ON (SHAPE-)WILF-EQUIVALENCE FOR WORDS

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Abstract. Stankova and West showed that for any non-negative integer s and any permutation γ of $\{4, 5, \dots, s+3\}$ there are as many permutations that avoid 231γ as there are that avoid 312γ. We extend this result to the setting of words.

1. Introduction

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation of $\{1, 2, \dots, n\}$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_r$ be a permutation of $\{1, 2, \dots, r\}$, $r \le n$. We say that the permutation π *contains the pattern* σ, if there are indices $1 \le i_1 < i_2 < \cdots < i_r \le n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_r}$ is in the same relative order as $\sigma_1\sigma_2\cdots\sigma_r$. Otherwise, π is said to *avoid the pattern* σ, or, alternatively, we say that π is σ*-avoiding.* As usual, we write S_n for the set of all permutations of $\{1, 2, \dots, n\}$, and $S_n(\sigma)$ for the set of permutations in S_n that avoid σ.

The enumeration of permutations which avoid certain patterns has been a flourishing research subject since the seminal article [10] of Simion and Schmidt, where this research subject was "defined." The reader is referred to [8] and [4, Chapters 4 and 5] for in-depth accounts of the enumeration of pattern avoiding permutations.

We define the *direct sum* $\sigma \oplus \tau$ of two permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_r \in S_r$ and $\tau = \tau_1\tau_2\cdots\tau_s \in S_s$ as

$$\sigma \oplus \tau = \sigma_1\sigma_2\cdots\sigma_r(\tau_1 + r)(\tau_2 + r)\cdots(\tau_s + r).$$

The starting point for the work on this paper was an email of Doron Zeilberger [12] to the second author saying[1]:

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[1]Notation is slightly adapted in order to conform with the notation of our paper.

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According to http://en.wikipedia.org/wiki/Permutation_pattern,
Backelin, West & Xin (2007) proved that for any permutation
beta and any positive integer k, the permutations
12..k "+" beta and k....21 "+" beta are Wilf equivalent,

and in 2002 Stankova and West proved that 231 "+" beta and
312 "+" beta are Wilf equivalent.

According to Vince Vatter and Jonathan Bloom you have nice
proofs at least of the first result, in "Growth Diagrams, ...
Ferrers Shapes".

My main question is whether it is true, and whether there
exists a proof, of the above two results generalized to words
with a_1 1's, a_2 2's, ..., a_m m's, where the original case is
a_1 = ... = a_m = 1.
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Here, two patterns σ and τ are said to be *Wilf-equivalent*, denoted by $\sigma \sim \tau$, if $|S_n(\sigma)| = |S_n(\tau)|$ for all positive integers n. What Zeilberger refers to are the following two results.

Theorem 1 ([2, Theorem 2.1])**.** *For all positive integers k, all non-negative integers s, and all patterns $\beta \in S_s$, the patterns $12 \cdots k \oplus \beta$ and $k \cdots 21 \oplus \beta$ are Wilf-equivalent.*

Theorem 2 ([11, Theorem 1])**.** *For all non-negative integers s and all patterns $\beta \in S_s$, the patterns $231 \oplus \beta$ and $312 \oplus \beta$ are Wilf-equivalent.*

In fact, these two theorems explain all instances of Wilf-equivalence (of single patterns) that are known up to this date, except for $1423 \sim 3142$ (and equivalent Wilf-equivalence relations arising from reversal and/or complementation of patterns; cf. [2, pp. 134/135]).

Now, clearly, the notion of pattern avoidance can be straightforwardly adapted to *words* over the alphabet of positive integers. Given a word $w = w_1 w_2 \cdots w_n$ and a word $x = x_1 x_2 \cdots x_r$ with the letters w_i and x_i taken from the positive integers, we say that the word w *contains the pattern* x if there are indices $1 \le i_1 < i_2 < \cdots < i_r \le n$ such that $w_{i_1} w_{i_2} \cdots w_{i_r}$ is in the same relative order as $x_1 x_2 \cdots x_r$. Otherwise, w is said to *avoid the pattern* x. Since in words there can be equal letters, by "same relative order" we mean that, if there is an equality $x_s = x_t$ then there must also hold the equality $w_{i_s} = w_{i_t}$. Thus, Zeilberger asks whether Theorems 1 and 2 extend to words with a given number of 1's, 2's, ..., m's, the theorems themselves then being the special case where one considers words with exactly one 1, exactly one 2, ..., and exactly one m.

As it turns out, the first question had already been answered by Jelínek and Mansour in [7]. The purpose of this note is to answer the second question.

Before we describe the corresponding results, it is however helpful to recall that the key concept behind the proofs of Theorems 1 and 2 is *shape-Wilf-equivalence*, a concept introduced in [2] (already non-explicitly present in [1]; see Section 2 for its definition). This is a stronger notion than Wilf-equivalence. In particular, if two permutations σ and τ are *shape-Wilf-equivalent*, then they are also *Wilf-equivalent*. The significance of shape-Wilf-equivalence is explained by the following proposition.

Proposition 3 ([2, Prop. 2.3])**.** *If the permutations β and γ are shape-Wilf-equivalent, then for every permutation δ, $\beta \oplus \delta$ and $\gamma \oplus \delta$ are also shape-Wilf-equivalent.*

In plain terms, given two shape-Wilf-equivalent permutations, by forming direct sums we can obtain a whole infinite family of pairs of shape-Wilf-equivalent permutations, which are automatically also pairs of Wilf-equivalent permutations. In particular, Theorems 1 and 2 follow from the following two results.

Theorem 4 ([2, Theorem 2.1]). *For every positive integer k, the permutations $12\cdots k$ and $k\cdots 21$ are shape-Wilf-equivalent.*

Theorem 5 ([11, Theorem 1]). *The permutations 231 and 312 are shape-Wilf-equivalent.*

In view of Proposition 3, one sees that in Theorems 1 and 2 "Wilf-equivalent" may actually be replaced by "shape-Wilf-equivalent."

In [7], Jelínek and Mansour show that Theorem 1 extends directly to the word setting. The key to prove this is again shape-Wilf-equivalence, here extended to words (see Section 2 for the definition; for the sake of the current discussion, it is only relevant to know that *shape-Wilf-equivalence* of x and y implies *strong Wilf-equivalence* of x and y, the latter meaning by definition that for all sequences $a_1, a_2, \ldots, a_m$ of positive integers there are as many words with a_i letters i, $i = 1, 2, \ldots, m$, and avoiding x as there are words with a_i letters i, $i = 1, 2, \ldots, m$, and avoiding y). In terms of this notion, Zeilberger's questions translate into the questions whether $12\cdots k \oplus \beta$ and $k\cdots 21 \oplus \beta$ are strongly Wilf-equivalent, and whether $231 \oplus \beta$ and $312 \oplus \beta$ are strongly Wilf-equivalent, for any pattern β.

Proposition 6 ([7, Lemma 2.1]). *Let x and y be two words with letters from $\{1, 2, \ldots, k\}$ and w another word. If x and y are shape-Wilf-equivalent for words then so are $x \oplus w$ and $y \oplus w$.*

For the definition of the direct sum of words see the end of Section 2.

Theorem 7 ([9, proof of Theorem 13, Eq. (4.6)]). *The words $12\cdots k$ and $k\cdots 21$ are shape-Wilf-equivalent for words.*

If these two results are combined, the answer to Zeilberger's first question is obtained.

Corollary 8 ([7, Fact 2.2]). *For all words w, the words $12\cdots k \oplus w$ and $k\cdots 21 \oplus w$ are shape-Wilf-equivalent for words.*

The situation is different for Theorems 2 and 5. The counterexamples in Section 3 demonstrate that Theorem 5 does not straightforwardly extend to the word setting, that is, that 231 and 312 are *not shape*-Wilf-equivalent for words. Moreover, our computer calculations strongly indicate that $231 \oplus \beta$ and $312 \oplus \beta$ are never Wilf-equivalent for words when β is a non-empty permutation, regardless of the exact notion of "Wilf-equivalence" that we consider (see Conjecture 10). On the other hand, it seems more difficult to avoid 312 than 231. This vague statement is made more precise in Conjecture 11.

Nevertheless, weak versions of Theorems 2 and 5 hold true. If we do not insist on the "strict" patterns 231 and 312 but instead, say, interpret an occurrence of the patterns 231 and 312 to mean that the '2' and the '3' could also "be equal", then it is possible to extend Theorems 2 and 5. Theorem 12 in Section 4 says that the *sets of patterns*

$\{231, 221\}$ and $\{312, 212\}$ are shape-Wilf-equivalent for words, and Theorem 13 in the same section says that $\{231, 121\}$ and $\{312, 211\}$ are shape-Wilf-equivalent for words.

Furthermore, although shape-Wilf-equivalence of 231 and 312 for words is not true, the (strong) Wilf-equivalence of 231 and 312 for words (and, actually, of all permutation patterns of length 3) is again known and follows as a special case from another result of Jelínek and Mansour.

Theorem 9 ([7, Lemma 2.4]). *For any k, all the patterns that consist of a single letter '1', a single letter '3' and $k-2$ letters '2' are strongly Wilf-equivalent.*

Before describing our results, we need to collect definitions and notation in the next section.

2. Definitions and notation

We recall from the introduction that $S_n(\sigma)$ denotes the set of all permutations of $\{1, 2, \ldots, n\}$ that avoid σ. We choose a similar notation in the word setting. We write $W^{(a_1,a_2,\ldots)}(x)$ for the set of all words consisting of a_i letters i, $i = 1, 2, \ldots$, and avoiding x. More generally, given a set Ω of words, we write $W^{(a_1,a_2,\ldots)}(\Omega)$ for the set of all words consisting of a_i letters i, $i = 1, 2, \ldots$, and avoiding all $x \in \Omega$. (For convenience, the set braces may be omitted in this notation.) Occasionally, we shall use the symbol $W_{n,m}(x)$ to denote the set of *all* words of length n that use letters from $\{1, 2, \ldots, m\}$ and avoid the word x.

Next we reformulate pattern avoidance of permutations in terms of matrices. Clearly, any permutation $\pi = \pi_1\pi_2\cdots\pi_n$ can be represented by the corresponding *permutation matrix* $M(\pi)$, which for us is the $n\times n$ matrix which contains a 1 in column j and row π_j, $j = 1, 2, \ldots, n$, and all other entries are zero. When we display a matrix, our convention is that rows are ordered from bottom to top, so that row 1 is the lowest row and row n the top-most. For example, the left of Figure 1 shows the matrix corresponding to the permutation 352164. Alternatively, we may represent an $n \times n$ permutation matrix as a 0-1-filling of a square arrangement of cells, with n cells in each row and in each column, such that each row and each column contain exactly one 1 and otherwise 0's. In the interest of better readability, instead of 1's we write X's, and we suppress the 0's, see the right of Figure 1.

$$M(352164) = \begin{pmatrix} 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}$$

				X	
	X				
					X
X					
		X			
			X		

Figure 1. Matrix representation of 352164

Clearly, a permutation π avoids the permutation σ if and only if $M(\pi)$ does not contain $M(\sigma)$ as a submatrix. For example, our permutation 352164 avoids 1324 but does not avoid 231.

Pattern avoidance in words can also be formulated in terms of matrices. We may represent a word $w = w_1 w_2 \cdots$ with a_1 letters 1, a_2 letters 2, ..., a_m letters m as a 0-1-filling of an $m \times (a_1 + a_2 + \cdots + a_m)$ rectangle, by placing a 1 into the j-th column and w_j-th row, $j = 1, 2, \dots, a_1 + a_2 + \cdots + a_m$, all other entries being 0. We denote this 0-1-filling by $M(w)$. For example, the word 213314242 is represented by the filling of Figure 2.

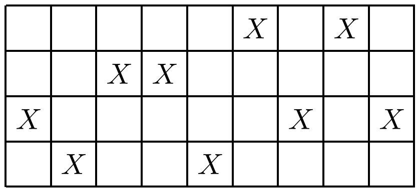

FIGURE 2. The 0-1-filling corresponding to 213314242

Similarly as for permutations, a word w avoids the word x if and only if $M(w)$ does not contain $M(x)$ as a submatrix. For example, our word 213314242 avoids 3112 but does not avoid 123.

We are now in the position to define (strong) Wilf-equivalence and shape-Wilf-equivalence for words. Let λ be a Ferrers shape, which is a left-justified arrangement of cells with the property that the row-lengths are non-increasing from bottom to top. (That is, we use the French convention when we represent Ferrers shapes.) As usual, we encode λ in terms of $(\lambda_1, \lambda_2, \dots)$ where λ_i is the length of the i-th row of λ (counted from bottom to top). We write $W_\lambda^{(a_1,a_2,\dots)}(x)$ for the set of all 0-1-fillings with *exactly one* 1 in each column, with a_i 1's in row i, $i = 1, 2, \dots$, and which avoid x. Here, for a 0-1-filling F of the Ferrers shape λ to avoid x means that there do not exist rows $r_1, r_2, \dots$ and columns $c_1, c_2, \dots$ such that the entries of F corresponding to these rows and columns form a matrix which is identical with $M(x)$. Phrased differently, the important point is that the *complete* matrix $M(x)$ is found as a submatrix in the filling. Figure 3 shows a 0-1-filling of shape $(10, 10, 10, 7, 4, 4)$ with exactly one 1 in each column. It avoids for example the pattern 4312.

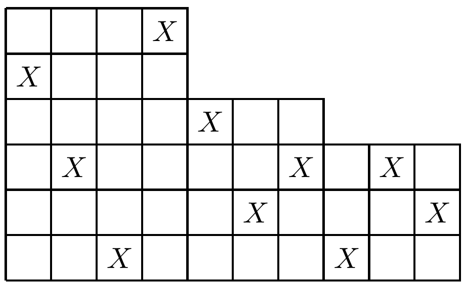

FIGURE 3. A 0-1-filling of shape $(10, 10, 10, 7, 4, 4)$

More generally, given a set Ω of words, we write $W_\lambda^{(a_1,a_2,\dots)}(\Omega)$ for the analogous set of 0-1-fillings all of which avoid all $x \in \Omega$.

We call two words x and y *strongly Wilf-equivalent for words* if

$$|W_{R(\mathbf{a})}^{(a_1,a_2,\dots)}(x)| = |W_{R(\mathbf{a})}^{(a_1,a_2,\dots)}(y)| \tag{2.1}$$

for all finite sequences $\mathbf{a} = a_1, a_2, \dots$ of positive integers, where $R(\mathbf{a})$ denotes the $m\times(a_1+a_2+\cdots)$ rectangle. Equivalently, in view of the earlier explained representations of words in terms of fillings, the words x and y are strongly Wilf-equivalent for words if and only if the number of words consisting of exactly a_i letters i, $i = 1, 2, \dots$, and avoiding x is the same as the number of words consisting of exactly a_i letters i, $i = 1, 2, \dots$, and avoiding y, for all finite sequences $a_1, a_2, \dots$ of positive integers. On the other hand, "ordinary" Wilf-equivalence of words x and y means that $|W_{n,m}(x)| = |W_{n,m}(y)|$ for all n and m.

We call two words x and y *shape-Wilf-equivalent for words* if

$$|W_{\lambda}^{(a_1,a_2,\dots)}(x)| = |W_{\lambda}^{(a_1,a_2,\dots)}(y)| \tag{2.2}$$

for all finite sequences $\mathbf{a} = a_1, a_2, \dots$ of positive integers and all shapes λ. We extend the notions of strong Wilf-equivalence and shape-Wilf-equivalence to *sets* of words, say Ω and Σ, by requiring that (2.1) respectively (2.2) hold with x replaced by Ω and y replaced by Σ for all finite sequences $a_1, a_2, \dots$ of positive integers and shapes λ.

If we restrict the above definitions to $a_1 = a_2 = \cdots = 1$, then they reduce to Wilf-equivalence and shape-Wilf-equivalence for permutations.

Finally, given a word $x = x_1x_2\cdots x_r$ with letters from $\{1, 2, \dots, m\}$ and another word $y = y_1y_2\cdots y_s$, we define the *direct sum* $x \oplus y$ by

$$x \oplus y = x_1x_2\cdots x_r(y_1+m)(y_2+m)\cdots(y_s+m).$$

3. Counterexamples

We have done calculations by computer for the word patterns 231 and 312. Consider the shapes $\lambda_1 = (5,5,4)$, $\lambda_2 = (5,5,5,4)$, and $\lambda_3 = (6,6,6,4)$. Table 1 presents the results of our computations for all possible sequences $\mathbf{a} = a_1, a_2, \dots$ of positive integers. (Recall that, since our 0-1-fillings always have exactly one 1 in each column, the sum of the a_i's must equal the length of the longest row of the shape. Thus, for λ_1 and λ_2 the sum of the a_i's must be 5, whereas for λ_3 the sum of the a_i's must be 6.)

$\mathbf{a}$	(221)	(212)	(122)	(311)	(131)	(113)
$\lvert W_{\lambda_1}^{\mathbf{a}}(231)\rvert$	18	15	15	13	13	8
$\lvert W_{\lambda_1}^{\mathbf{a}}(312)\rvert$	18	15	15	13	13	8

$\mathbf{a}$	(2111)	(1211)	(1121)	(1112)
$\lvert W_{\lambda_2}^{\mathbf{a}}(231)\rvert$	25	26	25	21
$\lvert W_{\lambda_2}^{\mathbf{a}}(312)\rvert$	25	25	26	21

$\mathbf{a}$	(1113)	(3111)	(1311)	(1131)	(2112)	(1212)	(1122)	(2211)	(2121)	(1221)
$\lvert W_{\lambda_3}^{\mathbf{a}}(231)\rvert$	20	42	40	42	42	39	42	52	54	52
$\lvert W_{\lambda_3}^{\mathbf{a}}(312)\rvert$	20	42	42	42	42	42	39	54	53	53

Table 1

We make the following observations.

(1) For λ_1 and arbitrary sequences $\mathbf{a} = a_1, a_2, a_3$ of positive integers, the sets $W_{\lambda_1}^{(a_1,a_2,a_3)}(231)$ and $W_{\lambda_1}^{(a_1,a_2,a_3)}(312)$ have the same cardinality.

(2) For λ_2, although the total numbers of 0-1-fillings with exactly one 1 in each column and at least one 1 in each row that avoid 231 respectively 312 are the same, there exist positive integers a_1, a_2, a_3, a_4 for which $|W_{\lambda_2}^{(a_1,a_2,a_3,a_4)}(231)|$ is different from $|W_{\lambda_2}^{(a_1,a_2,a_3,a_4)}(312)|$.

(3) For λ_3, even the total numbers do not match. Instead, as one lets a_1, a_2, a_3, a_4 grow, one finds more and more sequences of positive integers a_1, a_2, a_3, a_4 for which $|W_{\lambda_3}^{(a_1,a_2,a_3,a_4)}(231)|$ is different from $|W_{\lambda_3}^{(a_1,a_2,a_3,a_4)}(312)|$.

This establishes the following fact.

Fact. *The patterns* 231 *and* 312 *are not shape-Wilf-equivalent for words.*

Consequently, Proposition 6 cannot be applied to conclude that $231 \oplus \beta$ and $312 \oplus \beta$ are (strongly) Wilf-equivalent for words for arbitrary patterns β. However, the latter might still be true, at least for some patterns β. In order to clarify that point as well, we made more computer calculations.

First, we looked at the case where $\beta = 1$, that is, at words (and, more generally, 0-1-fillings) avoiding 2314 respectively 3124. We found that

$$|W_{7,5}(2314)| = 67853 \neq 67854 = |W_{7,5}(3124)|,$$

and, with $\lambda_4 = (7, 7, 7, 7, 7)$, the finer enumerations

$$|W_{\lambda_4}^{(1,2,1,2,1)}(2314)| = 908 \neq 909 = |W_{\lambda_4}^{(1,2,1,2,1)}(3124)|.$$

See Table 2 for more details.

$\mathbf{a}$	(31111)	(13111)	(11311)	(11131)	(11113)
$\lvert W_{\lambda_4}^{\mathbf{a}}(2314)\rvert$	640	640	640	635	640
$\lvert W_{\lambda_4}^{\mathbf{a}}(3124)\rvert$	640	640	640	635	640

$\mathbf{a}$	(22111)	(21211)	(21121)	(21112)	(12211)	(12121)	(12112)	(11221)	(11212)	(11122)
$\lvert W_{\lambda_4}^{\mathbf{a}}(2314)\rvert$	913	913	909	913	913	908	913	909	913	909
$\lvert W_{\lambda_4}^{\mathbf{a}}(3124)\rvert$	913	913	909	913	913	909	913	909	913	909

TABLE 2

Furthermore, we have

$$|W_{8,5}(2314)| = 310540 \neq 310563 = |W_{8,5}(3124)|.$$

Table 3 shows that, with $\lambda_5 = (8, 8, 8, 8, 8)$, in fact

$$|W_{\lambda_5}^{(a_1,a_2,a_3,a_4,a_5)}(2314)| \neq |W_{\lambda_5}^{(a_1,a_2,a_3,a_4,a_5)}(3124)|$$

for all possible sequences $(a_1, a_2, a_3, a_4, a_5)$ of positive integers.

$\mathbf{a}$	(13121)	(12131)	(11231)	(22121)	(12221)	(12122)	(11222)
$\lvert W_{\lambda_5}^{\mathbf{a}}(2314)\rvert$	2258	2245	2251	3162	3164	3163	3167
$\lvert W_{\lambda_5}^{\mathbf{a}}(3124)\rvert$	2263	2251	2250	3167	3167	3167	3168

TABLE 3

For the case $\beta = 12$, that is, for 23145- respectively 31245-avoiding words (0-1-fillings), we found that

$$|W_{8,6}(23145)| = |W_{8,6}(23154)| = 1640298 \neq 1640299 = |W_{8,6}(31245)| = |W_{8,6}(31254)|.$$

Here, the equalities follow from reversal and complementation of patterns and Corollary 8.

λ	(6664)	(88844)	(99933)	(99944)	(775333)	(777744)	(888664)	(888844)	(987654)
$\lvert W_\lambda(231)\rvert$	425	4443	6177	13435	70	1012	6232	6160	6183
$\lvert W_\lambda(312)\rvert$	429	4443	6177	13435	70	1012	6352	6160	6303

λ	(987655)	(987755)	(996644)	(997655)	(997744)	(997755)	(999663)	(999755)	(999944)
$\lvert W_\lambda(231)\rvert$	7301	9133	6130	14602	12870	18266	21549	30517	28036
$\lvert W_\lambda(312)\rvert$	7375	9213	6130	14750	12870	18426	21645	31185	28036

Table 4

The data seem to suggest the following.

Conjecture 10. *For all non-empty permutations β, the patterns $231 \oplus \beta$ and $312 \oplus \beta$ are not Wilf-equivalent for words.*

On the other hand, based on more computational data (see for example Table 4), we suspect that the following may be true.

Conjecture 11. *For every Ferrers shape λ, we have $|W_\lambda(231)| \le |W_\lambda(312)|$, where $W_\lambda(x)$ denotes the set of* all *0-1-fillings of shape λ avoiding x with exactly one 1 in each column.*

Remark. By going through the proof of Proposition 6 in [7], one sees that the validity of Conjecture 11 would imply that

$$|W_\lambda(231 \oplus \beta)| \le |W_\lambda(312 \oplus \beta)|$$

for all patterns β.

4. "Modified" shape-Wilf-equivalence of 231 and 312 for words

We now show that a shape-Wilf-equivalence result can be obtained if we do not insist on the "strictness" of the patterns 231 and 312. Namely, if in these patterns we include instances where there is equality between the letters corresponding to the '2' and the '3' in the patterns, or alternatively if we include instances where there is equality between the letters corresponding to the '1' and the '2' in the patterns, then shape-Wilf-equivalence results do hold. The precise formulations are given in the two theorems below.

Theorem 12. *The sets of patterns $\{231, 221\}$ and $\{312, 212\}$ are shape-Wilf-equivalent for words.*

Theorem 13. *The sets of patterns $\{231, 121\}$ and $\{312, 211\}$ are shape-Wilf-equivalent for words.*

Before we are able to prove these two theorems, we need to recall the essential ingredients of the bijective proof of Theorem 5 given by Bloom and Saracino in [3].

Shape-Wilf-equivalence for permutations (as defined in Section 2; see the paragraph after (2.2)) involves 0-1-fillings with *exactly one* 1 *in each column*, and with *exactly one* 1 *in each row* (the latter coming from the restriction $a_1 = a_2 = \cdots = 1$). We call such fillings *full rook placements* from now on.

Consider a Ferrers shape λ and a full rook placement R on it. For each vertex v (by which we mean a corner of a cell) along the right/up border of λ we assign an integer $I_R(v)$, which by definition is the length of the longest increasing chain of 1's in the region to the left and below of v. The right of Figure 4 shows the numbers $I_R(v)$ for the particular full rook placement R presented there, as well does the left of Figure 5. (At this point, the varying thickness of lines should be ignored.) Given a full rook placement R on λ, we denote the sequence of numbers $I_R(v)$, where v ranges over the vertices along the right/up border of λ by $I(R)$.

Bloom and Saracino show the following:

(1) A full rook placement R on λ that is 231-avoiding (as a 0-1-filling) is uniquely determined by $I(R)$ (see [3, Theorem 2]).
(2) The possible sequences $I(R)$, where R is a 231-avoiding full rook placement on λ, have a simple characterisation, by means of the so-called 231-*conditions* (see [3] for their definition).
(3) A full rook placement R on λ that is 312-avoiding (as a 0-1-filling) is uniquely determined by $I(R)$ (see [3, Theorem 2]).
(4) The possible sequences $I(R)$, where R is a 312-avoiding full rook placement on λ, have a simple characterisation, by means of the so-called 312-*conditions* (see [3] for their definition).
(5) A bijection α from 231- to 312-avoiding full rook placements of λ can be defined as follows: let R_1 be a 231-avoiding full rook placement on λ. For each vertex v along the right/up border of λ calculate the number

$$\begin{cases} 0, & \text{if } I_{R_1}(v) = 0, \\ N_{R_1}(v) - I_{R_1}(v) + 1, & \text{otherwise,} \end{cases} \tag{4.1}$$

where $N_R(v)$ denotes the total number of 1's in a full rook placement R in the region to the left and below of v. This defines a new sequence of non-negative numbers. Let R_2 be the uniquely determined 312-avoiding full rook placement corresponding to that sequence. The map $\alpha : R_1 \to R_2$ is a bijection.

It should be noted that the full rook placements on the right of Figure 4 and on the left of Figure 5 correspond to each other under the bijection α.

Proof of Theorem 12. Let λ be a Ferrers shape and $a_1, a_2, \dots$ a sequence of positive integers. We have to prove that $|W_\lambda^{(a_1,a_2,\dots)}(231, 221)| = |W_\lambda^{(a_1,a_2,\dots)}(312, 212)|$. We are going to achieve this by constructing a bijection between the sets $W_\lambda^{(a_1,a_2,\dots)}(231, 221)$ and $W_\lambda^{(a_1,a_2,\dots)}(312, 212)$.

Let $T_1 \in W_\lambda^{(a_1,a_2,\dots)}(231, 221)$. In general, the filling T_1 is not a full rook placement since it may contain several 1's in a row. However, we convert T into a full rook placement by replacing row i by a_i rows in the new filling, $i = 1, 2, \dots$, and rearranging the a_i 1's in an increasing fashion from bottom/left to top/right so that each of these 1's stays in its column but each of the a_i rows contains exactly one 1. For later reference, we call the region covered by these a_i rows the *i-th band*, and the obtained full rook placement R_1. It should be noted that the original filling is $\{231, 221\}$-avoiding if and only if the new filling — which necessarily is a full rook placement — is 231-avoiding. This construction is illustrated in Figure 4. The left of the figure shows a filling in $W^{(2,2,3,1,1,1)}_{(10,10,10,7,4,4)}(231, 221)$. The filling on the right shows the result of the above described conversion. In the figure, the separations between the original rows are indicated by thick lines, whereas the newly created rows are separated by thin lines. The resulting filling is in $W^{(1,1,\dots,1)}_{\lambda_1}(231)$, where $\lambda_1 = (10, 10, 10, 10, 10, 10, 10, 7, 4, 4)$.

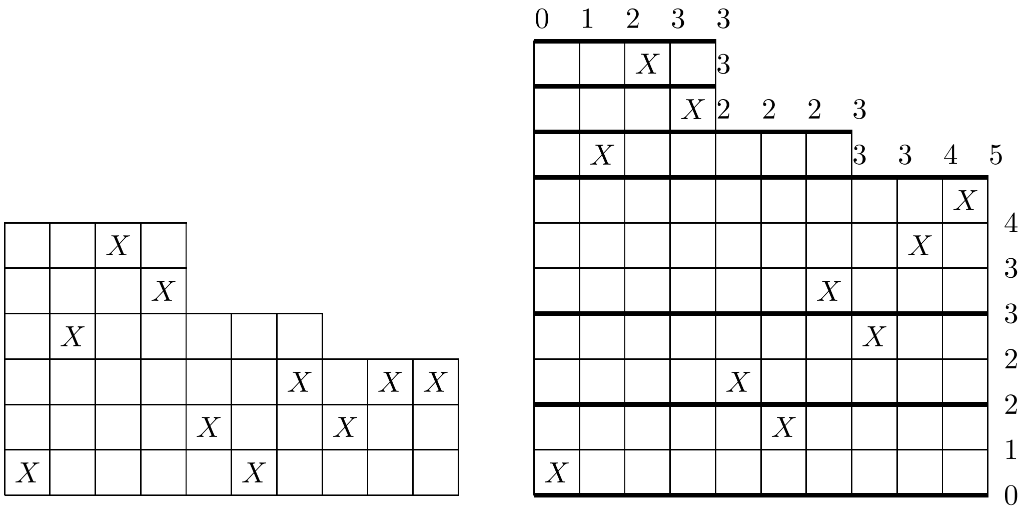

FIGURE 4. The blowup of a filling

We now apply the bijection α to R_1, to obtain a 312-avoiding full rook placement R_2. The left of Figure 5 shows the result when we apply α to the full rook placement on the right of Figure 4.

CLAIM 1. *In the i-th band, the 1's in R_2 are arranged in a decreasing fashion, from top/left to bottom/right, for each i.*

Finally, we shrink the i-th band back to a single row, putting a 1 in those columns where R_2 contained a 1. We denote the obtained filling by T_2. Since R_2 was 312-avoiding, the "compressed" filling will be $\{312, 212\}$-avoiding — if we take the above claim for granted. Thus, the filling T_2 is in $W^{(a_1,a_2,\dots)}_{\lambda}(312, 212)$. Figure 5 shows the above shrinking process applied to our running example.

From the construction, it is also obvious how the inverse mapping works. It involves however another claim.

CLAIM 2. *If the inverse of α is applied to a full rook placement with the property that, in the i-th band, the 1's are arranged in decreasing fashion, then we obtain a full rook placement where, in the i-th band, the 1's are arranged in increasing fashion, $i = 1, 2, \dots$.*

If we assume the truth of these two claims then the proof of Theorem 12 is complete.

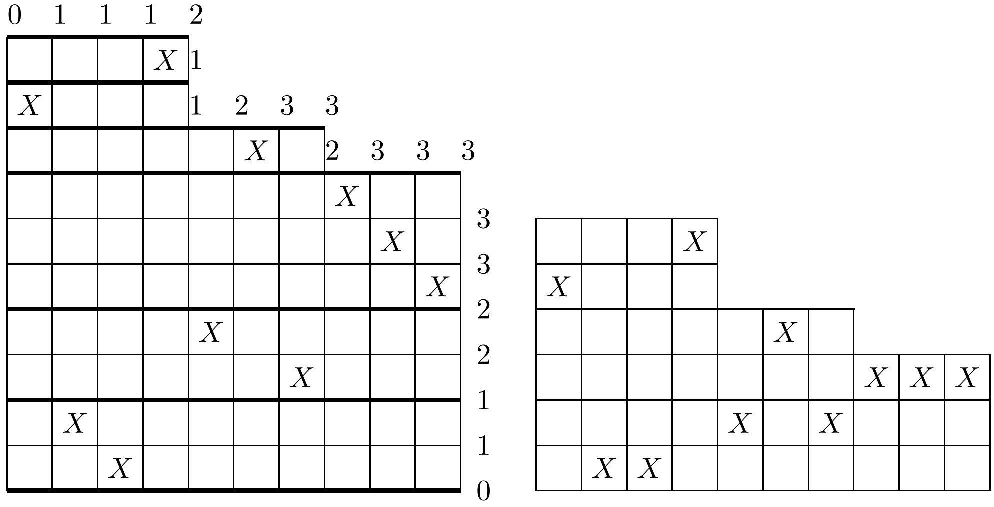

FIGURE 5. The shrinking of a full rook placement

We now prove Claim 1. We want to establish that, if R_1 is a 231-avoiding full rook placement with 1's arranged in increasing fashion in the i-th band, then $R_2 = \alpha(R_1)$ has 1's arranged in decreasing fashion in the i-th band, $i = 1, 2, \ldots$. Let us concentrate on the i-th band of R_1. We denote the vertices along the right border of the band by $u_0, u_1, \ldots, u_{a_i}$. In order to follow the next arguments, it might be helpful to look at Figure 6. The left of Figure 6 is meant to be the sketch of the i-th band (with $a_i = 4$). The 1's in the full rook placement R_1 are indicated by $X_1, X_2, \ldots$.

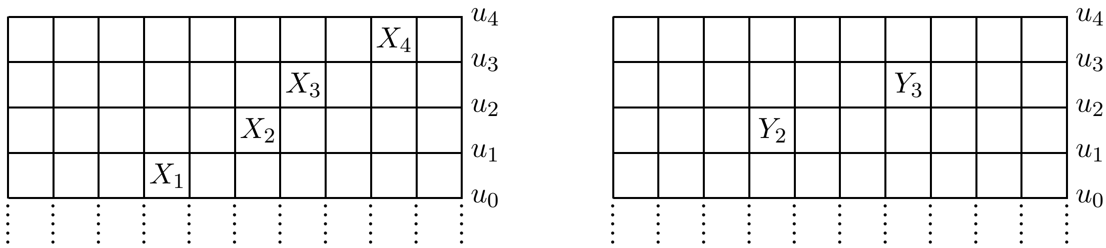

FIGURE 6. The i-th band

We claim that

$$I_{R_1}(u_{j+1}) = I_{R_1}(u_j) + 1, \quad \text{for } j = 1, 2, \ldots, a_i - 1. \tag{4.2}$$

Indeed, by definition, $I_{R_1}(u_1)$ is the length of the longest increasing chain of 1's in the region to the left and below of u_1. Since there cannot be any 1's in the region to the right and below of X_2 because R_1 is 231-avoiding, we must have $I_{R_1}(u_2) = I_{R_1}(u_1) + 1$. The same argument shows $I_{R_1}(u_3) = I_{R_1}(u_2) + 1$, etc., thus establishing our claim in (4.2).

In order to apply the bijection α, we have to calculate the numbers in (4.1), which then become lengths of increasing chains $I_{R_2}(v)$ for some 312-avoiding full rook placement R_2. Because of (4.2), we have

$$I_{R_2}(u_{j+1}) = I_{R_2}(u_j), \quad \text{for } j = 1, 2, \ldots, a_i - 1. \tag{4.3}$$

Let us suppose that R_2 has two 1's in the i-th band which are in increasing order. Without loss of generality, we may assume that these two 1's are located in neighbouring rows. The right of Figure 6 is meant to illustrate this situation, with Y_2 and Y_3 indicating the positions of these two 1's. Since R_2 is 312-avoiding, 1's in the region to the right and below of Y_2 must be arranged in decreasing order. Thus, together with Y_2, they form a decreasing chain. The number $I_{R_2}(u_2)$ gives the length of the longest increasing chain in the region to the left and below of u_2. Because of the above observed arrangement of 1's to the right and below of Y_2, one of the chains with length $I_{R_2}(u_2)$ is one which ends in Y_2. However, this implies that $I_{R_2}(u_3) = I_{R_2}(u_2) + 1$, in contradiction with (4.3). Thus, Y_2 and Y_3 cannot be arranged in increasing fashion. The same argument applies to any two neighbouring rows in the i-th band. Thus, we have proven Claim 1.

The only remaining task for the completion of the proof of the theorem is to verify Claim 2. This is however completely analogous and is left to the reader. □

Proof of Theorem 13. This can be proven in a fashion which is analogous to the proof of Theorem 12. The major difference is that, here, a $\{231, 121\}$-avoiding filling is converted into a 231-avoiding full rook placement by rearranging the a_i 1's in the i-th band in *decreasing* fashion, while a $\{312, 211\}$-avoiding filling is converted into a 312-avoiding full rook placement by rearranging the a_i 1's in the i-th band in *increasing* fashion. We leave the details to the reader. □

In view of Proposition 6 and the fact that it can easily be extended to *sets* of patterns, Theorems 12 and 13 entail the following corollaries.

Corollary 14. *For any word β, the sets of patterns $\{231 \oplus \beta, 221 \oplus \beta\}$ and $\{312 \oplus \beta, 212 \oplus \beta\}$ are shape-Wilf-equivalent for words.*

Corollary 15. *For any word β, the sets of patterns $\{231 \oplus \beta, 121 \oplus \beta\}$ and $\{312 \oplus \beta, 211 \oplus \beta\}$ are shape-Wilf-equivalent for words.*

Remark. It is interesting to note that $\{312, 212\}$-avoiding words appear in a "Catalan" context in [5]. Let $\mathbf{s} = (s_1, s_2, \dots, s_a)$ be an a-tuple of positive integers. In the above paper, Gessel and Stanley's [6] Stirling permutations are generalised there to Stirling $\mathbf{s}$-(multi)permutations, which by definition are — in our language here — 212-avoiding words consisting of s_1 letters 1, s_2 letters 2, …, and s_a letters a. It is shown (see [5, Theorem 2.1]) that 312-avoiding Stirling $\mathbf{s}$-permutations are in bijection with numerous other combinatorial objects subject to certain restrictions, including planted plane trees, lattice paths, non-crossing partitions, and dissections of polygons.

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