in $\widetilde{NC}_+(N)$. By Proposition C.5, we know that π must contain a central block, Z say. If $|Z| = 2$, then it is not difficult to see directly that π must arise from the corresponding (degenerate) case of Construction 1.

From now on we let $|Z| \geq 3$. We now consider $\kappa_A^j(\pi)$ for $j = 0, 1, \ldots$. At some point, $\kappa_A^j(Z)$ must contain N and 1. If in all such situations also $N-1$ is contained in Z, then $Z = \{1, 2, \ldots, N\}$ and there is nothing to prove. Otherwise, for some j, $\kappa_A^j(Z)$ will contain

N and 1 but not $N-1$. We write π' for $\kappa_A^j(\pi)$ and Z' for $\kappa_A^j(Z)$. By the definition of the pseudo-rotation κ_A, we see that the elements $N - \frac{N-2}{\rho}k$ must also be in Z' for $k = 1, 2, \dots, \rho$. In particular, we have $2 \in Z'$.

Let B a block of π' different from Z'. Its support does not contain N and 1, and, by the last observation above, we have $\mathrm{wd}(B) < (N-2)/\rho$. Hence, by the proof of Lemma C.1, the pseudo-rotation κ_A acts as ordinary rotation, except while moving across N and 1. In particular, $\kappa_A^{(N-2)/\rho}$ acts by (ordinarily) rotating B by $\frac{N-2}{\rho}$ units, except when we move across N and 1 where B is rotated by $\frac{N-2}{\rho} + 2$ units. Thus we have shown that Z' arises from Construction 1, and all other elements in the orbit of π arise by applying the pseudo-rotation κ_A several times.

(2) Let π be a 2-pseudo-rotationally invariant non-crossing partition in $\widetilde{NC}_+(N)$. If it contains a central block, then the arguments from Case (1) apply also here and show that π arises from Construction 1 possibly followed by several applications of κ_A.

If π does not contain a central block, then we run through the arguments of the proof of Proposition C.5. There, the restriction of the proposition that $\rho \geq 3$ enters only at exactly one point, namely in the chain of inequalities (C.1). So, if we want to find a 2-pseudo-rotationally invariant non-crossing partitions in $\widetilde{NC}_+(N)$, then it is found at this point of the proof of Proposition C.5. Thus, in the construction there (cf. the first and the third image in Figure 60) we must have $a = M + 1 = \frac{N-2}{2} + 1 = \frac{N}{2}$. However, examining the arguments in the proof of Proposition C.5 leading from the first to the third image in Figure 60, we see that these describe exactly how to generate an element of $\widetilde{NC}_+(N)$ by means of Construction 2.

This completes the proof. □

C.2. **Characterisation of pseudo-rotationally invariant positive non-crossing partitions in type B_n.** In the proof of Lemma C.6 below, we make use of the following notions, modifying notions from the previous subsection. Given N, we define a *cyclic interval* of $\{1, 2, \dots, N, \overline{1}, \overline{2}, \dots, \overline{N}\}$ to be an interval of the form $\{e, e+1, \dots, f-1, f\}$, where two successive elements in this set are successive in $\{1, 2, \dots, N, \overline{1}, \overline{2}, \dots, \overline{N}\}$ or are equal to $\overline{N}, 1$. Thus, for $N = 12$, the sets $\{3, 4, 5, 6\}$, $\{10, 11, 12, \overline{1}\}$, as well as the set $\{\overline{10}, \overline{11}, \overline{12}, 1, 2\}$ are cyclic intervals. Given an element π of $\widetilde{NC}(\pm N)$, the *width* of a block B of π, denoted by $\mathrm{wd}(B)$, is defined as the minimal cardinality of a cyclic interval containing B. Such a minimal-cardinality cyclic interval will also be called a *support* for B. (With this definition, a support of a block need not be unique; it is unique however for non-zero blocks.)

Lemma C.6. *Let π be a non-crossing partition in $\widetilde{NC}(\pm N)$ without zero block. Then the order of π under the action of κ_B is $N-1$.*

Proof. We consider all blocks which contain 1 in their support. Among these, let B be the one with largest width. Let the support of B be $[e, e+1, \dots, f]$. We must have $e < 1 \leq f$. Figure 61 illustrates the structure of such a non-crossing partition π.

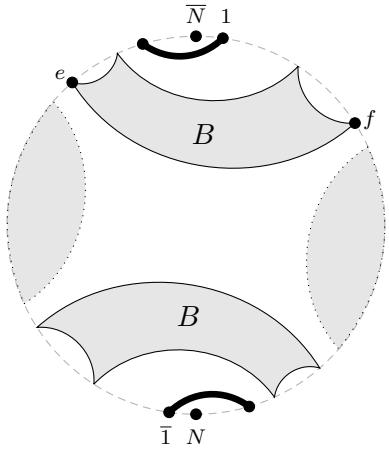

FIGURE 61. A non-crossing partition in $\widetilde{NC}(\pm N)$ without zero block

We start with the observation that the support of $\kappa_B^j(B)$ will have the form $[e+j,\dots]$, for $j = 0, 1, \dots, N-e$. (In less precise terms: the "first" element e of B is shifted by one unit during each of the first $N-e$ applications of κ_B.)

Next, we distinguish between two cases, depending on whether there is exactly one positive element in B or more than one. (It must contain *at least one* positive element.)

CASE 1. THE BLOCK B CONTAINS ONLY ONE POSITIVE ELEMENT. We consider first the case where $1 \in B$. If also $\overline{N} \in B$, then one application of κ_B maps B to a non-crossing partition, in which $\kappa_B(B)$ is a block that contains 1 *and* 2, that is, *two* positive elements. Thus, we are in Case 2 which is discussed below. If $\overline{N} \notin B$ (see Figure 62 for a schematic illustration), then the first application of κ_B will act on B as (ordinary) rotation. Also subsequent applications of κ_B may act as rotations, until another application of κ_B will have to be performed according to Case (1) in Theorem 4.6; see Figures 14 and 15. Then the block which arose from B will "lose" its "largest element" (its unique positive element), which is replaced by 1. Thus, we are back at the original situation of this case and we may continue by repeating the above arguments.

The only other possibility arises after $N-e$ applications of κ_B. Then we encounter the situation schematically illustrated in Figure 63. For the next application of κ_B we are in Case (2) of Theorem 4.6. This implies in particular that $\overline{1} \in \kappa_B^{N-e+1}(B)$. The "twin block" of $\kappa_B^{N-e+1}(B)$, that is, the block which consists of all the negatives of the elements of $\kappa_B^{N-e+1}(B)$, then contains 1 and it is its largest element. This is again the same situation as at the very beginning of this case. The above arguments apply and show that subsequent applications of κ_B either act as (ordinary) rotations, or the block that arose from the original block B "loses" its "first" element (the element closest to $\overline{N}$ in clockwise direction), which gets replaced by $\overline{1}$.

The important point is that at no point during the first $N-2$ applications of κ_B we obtain a structure that matches the original structure of π; cf. again Figures 62 and 63. Thus, we return to π after $N-1$ applications of κ_B, but not earlier.

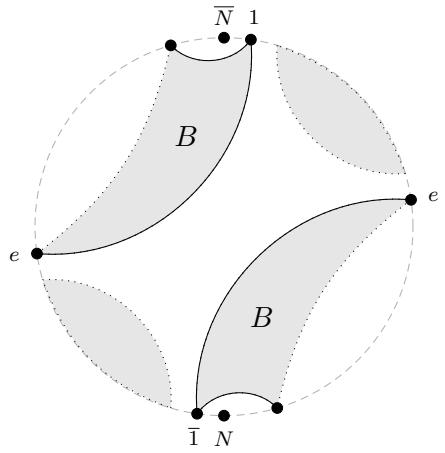

FIGURE 62. A non-crossing partition in $\widetilde{NC}(\pm N)$ without zero block

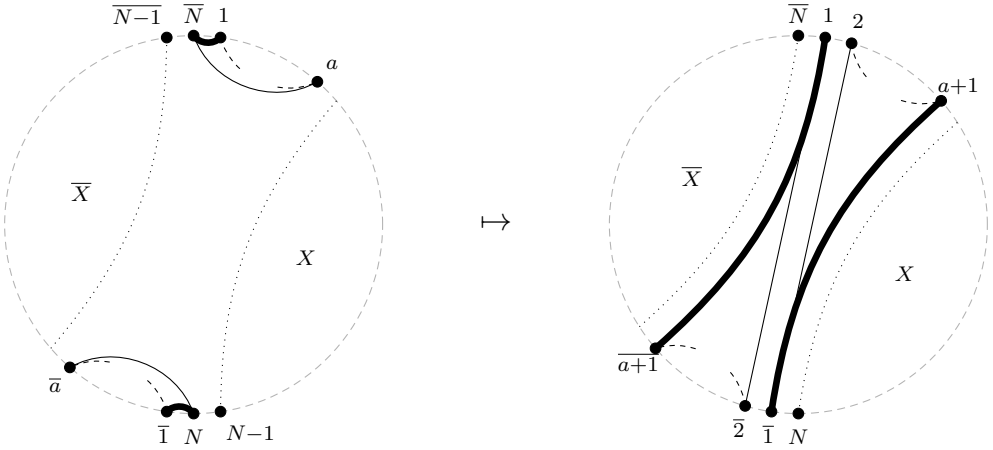

FIGURE 63. After $N - e$ applications of κ_B

CASE 2. THE BLOCK B CONTAINS AT LEAST TWO POSITIVE ELEMENTS. In this case, it is straightforward to see that, for $j = 0, 1, \ldots, N - e$, the support of $\kappa_B^j(B)$ equals $[e + j, e + j + 1, \ldots, f + j + 1]$. When we apply κ_B to $\kappa_B^{N-e}(\pi)$, then we are in Case (2) of Theorem 4.6. The remaining arguments are identical with the arguments of the last two paragraphs of the previous case. □

Proof of Theorem 4.14. It is not difficult to see from the analysis of what happens to a block under repeated application of the pseudo-rotation κ_B in Step 3 of the proof of Theorem 4.10 (given in Appendix A.2) that the Construction from Subsection 4.5 does indeed yield 2ν-pseudo-rotationally invariant positive non-crossing partitions of type B.

For the converse, let π be a 2ν-pseudo-rotationally invariant non-crossing partitions in $\widetilde{NC}(\pm N)$. We have to show that 2ν-pseudo-rotationally invariant positive non-crossing partitions of type B cannot arise in a different way. By Lemma C.6, we know that π must contain a zero block, Z say. Since a zero block in a type B non-crossing partition is unique, it must satisfy $\kappa_B^{(N-1)/\nu}(B) = B$, that is, it remains invariant under the pseudo-rotation $\kappa_B^{(N-1)/\nu}$.

If $|Z| = 2$, then it is not difficult to see directly that π must arise from the corresponding (degenerate) case of the Construction.

Now let $|Z| \geq 3$. For some j, the block $\kappa_B^j(Z)$ must contain 1. Since $|Z| \geq 3$, the degenerate case of the Construction never applies when applying κ_B to π. But then $1-\frac{N-1}{\nu}k$ must also be in $\kappa_B^j(Z)$. For $k = \nu$, we see that $1 - (N-1) = \overline{2}$ is an element of $\kappa_B^j(Z)$, and thus also 2. It is easy to see that $\kappa_B^{(N-1)/\nu}$ must act by (ordinarily) rotating the other blocks B of $\kappa_B^j(\pi)$ by $\frac{N-1}{\nu}$ units, except when we move across 1 where B is rotated by $\frac{N-1}{\nu}+1$ units. Thus we have shown that $\kappa_B^j(Z)$ arises from the Construction, and all other elements in the orbit of π arise by applying the pseudo-rotation κ_B several times. □

C.3. **Characterisation of pseudo-rotationally invariant positive non-crossing partitions in type D_n.**

Proof of Theorem 5.13. Let first π be a partition arising from Construction 1. It is in particular a partition with a zero block. If we remove the elements of the inner circle from π, then what remains is a partition in $\widetilde{NC}_+^{(m)}(B_{n-1})$ with a zero block which is 2ν-pseudo-rotationally invariant with respect to κ_B (cf. the Construction in Subsection 4.5). Since κ_D acts on the elements of the outer circle in that case in the same way as κ_B, the entire partition π is 2ν-pseudo-rotationally invariant with respect to κ_D.

Next, let π be a partition arising from Construction 2. Since in that case the partition of the elements of the inner circle is uniquely determined, we may as well ignore the elements of the inner circle. What remains is a partition in $\widetilde{NC}_+^{(m)}(B_{n-1})$ without zero block. By Theorem 4.10 with $N = m(n-1)$, this reduced partition is invariant under $\kappa_B^{m(n-1)-1}$. Since κ_D acts on the elements of the outer circle in that case in the same way as κ_B, the entire partition π is 2-pseudo-rotationally invariant with respect to κ_D.

Finally, let π be a partition arising from Construction 3. It is not difficult to see from an analysis in the spirit of the proof of Theorem 5.10 (given in Appendix A.3) that after $\big(m(n-1)-1\big)/\nu$ applications of κ_D we will obtain a partition that is the same as π, except that the elements on the inner circle may have changed their sign. In order to see what happened with the elements on the inner circle, we note that the elements of the inner circle of the block B in Construction 3 are rotated by $\frac{1}{\nu}\big(m(n-1)-1\big)+1$ units in counter-clockwise direction, the "+1" coming from the very first application of κ_D when an additional element on the inner circle is added to the block. After this number of applications of κ_D, before that rotated block (in clockwise direction) we find again the fundamental seed, possibly with changed sign for the elements of the inner circle. By definition, the fundamental seed

has x elements on the inner circle. Furthermore, we assume that it contains in total my elements. We know from the second assertion in Lemma 5.12 that $y = n/\nu$. Hence, after these $\big(m(n-1)-1\big)/\nu$ applications of κ_D, the "first" element on the inner circle of the fundamental seed is

$$\pm(mn-1)+\frac{1}{\nu}\big(m(n-1)-1\big)+1+x = \pm(mn-1)+(my-x-1)+1+x = \pm(mn-1)+my,$$

taken modulo $2m$ (the number of elements on the inner circle). In order to get the above equality we used (5.11). Consequently, if $y = n/\nu$ is even then this is $\pm(mn-1)$, which means that $\kappa_D^{(m(n-1)-1)/\nu}$ leaves π invariant. On the other hand, if $y = n/\nu$ is odd then this is $\mp(mn-1)$, which means that we need $2\big(m(n-1)-1\big)/\nu$ applications of κ_D until we return to π.

For the converse, let π be a ρ-pseudo-rotationally invariant non-crossing partitions in $\widetilde{NC}_+^{(m)}(D_n)$. We have to show that ρ-pseudo-rotationally invariant positive non-crossing partitions of type D cannot arise in a different way than from Constructions 1, 2, or 3 plus subsequent applications of κ_D. If, in π, the inner circle as a whole belongs to the zero block, then the action of κ_D on π can be described in the following way: let $\widehat{\pi}$ be the type B positive non-crossing partition that arises from π by removing the elements of the inner circle from the zero block of π. Then form $\kappa_B(\widehat{\pi})$. Finally add the elements of the inner circle to the zero block of $\kappa_B(\widehat{\pi})$. In other words: the pseudo-rotation κ_D acts as κ_B on the outer circle, while the inner circle is completely contained in the zero block anyway. By Theorem 4.14, it follows that π must arise from Construction 1.

If, in π, the inner circle decomposes into two blocks, then the action of κ_D on π can be described in the following way: let $\widehat{\pi}$ be the type B positive non-crossing partition that arises from π by ignoring the elements of the inner circle. Then form $\kappa_B(\widehat{\pi})$. Finally add the elements of the inner circle decomposed into two blocks in the unique way determined by $\kappa_B(\widehat{\pi})$ according to rule (D3) in Subsection 5.1. In other words: the pseudo-rotation κ_D acts as κ_B on the outer circle, while the inner circle is completely determined by the partition on the outer circle. By Lemma C.6, it follows that π must arise from Construction 2 and that it is 2-pseudo-rotationally invariant.

Finally, let π be an element of $\widetilde{NC}_+^{(m)}(D_n)$ that contains a bridging block and is ρ-pseudo-rotationally invariant, i.e., $\kappa_D^{2(m(n-1)-1)/\rho}(\pi) = \pi$. If ρ is even, then

$$\frac{2\big(m(n-1)-1\big)}{\rho} = \frac{\big(m(n-1)-1\big)}{\rho/2}$$

applications on π bring us back to π. By arguing in the spirit of the proof of Theorem 5.10 (given in Appendix A.3), it follows that π must arise from Construction 3 with $\nu = \rho/2$ (and possibly subsequent applications of κ_D).

On the other hand, let ρ be odd. Let λ be the multiplicative inverse of 2 modulo ρ. Then there exists a positive integer κ such that $2\lambda = 1 + \rho\kappa$. Thus, we have

$$\lambda \cdot \frac{2\big(m(n-1)-1\big)}{\rho} = \frac{2\lambda}{\rho}(m(n-1)-1) = \frac{\big(m(n-1)-1\big)}{\rho} + \kappa\big(m(n-1)-1\big). \quad \text{(C.2)}$$

Now, by assumption π is invariant under $\kappa_D^{2(m(n-1)-1)/\rho}$. Recall from the proof of Theorem 5.10 that $\kappa_D^{m(n-1)-1}$ applied to π yields structurally the same partition as π, except that the elements on the inner circle may have changed their sign. If these two facts are combined with Equation (C.2), then we see that $\kappa_D^{(m(n-1)-1)/\rho}$ applied to π yields structurally the same partition as π, except that the elements on the inner circle may have changed their sign. However, this means that π must arise from Construction 3 with $\nu = \rho$ (and possibly subsequent applications of κ_D).

This finishes the proof of the theorem. □

Appendix D. Proofs: Evaluations and non-negativity results for the positive q-Fuss–Catalan numbers

This appendix is devoted to the proofs of the three auxiliary lemmas in Lemmas 3.21, 3.22 and 5.19 giving evaluations and non-negativity results for the positive q-Fuß–Catalan numbers.

D.1. **An auxiliary result.** We start with a strengthening of the fact that *rational q-Catalan polynomials* (cf. [Arm24, Sec. 5]) have non-negative coefficients. The proof which we give follows largely the proofs in [And04, Thm. 2] and [RSW04, Prop. 10.1(iii)] of the above fact.

Lemma D.1. *Let a and b be positive integers. Then $\frac{[\gcd(a,b)]_q}{[a+b]_q}\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right]_q$ is a polynomial in q with non-negative integer coefficients.*

Proof. We first show that the expression in $\frac{[\gcd(a,b)]_q}{[a+b]_q}\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right]_q$ is a polynomial in q, and then derive the non-negativity of the coefficients from the fact that the q-binomial coefficient $\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right]_q$ is a reciprocal[10] and unimodal[11] polynomial in q. In order to show polynomiality, one recalls the well-known fact that

$$q^n - 1 = \prod_{d|n} C_d(q),$$

where $C_d(q)$ denotes the d^{th} cyclotomic polynomial in q. Consequently,

$$\frac{[\gcd(a,b)]_q}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q = \prod_{d=2}^{a+b-1} C_d(q)^{e_d},$$

with

$$e_d = \chi(d \mid \gcd(a,b)) + \left\lfloor \frac{a+b-1}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor - \left\lfloor \frac{b}{d} \right\rfloor, \tag{D.1}$$

[10] A polynomial $P(q)$ in q of degree d is called *reciprocal* if $P(q) = q^d P(1/q)$.

[11] A polynomial $P(q) = \sum_{i=0}^{d} p_i q^i$ is called *unimodal* if there is an integer r with $0 \le r \le d$ and $0 \le p_0 \le \cdots \le p_r \ge \cdots \ge p_d \ge 0$. It is well-known that q-binomial coefficients are unimodal; see [Sta99, Ex. 7.75.d].

where $\chi(\mathcal{S}) = 1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S}) = 0$ otherwise. Let us write $A = \{a/d\}$ and $B = \{b/d\}$, where $\{\alpha\} := \alpha - \lfloor\alpha\rfloor$ denotes the fractional part of α. Using this notation, Equation (D.1) becomes

$$e_d = \chi(d \mid \gcd(a,b)) + \left\lfloor A + B - \frac{1}{d} \right\rfloor.$$

This is clearly non-negative, unless $A = B = 0$. However, in that case we have $d \mid a$ and $d \mid b$, that is, $d \mid \gcd(a,b)$, so that e_d is non-negative also in this case.

Since we know that $\begin{bmatrix} a+b \\ a \end{bmatrix}_q$ is a reciprocal and unimodal polynomial in q of degree ab (see Footnote 11), the first $\lfloor ab/2 \rfloor$ coefficients of $(1-q^{\gcd(a,b)}) \begin{bmatrix} a+b \\ a \end{bmatrix}_q$ are non-negative, and therefore the same must be true for

$$\frac{1-q^{\gcd(a,b)}}{1-q^{a+b}} \begin{bmatrix} a+b \\ a \end{bmatrix}_q = \frac{[\gcd(a,b)]_q}{[a+b]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q.$$

However, we already know that the last expression is actually a *polynomial* in q, of degree $ab-(a+b)+\gcd(a,b)$, and it is as well a reciprocal polynomial. Hence, also the remaining coefficients are non-negative, which is what we wanted to prove. □

D.2. **Polynomiality with non-negative coefficients.**

Proof of Lemma 3.21. Expression (3.16) is the special case of (3.17) where $s = 0$, and consequently does not need to be treated separately.

In order to establish polynomiality of Expression (3.17), we rewrite the expression (3.17) in the form

$$\frac{[mn-n+s]_q}{[\gcd(mn-1,n-s-1)]_q\,[\gcd(n,s)]_q} \tag{D.2a}$$

$$\times \frac{[\gcd(n,s)]_q}{[n]_q} \begin{bmatrix} n \\ s \end{bmatrix}_q \tag{D.2b}$$

$$\times \frac{[\gcd(mn-1,n-s-1)]_q}{[mn-1]_q} \begin{bmatrix} mn-1 \\ n-s-1 \end{bmatrix}_q. \tag{D.2c}$$

We note that both $\gcd(n,s)$ and $\gcd(mn-1,n-s-1)$ divide $mn-n+s$. Moreover, we claim that the two numbers are coprime. For, if there is a d dividing both $\gcd(n,s)$ and $\gcd(mn-1,n-s-1)$, then we infer that $d \mid (n-1)$, which, in combination with $d \mid n$, means that actually $d = 1$. It is then not difficult to see that the quotient in (D.2a) is a polynomial in q (see e.g. [KM13, Lemma 5]). Clearly, Lemma D.1 implies that both (D.2b) and (D.2c) are also polynomials.

Finally, to see that (3.17) is not only a polynomial, but also has non-negative coefficients, we proceed as in the proof of Lemma D.1. Since products of reciprocal and unimodal polynomials are again reciprocal and unimodal (see [And76, Thm. 3.9]), the polynomials

$$\begin{bmatrix} n \\ s \end{bmatrix}_q \quad \text{and} \quad \begin{bmatrix} mn-1 \\ n-s-1 \end{bmatrix}_q$$

are reciprocal and unimodal. The first one is of degree $D_1 := s(n-s)$, while the second is of degree $D_2 := (n-s-1)(mn-n+s)$. As a consequence, the first $\lfloor D_1/2 \rfloor$ coefficients of $(1-q)\begin{bmatrix} n \\ s \end{bmatrix}_q$ and the first $\lfloor D_2/2 \rfloor$ coefficients of

$$(1-q^{mn-n+s})\begin{bmatrix} mn-1 \\ n-s-1 \end{bmatrix}_q$$

are non-negative. But then the first $\lfloor D_1/2 \rfloor + \lfloor D_2/2 \rfloor$ coefficients of the series

$$\frac{(1-q)(1-q^{mn-n+s})}{(1-q^{mn-1})(1-q^n)}\begin{bmatrix} n \\ s \end{bmatrix}_q \begin{bmatrix} mn-1 \\ n-s-1 \end{bmatrix}_q,$$

are also non-negative. However, we know that the last expression (which agrees with (3.17)) is actually a *polynomial* in q, of degree $D_1 + D_2 - (n-1) - (n-s-1)$, and it is as well a reciprocal polynomial. Hence, also the remaining coefficients are non-negative, which is what we wanted to prove.

The claims about (3.18) can be established in a completely analogous fashion. To prove polynomiality, we rewrite (3.18) in the form

$$\frac{[mn-b_1-b_2-\cdots-b_n]_q}{[\gcd(b_1,b_2,\dots,b_n)]_q\,[\gcd(mn-1,b_1+b_2+\cdots+b_n-1)]_q} \tag{D.3a}$$

$$\times \frac{[\gcd(b_1,b_2,\dots,b_n)]_q}{[b_1+b_2+\cdots+b_n]_q}\begin{bmatrix} b_1+b_2+\cdots+b_n \\ b_1,b_2,\dots,b_n \end{bmatrix}_q \tag{D.3b}$$

$$\times \frac{[\gcd(mn-1,b_1+b_2+\cdots+b_n-1)]_q}{[mn-1)]_q}\begin{bmatrix} mn-1 \\ b_1+b_2+\cdots+b_n-1 \end{bmatrix}_q. \tag{D.3c}$$

It follows from (3.15) that $\gcd(b_1,b_2,\dots,b_n)$ divides $(mn-b_1-b_2-\cdots-b_n)$. Moreover, one sees directly that $\gcd(mn-1,b_1+b_2+\cdots+b_n-1)$ divides $(mn-b_1-b_2-\cdots-b_n)$. Furthermore, if there would be a d dividing both $\gcd(b_1,b_2,\dots,b_n)$ and $\gcd(mn-1,b_1+b_2+\cdots+b_n-1)$, then (3.15) would imply that $d \mid n$, so that, if this is combined with $d \mid (mn-1)$, we see that we must have $d=1$. As earlier, as a consequence, the fraction (D.3a) is a polynomial in q. That (D.3b) is a polynomial in q can be seen by proceeding along the lines of the proof of Lemma D.1. We leave it to the reader to work out the details. In its turn, Lemma D.1 implies polynomiality of (D.3c). This establishes the polynomiality of (3.18). Non-negativity of the coefficients follows now as before of those of (3.17). We omit the details for the sake of brevity. □

D.3. **Evaluations at roots of unity.**

Proof of Lemma 3.22. We have

$$\frac{1}{[n]_q}\begin{bmatrix} (m+1)n-2 \\ n-1 \end{bmatrix}_q = \frac{(1-q^{(m+1)n-2})(1-q^{(m+1)n-3})\cdots(1-q^{mn})}{(1-q^n)(1-q^{n-1})\cdots(1-q^2)}.$$

In order to perform the limit $q \to \omega_\rho = e^{2\pi i\sigma/(mn-2)} = e^{2\pi i/\rho}$, we use the simple fact

$$\lim_{q\to\omega_\rho} \frac{1-q^\alpha}{1-q^\beta} = \begin{cases} \frac{\alpha}{\beta}, & \text{if } \alpha \equiv \beta \equiv 0 \pmod{\rho}, \\ 1, & \text{otherwise,} \end{cases} \tag{D.4}$$

where α, β are non-negative integers such that $\alpha \equiv \beta \pmod{\rho}$. Since $\rho \mid (mn-2)$, the claimed limit in (3.19) is obtained straightforwardly.

In order to see (3.20), we write

$$\frac{1}{[n]_q} \begin{bmatrix} mn-2 \\ n-1 \end{bmatrix}_q = \frac{(1-q^{mn-2})(1-q^{mn-3})\cdots(1-q^{mn-n})}{(1-q^n)(1-q^{n-1})\cdots(1-q^2)}. \tag{D.5}$$

Here, we first check in which cases the number of factors in the numerator which vanish when $q \to e^{2\pi i\sigma/(mn-2)} = e^{2\pi i/\rho}$ does not exceed the number of those in the denominator, because otherwise the limit of the left-hand side of (D.5) as $q \to e^{2\pi i\sigma/(mn-2)}$ is zero. Now, the difference of the numbers of vanishing factors in the numerator and the denominator is

$$\left\lfloor \frac{mn-2}{\rho} \right\rfloor - \left\lfloor \frac{mn-n-1}{\rho} \right\rfloor - \left\lfloor \frac{n}{\rho} \right\rfloor = - \left\lfloor \frac{-n+1}{\rho} \right\rfloor - \left\lfloor \frac{n}{\rho} \right\rfloor.$$

Writing $N = \{n/\rho\}$ for the fractional part of n/ρ, this reduces to $-\left\lfloor -N + \frac{1}{\rho} \right\rfloor$, which can only be zero if $N = 0$ or $N = 1/\rho$, or, equivalently, if $n \equiv 0, 1 \pmod{\rho}$. If $n \equiv 0 \pmod{\rho}$, then, because of the assumption $\rho \mid (mn-2)$, we infer $\rho = 2$. The first alternative on the right-hand side of (3.20) follows immediately using (D.4). On the other hand, if $n \equiv 1 \pmod{\rho}$ then the second alternative on the right-hand side of (3.20) follows again from (D.4).

Next we consider (3.21). We rewrite the expression on the left-hand side in an analogous fashion as product/quotient of factors of the form $1-q^j$. Then, again, we count how many of these factors vanish when $q \to e^{2\pi i\sigma/(mn-2)} = e^{2\pi i/\rho}$. Letting $N = \{(n-1)/\rho\}$, $S = \{s/\rho\}$, and remembering that $\rho \mid (mn-2)$, the difference of the numbers of vanishing factors in the numerator and in the denominator equals

$$-\left\lfloor N - S + \frac{1}{\rho} \right\rfloor - \lfloor N - S \rfloor - \lfloor -N + S \rfloor = -\left\lfloor N - S + \frac{1}{\rho} \right\rfloor + 1 - \chi\big(\rho \mid (n-s-1)\big). \tag{D.6}$$

Now there are three cases to distinguish.

Case 1: $\left\lfloor N - S + \frac{1}{\rho} \right\rfloor = 1$. This means that $N = \frac{\rho-1}{\rho}$ and $S = 0$, that is, $\rho \mid n$ and $\rho \mid s$. In combination with $\rho \mid (mn-2)$, it follows that $\rho = 2$. A further consequence is that $2 = \rho \nmid (n-s-1)$, so that (D.6) is satisfied. The result of the substitution on the left-hand side of (3.21) is easily seen to be as asserted in the second alternative on the right-hand side.

Case 2: $\left\lfloor N - S + \frac{1}{\rho} \right\rfloor = 0$. Another way to describe the defining condition of this case is to say that $\{n/\rho\} \geq \{s/\rho\}$ and not both $\rho \mid n$ and $\rho \mid s$. In order (D.6) to be satisfied, we must have $\rho \mid (n-s-1)$, implying that $\{s/\rho\} = \{(n-1)/\rho\} \leq \{n/\rho\}$, and that both

$\rho \mid n$ and $\rho \mid s$ is impossible. The result of the substitution on the left-hand side of (3.21) is as asserted in the first alternative on the right-hand side.

CASE 3: $\left\lfloor N - S_1 + \frac{1}{\rho} \right\rfloor = -1$. Here, the expression (D.6) is always positive, so that the left-hand side of (3.21) vanishes.

Finally we address (3.22). As before, we rewrite the expression on the left-hand side as product/quotient of factors of the form $1 - q^j$. Subsequently we count how many of these factors vanish when $q \to \omega_\rho = e^{2\pi i\sigma/(mn-2)} = e^{2\pi i/\rho}$. Letting $B_i = \{b_i/\rho\}$, $i = 1, 2, \dots, n$, and again remembering that $\rho \mid (mn-2)$, the difference of the numbers of vanishing factors in the numerator and in the denominator equals

$$- \left\lfloor - \sum_{k=1}^{n} B_k + \frac{1}{\rho} \right\rfloor. \tag{D.7}$$

Again we distinguish three cases.

CASE 1: $\sum_{k=1}^{n} B_k = 0$. In other words, we have $B_k = 0$ for all k, or, equivalently, $\rho \mid b_k$ for all k. Because of (3.15), this implies that also $\rho \mid n$. From $\rho \mid (mn-2)$ it then follows that $\rho \mid 2$, that is, that $\rho = 2$. The result of the substitution on the left-hand side of (3.22) is as asserted in the second alternative on the right-hand side.

CASE 2: $\sum_{k=1}^{n} B_k = \frac{1}{\rho}$. Here we have $B_k = 0$ for all k but one; let j be the exception. More explicitly, we have $\rho \mid b_k$ for all $k \neq j$ and $b_j \equiv 1 \pmod{\rho}$. The result of the substitution on the left-hand side of (3.22) is as asserted in the first alternative on the right-hand side.

CASE 3: $\sum_{k=1}^{n} B_k > \frac{1}{\rho}$. Then $\left\lfloor - \sum_{k=1}^{n} B_k + \frac{1}{\rho} \right\rfloor < 0$, and consequently the sum in (D.7) is positive, so that the left-hand side of (3.22) vanishes.

This completes the proof of the lemma. □

Proof of Lemma 4.18. We have

$$\begin{bmatrix} (m+1)n-1 \\ n \end{bmatrix}_{q^2} = \frac{(1-q^{2(m+1)n-2})(1-q^{2(m+1)n-4})\cdots(1-q^{2mn})}{(1-q^{2n})(1-q^{2n-2})\cdots(1-q^2)}.$$

In order to perform the limit $q \to \omega_\rho = e^{2\pi i\sigma/(2mn-2)} = e^{2\pi i/\rho}$, we use again the simple fact

$$\lim_{q\to\omega_\rho} \frac{1-q^\alpha}{1-q^\beta} = \begin{cases} \frac{\alpha}{\beta}, & \text{if } \alpha \equiv \beta \equiv 0 \pmod{\rho}, \\ 1, & \text{otherwise,} \end{cases} \tag{D.8}$$

where α, β are non-negative integers such that $\alpha \equiv \beta \pmod{\rho}$. Since $\rho \mid (mn-1)$, the claimed limit in (4.19) is obtained without much effort.

In order to see (4.20), we write

$$\begin{bmatrix} mn-1 \\ n \end{bmatrix}_{q^2} = \frac{(1-q^{2mn-2})(1-q^{2mn-4})\cdots(1-q^{2mn-2n})}{(1-q^{2n})(1-q^{2n-2})\cdots(1-q^2)}. \tag{D.9}$$

Here, we first check in which cases the number of factors in the numerator which vanish when $q \to e^{2\pi i\sigma/2(mn-1)} = e^{2\pi i/\rho}$ does not exceed the number of those in the denominator, because otherwise the limit of the left-hand side of (D.9) as $q \to e^{2\pi i\sigma/2(mn-1)} = e^{2\pi i/\rho}$ is zero.

If ρ is even, the difference of the numbers of vanishing factors in the numerator and the denominator is

$$\left\lfloor \frac{2(mn-1)}{\rho} \right\rfloor - \left\lfloor \frac{2(mn-n-1)}{\rho} \right\rfloor - \left\lfloor \frac{2n}{\rho} \right\rfloor = - \left\lfloor \frac{-2n}{\rho} \right\rfloor - \left\lfloor \frac{2n}{\rho} \right\rfloor .$$

Writing $N = \{2n/\rho\}$ for the fractional part of $2n/\rho$, this reduces to $-\lfloor -N \rfloor$, which can only by zero if $N = 0$. This means that $2n \equiv 0 \pmod{\rho}$. In this case, because of the assumption $\rho \mid 2(mn-1)$, we infer $\rho = 2$. The first alternative on the right-hand side of (4.20) follows immediately using (D.8).

If ρ is odd, the difference of the numbers of vanishing factors in the numerator and the denominator is

$$\left\lfloor \frac{mn-1}{\rho} \right\rfloor - \left\lfloor \frac{mn-n-1}{\rho} \right\rfloor - \left\lfloor \frac{n}{\rho} \right\rfloor = - \left\lfloor \frac{-n}{\rho} \right\rfloor - \left\lfloor \frac{n}{\rho} \right\rfloor .$$

Writing $N = \{n/\rho\}$ for the fractional part of n/ρ, this reduces to $-\lfloor -N \rfloor$, which can only by zero if $N = 0$. This means that $n \equiv 0 \pmod{\rho}$. In this case, because of the assumption $\rho \mid 2(mn-1)$, we infer $\rho = 1$. The first alternative on the right-hand side of (4.20) follows immediately using (D.8).

Next we consider (4.21). We rewrite the expression on the left-hand side in an analogous fashion as product/quotient of factors of the form $1 - q^j$. Then, again, we count how many of these factors vanish when $q \to e^{2\pi i\sigma/(mn-1)} = e^{2\pi i/\rho}$.

If ρ is even, then let $N = \{2n/\rho\}$ and $S = \{2s/\rho\}$. Remembering that $\rho \mid (mn-1)$, the difference of the numbers of vanishing factors in the numerator and in the denominator equals

$$-2\lfloor N - S \rfloor - \lfloor -N + S \rfloor . \tag{D.10}$$

If $N < S$, then this expression equals 2. If $N > S$, then it is 1. Only if $N = S$, or, equivalently, $2n \equiv 2s \pmod{\rho}$, the expression (D.10) vanishes. The first alternative on the right-hand side of (4.21) follows immediately using (D.8).

If ρ is odd, then we let $N = \{n/\rho\}$ and $S = \{s/\rho\}$. Here, the difference of the numbers of vanishing factors in the numerator and in the denominator equals again (D.10), however with the modified meanings of N and S that we consider here. The remaining steps are completely analogous to the ones in the previous paragraph, and they lead to the second alternative in (4.21).

Finally we address (4.22). As before, we rewrite the expression on the left-hand side as product/quotient of factors of the form $1 - q^j$. Subsequently we count how many of these factors vanish when $q \to \omega_\rho = e^{2\pi i\sigma/(mn-1)} = e^{2\pi i/\rho}$.

If ρ is even, then write $B_i = \{2b_i/\rho\}$, $i = 1, 2, \dots, n$. Again remembering that $\rho \mid (mn-1)$, we see that the difference of the numbers of vanishing factors in the numerator

and in the denominator equals

$$-\left\lfloor -\sum_{k=1}^{n} B_k \right\rfloor. \tag{D.11}$$

This expression can only be zero if $\sum_{k=1}^{n} B_k = 0$, that is, if $B_k = 0$ for all k. Equivalently, we have $\rho \mid 2b_k$ for all k. The result of the substitution on the left-hand side of (4.22) is as asserted in the first alternative on the right-hand side.

If ρ is odd, then we let $B_i = \{b_i/\rho\}$, $i = 1, 2, \dots, n$. Here, the difference of the numbers of vanishing factors in the numerator and in the denominator turns out to again equal (D.11), with the modified meaning of the B_i's, however. The remaining steps are completely analogous to the ones in the previous paragraph, and they lead to the second alternative in (4.22).

This completes the proof of the lemma. □

Proof of Lemma 5.19. We have

$$\begin{aligned}&\frac{[2m(n-1)+n-2]_q}{[n]_q}\begin{bmatrix}(m+1)(n-1)-1\\ n-1\end{bmatrix}_{q^2}\\ &\quad=\frac{(1-q^{2m(n-1)+n-2})}{(1-q^n)}\cdot\frac{(1-q^{2(m+1)(n-1)-2})(1-q^{2(m+1)(n-1)-4})\cdots(1-q^{2m(n-1)})}{(1-q^{2n-2})\cdots(1-q^4)(1-q^2)}.\end{aligned}$$

In order to perform the limit $q \to \omega_\rho = e^{2\pi i\sigma/(2m(n-1)-2)} = e^{2\pi i/\rho}$, we use again the simple fact

$$\lim_{q\to\omega_\rho}\frac{1-q^\alpha}{1-q^\beta}=\begin{cases}\frac{\alpha}{\beta}, & \text{if } \alpha\equiv\beta\equiv 0 \pmod{\rho},\\ 1, & \text{otherwise,}\end{cases} \tag{D.12}$$

where α, β are non-negative integers such that $\alpha \equiv \beta \pmod{\rho}$. Since $\rho \mid (2m(n-1)-2)$, the claimed limit in (5.19) is obtained without much effort.

In order to see (5.20), we write

$$\begin{aligned}&\frac{[2m(n-1)-n]_q}{[n]_q}\begin{bmatrix}m(n-1)-1\\ n-1\end{bmatrix}_{q^2}\\ &\quad=\frac{(1-q^{2m(n-1)-n})}{(1-q^n)}\cdot\frac{(1-q^{2m(n-1)-2})(1-q^{2m(n-1)-4})\cdots(1-q^{2m(n-1)-2(n-1)})}{(1-q^{2n-2})\cdots(1-q^4)(1-q^2)}.\end{aligned} \tag{D.13}$$

Here, we first check in which cases the number of factors in the numerator which vanish when $q \to e^{2\pi i\sigma/(2m(n-1)-2)} = e^{2\pi i/\rho}$ does not exceed the number of those in the denominator, because otherwise the limit of the left-hand side of (D.13) as $q \to e^{2\pi i\sigma/(2m(n-1)-2)} = e^{2\pi i/\rho}$ is zero.

If ρ is even, the difference of the numbers of vanishing factors in the numerator and the denominator is

$$\chi\big(\rho \mid (2m(n-1)-n)\big)-\chi(\rho \mid n)+\left\lfloor \frac{2(m(n-1)-1)}{\rho} \right\rfloor-\left\lfloor \frac{2(m(n-1)-n)}{\rho} \right\rfloor-\left\lfloor \frac{2(n-1)}{\rho} \right\rfloor$$
$$=\chi\big(\rho \mid (2-n)\big)-\chi(\rho \mid n)-\left\lfloor \frac{-2(n-1)}{\rho} \right\rfloor-\left\lfloor \frac{2(n-1)}{\rho} \right\rfloor.$$

Writing $N=\{2(n-1)/\rho\}$ for the fractional part of $2(n-1)/\rho$, this reduces to

$$\chi\big(\rho \mid (2-n)\big)-\chi(\rho \mid n)-\lfloor -N \rfloor. \tag{D.14}$$

We now distinguish several cases.

Case 1: $\rho=2$. In this case, we have $N=0$, and, regardless whether n is even or odd, the expression in (D.14) vanishes. The result of the limit $q\to\omega_\rho$ of (D.13) is as asserted in the first and the second alternative on the right-hand side of (5.20), respectively.

Case 2: $\rho\geq 4$ and $N=0$. The condition $N=0$ means that $\rho \mid 2(n-1)$. Together with $\rho \mid (2m(n-1)-2)$, this entails that $\rho=2$, a contradiction to $\rho\geq 4$.

Case 3: $\rho\geq 4$, $N>0$ and $\rho \mid n$. Here, the expression (D.14) vanishes. The result of the limit $q\to\omega_\rho$ of (D.13) is as asserted in the third alternative on the right-hand side of (5.20).

In all other cases, the expression (D.14) is strictly positive. The limit $q\to\omega_\rho$ of (D.13) is therefore zero.

Now we treat the case where ρ is odd. The difference of the numbers of vanishing factors in the numerator and the denominator of (D.13) is

$$\chi\big(\rho \mid (2m(n-1)-n)\big)-\chi(\rho \mid n)+\left\lfloor \frac{m(n-1)-1}{\rho} \right\rfloor-\left\lfloor \frac{m(n-1)-n}{\rho} \right\rfloor-\left\lfloor \frac{n-1}{\rho} \right\rfloor$$
$$=\chi\big(\rho \mid (2m(n-1)-n)\big)-\chi(\rho \mid n)-\left\lfloor \frac{-(n-1)}{\rho} \right\rfloor-\left\lfloor \frac{n-1}{\rho} \right\rfloor.$$

Writing $N=\{(n-1)/\rho\}$ for the fractional part of $(n-1)/\rho$, this reduces to

$$\chi\big(\rho \mid (2-n)\big)-\chi(\rho \mid n)-\lfloor -N \rfloor. \tag{D.15}$$

Again, we distinguish several cases.

Case 1: $N=0$. The condition $N=0$ means that $\rho \mid (n-1)$. Together with $\rho \mid (2m(n-1)-2)$, this entails that $\rho=1$. The result of the limit $q\to\omega_\rho$ of (D.13) is as asserted in the fourth alternative on the right-hand side of (5.20).

Case 2: $N>0$ and $\rho \mid n$. Here, the expression (D.15) vanishes. The result of the limit $q\to\omega_\rho$ of (D.13) is as asserted in the fifth alternative on the right-hand side of (5.20).

In all other cases, the expression (D.15) is strictly positive. The limit $q\to\omega_\rho$ of (D.13) is therefore zero.

This completes the proof of the lemma. □

Appendix E. Proofs: Cyclic sieving for dihedral and exceptional groups

In this appendix, we prove Theorem 6.15. We treat the cyclic sieving for the dihedral groups explicitly in Appendix E.1 and the exceptional groups in detail in Appendix E.2, each separately for $\widetilde{\mathsf{K}}_+$ and for K_+.

In each situation, we exactly follow the approach detailed in the paragraph before Theorem 6.15, based on the results in Section 6.

E.1. **Cyclic sieving for the dihedral groups.** We consider the situation for the dihedral group $I_2(h)$. The degrees are $2, h$, and we hence have

$$\mathrm{Cat}_+^{(m)}(I_2(h);q) = \frac{[hm+(h-2)]_q\,[hm]_q}{[h]_q\,[2]_q}. \tag{E.1}$$

E.1.1. *Cyclic sieving for* $\widetilde{\mathsf{K}}_+$. Given $(m+1)h-2 = hm+(h-2) = k\cdot\rho$, let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.1) occur:

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(I_2(h);q) = \mathrm{Cat}_+^{(m)}(I_2(h)), \qquad \text{if } \rho=1, \tag{E.1a}$$

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(I_2(h);q) = \mathrm{Cat}_+^{(m)}(I_2(h)), \qquad \text{if } h\equiv 0 \pmod 2,\ \rho=2, \tag{E.1b}$$

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(I_2(h);q) = \tfrac{(m+1)h}{2}-1, \qquad \text{if } m\equiv h\equiv 1 \pmod 2,\ \rho=2, \tag{E.1c}$$

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(I_2(h);q) = 0, \qquad \text{otherwise.} \tag{E.1d}$$

To compute the invariant subwords, we distinguish the two situations where h is even or odd.

First, let $h\equiv 0 \pmod 2$. Then ρ is even and $\rho/2$ must divide $n=2$, so $\rho\in\{2,4\}$. The case where $\rho=2$ is clear, and it only remains to consider the case where $\rho=4$. In this case, $\mathbf{inv}_c$ consists of the h reflections and the corresponding word of h alternating copies of the two reflections s and t has even length (so that it starts with s and ends with t). The subwords of $\mathbf{inv}_c$ are given by any two consecutive reflections in cyclically reverse order. For example, in type $I_2(6)$, we have

$$\mathbf{inv}_c = \mathrm{r}_1\mathrm{r}_2\mathrm{r}_3\mathrm{r}_4\mathrm{r}_5\mathrm{r}_6 = s, sts, ststs, tstst, tst, t,$$

so that the reduced $\mathcal{R}$-words for the Coxeter element $c=st$ are

$$\mathrm{r}_1\mathrm{r}_6 = \mathrm{r}_2\mathrm{r}_1 = \mathrm{r}_3\mathrm{r}_2 = \mathrm{r}_4\mathrm{r}_3 = \mathrm{r}_5\mathrm{r}_4 = \mathrm{r}_6\mathrm{r}_5.$$

The corresponding $\mathcal{S}$-word is $\mathrm{c}_A\mathrm{c}_B\mathrm{c}_A\mathrm{c}_B\mathrm{c}_A\mathrm{c}_B = \mathrm{ststst}$ with $\mathrm{c}_A=\mathrm{s}$ and $\mathrm{c}_B=\mathrm{t}$. We thus see that all factorisations of $\mathrm{c}=\mathrm{st}$ use one letter from c_A and one letter from c_B, as above for $I_2(6)$. However, by Lemma 6.8(ii), none of these factorisations can correspond to an $\rho/2$-fold symmetry.

Second, let $h \equiv 1 \pmod 2$. Then ρ must divide $n = 2$, so $\rho \in \{1, 2\}$. The case where $\rho = 1$ is clear, and it only remains to consider the case where $\rho = 2$. In this case, $\mathbf{inv}_c$ consists of the h reflections and the corresponding word of h alternating copies of the two reflections s and t has odd length (so that it starts with s and ends with s). The subwords of $\mathbf{inv}_c$ are given by any two consecutive reflections in cyclically reverse order. For example, in type $I_2(5)$, we have

$$\mathbf{inv}_c = \mathrm{r}_1\mathrm{r}_2\mathrm{r}_3\mathrm{r}_4\mathrm{r}_5 = s, sts, ststs, tst, t,$$

so that the reduced $\mathcal{R}$-words for the Coxeter element $c = st$ are

$$\mathrm{r}_1\mathrm{r}_5 = \mathrm{r}_2\mathrm{r}_1 = \mathrm{r}_3\mathrm{r}_2 = \mathrm{r}_4\mathrm{r}_3 = \mathrm{r}_5\mathrm{r}_4.$$

The corresponding $\mathcal{S}$-word is $\mathrm{c}_A\mathrm{c}_B\mathrm{c}_A\mathrm{c}_B\mathrm{c}_A = \mathrm{ststs}$ with $\mathrm{c}_A = \mathrm{s}$ and $\mathrm{c}_B = \mathrm{t}$. From Lemma 6.2, we obtain that 2 divides $(m+1)h - 2$ if and only if $m \equiv 1 \pmod 2$, and we therefore write $m = 2a + 1$ and finally get

$$i_{\mathrm{fix}} = \tfrac{1}{2}\big((1+1)h - 2\big) + 1 = h.$$

Thus, as all factorisations are invariant in this case, we finally sum over all factorisations to obtain that for $m = 2a+1$, the number of invariant subwords of $\mathbf{inv}_c^m \mathbf{inv}_c^L$ whose product is $c = st$ is given by

$$\binom{a}{1} + (h-1)\binom{a+1}{1} = h(a+1) - 1,$$

which equals (E.1c) with $m = 2a + 1$, as desired.

E.1.2. *Cyclic sieving for* K_+. Given $mh - 2 = k \cdot \rho$, let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.1) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(I_2(h); q) = \mathrm{Cat}_+^{(m)}(I_2(h)), \qquad \text{if } \rho = 1, \tag{E.1e}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(I_2(h); q) = \mathrm{Cat}_+^{(m)}(I_2(h)), \qquad \text{if } h \equiv 0 \pmod 2,\ \rho = 2, \tag{E.1f}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(I_2(h); q) = \tfrac{mh}{2}, \qquad \text{if } m \not\equiv h \equiv 1 \pmod 2,\ \rho = 2, \tag{E.1g}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(I_2(h); q) = 1, \qquad \text{otherwise.} \tag{E.1h}$$

This situation can be treated using the obtained counting formula for $\widetilde{\mathsf{K}}_+$ in Appendix E.1.1: the cases $\rho = 1$ and $\rho = 2$ with $h \equiv 0 \pmod 2$ are clear. So we only need to consider the cases $\rho = 2$ for $h \equiv 1 \pmod 2$ and $\rho = 4$ for $h \equiv 0 \pmod 4$. But in both situations there is no single reflection that is invariant itself, so we have the same cases as in the previous situation with the parameter m shifted to $m - 1$, and with the empty factorisation (which is always invariant) added to both situations. In the case where $h \equiv 1 \pmod 2$ and $\rho = 4$, this is the shift from 0 (no invariant factorisation) to 1 (only the empty invariant factorisation). In the case where $h \equiv 0 \pmod 2$ and $\rho = 2$, this is the shift from $\frac{(m+1)h}{2} - 1$ (factorisation of length 2) to $\frac{mh}{2}$ (factorisation of length 2 plus the empty invariant factorisation).

E.2. **Cyclic sieving for the exceptional groups.** For our two positive Kreweras maps, we determine in detail the parameters in the considerations detailed at the end of Section 6. These are then used in the algorithm implemented in the computer algebra system SageMath[12].

We remark that we do not explicitly mention the *otherwise* case (being 0 for $\widetilde{\mathsf{K}}_+$ and 1 for K_+, respectively) in each situation, but emphasise that these have all been computationally verified. Moreover, we list the computed indices of invariant factorisations only in several small example situations.

E.2.1. *Cyclic sieving for* $\widetilde{\mathsf{K}}_+$. In this situation, we treat factorisations $\mathrm{r}_1 \dots \mathrm{r}_n$ of the bipartite Coxeter element $c = c_L c_R$ that are invariant under appropriate k^{th} powers of $\widetilde{\mathsf{K}}_+$ with $(m+1)h - 2 = k \cdot \rho$. From Theorem 2.17, we see that the order of $\widetilde{\mathsf{K}}_+$ equals $k \cdot \rho$ if $\psi \equiv \varepsilon$ and $k \cdot \rho/2$ if $\psi \not\equiv \varepsilon$. We can thus assume that ρ is even if $\psi \not\equiv \varepsilon$. Moreover, we have seen in Proposition 6.12 that we only need to consider parameters ρ such that ρ (if $\psi \not\equiv \varepsilon$) or, respectively, $\rho/2$ (if $\psi \equiv \varepsilon$) divides n.

Case H_3**.** The degrees are $2, 6, 10$, and $(m+1)h - 2 = 10m + 8 = k \cdot \rho$. We hence have

$$\mathrm{Cat}_+^{(m)}(H_3; q) = \frac{[10m+8]_q\,[10m+4]_q\,[10m]_q}{[10]_q\,[6]_q\,[2]_q}. \tag{E.2}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.2) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(H_3; q) = \mathrm{Cat}_+^{(m)}(H_3), \qquad \text{if } \rho \in \{1, 2\}, \tag{E.2a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(H_3; q) = \tfrac{5m+4}{3}, \qquad \text{if } m \equiv 1 \pmod 3,\ \rho \in \{3, 6\}, \tag{E.2b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(H_3; q) = 0, \qquad \text{otherwise.} \tag{E.2c}$$

We may assume that ρ is even. In this case $\rho/2$ divides $n = 3$, implying $\rho \in \{2, 6\}$.

The case $\rho = 2$ is computed using all ascent-descent configurations of the 50 factorisations of c, giving, with $m = a$,

$$21\binom{a+2}{3} + 28\binom{a+1}{3} + \binom{a}{3} = \mathrm{Cat}_+^{(m)}(H_3).$$

The case $\rho = 6$ is given by the parameters

$$y = 2, \quad m' = 1, \quad i_{\text{fix}} = 10.$$

There are 5 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. As positions of letters in $\mathbf{inv}_c$ (which lists the 15 reflections in the reflection ordering), these are given by

$$\big\{(2, 11, 5), (5, 14, 8), (8, 2, 11), (11, 5, 14), (14, 8, 2)\big\}.$$

[12]The computations are available upon request.

Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$3\binom{a+1}{1}+2\binom{a}{1}=5a+3,$$

With $m'=1$, we obtain $m=3a+1$, which, by Equation (E.2b), leads to

$$\tfrac{1}{3}(5m+4)=\tfrac{1}{3}(15a+9)=5a+3,$$

as desired.

Case H_4. The degrees are $2,12,20,30$, and $(m+1)h-2=30m+28=k\cdot\rho$. We hence have

$$\mathrm{Cat}_+^{(m)}(H_4;q)=\frac{[30m+28]_q\,[30m+18]_q\,[30m+10]_q\,[30m]_q}{[30]_q\,[20]_q\,[12]_q\,[2]_q}. \tag{E.3}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.3) occur:

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(H_4;q)=\mathrm{Cat}_+^{(m)}(H_4), \qquad \text{if } \rho\in\{1,2\}, \tag{E.3a}$$

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(H_4;q)=\tfrac{(15m+14)m}{4}, \qquad \text{if } m\equiv 0 \pmod 2,\ \rho=4, \tag{E.3b}$$

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(H_4;q)=0, \qquad \text{otherwise.} \tag{E.3c}$$

We may assume that ρ is even. In this case $\rho/2$ divides $n=4$, implying $\rho\in\{2,4,8\}$.

The case $\rho=2$ is computed using all ascent-descent configurations of the 1150 factorisations of c, giving, with $m=a$,

$$232\binom{a+3}{4}+842\binom{a+2}{4}+275\binom{a+1}{4}+\binom{a}{4}=\mathrm{Cat}_+^{(m)}(H_4).$$

The case $\rho=4$ is given by the parameters

$$y=2,\quad m'=0,\quad i_{\text{fix}}=29.$$

There are 30 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. As positions of letters in $\mathbf{inv}_c$ (which lists the 60 reflections in the reflection ordering), these are given by

$$\left\{\begin{array}{ccccc}
(4,44,32,12), & (4,52,32,20), & (8,48,36,16), & (8,56,36,24), & (12,52,40,20),\\
(12,60,40,28), & (16,4,44,32), & (16,56,44,24), & (20,8,48,36), & (20,60,48,28),\\
(24,4,52,32), & (24,12,52,40), & (28,8,56,36), & (28,16,56,44), & (32,12,60,40),\\
(32,20,60,48), & (36,16,4,44), & (36,24,4,52), & (40,20,8,48), & (40,28,8,56),\\
(44,24,12,52), & (44,32,12,60), & (48,28,16,56), & (48,36,16,4), & (52,32,20,60),\\
(52,40,20,8), & (56,36,24,4), & (56,44,24,12), & (60,40,28,8), & (60,48,28,16)
\end{array}\right\}$$

Their ascent-descent sequences imply that the number of $(\widetilde{\mathsf{K}}_+)^k$-invariant subwords as

$$22\binom{a+1}{2}+8\binom{a}{2}=(15a+7)a,$$

With $m' = 0$, we obtain $m = 2a$, which, by Equation (E.3b), leads to

$$\tfrac{1}{4}(15m + 14)m = \tfrac{1}{4}(30a + 14)2a = (15a + 7)a,$$

as desired.

The case $\rho = 8$ is given by the parameters

$$y = 2, \quad m' = 2, \quad i_{\text{fix}} = 45.$$

In this case, there are no factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12, in agreement with Equation (E.3c).

Case F_4. The degrees are $2, 6, 8, 12$, and $(m+1)h - 2 = 12m + 10 = k \cdot \rho$. We hence have

$$\mathrm{Cat}_+^{(m)}(F_4; q) = \frac{[12m+10]_q\,[12m+6]_q\,[12m+4]_q\,[12m]_q}{[12]_q\,[8]_q\,[6]_q\,[2]_q}. \tag{E.4}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.4) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(F_4; q) = \mathrm{Cat}_+^{(m)}(F_4), \qquad \text{if } \rho \in \{1, 2\}, \tag{E.4a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(F_4; q) = 0, \qquad \text{otherwise.} \tag{E.4b}$$

We may assume that ρ is even. In this case $\rho/2$ divides $n = 4$, implying $\rho \in \{2, 4, 8\}$.

The case $\rho = 2$ is computed using all ascent-descent configurations of the 432 factorisations of c, giving, with $m = a$,

$$66\binom{a+3}{4} + 265\binom{a+2}{4} + 100\binom{a+1}{4} + \binom{a}{4} = \mathrm{Cat}_+^{(m)}(F_4).$$

The cases $\rho \in \{4, 8\}$ reduce to Lemma 6.2(1).

Case E_6. The degrees are $2, 5, 6, 8, 9, 12$, and $(m+1)h - 2 = 12m + 10 = k \cdot \rho$. We hence have

$$\mathrm{Cat}_+^{(m)}(E_6; q) = \frac{[12m+10]_q\,[12m+7]_q\,[12m+6]_q\,[12m+4]_q\,[12m+3]_q\,[12m]_q}{[12]_q\,[9]_q\,[8]_q\,[6]_q\,[5]_q\,[2]_q}. \tag{E.5}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.5) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_6; q) = \mathrm{Cat}_+^{(m)}(E_6), \qquad \text{if } \rho = 1, \tag{E.5a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_6; q) = \tfrac{(6m+5)(3m+1)(2m+1)m}{2}, \qquad \text{if } \rho = 2, \tag{E.5b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_6; q) = 0, \qquad \text{otherwise.} \tag{E.5c}$$

In this case ρ divides $n = 6$, implying $\rho \in \{1, 2, 3, 6\}$.

The case $\rho = 1$ is computed using all ascent-descent configurations of the 41,472 factorisations of c, giving, with $m = a$,

$$418\binom{a+5}{6} + 8141\binom{a+4}{6} + 21308\binom{a+3}{6} + 10778\binom{a+2}{6} + 826\binom{a+1}{6} + \binom{a}{6} = \mathrm{Cat}_+^{(m)}(E_6).$$

The case $\rho = 2$ is given by the parameters

$$y = 2, \quad m' = 0, \quad i_{\mathrm{fix}} = 31.$$

There are 432 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. Their ascent-descent sequences imply that the number of $(\widetilde{\mathsf{K}}_+)^k$-invariant subwords as

$$66\binom{a+3}{4} + 265\binom{a+2}{4} + 100\binom{a+1}{4} + 1\binom{a}{4} = \tfrac{1}{2}(6a+5)(3a+1)(2a+1)a.$$

With $m' = 0$, we obtain $m = a$, which yields Equation (E.5b), as desired.

The cases $\rho \in \{3, 6\}$ reduce to Lemma 6.2(1).

Case E_7. The degrees are $2, 6, 8, 10, 12, 14, 18$, and $(m+1)h - 2 = 18m + 16 = k \cdot \rho$. We hence have

$$\mathrm{Cat}_+^{(m)}(E_7; q) = \frac{[18m+16]_q\,[18m+12]_q\,[18m+10]_q}{[18]_q\,[14]_q\,[12]_q} \times \frac{[18m+8]_q\,[18m+6]_q\,[18m+4]_q\,[18m]_q}{[10]_q\,[8]_q\,[6]_q\,[2]_q}. \tag{E.6}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.6) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_7; q) = \mathrm{Cat}_+^{(m)}(E_7), \qquad \text{if } \rho \in \{1, 2\}, \tag{E.6a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_7; q) = \tfrac{9m+8}{7}, \qquad \text{if } m \equiv 3 \pmod 7,\ \rho \in \{7, 14\}, \tag{E.6b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_7; q) = 0, \qquad \text{otherwise.} \tag{E.6c}$$

We may assume that ρ is even. In this case $\rho/2$ divides $n = 7$, implying $\rho \in \{2, 14\}$.

The case $\rho = 2$ is computed using all ascent-descent configurations of the 1,062,882 factorisations of c, giving, with $m = a$,

$$2431\binom{a+6}{7} + 85800\binom{a+5}{7} + 412764\binom{a+4}{7} + 446776\binom{a+3}{7} + 110958\binom{a+2}{7} + 4152\binom{a+1}{7} + \binom{a}{7} = \mathrm{Cat}_+^{(m)}(E_7).$$

The case $\rho = 14$ is given by the parameters

$$y = 2, \quad m' = 3, \quad i_{\text{fix}} = 36.$$

There are 9 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. As positions of letters in $\mathbf{inv}_c$ (which lists the 63 reflections in the reflection ordering), these are given by

$$\left\{ \begin{array}{lll} (7,42,14,49,21,56,28), & (14,49,21,56,28,63,35), & (21,56,28,63,35,7,42), \\ (28,63,35,7,42,14,49), & (35,7,42,14,49,21,56), & (42,14,49,21,56,28,63), \\ (49,21,56,28,63,35,7), & (56,28,63,35,7,42,14), & (63,35,7,42,14,49,21) \end{array} \right\}$$

Their ascent-descent sequences imply that the number of $(\widetilde{\mathsf{K}}_+)^k$-invariant subwords as

$$5\binom{a+1}{1} + 4\binom{a}{1} = 9a + 5,$$

With $m' = 3$, we obtain $m = 7a + 3$, which, by Equation (E.5b), leads to

$$\tfrac{1}{7}(9m + 8) = \tfrac{1}{7}(63a + 35) = 9a + 5,$$

as desired.

Case E_8. The degrees are $2, 8, 12, 14, 18, 20, 24, 30$, and $(m+1)h - 2 = 30m + 28 = k \cdot \rho$. We hence have

$$\mathrm{Cat}_+^{(m)}(E_8; q) = \frac{[30m+28]_q\,[30m+22]_q\,[30m+18]_q\,[30m+16]_q}{[30]_q\,[24]_q\,[20]_q\,[18]_q} \times \frac{[30m+12]_q\,[30m+10]_q\,[30m+6]_q\,[30m]_q}{[14]_q\,[12]_q\,[8]_q\,[2]_q}. \tag{E.7}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.7) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_8; q) = \mathrm{Cat}_+^{(m)}(E_8), \qquad \text{if } \rho \in \{1, 2\}, \tag{E.7a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_8; q) = \tfrac{(15m+14)(15m+8)(5m+2)m}{64}, \qquad \text{if } m \equiv 0 \pmod 2,\ \rho = 4, \tag{E.7b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_8; q) = \tfrac{(15m+14)(5m+2)}{16}, \qquad \text{if } m \equiv 2 \pmod 4,\ \rho = 8, \tag{E.7c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_8; q) = 0, \qquad \text{otherwise.} \tag{E.7d}$$

We may assume that ρ is even. In this case $\rho/2$ divides $n = 8$, implying $\rho \in \{2, 4, 8, 16\}$.

The case $\rho = 2$ is computed using all ascent-descent configurations of the 37,968,750 factorisations of c (which we omit here).

The case $\rho = 4$ is given by the parameters

$$y = 2, \quad m' = 0, \quad i_{\text{fix}} = 57.$$

There are 6,750 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. Their ascent-descent sequences imply that the number of $(\widetilde{\mathsf{K}}_+)^k$-invariant subwords as

$$627\binom{a+3}{4}+3784\binom{a+2}{4}+2251\binom{a+1}{4}+88\binom{a}{4}=\tfrac{1}{4}(15a+7)(15a+4)(5a+1)a.$$

With $m'=0$, we obtain $m=2a$, which, by Equation (E.7b), leads to

$$\begin{aligned}\tfrac{1}{64}(15m+14)(15m+8)(5m+2)m&=\tfrac{1}{64}(30a+14)(30a+8)(10a+2)2a\\&=\tfrac{1}{4}(15a+7)(15a+4)(5a+1)a,\end{aligned}$$

as desired.

The case $\rho=8$ is given by the parameters

$$y=2,\quad m'=2,\quad i_{\mathrm{fix}}=89.$$

There are 150 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. Their ascent-descent sequences imply that the number of $(\widetilde{\mathsf{K}}_+)^k$-invariant subwords as

$$33\binom{a+2}{4}+109\binom{a+1}{4}+8\binom{a}{4}=(15a+11)(5a+3).$$

With $m'=2$, we obtain $m=4a+2$, which, by Equation (E.7c), leads to

$$\tfrac{1}{16}(15m+14)(5m+2)=\tfrac{1}{16}(60a+44)(20a+12)=(15a+11)(5a+3),$$

as desired.

The case $\rho=16$ is given by the parameters

$$y=2,\quad m'=6,\quad i_{\mathrm{fix}}=105.$$

In this case, there are no factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12, in agreement with Equation (E.7d).

E.2.2. *Cyclic sieving for* K_+. In this situation, we treat factorisations $\mathrm{r}_1\dots\mathrm{r}_{n'}$ that can be extended to factorisations of the bipartite Coxeter element $c=c_Lc_R$ and that are invariant under appropriate k^{th} powers of K_+ with $(m+1)h-2=k\cdot\rho$. From Theorem 2.17, we see that the order of K_+ equals $k\cdot\rho$ if $\psi\equiv\varepsilon$ and $k\cdot\rho/2$ if $\psi\not\equiv\varepsilon$. We can thus assume that ρ is even if $\psi\not\equiv\varepsilon$. Moreover, we have seen in Proposition 6.12 that we need to consider all parameters ρ such that ρ (if $\psi\not\equiv\varepsilon$) or, respectively, $\rho/2$ (if $\psi\equiv\varepsilon$) divides some $n'\le n$, i.e., all parameters ρ such that ρ or, respectively, $\rho/2$ is less than or equal to n. Otherwise, there will only be the invariant trivial factorisation.

Case H_3. The degrees are $2, 6, 10$, and $mh - 2 = 10m - 2 = k \cdot \rho$. We hence have

$$\mathrm{Cat}_+^{(m)}(H_3; q) = \frac{[10m+8]_q\,[10m+4]_q\,[10m]_q}{[10]_q\,[6]_q\,[2]_q}. \tag{E.8}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.8) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(H_3; q) = \mathrm{Cat}_+^{(m)}(H_3), \qquad \text{if } \rho \in \{1, 2\}, \tag{E.8a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(H_3; q) = \tfrac{5m+2}{3}, \qquad \text{if } m \equiv 2 \pmod 3,\ \rho \in \{3, 6\}, \tag{E.8b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(H_3; q) = 1, \qquad \text{otherwise.} \tag{E.8c}$$

We may assume that ρ is even. In this case $\rho/2 \le n = 3$, implying $\rho \in \{2, 4, 6\}$.

The case $\rho = 2$ is computed using all ascent-descent configurations of the 116 factorisations that can be extended to factorisations of c, giving, with $m = a + 1$,

$$21\binom{a+2}{3} + 28\binom{a+1}{3} + \binom{a}{3} + 8\binom{a+2}{2} + 39\binom{a+1}{2} + 3\binom{a}{2} + 12\binom{a+1}{1} + 3\binom{a}{1} + \binom{a}{0} = \mathrm{Cat}_+^{(m)}(H_3).$$

The case $\rho = 4$ is given by the parameters

$$y = 2, \quad m' = 1, \quad i_{\text{fix}} = 7.$$

In this case, there is only the trivial factorisation that satisfies the properties in Proposition 6.12, in agreement with Equation (E.8c).

The case $\rho = 6$ is given by the parameters

$$y = 2, \quad m' = 2, \quad i_{\text{fix}} = 10.$$

There are 6 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. As positions of letters in $\mathbf{inv}_c$ (which lists the 15 reflections in the reflection ordering), these are given by the empty factorisation and the 5 factorisations of c listed in the above considerations for type H_3. Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$3\binom{a+1}{1} + 2\binom{a}{1} + \binom{a}{0} = 5a + 4$$

With $m' = 2$, we obtain $m = 3a + 2$, which, by Equation (E.8b), leads to

$$\tfrac{1}{3}(5m + 2) = \tfrac{1}{3}(15a + 12) = 5a + 4,$$

as desired.

Case H_4. The degrees are $2, 12, 20, 30$, and $mh - 2 = 30m - 2 = k \cdot \rho$. We hence have

$$\mathrm{Cat}_+^{(m)}(H_4; q) = \frac{[30m+28]_q\,[30m+18]_q\,[30m+10]_q\,[30m]_q}{[30]_q\,[20]_q\,[12]_q\,[2]_q}. \tag{E.9}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.9) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(H_4; q) = \mathrm{Cat}_+^{(m)}(H_4), \qquad \text{if } \rho \in \{1, 2\}, \tag{E.9a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(H_4; q) = \tfrac{(5m+3)(3m+1)}{4}, \qquad \text{if } m \equiv 1 \bmod 2,\ \rho = 4, \tag{E.9b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(H_4; q) = 1, \qquad \text{otherwise.} \tag{E.9c}$$

We may assume that ρ is even. In this case $\rho/2 \le n = 4$, implying $\rho \in \{2, 4, 6, 8\}$.

The case $\rho = 2$ is computed using all ascent-descent configurations of the 3,226 factorisations that can be extended to factorisations of c, giving, with $m = a + 1$,

$$\begin{aligned} 232\binom{a+3}{4} + 842\binom{a+2}{4} + 275\binom{a+1}{4} + \binom{a}{4} + 42\binom{a+3}{3} \\ + 760\binom{a+2}{3} + 544\binom{a+1}{3} + 4\binom{a}{3} + 133\binom{a+2}{2} + 326\binom{a+1}{2} \\ + 6\binom{a}{2} + 56\binom{a+1}{1} + 4\binom{a}{1} + \binom{a}{0} = \mathrm{Cat}_+^{(m)}(H_4). \end{aligned}$$

The case $\rho = 4$ is given by the parameters

$$y = 2, \quad m' = 1, \quad i_{\text{fix}} = 29.$$

There are 46 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. As positions of letters in $\mathbf{inv}_c$ (which lists the 60 reflections in the reflection ordering), these are given by the empty factorisation, the 30 factorisations of c listed in the above considerations for type H_4, and

$$\left\{ \begin{array}{ccccc} (4,32), & (8,36), & (12,40), & (16,44), & (20,48), \\ (24,52), & (28,56), & (32,60), & (36,4), & (40,8), \\ (44,12), & (48,16), & (52,20), & (56,24), & (60,28) \end{array} \right\}$$

Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$22\binom{a+1}{2} + 8\binom{a}{2} + 7\binom{a+1}{1} + 8\binom{a}{1} + \binom{a}{0} = (5a+4)(3a+2)$$

With $m' = 1$, we obtain $m = 2a + 1$, which, by Equation (E.9b), leads to

$$\tfrac{1}{4}(5m+3)(3m+1) = \tfrac{1}{4}(10a+8)(6a+4) = (5a+4)(3a+2),$$

as desired.

The case $\rho = 6$ reduces to Lemma 6.2(1).

The case $\rho = 8$ is given by the parameters

$$y = 2, \quad m' = 3, \quad i_{\text{fix}} = 45.$$

In this case, there is only the trivial factorisation that satisfies the properties in Proposition 6.12, in agreement with Equation (E.9c).

Case F_4. The degrees are $2, 6, 8, 12$, and $mh - 2 = 12m - 2 = k \cdot \rho$. We hence have

$$\operatorname{Cat}_+^{(m)}(F_4; q) = \frac{[12m+10]_q\,[12m+6]_q\,[12m+4]_q\,[12m]_q}{[12]_q\,[8]_q\,[6]_q\,[2]_q}. \tag{E.10}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.10) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}_+^{(m)}(F_4; q) = \operatorname{Cat}_+^{(m)}(F_4), \qquad \text{if } \rho \in \{1, 2\}, \tag{E.10a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}_+^{(m)}(F_4; q) = 1, \qquad \text{otherwise.} \tag{E.10b}$$

We may assume that ρ is even. In this case $\rho/2 \le n = 4$, implying $\rho \in \{2, 4, 6, 8\}$.

The case $\rho = 2$ is computed using all ascent-descent configurations of the 1,045 factorisations that can be extended to factorisations of c, giving, with $m = a + 1$,

$$\begin{aligned} 66\binom{a+3}{4} + 265\binom{a+2}{4} + 100\binom{a+1}{4} + \binom{a}{4} + 10\binom{a+3}{3} \\ + 224\binom{a+2}{3} + 194\binom{a+1}{3} + 4\binom{a}{3} + 35\binom{a+2}{2} + 115\binom{a+1}{2} \\ + 6\binom{a}{2} + 20\binom{a+1}{1} + 4\binom{a}{1} + \binom{a}{0} = \operatorname{Cat}_+^{(m)}(F_4). \end{aligned}$$

The cases $\rho \in \{4, 6, 8\}$ reduce to Lemma 6.2(1).

Case E_6. The degrees are $2, 5, 6, 8, 9, 12$, and $mh - 2 = 12m - 2 = k \cdot \rho$. We hence have

$$\operatorname{Cat}_+^{(m)}(E_6; q) = \frac{[12m+10]_q\,[12m+7]_q\,[12m+6]_q\,[12m+4]_q\,[12m+3]_q\,[12m]_q}{[12]_q\,[9]_q\,[8]_q\,[6]_q\,[5]_q\,[2]_q}. \tag{E.11}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.11) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}_+^{(m)}(E_6; q) = \operatorname{Cat}_+^{(m)}(E_6), \qquad \text{if } \rho = 1, \tag{E.11a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}_+^{(m)}(E_6; q) = \tfrac{(6m+5)(3m+1)(2m+1)m}{2}, \qquad \text{if } \rho = 2, \tag{E.11b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}_+^{(m)}(E_6; q) = \tfrac{12m+3}{5}, \qquad \text{if } m \equiv 1 \pmod 5,\ \rho = 5, \tag{E.11c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}_+^{(m)}(E_6; q) = 1, \qquad \text{otherwise.} \tag{E.11d}$$

In this case $\rho \le n = 6$, implying $\rho \in \{1, 2, 3, 4, 5, 6\}$.

The case $\rho = 1$ is computed using all ascent-descent configurations of the 104,653 factorisations that can be extended to factorisations of c, giving, with $m = a + 1$,

$$\begin{aligned}
&418\binom{a+5}{6} + 814\binom{a+4}{6} + 21308\binom{a+3}{6} + 10778\binom{a+2}{6} \\
&\quad + 826\binom{a+1}{6} + \binom{a}{6} + 7\binom{a+5}{5} + 2466\binom{a+4}{5} + 18258\binom{a+3}{5} \\
&\quad + 18272\binom{a+2}{5} + 2463\binom{a+1}{5} + 6\binom{a}{5} + 70\binom{a+4}{4} + 4039\binom{a+3}{4} \\
&\quad + 10375\binom{a+2}{4} + 278\binom{a+1}{4} + 15\binom{a}{4} + 175\binom{a+3}{3} + 2226\binom{a+2}{3} \\
&\quad + 1467\binom{a+1}{3} + 20\binom{a}{3} + 135\binom{a+2}{2} + 354\binom{a+1}{2} + 15\binom{a}{2} \\
&\quad + 30\binom{a+1}{1} + 6\binom{a}{1} + \binom{a}{0} = \mathrm{Cat}_+^{(m)}(E_7).
\end{aligned}$$

The case $\rho = 2$ is given by the parameters

$$y = 2, \quad m' = 1, \quad i_{\mathrm{fix}} = 31.$$

There are 1045 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$\begin{aligned}
&66\binom{a+3}{4} + 265\binom{a+2}{4} + 100\binom{a+1}{4} + \binom{a}{4} + 10\binom{a+3}{3} \\
&\quad + 224\binom{a+2}{3} + 194\binom{a+1}{3} + 4\binom{a}{3} + 35\binom{a+2}{2} + 115\binom{a+1}{2} \\
&\quad + 6\binom{a}{2} + 20\binom{a+1}{1} + 4\binom{a}{1} + \binom{a}{0} = \frac{(6a+11)(3a+4)(2a+3)(a+1)}{2}.
\end{aligned}$$

With $m' = 1$, we obtain $m = a + 1$, which, by Equation (E.11b), leads to

$$\tfrac{1}{2}(6m+5)(3m+1)(2m+1)m = \tfrac{1}{2}(6a+11)(3a+4)(2a+3)(a+1),$$

as desired.

The cases $\rho \in \{3, 4\}$ reduce to Lemma 6.2(1).

The case $\rho = 5$ is given by the parameters

$$y = 1, \quad m' = 1, \quad i_{\mathrm{fix}} = 7.$$

There are 13 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. As positions of letters in $\mathbf{inv}_c$ (which lists the 36 reflections in the

reflection ordering), these are given by the empty factorisation and

$$\left\{\begin{array}{cccc}(1,8,13,20,25), & (2,7,14,19,26), & (7,14,19,26,31), & (8,13,20,25,32),\\ (13,20,25,32,2), & (14,19,26,31,1), & (19,26,31,1,8), & (20,25,32,2,7),\\ (25,32,2,7,14), & (26,31,1,8,13), & (31,1,8,13,20), & (32,2,7,14,19)\end{array}\right\}$$

Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$2\binom{a+1}{1}+10\binom{a}{1}+\binom{a}{0}=12a+3.$$

With $m'=1$, we obtain $m=5a+1$, which, by Equation (E.11c), leads to

$$\tfrac{1}{5}(12m+3)=\tfrac{1}{5}(60a+15)=12a+3,$$

as desired.

The case $\rho=6$ reduces to Lemma 6.2(1).

Case E_7. The degrees are $2,6,8,10,12,14,18$, and $mh-2=18m-2=k\cdot\rho$. We hence have

$$\mathrm{Cat}_+^{(m)}(E_7;q)=\frac{[18m+16]_q\,[18m+12]_q\,[18m+10]_q}{[18]_q\,[14]_q\,[12]_q}\times\frac{[18m+8]_q\,[18m+6]_q\,[18m+4]_q\,[18m]_q}{[10]_q\,[8]_q\,[6]_q\,[2]_q}. \tag{E.12}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.12) occur:

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(E_7;q)=\mathrm{Cat}_+^{(m)}(E_7), \qquad \text{if } \rho\in\{1,2\}, \tag{E.12a}$$

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(E_7;q)=\tfrac{(9m+5)(3m+1)}{8}, \qquad \text{if } m\equiv 1 \pmod 2,\ \rho=4, \tag{E.12b}$$

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(E_7;q)=\tfrac{9m+3}{4}, \qquad \text{if } m\equiv 1 \pmod 4,\ \rho=8, \tag{E.12c}$$

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(E_7;q)=\tfrac{9m+4}{5}, \qquad \text{if } m\equiv 4 \pmod 5,\ \rho\in\{5,10\}, \tag{E.12d}$$

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(E_7;q)=\tfrac{9m+6}{7}, \qquad \text{if } m\equiv 4 \pmod 7,\ \rho\in\{7,14\}, \tag{E.12e}$$

$$\lim_{q\to\zeta}\mathrm{Cat}_+^{(m)}(E_7;q)=1, \qquad \text{otherwise.} \tag{E.12f}$$

We may assume that ρ is even. In this case $\rho/2\le n=7$, implying $\rho\in\{2,4,6,8,10,12,14\}$.

The case $\rho=2$ is computed using all ascent-descent configurations of the 2,702,332 factorisations that can be extended to factorisations of c (which we omit here).

The case $\rho=4$ is given by the parameters

$$y=2,\quad m'=1,\quad i_{\text{fix}}=29.$$

There are 37 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. As positions of letters in $\mathbf{inv}_c$ (which lists the 63 reflections in the reflection ordering), these are given by the empty factorisation and

$$\left\{\begin{array}{ccccc} (1,29),(8,36),(15,43),(22,50),(29,57),(36,1),(43,8),(50,15),(57,22), & & & & \\ (1,15,29,43), & (1,43,29,8), & (1,50,29,15), & (8,22,36,50), & (8,50,36,15), \\ (8,57,36,22), & (15,1,43,29), & (15,29,43,57), & (15,57,43,22), & (22,1,50,29), \\ (22,8,50,36), & (22,36,50,1), & (29,8,57,36), & (29,15,57,43), & (29,43,57,8), \\ (36,15,1,43), & (36,22,1,50), & (36,50,1,15), & (43,22,8,50), & (43,29,8,57), \\ (43,57,8,22), & (50,1,15,29), & (50,29,15,57), & (50,36,15,1), & (57,8,22,36), \\ & & (57,36,22,1),(57,43,22,8) & & \end{array}\right\}$$

Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$2\binom{a+2}{2}+20\binom{a+1}{2}+5\binom{a}{2}+4\binom{a+1}{1}+5\binom{a}{1}+\binom{a}{0}=\frac{(9a+7)(3a+2)}{2}.$$

With $m'=1$, we obtain $m=2a+1$, which, by Equation (E.12b), leads to

$$\tfrac{1}{8}(9m+5)(3m+1)=\tfrac{1}{8}(18a+14)(6a+4)=\tfrac{1}{2}(9a+7)(3a+2),$$

as desired.

The case $\rho=6$ reduce to Lemma 6.2(1).

The case $\rho=8$ is given by the parameters

$$y=2,\quad m'=1,\quad i_{\mathrm{fix}}=15.$$

There are 10 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. As positions of letters in $\mathbf{inv}_c$ (which lists the 63 reflections in the reflection ordering), these are given by the empty factorisation and

$$\left\{\begin{array}{ccc} (1,15,29,43), & (8,22,36,50), & (15,29,43,57), \\ (22,36,50,1), & (29,43,57,8), & (36,50,1,15), \\ (43,57,8,22), & (50,1,15,29), & (57,8,22,36) \end{array}\right\}$$

Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$2\binom{a+1}{1}+7\binom{a}{1}+\binom{a}{0}=9a+3.$$

With $m'=1$, we obtain $m=4a+1$, which, by Equation (E.12c), leads to

$$\tfrac{1}{4}(9m+3)=\tfrac{1}{4}(36a+12)=9a+3,$$

as desired.

The case $\rho=10$ is given by the parameters

$$y=2,\quad m'=4,\quad i_{\mathrm{fix}}=50.$$

There are 10 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. As positions of letters in $\mathbf{inv}_c$ (which lists the 63 reflections in the reflection ordering), these are given by the empty factorisation and

$$\left\{\begin{array}{ccc}(7,56,42,28,14), & (14,63,49,35,21), & (21,7,56,42,28),\\ (28,14,63,49,35), & (35,21,7,56,42), & (42,28,14,63,49),\\ (49,35,21,7,56), & (56,42,28,14,63), & (63,49,35,21,7)\end{array}\right\}$$

Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$7\binom{a+1}{1}+2\binom{a}{1}+\binom{a}{0}=9a+8.$$

With $m'=4$, we obtain $m=5a+4$, which, by Equation (E.12d), leads to

$$\tfrac{1}{5}(9m+4)=\tfrac{1}{5}(45a+40)=9a+8,$$

as desired.

The cases $\rho=12$ reduces to Lemma 6.2(1).

The case $\rho=14$ is given by the parameters

$$y=2,\quad m'=4,\quad i_{\mathrm{fix}}=36.$$

There are 10 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. As positions of letters in $\mathbf{inv}_c$ (which lists the 63 reflections in the reflection ordering), these are given by the empty factorisation and

$$\left\{\begin{array}{ccc}(7,42,14,49,21,56,28), & (14,49,21,56,28,63,35), & (21,56,28,63,35,7,42),\\ (28,63,35,7,42,14,49), & (35,7,42,14,49,21,56), & (42,14,49,21,56,28,63),\\ (49,21,56,28,63,35,7), & (56,28,63,35,7,42,14), & (63,35,7,42,14,49,21)\end{array}\right\}$$

Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$5\binom{a+1}{1}+4\binom{a}{1}+\binom{a}{0}=9a+6.$$

With $m'=4$, we obtain $m=7a+4$, which, by Equation (E.12e), leads to

$$\tfrac{1}{7}(9m+6)=\tfrac{1}{7}(63a+42)=9a+6,$$

as desired.

Case E_8. The degrees are $2,8,12,14,18,20,24,30$, and $mh-2=30m-2=k\cdot\rho$. We hence have

$$\mathrm{Cat}_+^{(m)}(E_8;q)=\frac{[30m+28]_q\,[30m+22]_q\,[30m+18]_q\,[30m+16]_q}{[30]_q\,[24]_q\,[20]_q\,[18]_q}\\ \times\frac{[30m+12]_q\,[30m+10]_q\,[30m+6]_q\,[30m]_q}{[14]_q\,[12]_q\,[8]_q\,[2]_q}. \quad \text{(E.13)}$$

Let ζ be a primitive ρ^{th} root of unity. The following cases on the right-hand side of (E.13) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_8;q) = \mathrm{Cat}_+^{(m)}(E_8), \qquad \text{if } \rho\in\{1,2\}, \tag{E.13a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_8;q) = \tfrac{(15m+11)(5m+3)(5m+1)(3m+1)}{64}, \qquad \text{if } m\equiv 1 \pmod 2,\ \rho=4, \tag{E.13b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_8;q) = \tfrac{(15m+11)(5m+1)}{16}, \qquad \text{if } m\equiv 3 \pmod 4,\ \rho=8, \tag{E.13c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_8;q) = \tfrac{15m+6}{7}, \qquad \text{if } m\equiv 1 \pmod 7,\ \rho\in\{7,14\}, \tag{E.13d}$$

$$\lim_{q\to\zeta} \mathrm{Cat}_+^{(m)}(E_8;q) = 1, \qquad \text{otherwise.} \tag{E.13e}$$

We may assume that ρ is even. In this case $\rho/2\le n=8$, implying $\rho\in\{2,4,6,8,10,12,14,16\}$.

The case $\rho=2$ is computed using all ascent-descent configurations of the 97,126,171 factorisations that can be extended to factorisations of c (which we omit here).

The case $\rho=4$ is given by the parameters

$$y=2,\quad m'=1,\quad i_{\text{fix}}=57.$$

There are 10,771 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$\begin{aligned}
&627\binom{a+3}{4}+3784\binom{a+2}{4}+2251\binom{a+1}{4}+88\binom{a+0}{4}\\
&\quad+14\binom{a+3}{3}+1198\binom{a+2}{3}+1987\binom{a+1}{3}+176\binom{a+0}{3}\\
&\quad+42\binom{a+2}{2}+446\binom{a+1}{2}+112\binom{a+0}{2}+21\binom{a+1}{1}\\
&\quad+24\binom{a+0}{1}+1\binom{a+0}{0}=\tfrac14(15a+13)(5a+4)(5a+3)(3a+2).
\end{aligned}$$

With $m'=1$, we obtain $m=2a+1$, which, by Equation (E.13b), leads to

$$\begin{aligned}
\tfrac{1}{64}(15m+11)(5m+3)(5m+1)(3m+1)&=\tfrac{1}{64}(30a+26)(10a+8)(10a+6)(6a+4)\\
&=\tfrac14(15a+13)(5a+4)(5a+3)(3a+2),
\end{aligned}$$

as desired.

The case $\rho=6$ reduce to Lemma 6.2(1).

The case $\rho=8$ is given by the parameters

$$y=2,\quad m'=3,\quad i_{\text{fix}}=89.$$

There are 181 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$33\binom{a+2}{2} + 109\binom{a+1}{2} + 8\binom{a+0}{2} + 22\binom{a+1}{1} + 8\binom{a+0}{1} + 1\binom{a+0}{0} = (15a+14)(5a+4).$$

With $m' = 3$, we obtain $m = 4a + 3$, which, by Equation (E.13c), leads to

$$\tfrac{1}{16}(15m+11)(5m+1) = \tfrac{1}{16}(60a+56)(20a+16) = (15a+14)(5a+4),$$

as desired.

The cases $\rho \in \{10, 12\}$ reduce to Lemma 6.2(1).

The case $\rho = 14$ is given by the parameters

$$y = 2, \quad m' = 1, \quad i_{\text{fix}} = 17.$$

There are 16 factorisations of the bipartite Coxeter element that satisfy the properties in Proposition 6.12. As positions of letters in $\mathbf{inv}_c$ (which lists the 120 reflections in the reflection ordering), these are given by the empty factorisation and

$$\left\{ \begin{array}{ccc} (4,20,36,52,68,84,100), & (12,28,44,60,76,92,108), & (20,36,52,68,84,100,116), \\ (28,44,60,76,92,108,4), & (36,52,68,84,100,116,12), & (44,60,76,92,108,4,20), \\ (52,68,84,100,116,12,28), & (60,76,92,108,4,20,36), & (68,84,100,116,12,28,44), \\ (76,92,108,4,20,36,52), & (84,100,116,12,28,44,60), & (92,108,4,20,36,52,68), \\ (100,116,12,28,44,60,76), & (108,4,20,36,52,68,84), & (116,12,28,44,60,76,92) \end{array} \right\}$$

Their ascent-descent sequences imply that the number of $(\mathsf{K}_+)^k$-invariant subwords as

$$2\binom{a+1}{1} + 13\binom{a}{1} + \binom{a}{0} = 15a + 3.$$

With $m' = 1$, we obtain $m = 7a + 1$, which, by Equation (E.13d), leads to

$$\tfrac{1}{7}(15m+6) = \tfrac{1}{7}(105a+21) = 15a+3,$$

as desired.

The case $\rho = 16$ is given by the parameters

$$y = 2, \quad m' = 7, \quad i_{\text{fix}} = 105.$$

In this case, there is only the trivial factorisation that satisfies the properties in Proposition 6.12, in agreement with Equation (E.13e).

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