A BIJECTION BETWEEN PROCTOR'S AND SUNDARAM'S ODD ORTHOGONAL TABLEAUX

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Abstract. An explicit bijection between Proctor's odd orthogonal tableaux and Sundaram's odd orthogonal tableaux is given.

1. Introduction

Over the last twenty years many enumerative combinatorialists have been interested in identities which count the number of tableaux or plane partitions of a given type. Sometimes two families of tableaux or plane partitions have the same counts. But it seems to be very rare to find bijections between two such families. In this paper we use the Garsia–Milne technique to generate a bijection in such a situation.

This situation is the following. Proctor [10, 11] and Sundaram [14] defined *different* tableaux that both enumerate the odd orthogonal characters. To be more precise, the generating function with respect to a certain weight for either Proctor's tableaux of shape λ or Sundaram's tableaux of shape λ equals the irreducible character of $SO\,(2n+1)$ indexed by λ. Since the generating functions are equal, there must exist a weight-preserving bijection between the two sets of tableaux. The purpose of this paper is to construct an *explicit* such bijection.

Our bijection bases on the fact that both types of tableaux "generate" Jacobi–Trudi-type identities for their generating functions, as shown in [2]. In fact, the bijection combines mappings from [2] with the involution principle of Garsia and Milne [3].

We remark that lately there has been considerable interest in combinatorial descriptions of classical group characters. Such descriptions evolved mainly from either algebraic combinatorics or algebraic geometry, culminating in Proctor's [11] and Littelmann's [8] papers, respectively, which can also serve as reference papers. However,

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the relations between the various combinatorial descriptions that have been developed are in most cases unclear, except for the general linear characters where all descriptions are more or less the same. In this paper we relate two combinatorial descriptions for orthogonal characters which come from the algebraic combinatorics side. Recently, Sheats [12] related the symplectic tableaux of King and El-Sharkaway [6] and the symplectic tableaux of DeConcini [1], the first coming from the algebraic combinatorics side, the second coming from the algebraic geometry side. But there is still much more work to be done to understand the relations, in particular between the descriptions that come from different sides.

Our paper is organized as follows. In section 2 we define the odd orthogonal tableaux of Proctor and Sundaram and their weights. Since we use the involution principle we briefly review it in section 3. Then, in section 4 we give a rough outline of our bijection. The details of the underlying involutions are provided in section 5. An elaborate example is the subject of section 6. As indicated at the beginning, the ideas behind the involutions in section 5 stem from lattice path proofs for Jacobi–Trudi-type identities for the irreducible characters of $SO(2n+1)$ given in [2]. This background is explained in section 7. It may be helpful to read this section immediately after having digested what Proctor tableaux and Sundaram tableaux are and how the Garsia–Milne involution principle works, since it gives an idea of the "program" for our bijection, not requiring that the reader is familiar with [2]. Finally, in section 8 we comment on bijections between some more odd orthogonal tableaux, due to Proctor [11] and King and Welsh [7].

2. Proctor's and Sundaram's odd orthogonal tableaux

For readers convenience, we recall some basic concepts. An r-tuple $\lambda = (\lambda_1, \dots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ is called a *partition of length* r. We denote the length of the partition λ by $\ell(\lambda)$. The *Ferrers board* of λ is an array of cells with $\ell(\lambda)$ left-justified rows and λ_i cells in row i. The *conjugate of* λ is the partition $(\lambda'_1, \dots, \lambda'_{\lambda_1})$ where λ'_j is the length of the j-th column in the Ferrers board of λ.

We shall consider fillings of the cells of a Ferrers board with positive integers. Given a filling T we write $T_{i,j}$ for the entry in cell (i,j). Also, we write $T_{.,j}$ for the entries in the j-th column of T. The notation $T_{.,j}$ has to be understood in a suggestive sense, for instance, $\{T_{.,j} \leq p\}$ means the multiset of all entries in the j-th column of T that are at most p.

2.1. Sundaram tableaux. Now let λ be a partition with $\ell(\lambda) \leq n+1$. A $(2n+1)$-*Sundaram tableau of shape* λ is a filling T of the cells of the Ferrers board of λ with elements from $\{1, 2, \dots, 2n, \infty\}$ such that

(1) entries along rows are weakly increasing, each row is allowed to contain only at most one entry ∞,
(2) entries along columns are weakly increasing, and strictly increasing with respect to the finite entries,

(3) all entries in row i are at least $2i-1$.

Obviously it suffices to require (3) to hold for the first element in each row only, since rows are weakly increasing. So (3) is equivalent with

$$T_{i,1} \geq 2i-1 \quad \text{for } 1 \leq i \leq \ell(\lambda). \tag{2.1}$$

Figure 1 shows an example of a 9-Sundaram tableau of shape $(5,4,3,1,1)$.

Example of a 9-Sundaram tableau

1	4	4	6	7
4	5	5	6	
5	7	∞		
∞				
∞				

Figure 1

2.2. Proctor tableaux. Let λ be a partition. A $(2n+1)$-*Proctor tableau of shape* λ is a filling of the cells of the Ferrers board of λ with elements from $\{1,2,\dots,2n,2n+1\}$ such that

(1) entries along rows are weakly increasing,
(2) entries along columns are strictly increasing,
(3) the $(2c)$-th orthogonal condition is satisfied for all c (see below), and the $(2n+1)$-st orthogonal condition is satisfied,
(4) the $(2c)$-th protection condition is satisfied for all c (see below).

A filling T *satisfies the p-th orthogonal condition* if the number of entries less or equal than p in the *first two* columns of T is less or equal to p. Stated more formally, this is

$$\left|\{T_{\cdot,1} \leq p\}\right| + \left|\{T_{\cdot,2} \leq p\}\right| \leq p, \tag{2.2}$$

where $|S|$ denotes the cardinality of a multiset S, as usual.

A filling T *satisfies the $(2c)$-th protection condition* if whenever $i+j=2c$ and $T_{i,1}=2c-1$ and $T_{j,2}=T_{j,3}=\cdots=T_{j,h-1}=2c-1$ and $T_{j,h}=2c$, then we have $T_{j-1,h}=2c-1$. Stated in a more formal manner, this is

$$\begin{aligned} &i+j=2c \text{ and } T_{i,1}=2c-1\\ &\qquad \text{and } T_{j,2}=\cdots=T_{j,h-1}=2c-1 \text{ (for } h\geq 2)\\ &\qquad \text{and } T_{j,h}=2c\\ &\text{implies } T_{j-1,h}=2c-1. \end{aligned} \tag{2.3}$$

The filling in Figure 2 satisfies the 6-th protection condition if and only if 5 is inserted instead of $*$ (choose $i=4$ and $j=2$). In fact, if $*=5$ then it is a 9-Proctor tableau of shape $(5,4,3,1)$.

Example of a 9-Proctor tableau, provided $*=5$

1	4	4	$*$	6
2	5	5	6	
3	7	8		
5				

Figure 2

Note that by the $(2n+1)$-st orthogonal condition for every $(2n+1)$-Proctor tableau of shape λ we must have $\lambda_1'+\lambda_2'\le 2n+1$.

2.3. Odd orthogonal characters. For both types of $(2n+1)$-orthogonal tableaux just introduced we define weights in the following way (the definition is the same for both types):

$$w\left(T\right):=\prod_{i=1}^{n} x_i^{\#(2i-1\in T)-\#(2i\in T)}. \tag{2.4}$$

With this definition of weights, the irreducible character of the orthogonal group $SO\left(2n+1\right)$ indexed by λ is given as

$$so_{2n+1}\left(\lambda,\mathbf{x}\right)=\sum_{T} w\left(T\right), \tag{2.5}$$

where the sum is over all $(2n+1)$-Proctor tableaux T of shape λ, or *equivalently* over all $(2n+1)$-Sundaram tableaux T of shape λ. Hence there exists a weight-preserving bijection between odd orthogonal Proctor tableaux and odd orthogonal Sundaram tableaux of the same shape. The purpose of this work is to construct such a bijection explicitly.

3. The involution principle of Garsia and Milne

Since we are going to rely on the involution principle of Garsia and Milne [3] (see also [13, sec. 4.6]), we recall it first, for convenience.

Suppose we are given two finite sets $\mathbb{S}$ and $\mathbb{P}$ with *signed* weight functions $w_{\mathbb{S}}$, $w_{\mathbb{P}}$, respectively. Denote the subsets of positively/negatively signed elements of $\mathbb{S}$ and $\mathbb{P}$ by $\mathbb{S}^+/\mathbb{S}^-$ and $\mathbb{P}^+/\mathbb{P}^-$, respectively. Let there be sets $G_{\mathbb{S}}$ and $G_{\mathbb{P}}$ of "good" elements contained in the respective positive sets, i.e. $G_{\mathbb{S}}\subseteq\mathbb{S}^+$ and $G_{\mathbb{P}}\subseteq\mathbb{P}^+$. Denote the complementary sets of "bad" elements by $B_{\mathbb{S}}:=\mathbb{S}\setminus G_{\mathbb{S}}$ and $B_{\mathbb{P}}:=\mathbb{P}\setminus G_{\mathbb{P}}$. Suppose there is a *weight- and sign-preserving* bijection $\mathfrak{a}$ between $\mathbb{S}$ and $\mathbb{P}$, $\mathfrak{a}:\mathbb{S}\mapsto\mathbb{P}$. Furthermore, suppose that there are *weight-preserving but sign-reversing involutions*

$\mathfrak{i}_{\mathbb{S}}$ and $\mathfrak{i}_{\mathbb{P}}$ on the respective sets of bad elements, $\mathfrak{i}_{\mathbb{S}} : B_{\mathbb{S}} \mapsto B_{\mathbb{S}}$ and $\mathfrak{i}_{\mathbb{P}} : B_{\mathbb{P}} \mapsto B_{\mathbb{P}}$. (One can think of these involutions as "cancelling" all the bad elements.) Then there must be a weight-preserving bijection between the sets $G_{\mathbb{S}}$ and $G_{\mathbb{P}}$ of respective good elements. And an explicit such bijection, g say, can be constructed in the following algorithmic way. Start with some element $x \in G_{\mathbb{S}}$. Then $g(x)$ is given by $(\mathfrak{a} \circ \mathfrak{i}_{\mathbb{S}} \circ \mathfrak{a}^{-1} \circ \mathfrak{i}_{\mathbb{P}})^n \circ \mathfrak{a}(x)$, for some $n \geq 0$ (depending on x). To be precise, n is the minimal nonnegative integer such that $(\mathfrak{a} \circ \mathfrak{i}_{\mathbb{S}} \circ \mathfrak{a}^{-1} \circ \mathfrak{i}_{\mathbb{P}})^n \circ \mathfrak{a}(x)$ is in $G_{\mathbb{P}}$. (In fact, n is unique, since $\mathfrak{i}_{\mathbb{P}}$ is not defined for an element of $G_{\mathbb{P}}$.) For a proof see [3] or [13, sec. 4.6].

Illustration of involution principle

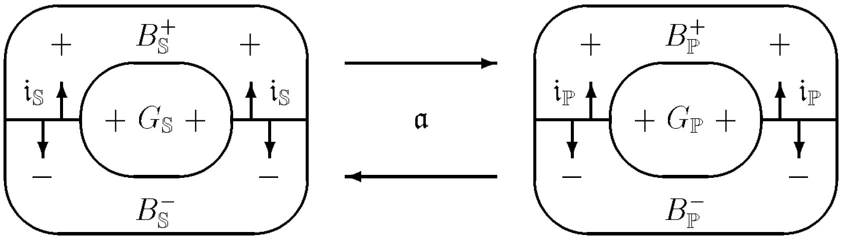

Figure 3

4. Outline of the bijection

The reason that our bijection is not straight-forward but employs the involution principle, is that $(2n+1)$-Proctor tableaux and $(2n+1)$-Sundaram tableaux are rather "far away from one another". The first obvious difference is the set of entries. This is, of course, very easy to remove. Just replace $2n+1$ by ∞ in every $(2n+1)$-Proctor tableau. Since neither entries $2n+1$ nor entries ∞ do contribute to the tableau's weight, this certainly is a weight-preserving conversion. From now on we shall consider these converted tableaux and, by abuse of terminology, call them also $(2n+1)$-Proctor tableaux.

In the following, let λ be a fixed partition with $\lambda_1' + \lambda_2' \leq 2n+1$ and $\ell(\lambda) \leq n+1$. (Recall that $\lambda_1' + \lambda_2' \leq 2n+1$ is needed to have well-defined $(2n+1)$-Proctor tableaux, and $\ell(\lambda) \leq n+1$ is needed to have well-defined $(2n+1)$-Sundaram tableaux.) Basically, in view of section 3, the only thing we have to do is to specify the sets $\mathbb{S}$, $\mathbb{P}$, their weights $w_{\mathbb{S}}$, $w_{\mathbb{P}}$, the sets $G_{\mathbb{S}}$, $G_{\mathbb{P}}$ of good elements, the bijection $\mathfrak{a}$, and the involutions $\mathfrak{i}_{\mathbb{S}}$, $\mathfrak{i}_{\mathbb{P}}$ on the sets of bad elements.

Of course, the sets $G_{\mathbb{S}}$ and $G_{\mathbb{P}}$ of good elements are the set of $(2n+1)$-Sundaram tableaux of shape λ and the set of $(2n+1)$-Proctor tableaux of shape λ, respectively. Next we have to embed $G_{\mathbb{S}}$ and $G_{\mathbb{P}}$ into the bigger sets $\mathbb{S}$ and $\mathbb{P}$. To describe these sets we have to set up some more terminology.

Consider our partition λ and think of the Ferrers board of λ in the following way: The lower edge forms a sequence of λ_1 "steps". Columns may be viewed as "heaps of cells" stacked on top of each "step". The number of cells in the heap stacked on the i-th step is precisely λ_i' (by definition the length of column i). Now define a *generalized Ferrers board of* λ to consist of λ_1 columns *of arbitrary non-negative length*, stacked on the "steps" of the Ferrers board of λ. Figure 4 gives an example of this concept, where $\lambda = (4, 3, 2)$. Figure 4.a shows the Ferrers board of $(4, 3, 2)$, while Figure 4.b shows a generalized Ferrers board of $(4, 3, 2)$ with column-lengths $1, 5, 1, 0$. By the rows of such a generalized Ferrers board of λ we mean the rows (in the usual sense) that contain at least one cell of the board (but may contain "holes", too). The bottom-most row gets number λ_1', the other rows are numbered "from bottom to top", as indicated in Figure 4.

a. Ferrers board of $(4, 3, 2)$ b. A generalized Ferrers board of $(4, 3, 2)$

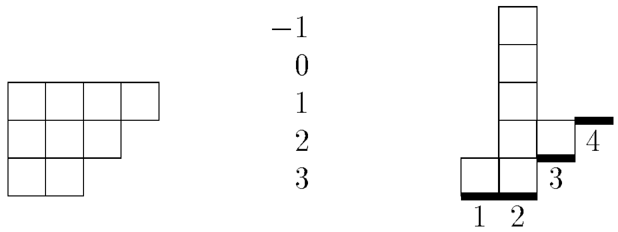

Figure 4

Let $\sigma \in \mathfrak{S}_{\lambda_1}$ ($\mathfrak{S}_{\lambda_1}$ denotes the symmetric group of order λ_1) be an arbitrary permutation and $\varepsilon \in \{-1, +1\}^{\lambda_1}$ be an arbitrary vector of $+1$'s and -1's. The *generalized Ferrers board with parameters* $(\lambda, \sigma, \varepsilon)$ is the generalized Ferrers board F of λ where the length of the j-th column of F, denoted by $\mathrm{cl}_j(F)$, is given by

$$\mathrm{cl}_j(F) = \lambda_j' - j + \sigma(j) \cdot \varepsilon_{\sigma(j)}. \tag{4.1}$$

An example with $\lambda = (4, 3, 2)$, $\sigma = 1423$, and $\varepsilon = (-1, 1, 1, 1)$ is displayed in Figure 4.b. Note that for $\sigma = \mathrm{id}$ and $\varepsilon = (1, 1 \dots, 1)$ the generalized Ferrers board with parameters $(\lambda, \sigma, \varepsilon)$ is precisely the (ordinary) Ferrers board of λ.

Our next definition concerns fillings of the cells of generalized Ferrers boards. A $(2n + 1)$-*tabloid* T *with parameters* $(\lambda, \sigma, \varepsilon)$ (we shall frequently write $T(\lambda, \sigma, \varepsilon)$ for short) is a filling of the cells of the generalized Ferrers board with parameters $(\lambda, \sigma, \varepsilon)$ with elements from $\{1, 2 \dots, 2n, \infty\}$ such that columns are weakly increasing, and *strictly* increasing with respect to the finite entries. So, entry ∞ may occur more than once in any column, occupying the bottom entries. When we display a tabloid we indicate the corresponding permutation σ by writing $\sigma(j)$ beneath the j-th column, and the corresponding sign vector ε by writing the sign of $\varepsilon_{\sigma(j)}$ beneath the j-th column. For example, the second tabloid in the second row of Figure 9 is a tabloid with parameters $\big((3, 3, 2, 2, 1, 1), 132, (-1, +1, +1)\big)$.

The *signed weight* of a tabloid $T(\lambda, \sigma, \varepsilon)$ is defined by

$$w(T(\lambda, \sigma, \varepsilon)) := \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\varepsilon) \cdot \prod_{i=1}^{n} x_i^{\#(2i-1 \in T) - \#(2i \in T)}, \tag{4.2}$$

where $\operatorname{sgn}(\varepsilon) := \prod_{k=1}^{\lambda_1} \varepsilon_k$.

Now we are in the position to say what the sets $\mathbb{S}$ and $\mathbb{P}$ are. We define the set $\mathbb{S}$, and also the set $\mathbb{P}$, to be the set of all $(2n+1)$-tabloids $T(\lambda, \sigma, \varepsilon)$, where σ varies over $\mathfrak{S}_{\lambda_1}$ and ε varies over $\{-1, +1\}^{\lambda_1}$ arbitrarily. For notational reasons we consider $\mathbb{S}$ and $\mathbb{P}$ as two different sets. Observe that the set $G_{\mathbb{S}}$ of all $(2n+1)$-Sundaram tableaux of shape λ, viewed as tabloids with parameters $\sigma = \mathrm{id}$ and $\varepsilon = (1, 1, \ldots, 1)$, is indeed contained in $\mathbb{S}$, as well as the set $G_{\mathbb{P}}$ of $(2n+1)$-Proctor tableaux of shape λ, viewed as tabloids with parameters $\sigma = \mathrm{id}$ and $\varepsilon = (1, 1, \ldots, 1)$, is contained in $\mathbb{P}$. Of course, the weights $w_{\mathbb{S}}$ and $w_{\mathbb{P}}$ are given by w in (4.2).

Since $\mathbb{S}$ and $\mathbb{P}$ are the same set, we do not have to say much about the bijection $\mathfrak{a}$. It is simply the identity mapping. Note that this reduces the bijection g given by the involution principle to $x \mapsto (\mathfrak{i}_{\mathbb{S}} \circ \mathfrak{i}_{\mathbb{P}})^n(x)$, for some n depending on x. In plain words, $\mathfrak{i}_{\mathbb{P}}$ and $\mathfrak{i}_{\mathbb{S}}$ are applied alternatively to x, as long as possible.

Finally we have to specify the involutions $\mathfrak{i}_{\mathbb{S}}$ and $\mathfrak{i}_{\mathbb{P}}$ on the sets $B_{\mathbb{S}} = \mathbb{S} \setminus G_{\mathbb{S}}$ and $B_{\mathbb{P}} = \mathbb{P} \setminus G_{\mathbb{P}}$ of bad elements, respectively. This is done in full detail in the next section. Actually, we partition $B_{\mathbb{S}}$ into two disjoint subsets $B_{\mathbb{S}}^1$ and $B_{\mathbb{S}}^2$ and construct involutions $\mathfrak{i}_{\mathbb{S}}^1$ and $\mathfrak{i}_{\mathbb{S}}^2$ for each of these. The involution $\mathfrak{i}_{\mathbb{S}}$ then is simply the union of these two "subinvolutions". Similarly, we partition $B_{\mathbb{P}}$ into three disjoint subsets $B_{\mathbb{P}}^1$, $B_{\mathbb{P}}^2$, $B_{\mathbb{P}}^3$ and construct involutions $\mathfrak{i}_{\mathbb{P}}^1$, $\mathfrak{i}_{\mathbb{P}}^2$, $\mathfrak{i}_{\mathbb{P}}^3$ for each of these. The involution $\mathfrak{i}_{\mathbb{P}}$ then is the union of these three "subinvolutions".

5. The involutions

We start with the involution $\mathfrak{i}_{\mathbb{S}} : B_{\mathbb{S}} \mapsto B_{\mathbb{S}}$, since it is much easier to describe.

5.1. THE INVOLUTION $\mathfrak{i}_{\mathbb{S}}$. As promised in the previous section, we partition $B_{\mathbb{S}}$ into two disjoint subsets $B_{\mathbb{S}}^1$ and $B_{\mathbb{S}}^2$, and construct involutions $\mathfrak{i}_{\mathbb{S}}^1$ and $\mathfrak{i}_{\mathbb{S}}^2$ on each of these. To define $B_{\mathbb{S}}^1$ and $B_{\mathbb{S}}^2$ we introduce two types of *violations*. Then, given some tabloid in $B_{\mathbb{S}}$, we look at all the violations in the tabloid (there will be at least one), and pick up the "minimal" one (in a certain sense). If this violation is of the first type then the tabloid belongs to $B_{\mathbb{S}}^1$, otherwise it belongs to $B_{\mathbb{S}}^2$.

The first type of violation stems from (2.1). We define a *Sundaram violation in row i* of a $(2n+1)$-tabloid T to be

$$T_{i,1} < 2i - 1 \text{ or cell } (i, 1) \text{ does not belong to } T\ . \tag{5.1}$$

To motivate the second type of violation, recall that for Sundaram tabloids we required that columns are weakly increasing, and *strictly* increasing with respect to the finite entries (so that entry ∞ is allowed to occur more than once in any column). However, we did not say anything about the rows. So, tabloids might very well have nonincreasing rows or more than one entry ∞ in some row, both of which is forbidden for Sundaram tableaux. This leads us to define a *violation of row monotonicity in row i and column j* of a $(2n+1)$-tabloid T to be the following:

(1) cell (i,j) belongs to T, $j<\lambda_1$ and $i\le\lambda'_{j+1}$,
(2) either $T_{i,j}>T_{i,j+1}$,
or $T_{i,j}=T_{i,j+1}=\infty$,
or cell $(i,j+1)$ does not belong to T.

Now, given a $(2n+1)$-tabloid T in $B_{\mathbb{S}}$, choose the maximal row, row I say, containing either a Sundaram or a row monotonicity violation, and choose the minimal column, column J say, which contains a violation in row I. (Of course, Sundaram violations always occur in column 1.) If in this row and column we have a Sundaram violation, then T belongs to $B^1_{\mathbb{S}}$, otherwise T belongs to $B^2_{\mathbb{S}}$.

There is one thing to check: We must show that *all* tabloids in $B_{\mathbb{S}}$ do in fact contain either a Sundaram or a row monotonicity violation. Note that the tabloids in $G_{\mathbb{S}}=\mathbb{S}\setminus B_{\mathbb{S}}$ (basically the Sundaram tableaux) all have $\lambda=\mathrm{id}$ and $\varepsilon=(1,1,\dots,1)$ and no monotonicity violations and no Sundaram violations. Therefore it suffices to show that there are *no* tabloids in $\mathbb{S}$ with $\lambda\ne\mathrm{id}$ *or* $\varepsilon\ne(1,1,\dots,1)$ that *do not* exhibit any of the above violations.

Suppose there is a tabloid $T(\lambda,\sigma,\varepsilon)$ with *no* monotonicity violation. In particular, there is *no* entry $T_{i,j}$, with $j<\lambda_1$ and $i\le\lambda'_{j+1}$, for which cell $(i,j+1)$ does not belong to T. This implies that the *upper edge* of the generalized Ferrers board of λ (forming the shape of T) consists *only of ascending steps*. In terms of ε and σ this means the following: Let $\varepsilon_{i_1}=\dots=\varepsilon_{i_k}=-1$, $\varepsilon_{j_1}=\dots=\varepsilon_{j_l}=1$, $i_1<i_2<\dots i_k$, $j_1<j_2\dots<j_l$, $k+l=\lambda_1$. Then because of (4.1) we must have

$$\begin{gathered}\sigma(1)=i_k,\ \sigma(2)=i_{k-1},\dots,\ \sigma(k)=i_1,\\ \sigma(k+1)=j_1,\ \sigma(k+2)=j_2,\dots,\ \sigma(\lambda_1)=j_l.\end{gathered}\tag{5.2}$$

Hence, if *some* $\varepsilon_j=-1$, then also $\varepsilon_{\sigma(1)}=-1$. But then cell $(1,1)$ does not belong to T, which is a Sundaram violation. On the other hand, if all $\varepsilon_j=1$, then we must have $\sigma=\mathrm{id}$ by (5.2).

Finally we define the involutions $\mathfrak{i}^1_{\mathbb{S}}$ on $B^1_{\mathbb{S}}$ and $\mathfrak{i}^2_{\mathbb{S}}$ on $B^2_{\mathbb{S}}$. We start with the definition of $\mathfrak{i}^1_{\mathbb{S}}$. This mapping relies on an operation which we call *inversion up to $2c$ of a column j* of a tabloid T, where c is some positive integer $\le n$. (Also $\mathfrak{i}^3_{\mathbb{P}}$ will rely on this operation.) This is defined as follows. If for some p, $1\le p\le c$, *both* entries $2p-1$ and $2p$ occur in column j then *remove* both of them. If *neither* $2p-1$ *nor* $2p$ is in column j then *insert* both of them. Apart from that, let the remaining entries remain unchanged (these are the ones where either $2p-1$ or $2p$ are in column j, but not both). Of course, put all the obtained entries, old and new, into increasing order to form the new column j. Figure 5 shows an example of an inversion up to $2c$ with $c=4$. Observe that inversion up to $2c$ is a weight-preserving operation, since a pair of entries $2k-1,2k$ contributes $x_kx_k^{-1}=1$ to the weight of the tableau, hence it can be removed or inserted without change of the weight. Observe further that it is an involution. Another application to the same column j restores the original situation.

Example of the "inversion up to $2c$"

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 6\\ \hline\hline 9\\ \hline\end{array} \quad\longrightarrow\quad \begin{array}{|c|}\hline 3\\ \hline 4\\ \hline 6\\ \hline 7\\ \hline 8\\ \hline\hline 9\\ \hline\end{array}$$

Figure 5

For later use, we have to compute the effect of inversion up to $2c$ on the length of a column. Denote the tabloid that is obtained from tabloid T by inversion up to $2c$ of column j by $\overline{T}$. Also denote the lengths of the j-th column of T and $\overline{T}$ by $\mathrm{cl}_j(T)$ and $\mathrm{cl}_j(\overline{T})$, respectively. Then it is obvious that

$$\left|\{T_{\cdot,j} \leq 2c\}\right| + \left|\{\overline{T}_{\cdot,j} \leq 2c\}\right| = 2c. \tag{5.3}$$

Hence the difference of column lengths is

$$\mathrm{cl}_j(\overline{T}) - \mathrm{cl}_j(T) = 2c - 2\cdot\left|\{T_{\cdot,j} \leq 2c\}\right|. \tag{5.4}$$

Now we construct the involution $\mathfrak{i}^1_{\mathbb{S}}$ on $B^1_{\mathbb{S}}$. Let $T(\lambda,\sigma,\varepsilon)$ be a tabloid in $B^1_{\mathbb{S}}$, with the Sundaram violation in row I that was selected before. Invert column 1 up to $2I-2$, leave σ unchanged, and replace $\varepsilon_{\sigma(1)}$ by $-\varepsilon_{\sigma(1)}$. Let $\overline{T}$ be the new tabloid. For examples, see operations 2, 6, 10, 14 and 18 in Figure 9.

We have to check that $\mathfrak{i}^1_{\mathbb{S}}$ is well-defined, and a weight-preserving and sign-reversing involution. For being well-defined, we have to check that $\overline{T}$ lies in $B^1_{\mathbb{S}}$, i.e, that it lies in $\mathbb{S}$ and contains a Sundaram violation that is minimal with respect to our selection scheme. To show that $\overline{T}$ lies in $\mathbb{S}$, we have to show that column j has the correct length with respect to (4.1), for all j. Since only the first column was changed it suffices to check the length of the first column of $\overline{T}$. The length of the first column of T is $\lambda'_1 - 1 + \sigma(1)\cdot\varepsilon_{\sigma(1)}$, hence

$$\left|\{T_{\cdot,1} \leq 2I-2\}\right| = I - 1 + \sigma(1)\,\varepsilon_{\sigma(1)}.$$

According to (5.4), we therefore have

$$\mathrm{cl}_1(\overline{T}) = \mathrm{cl}_1(T) - 2\sigma(1)\,\varepsilon_{\sigma(1)} = \lambda'_1 - 1 + \sigma(1)\,(-\varepsilon_{\sigma(1)}),$$

which is exactly what we need. That $\mathfrak{i}^1_{\mathbb{S}}$ reverses sign is obvious since it changes $\operatorname{sgn}(\varepsilon)$ and does not change $\operatorname{sgn}(\sigma)$. It is weight-preserving since inversion up to $2c$ has these properties. Moreover, it is clear that $\overline{T}$ again satisfies $\overline{T}_{I,1} < 2I-1$ or that cell $(I,1)$ does not belong to $\overline{T}$. And, I is maximal with respect to this property. Therefore, if $\mathfrak{i}^1_{\mathbb{S}}$ is applied to $\overline{T}$ then we recover T, since it means applying inversion up to $2I-2$ to the first column again, and this restores the original object, as we already observed. Hence, $\mathfrak{i}^1_{\mathbb{S}}$ is an involution.

Now we define $\mathfrak{i}^2_{\mathbb{S}}$. Let $T(\lambda,\sigma,\varepsilon)$ be a $(2n+1)$-tabloid in $B^2_{\mathbb{S}}$ with the violation of row monotonicity in row I and column J that were selected above. Again, think of a tabloid T as heaps of cells (making up the columns) side by side. Now, interchange the heap "above and including $T_{I-1,J}$" and the heap "above and including $T_{I,J+1}$", replace σ by $\sigma' := \sigma\circ(J,J+1)$, and leave ε unchanged. This yields some new object $\mathfrak{i}^2_{\mathbb{S}}(T) = T'(\lambda,\sigma',\varepsilon)$. This operation is shown with some examples in Figure 6, only displaying the involved columns. See operations 4, 8 and 16 in Figure 9 for more elaborate examples. Note that the effect on the numbers beneath the displayed tabloid that encode the permutation σ is that those beneath the J-th and the $(J+1)$-st column are simply interchanged, the same being true for the the $+$'s and $-$'s encoding ε.

Example of the operation defined for violations of row monotonicity

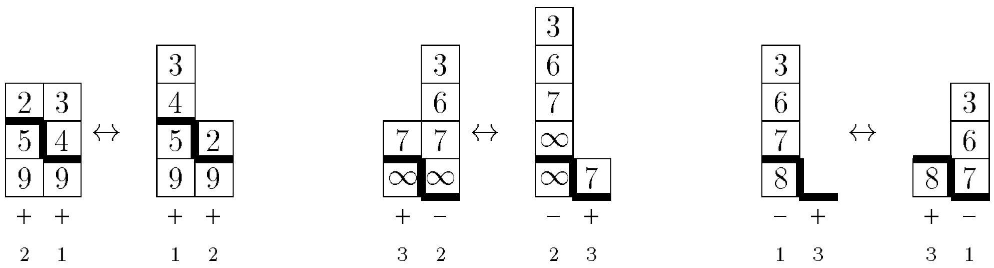

Figure 6

We have to check that our new object $T'(\lambda,\sigma',\varepsilon)$ is again a $(2n+1)$-tabloid in $B^2_{\mathbb{S}}$, and that $\mathfrak{i}^2_{\mathbb{S}}$ is a weight-preserving and sign-reversing involution. First, T' has the correct column length with respect to (4.1) and is therefore an element of $\mathbb{S}$. For, by construction we have

$$\begin{aligned}\operatorname{cl}_J(T') &= (\lambda'_J - I + 1) + \Big(\operatorname{cl}_{J+1}(T) - \big(\lambda'_{J+1} - I\big)\Big)\\ &= \lambda'_J - J + \sigma(J+1)\cdot\varepsilon_{\sigma(J+1)}\\ &= \lambda'_J - J + \sigma'(J)\cdot\varepsilon_{\sigma'(J)},\end{aligned}$$

and an analogous computation yields that column $J+1$ has the correct length, too. Second, T' again shows a violation of row monotonicity in the same row I and same column J (just look at the pictures of Figure 6). Besides, since the entry in cell (I,J)

and entries to the left or in lower rows were not changed, this violation is again the "minimal" with respect to our selection scheme. Therefore $T'(\lambda, \sigma', \varepsilon)$ is an element of $B^2_{\mathbb{S}}$.

The observation that $T'(\lambda, \sigma', \varepsilon)$ shows a violation of row monotonicity at the same place as $T(\lambda, \sigma, \varepsilon)$, implies that if $\mathfrak{i}^2_{\mathbb{S}}$ is applied to $T'(\lambda, \sigma, \varepsilon)$ then one recovers $T(\lambda, \sigma, \varepsilon)$. Therefore $\mathfrak{i}^2_{\mathbb{S}}$ is an involution on $B^2_{\mathbb{S}}$. That $\mathfrak{i}^2_{\mathbb{S}}$ is weight-preserving is obvious. It reverses sign since ε is not changed and σ and σ' are related by a transposition and hence have different sign.

5.2. The involution $\mathfrak{i}_{\mathbb{P}}$. Proctor tableaux are somewhat more complicated than Sundaram tableaux, and so is the corresponding involution $\mathfrak{i}_{\mathbb{P}}$ on $B_{\mathbb{P}}$. As promised, we shall partition the set $B_{\mathbb{P}}$ into three disjoint subsets $B^1_{\mathbb{P}}$, $B^2_{\mathbb{P}}$, and $B^3_{\mathbb{P}}$ and construct involutions $\mathfrak{i}^1_{\mathbb{P}}$, $\mathfrak{i}^2_{\mathbb{P}}$, and $\mathfrak{i}^3_{\mathbb{P}}$ for each of these, respectively. The involution $\mathfrak{i}_{\mathbb{P}}$ then is the union of these three subinvolutions.

First, we define the set $B^1_{\mathbb{P}}$. $B^1_{\mathbb{P}}$ consists of all $(2n+1)$-tabloids with more than $2\sigma(j)-1$ entries ∞ in column j or with $\varepsilon_{\sigma(j)} = -1$ for some j. (In terms of the preceding subsection these properties are considered as certain *violations* of "Proctor-tableau-ness".) In symbols,

(5.5)
$$B^1_{\mathbb{P}} := \{T(\lambda, \sigma, \varepsilon) \in B_{\mathbb{P}} : \Big(\exists j : \Big|\{T_{\cdot,j} = \infty\}\Big| > 2\sigma(j) - 1\Big) \text{ or } \Big(\exists j : \varepsilon_{\sigma(j)} = -1\Big)\}.$$

We define operation $\mathfrak{i}^1_{\mathbb{P}} : B^1_{\mathbb{P}} \to B^1_{\mathbb{P}}$ as follows: Given some $T \in B^1_{\mathbb{P}}$, search for J *minimal* with $\Big|\{T_{\cdot,J} = \infty\}\Big| > 2\sigma(J) - 1$ or $\varepsilon_{\sigma(J)} = -1$. If $\varepsilon_{\sigma(J)} = -1$, then insert $2\sigma(J)$ entries ∞ at the bottom of column J, else remove $2\sigma(J)$ entries ∞ from column J. In both cases, leave σ unchanged, and replace $\varepsilon_{\sigma(J)}$ by $-\varepsilon_{\sigma(J)}$. For elaborate examples, see operations 1, 7, 9, 15 and 17 in Figure 9.

We have to check that this operation is a weight-preserving and sign-reversing involution on $B^1_{\mathbb{P}}$. $\mathfrak{i}^1_{\mathbb{P}}$ is clearly an involution, and it is easy to see that the combined effect of insertion/removal of entries in column J together with the change of $\varepsilon_{\sigma(J)}$ yields a new column of the correct length (with respect to (4.1)), which either contains more than $2\sigma(J)-1$ entries ∞ or has $\varepsilon_{\sigma(J)} = -1$. So for every $T \in B^1_{\mathbb{P}}$ we have $\mathfrak{i}^1_{\mathbb{P}}(T) \in B^1_{\mathbb{P}}$. $\mathfrak{i}^1_{\mathbb{P}}$ clearly is weight-preserving since entries ∞ do not contribute to the weight at all. It is sign-reversing since $\operatorname{sgn}(\varepsilon)$ is changed and σ is left unchanged.

Next we define set $B^2_{\mathbb{P}}$. $B^2_{\mathbb{P}}$ consists of all $(2n+1)$-tabloids *not in* $B^1_{\mathbb{P}}$ where the number of entries ∞ is greater than 1 and less than $2\sigma(J)-2$ in some column J. (In terms of the preceding subsection, this property is another *violation* of "Proctor-tableau-ness".) In symbols,

$$B^2_{\mathbb{P}} := \{T(\lambda, \sigma, \varepsilon) \in B_{\mathbb{P}} \setminus B^1_{\mathbb{P}} : \Big(\exists J : \Big|\{T_{\cdot,J} = \infty\}\Big| \in [2, 2\sigma(J) - 3]\Big)\}. \tag{5.6}$$

(Observe, that ε actually equals $(1, 1, \ldots, 1)$ for all elements of $B^2_{\mathbb{P}}$.)

For this set we define mapping $\mathfrak{i}^2_{\mathbb{P}}$ as follows: Given some $T \in B^2_{\mathbb{P}}$, search for column J *with* $\sigma(J)$ *minimal*, where $\left|\{T_{\cdot,J} = \infty\}\right| \in [2, 2\sigma(J) - 3]$. Write l for $\left|\{T_{\cdot,J} = \infty\}\right|$. Suppose, l lies in the set

$$S_k := \{2k, 2k+1\} \cup \{2\sigma(J) - 2k - 2, 2\sigma(J) - 2k - 1\}$$

for some k, $1 \le k \le \left\lceil \frac{\sigma(J)}{2} \right\rceil - 1$. (It is obvious that the sets S_k, $1 \le k \le \left\lceil \frac{\sigma(J)}{2} \right\rceil - 1$, form a partition of the set $\{2, 3, \dots, 2\sigma(J) - 3\}$, hence k is uniquely determined.) We can deduce immediately that

$$\sigma(J) - 2k \ge \sigma(J) - 2\left\lceil \frac{\sigma(J)}{2} \right\rceil + 2 \ge 1.$$

Therefore, there must exist an index S with $\sigma(S) = \sigma(J) - 2k$. Now remove $2k$ entries ∞ from column J, insert them in column S, replace σ by $\sigma' := \sigma \circ (J, S)$, and leave ε unchanged. This yields the new object $T'(\lambda, \sigma', \varepsilon)$. For elaborate examples see operations 3 and 5 in Figure 9.

We have to check that this operation is a weight-preserving and sign-reversing involution on $B^2_{\mathbb{P}}$. It is easy to check that $\mathfrak{i}^2_{\mathbb{P}}(T) = T'(\lambda, \sigma', \varepsilon)$ has the correct column lengths with respect to (4.1) by using the fact that $\varepsilon_i = 1$ for all i. The number $\left|\{T_{\cdot,S} = \infty\}\right|$ of entries ∞ in column S (before application of mapping $\mathfrak{i}^2_{\mathbb{P}}$) lies in $[0, 1] \cup [2\sigma(S) - 2, 2\sigma(S) - 1]$, since otherwise J were not the column with $\sigma(J)$ minimal (or T would not belong to $B^2_{\mathbb{P}}$ at all). Since $\sigma(S) = \sigma(J) - 2k$, we have

$$\begin{aligned} \left|\{T'_{\cdot,S} = \infty\}\right| &\in [2k, 2k+1] \cup [2\sigma(S) + 2k - 2, 2\sigma(S) + 2k - 1] \\ &= [2k, 2k+1] \cup [2\sigma(J) - 2k - 2, 2\sigma(J) - 2k - 1] \\ &= [2k, 2k+1] \cup [2\sigma'(S) - 2k - 2, 2\sigma'(S) - 2k - 1], \end{aligned}$$

and

$$\begin{aligned} \left|\{T'_{\cdot,J} = \infty\}\right| &\in [0, 1] \cup [2\sigma(J) - 4k - 2, 2\sigma(J) - 4k - 1] \\ &= [0, 1] \cup [2\sigma'(J) - 2, 2\sigma'(J) - 1]. \end{aligned}$$

All the columns other than columns J and S were not changed at all. Therefore T' is again an element of $B^2_{\mathbb{P}}$. Besides, in the set of all columns of T' having the defining property of (5.6), column S is obviously the one with $\sigma'(S)$ minimal. So, by applying $\mathfrak{i}^2_{\mathbb{P}}$ to $T'(\lambda, \sigma', \varepsilon)$ we would recover $T(\lambda, \sigma, \varepsilon)$. Hence, $\mathfrak{i}^2_{\mathbb{P}}$ is an involution. $\mathfrak{i}^2_{\mathbb{P}}$ is clearly weight-preserving, since moving around entries ∞ has no influence on the weight. It is sign-reversing, since the the sign of σ and σ' is different and ε was left unchanged.

The "remaining" tabloids $B_{\mathbb{P}} \setminus (B^1_{\mathbb{P}} \cup B^2_{\mathbb{P}})$ constitute our set $B^3_{\mathbb{P}}$. They are $(2n+1)$-tabloids $T(\lambda, \sigma, \varepsilon)$ with $\varepsilon = (1, 1, \dots, 1)$ (since otherwise T were in $B^1_{\mathbb{P}}$) and $\left|\{T_{\cdot,j} = \infty\}\right| \in [0, 1] \cup [2\sigma(j) - 2, 2\sigma(j) - 1]$ (since otherwise T were in $B^1_{\mathbb{P}}$ or in $B^2_{\mathbb{P}}$) that are no $(2n+1)$-Proctor tableaux. Instead of directly working with these tabloids, it is more convenient to map them by a weight- and sign-preserving bijection, P say, to another set, $\overline{B_{\mathbb{P}}^3}$ say. $\overline{B_{\mathbb{P}}^3}$ consists of fillings of slightly different generalized Ferrers boards of λ. If σ is a permutation in $\mathfrak{S}_{\lambda_1}$ and ε is a sign-vector in $\{1\} \times \{1,-1\}^{\lambda_1 - 1}$ (note the difference to the previous definition of vector ε) then for $\overline{B_{\mathbb{P}}^3}$ we consider generalized Ferrers boards F of λ where the length of the j-th column of F is given by

$$\mathrm{cl}_j(F) = \lambda'_j - j + (\sigma(j) - 1)\, \varepsilon_{\sigma(j)} + 1. \tag{5.7}$$

(Note the difference to (4.1)). By definition, $\overline{B_{\mathbb{P}}^3}$ consists of all fillings T of a generalized Ferrers board F with parameters $(\lambda, \sigma, \varepsilon)$, but with column-lengths according to (5.7), with elements from $\{1, 2, \dots, 2n, \infty\}$, that are strictly increasing along columns and that are no Proctor tableaux. (Note that column-strictness implies that in particular there is at most *one* entry ∞ in any column.) To differentiate from our previous tabloids, we call objects in $\overline{B_{\mathbb{P}}^3}$ *new* $(2n+1)$*-tabloids*. The definition of signed weight for a tabloid in $\overline{B_{\mathbb{P}}^3}$ is quite similar to (4.2),

$$w(T(\lambda, \sigma, \varepsilon)) := \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_i^{\#(2i-1 \in T) - \#(2i \in T)}. \tag{5.8}$$

Note, however, that in contrast to (4.2) vector ε does not contribute to the sign here.

The bijection P between $B_{\mathbb{P}}^3$ and $\overline{B_{\mathbb{P}}^3}$ is defined as follows. Consider some tabloid $T(\lambda, \sigma, \varepsilon) \in B_{\mathbb{P}}^3$. As mentioned above, we have $\varepsilon = (1, 1, \dots, 1)$ for all tabloids in $B_{\mathbb{P}}^3$. Now, for *every* column j with $\sigma(j) > 1$ and $\left|\{T_{\cdot,j} = \infty\}\right| \geq 2\sigma(j) - 2$, remove $2\sigma(j) - 2$ entries ∞ and set $\varepsilon_{\sigma(j)} = -1$. For an example, see operation 11 in Figure 9. This is indeed a bijection. The inverse mapping is easily described. Given a tabloid $T(\lambda, \sigma, \varepsilon) \in \overline{B_{\mathbb{P}}^3}$, insert $2\sigma(j) - 2$ entries ∞ in every column j with $\varepsilon_{\sigma(j)} = -1$.

Recalling the definitions of signed weights (4.2) and (5.8) as well as the fact that entries ∞ do not contribute to the weight, it is easy to see that P is a weight- and sign-preserving bijection between $B_{\mathbb{P}}^3$ and $\overline{B_{\mathbb{P}}^3}$.

So the remaining task is to give a weight-preserving and sign-reversing involution on $\overline{B_{\mathbb{P}}^3}$, which is actually the most intricate part.

Similar to the definition of $\mathfrak{i}_{\mathbb{S}}$, we first define a list of *violations* for new tabloids. Then, among all violations we pick up the "minimal" (in some sense), and finally we apply a certain operation which depends on the type of violation that we are considering.

To motivate our definitions of violations, observe that $\overline{B_{\mathbb{P}}^3}$ consists precisely of those new $(2n+1)$-tabloids (T, σ, ε) where either $\sigma \neq \mathrm{id}$ or $\varepsilon \neq (1, 1, \dots, 1)$ or there is some violation of row monotonicity or some violation of the orthogonal or protection conditions.

We define a *violation of row monotonicity at p* of a new $(2n+1)$-tabloid T to be:

(1) cell (i,j) belongs to T, $j<\lambda_1$, $i\le\lambda'_{j+1}$, and $T_{i,j}=p$,

(2) either $\quad T_{i,j}>T_{i,j+1}$,
or $\quad$ cell $(i,j+1)$ does not belong to T.

(Note that here we *do not* consider two entries ∞ in the same row as a violation since this is not forbidden for Proctor tableaux.)

The other types of violations (and their namings) stem from [2] and come from the underlying lattice path interpretations given there. This is commented in section 7.

We say that a new $(2n+1)$-tabloid $T(\lambda,\sigma,\varepsilon)$ contains an *even reflected intersection at* $2c$, for some c with $1\le c\le n$, if

$$\left|\{T_{\cdot,1}\le 2c\}\right|+\left|\{T_{\cdot,2}\le 2c\}\right|=2c+(\sigma(1)-1)\,\varepsilon_{\sigma(1)}+(\sigma(2)-1)\,\varepsilon_{\sigma(2)}. \tag{5.9}$$

For an example consider the second tabloid in the third row of Figure 9. It contains an even reflected intersection at 6.

We say that a new $(2n+1)$-tabloid $T(\lambda,\sigma,\varepsilon)$ contains an *odd reflected crossing at* $2c-1$, for some c with $1\le c\le n$, if the first two columns of T contain neither $2c-1$ nor $2c$ and

$$\left|\{T_{\cdot,1}\le 2c-2\}\right|+\left|\{T_{\cdot,2}\le 2c-2\}\right|=2c+(\sigma(1)-1)\,\varepsilon_{\sigma(1)}+(\sigma(2)-1)\,\varepsilon_{\sigma(2)}-1.$$

The right-most tabloid in Figure 8 contains an odd reflected crossing at 6.

And, we say that a new tabloid $T(\lambda,\sigma,\varepsilon)$ contains an *odd reflected violation at* $2c-1$, for some c with $1\le c\le n$, if the following specialization of (2.3) holds (the additions are underlined in the text; note that they imply that there is *no* even reflected intersection at $2c$ or $2c-2$):

$$\begin{aligned}i+j=2c \text{ and } T_{i,1}=2c-1 &\underline{\text{ and } T_{i+1,1}>2c}\\ &\text{and } T_{j,2}=\cdots=T_{j,h-1}=2c-1 \text{ (for } h\ge 2)\\ &\underline{\text{and } T_{j+1,2}=\cdots=T_{j+1,h-1}>2c \text{ (for } h\ge 2)}\\ &\text{and } T_{j,h}=2c\underline{\text{ and } T_{j-1,h}<2c-1}.\end{aligned}$$

The left-most tabloid in Figure 8 contains an odd reflected violation at 6 (choose $i=4$, $j=2$, $h=3$).

The important fact is that every element of $\overline{B^3_{\mathbb{P}}}$ contains either a row monotonicity violation, or an even reflected intersection, or an odd reflected crossing, or an odd reflected violation. For, suppose that there is $T(\lambda,\sigma,\varepsilon)$ in $\overline{B^3_{\mathbb{P}}}$ containing none of these violations. Then we claim that there holds

$$\left|\{T_{\cdot,1}\le 2c\}\right|+\left|\{T_{\cdot,2}\le 2c\}\right|<2c+(\sigma(1)-1)\,\varepsilon_{\sigma(1)}+(\sigma(2)-1)\,\varepsilon_{\sigma(2)}, \tag{5.10}$$

for $c=0,1,\dots,n$. (Note that this inequality generalizes the orthogonal condition (2.2) for $p=2c$, to which it reduces for $\sigma=\mathrm{id}$ and $\varepsilon=(1,1,\dots,1)$.) In fact, these inequalities follow from the observation that

$$\begin{aligned}\Big|\{T_{\cdot,1} \le \infty\}\Big| + \Big|\{T_{\cdot,2} \le \infty\}\Big| &= \mathrm{cl}_1(T) + \mathrm{cl}_2(T)\\ &= \lambda_1' - 1 + (\sigma(1)-1)\varepsilon_{\sigma(1)} + 1 + \lambda_2' - 2 + (\sigma(2)-1)\varepsilon_{\sigma(2)} + 1 \qquad \text{(by (5.7))}\\ &< 2n + 1 + (\sigma(1)-1)\varepsilon_{\sigma(1)} + (\sigma(2)-1)\varepsilon_{\sigma(2)} \qquad \text{(since } \lambda_1' + \lambda_2' \le 2n+1\text{)},\end{aligned}$$

and a downward induction on c which uses that T contains neither an even intersection nor an odd reflected crossing. Since T does not contain any row monotonicity violation by assumption, the arguments leading to (5.2) also apply here. Therefore, if there is some $\varepsilon_j = -1$ then we must have $\varepsilon_{\sigma(1)} = -1$ by (5.2). The generalized orthogonal condition (5.10) with $c = 0$ then enforces $\varepsilon_{\sigma(2)} = 1$. Therefore $\sigma(2) = 1$ by (5.2), and so (5.10) does not hold for $c = 0$, which is a contradiction. Hence we must have $\varepsilon = (1, 1, \ldots, 1)$ which in turn implies $\sigma = \mathrm{id}$, again by (5.2). Summarizing, T is a new $(2n+1)$-tabloid with parameters $\sigma = \mathrm{id}$ and $\varepsilon = (1, 1, \ldots, 1)$ (which by (5.7) means that it is a filling of the Ferrers board of λ), that satisfies the $(2c)$-th orthogonal condition (since it satisfies (5.10)) and the $(2c)$-th protection condition (since it does not contain any odd reflected violation), for all c. In short, T is a $(2n+1)$-Proctor tableau, which is a contradiction since Proctor tableaux are excluded from $\overline{B^3_{\mathbb{P}}}$.

Finally we turn to the definition of $\mathfrak{i}^3_{\mathbb{P}}$ on $\overline{B^3_{\mathbb{P}}}$. Let $T(\lambda, \sigma, \varepsilon)$ be in $\overline{B^3_{\mathbb{P}}}$. Among all the row monotonicity violations, even reflected intersections, odd reflected crossings, odd reflected violations, pick out these which are at p with p maximal, and among all these pick the ones in the minimal column. If we encounter a row monotonicity violation among the latter, then $\mathfrak{i}^3_{\mathbb{P}}$ acts in the same way as $\mathfrak{i}^2_{\mathbb{S}}$ did. In the same way as it was done for $\mathfrak{i}^2_{\mathbb{S}}$ and the column-lengths (4.1), it is checked that this operation takes care of the (new) column lengths (5.7). Besides, the image under this operation will again contain a row monotonicity violation, at the same p and in the same column. Hence, by this operation we obtain an element of $\overline{B^3_{\mathbb{P}}}$, and by applying $\mathfrak{i}^3_{\mathbb{P}}$ again we would recover our original tabloid.

Example for the operation in case of an even reflected intersection

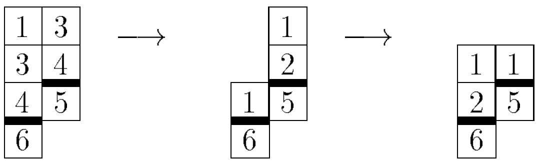

Figure 7

If we do not encounter a row monotonicity violation but an even intersection at $2c$, or one of the other two violations at $2c-1$, then the *basic* idea for the definition of $\mathfrak{i}_{\mathbb{P}}^3$ is as follows: Apply inversion up to $2c$ to both columns 1 and 2 and *interchange* the two "inverted heaps", replace σ by $\overline{\sigma} := \sigma \circ (1,2)$, and replace $\varepsilon_{\sigma(1)}$ and $\varepsilon_{\sigma(2)}$ by their negatives, except that ε_1 is not changed. (ε_1 has to be 1 by definition of $\overline{B_{\mathbb{P}}^3}$.) This operation is weight-preserving since inversion up to $2c$ is, and it is sign-reversing since σ and $\overline{\sigma}$ differ in sign and the weight (5.8) does not depend on vector ε.

Figure 7 gives an example for this operation with $2c = 4$.

In the case of an even reflected intersection we apply this idea without further modification, thus obtaining $\overline{T}(\lambda, \overline{\sigma}, \overline{\varepsilon})$. For an elaborate example, see operation 12 in Figure 9.

We have to check that $\overline{T}(\lambda, \overline{\sigma}, \overline{\varepsilon})$ is in $\overline{B_{\mathbb{P}}^3}$ and that application of $\mathfrak{i}_{\mathbb{P}}^3$ to $\overline{T}(\lambda, \overline{\sigma}, \overline{\varepsilon})$ would give back $T(\lambda, \sigma, \varepsilon)$. First, $\overline{T}$ has the correct column lengths with respect to (5.7) since for $\{i,j\} = \{1,2\}$ we have

$$\begin{aligned} \mathrm{cl}_i(\overline{T}) &= \lambda_i' - i + (\sigma(i) - 1)\,\varepsilon_{\sigma(i)} + 1 - \left|\{T_{.,i} \le 2c\}\right| + 2c - \left|\{T_{.,j} \le 2c\}\right| \\ &= \lambda_i' - i + (\sigma(i) - 1)\,\varepsilon_{\sigma(i)} + 1 - \Big((\sigma(i) - 1)\,\varepsilon_{\sigma(i)} + (\sigma(j) - 1)\,\varepsilon_{\sigma(j)}\Big) \\ &= \lambda_i' - i + (\sigma(j) - 1)\,(-\varepsilon_{\sigma(j)}) + 1 \\ &= \lambda_i' - i + (\overline{\sigma}(i) - 1)\,\overline{\varepsilon}_{\overline{\sigma}(i)} + 1. \end{aligned}$$

Note that the exceptional case of ε_1 (which must always equal 1) does not matter in this (and the following) computation, since ε_1 is always multiplied by $(1-1) = 0$. Moreover,

$$\begin{aligned} \left|\{\overline{T}_{.,1} \le 2c\}\right| + \left|\{\overline{T}_{.,2} \le 2c\}\right| &= 4c - (2c + (\sigma(1) - 1)\,\varepsilon_{\sigma(1)} + (\sigma(2) - 1)\,\varepsilon_{\sigma(2)}) \\ &= 2c + (\overline{\sigma}(1) - 1)\,\overline{\varepsilon}_{\overline{\sigma}(1)} + (\overline{\sigma}(2) - 1)\,\overline{\varepsilon}_{\overline{\sigma}(2)}, \end{aligned}$$

which says that in $\overline{T}$ we encounter an even reflected intersection again, for the same value $2c$. Hence, $\overline{T}(\lambda, \overline{\sigma}, \overline{\varepsilon})$ is in $\overline{B_{\mathbb{P}}^3}$. Besides, this operation did not introduce a row monotonicity violation at $2c$ in the first column. For, suppose that l is minimal such that $T_{l,1} > 2c$. Then we must have $T_{l,2} \ge T_{l,1} > 2c$ because T does not contain a row monotonicity violation at some integer $> 2c$. Now, if we assume that $T_{l-1,2} \le 2c$, then by using (5.7) we obtain for the number of entries $\le 2c$ in the first two columns of T

$$\left|\{T_{.,1} \le 2c\}\right| + \left|\{T_{.,2} \le 2c\}\right| = l - 1 + (\sigma(1) - 1)\varepsilon_{\sigma(1)} + l - 2 + (\sigma(2) - 1)\varepsilon_{\sigma(2)},$$

which contradicts (5.9) in parity. Hence we have $T_{l-1,2} > 2c$. So our operation cannot introduce a row monotonicity violation at $2c$ in the first column of $\overline{T}$. Therefore, by applying $\mathfrak{i}_{\mathbb{P}}^3$ to $\overline{T}(\lambda, \overline{\sigma}, \overline{\varepsilon})$ we recover $T(\lambda, \sigma, \varepsilon)$.

In case of odd reflected violations and odd reflected crossings, the basic idea of the operation as explained above is modified in the following way. Suppose we encounter an odd reflected violation at $2c$. First, set $T_{i,1} = 2c$ and, to compensate for the according change in weight (with respect to (5.8)), set $T_{j,h} = 2c - 1$. This amounts to an interchange of entries $2c$ and $2c-1$ in T which *by assumption* does not violate the strict monotonicity in *columns*. Moreover, this interchange clearly preserves weight. Next, *insert* $2c-1$ in column 1 and $2c$ in column 2 (in the appropriate rows). This is possible *by assumption* without violating column-strictness. Again, this insertion does not change weight. Now apply the operation from above. Figure 8 gives an example for this operation with $2c=6$, $i=4$, $j=2$, $h=3$.

Conversion of odd reflected violation into odd reflected crossing

$$
\begin{array}{ccc}
2 & 3 & 4 \\
3 & \boxed{5} & \boxed{6} \\
4 & 7 & 7 \\
\boxed{5} & & \\
+ & + & + \\
1 & 2 & 3
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc}
2 & 3 & 4 \\
3 & 5 & 5 \\
4 & 7 & 7 \\
6 & & \\
 & & \\
 & &
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc}
2 & 3 & \\
3 & 5 & 4 \\
4 & 6 & 5 \\
5 & 7 & 7 \\
6 & & \\
 & &
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc}
 & & 4 \\
1 & 2 & 5 \\
2 & 7 & 7 \\
3 & & \\
- & + & + \\
2 & 1 & 3
\end{array}
$$

Figure 8

Again, we have to check that this operation gives us an element of $\overline{B^3_{\mathbb{P}}}$. To see this, we compute the resulting column lengths. For $\{i,j\}=\{1,2\}$ we have

$$
\begin{aligned}
\mathrm{cl}_i(\overline{T}) &= \lambda'_i - i + (\sigma(i)-1)\,\varepsilon_{\sigma(i)} + 1 - \left|\{T_{.,i} \le 2c\}\right| + 2c - \left(\left|\{T_{.,j} \le 2c\}\right| + 1\right) \\
&= \lambda'_i - i + (\sigma(i)-1)\,\varepsilon_{\sigma(i)} + 1 - \left((\sigma(i)-1)\,\varepsilon_{\sigma(i)} + (\sigma(j)-1)\,\varepsilon_{\sigma(j)} - 1\right) - 1 \\
&= \lambda'_i - i + (\sigma(j)-1)\,(-\varepsilon_{\sigma(j)}) + 1. \\
&= \lambda'_i - i + (\overline{\sigma}(i)-1)\,\overline{\varepsilon}_{\overline{\sigma}(i)} + 1.
\end{aligned}
$$

Observe that neither column 1 nor column 2 of the resulting tableau $\overline{T}$ contains $2c-1$ or $2c$. Hence $\left|\{\overline{T}_{.,1} \le 2c\}\right| + \left|\{\overline{T}_{.,2} \le 2c\}\right| = \left|\{\overline{T}_{.,1} \le 2c-2\}\right| + \left|\{\overline{T}_{.,2} \le 2c-2\}\right|$ and so

$$
\begin{aligned}
&\left|\{\overline{T}_{.,1} \le 2c-2\}\right| + \left|\{\overline{T}_{.,2} \le 2c-2\}\right| \\
&\qquad = 2c - \left|\{T_{.,1} \le 2c\}\right| - 1 + 2c - \left|\{T_{.,2} \le 2c\}\right| - 1 \\
&\qquad = 2c - (\sigma(1)-1)\,\varepsilon_{\sigma(1)} - (\sigma(2)-1)\,\varepsilon_{\sigma(2)} - 1 \\
&\qquad = 2c + (\overline{\sigma}(1)-1)\,\overline{\varepsilon}_{\overline{\sigma}(1)} + (\overline{\sigma}(2)-1)\,\overline{\varepsilon}_{\overline{\sigma}(2)} - 1.
\end{aligned}
$$

So this operation converted an odd reflected violation at $2c$ into an odd reflected crossing at $2c$. Hence, $\overline{T}(\lambda, \overline{\sigma}, \overline{\varepsilon})$ is indeed in $\overline{B^3_{\mathbb{P}}}$. Also, this operation did not introduce a row monotonicity violation at $2c-1$ in the first column (there is no entry $2c-1$ in the first column of $\overline{T}$), and, of course, no even intersection at $2c-1$ (which simply does not exist). Therefore this odd reflected crossing in $\overline{T}$ is the “minimal” violation according to our selection scheme. Since $\mathfrak{i}_{\mathbb{P}}^3$ should become an involution, the definition of $\mathfrak{i}_{\mathbb{P}}^3$ for odd reflected crossings has to be the inverse of the above described conversion of an odd reflected violation into an odd reflected crossing. A close look shows that this operation can indeed be reversed. This completes the definition of the involution $\mathfrak{i}_{\mathbb{P}}^3$ and hence of $\mathfrak{i}_{\mathbb{P}}$.

6. The example

After these sometimes tricky constructions time has come to have a look at a concrete example, worked out in detail. In Figure 9 we start with a 13-Sundaram tableau of shape $\lambda = (3,3,2,2,1,1)$ ($\lambda' = (6,4,2)$), and apply the algorithmic bijection until we finally reach the corresponding 13-Proctor tableau. The algorithm consists of alternatively applying $\mathfrak{i}_{\mathbb{P}}$ and $\mathfrak{i}_{\mathbb{S}}$, as long as possible. Each step is labelled by the corresponding subinvolution. All the time, we keep track of permutation σ and vector ε by writing $\sigma(j)$ as well as the sign of $\varepsilon_{\sigma(j)}$ below column j. Moreover, the violation that is responsible for the respective involution is graphically marked with thick lines.

7. The background

In this section we explain the background behind our bijection. As indicated in the introduction, the basic source for the operations of the previous section is the paper [2]. In [2, sec. 8] it was shown how to prove combinatorially that the generating function $\sum w(T)$ for all $(2n+1)$-Sundaram tableaux T of shape λ equals the determinant

$$\left|\sum_{k\ge 0} e_{\lambda_i'-i+j-k}(\mathbf{x}) - \sum_{k\ge 0} e_{\lambda_i'-i-j-k}(\mathbf{x})\right|_{\lambda_1\times\lambda_1}, \tag{7.1}$$

where $\mathbf{x} = (x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1})$ and $e_m(z_1, z_2, \dots, z_s)$ denotes the m-th elementary symmetric function in $z_1, z_2, \dots, z_s$ (cf. [9, p. 12]). Also in [2] it was shown how to prove combinatorially that the generating function $\sum w(T)$ for all $(2n+1)$-Proctor tableaux T of shape λ equals the determinant

$$\left|e_{\lambda_i'-i+1}(\mathbf{x},1) \quad \vdots \quad e_{\lambda_i'-i+j}(\mathbf{x},1) + e_{\lambda_i'-i-j+2}(\mathbf{x},1)\right|_{\lambda_1\times\lambda_1}, \tag{7.2}$$

where $e_m(\mathbf{x},1) = e_m(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, 1) = e_m(\mathbf{x}) + e_{m-1}(\mathbf{x})$. The notation of the determinant means that the first expression gives the entries of the first column and the second the entries for the remaining columns, $j \ge 2$. Actually, by (2.5) both determinants equal the odd orthogonal character $so_{2n+1}(\lambda, \mathbf{x})$. That they are the same can also be shown directly by first subtracting the sums in (7.1), thereby

Example for the algorithmic bijection

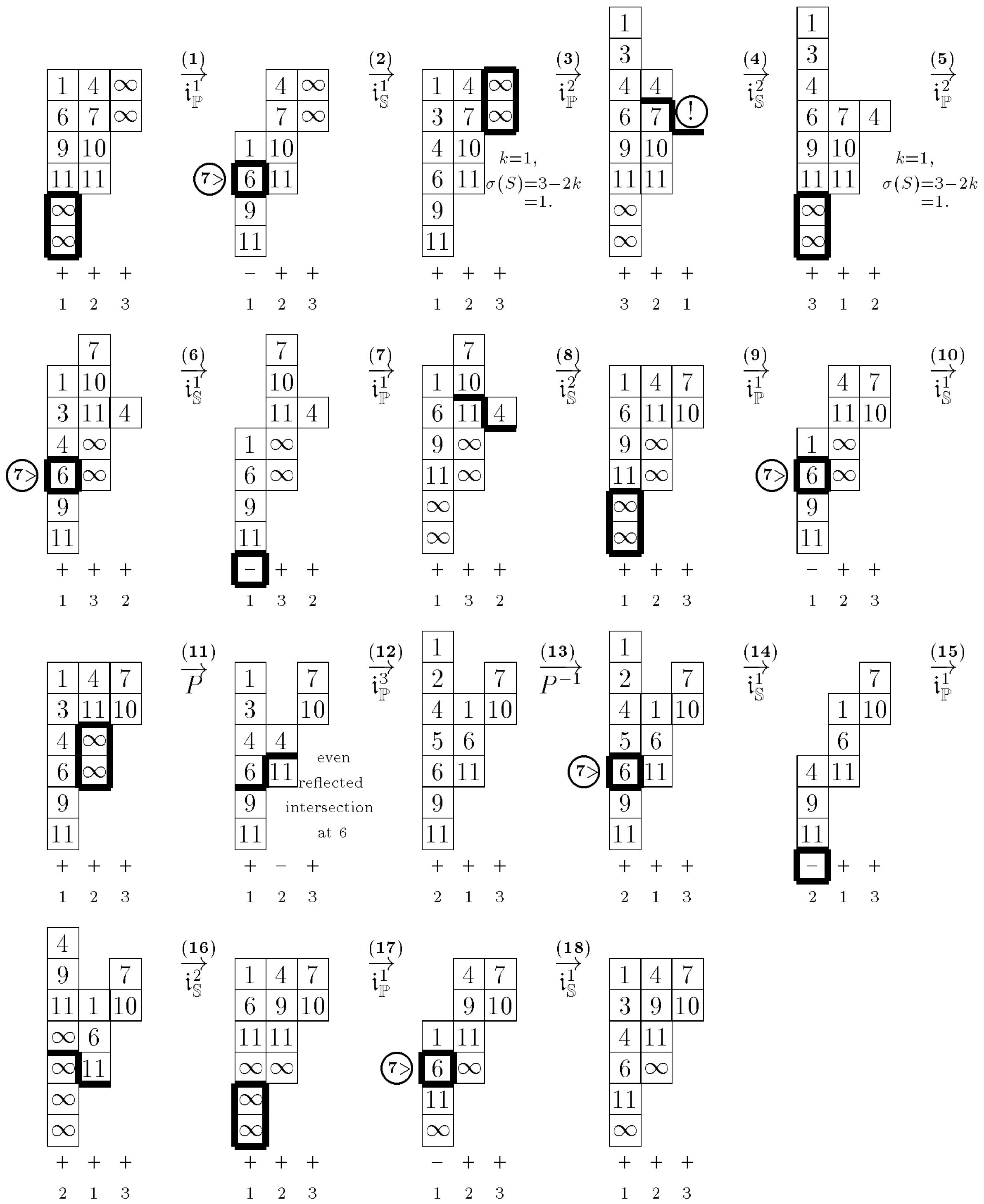

Figure 9

writing (7.1) as

$$\left|\sum_{k=1}^{2j} e_{\lambda_i'-i-j+k}(\mathbf{x})\right|_{\lambda_1\times\lambda_1} = \left|\sum_{k=1}^{j} e_{\lambda_i'-i-j+2k}(\mathbf{x},1)\right|_{\lambda_1\times\lambda_1},$$

and then subtracting column 1 from columns 2 through λ_1, column 2 from columns 3 through λ_1, etc. This results precisely in (7.2).

Now, here is the idea for our bijection between $(2n+1)$-Sundaram tableaux and $(2n+1)$-Proctor tableaux of the same shape λ: Expand the determinant (7.1) and interpret the resulting sum of (signed) monomials as a generating function for certain objects, $O_{\mathbb{S}}$ say. The $(2n+1)$-Sundaram tableaux of shape λ will appear in $O_{\mathbb{S}}$ since (7.1) equals their generating function. Subsequently, find a weight-preserving and sign-reversing involution that cancels all the remaining objects in $O_{\mathbb{S}}$. Likewise, expand the determinant (7.2) and interpret the resulting sum of (signed) monomials as a generating function for certain objects. In fact, as we saw by the reduction of (7.1) to (7.2), the determinant (7.2), when expanded, is a generating function for a *subset* of $O_{\mathbb{S}}$. The $(2n+1)$-Proctor tableaux of shape λ will appear in this subset since (7.2) equals their generating function. Subsequently, find a weight-preserving and sign-reversing involution that cancels all the remaining objects in this subset. And also, find a weight-preserving and sign-reversing involution that cancels all the objects in $O_{\mathbb{S}}$ that are not in this subset. (I.e., translate the algebraic steps that lead from (7.1) to (7.2) into combinatorics.) Provided these involutions are found, the involution principle (see section 3) applies and gives us an explicit bijection between $(2n+1)$-Sundaram tableaux and $(2n+1)$-Proctor tableaux of the same shape λ.

The proof for (7.1) in [2] consisted exactly of expanding the determinant, interpreting the resulting sum as a generating function for certain families of lattice paths, and finally cancelling those families of lattice paths that did not correspond to Sundaram tableaux by a weight-preserving and sign-reversing involution. The set $\mathbb{S}$ $(=\mathbb{P})$ defined in section 4 (paragraph after (4.2)) is the tabloid translation of the families of lattice paths. (Each column of a tabloid corresponds to a lattice path. The entries of the columns correspond to the labels of the horizontal steps in the lattice path.) The involution in [2] consisted of two different operations, one being the usual Gessel–Viennot mapping [4, 5], corresponding to our $\mathfrak{i}_{\mathbb{S}}^2$ (see section 5.1), and one being a modified reflection of paths, corresponding to our $\mathfrak{i}_{\mathbb{S}}^1$ (see section 5.1).

Likewise, the proof for (7.2) in [2] consisted exactly of expanding the determinant, interpreting the resulting sum as a generating function for certain families of lattice paths, and finally cancelling those families of lattice paths that did not correspond to Proctor tableaux by a weight-preserving and sign-reversing involution. The families of lattice paths correspond to the union $\overline{B_{\mathbb{P}}^3}\cup\{(2n+1)\text{-Proctor tableaux of shape }\lambda\}$. The involution in [2] consisted of two different operations, too. One was the usual Gessel–Viennot mapping, corresponding to our $\mathfrak{i}_{\mathbb{P}}^3$ (see section 5.2) in the case of a row monotonicity violation. The other one was called "modified reflection-Gessel–

Viennot mapping" and mapped even intersections to even intersections and odd reflected violations to odd reflected crossings. This operation corresponds to our $\mathfrak{i}^3_{\mathbb{P}}$ (see section 5.2) in the case of the other violations. (Actually, in [2] we even had to modify this mapping because we did not want it to introduce points of intersections. Fortunately, this is not necessary here.) Finally, the subtraction of the sums in the entries in (7.1) and the subsequent column operations that transformed (7.1) into (7.2) are combinatorially modelled by $\mathfrak{i}^1_{\mathbb{P}}$ and $\mathfrak{i}^2_{\mathbb{P}}$ (see section 5.2), respectively.

8. Concluding remarks

In [11, sec. 6] Proctor defined 4 different types of odd orthogonal tableaux, one type being due to King and Welsh [7]. Let us call them odd orthogonal tableaux of the first, second, third, and fourth kind (as was done in [2]). The first kind is the one considered in the previous sections. King and Welsh [7, App. A, p. 276f] gave a bijection between the tableaux of first and second kind, and an analogous argument provides a bijection between the tableaux of third and fourth kind (see [2, sec. 7, next-to-last paragraph]). We remark that it is not difficult to construct an explicit bijection between the tableaux of first and third kind using the involution principle again, by embedding both types of tableaux in a set of tabloids with new column lengths (5.7) and using the involution $\mathfrak{i}^3_{\mathbb{P}}$ and a variation of it. Thus all five types of odd orthogonal tableaux (including Sundaram tableaux) are related by an explicit bijection.

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