COUNTING TRIANGULATIONS OF SOME CLASSES OF SUBDIVIDED CONVEX POLYGONS

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Abstract. We compute the number of triangulations of a convex k-gon each of whose sides is subdivided by $r-1$ points. We find explicit formulas and generating functions, and we determine the asymptotic behaviour of these numbers as k and/or r tend to infinity. We connect these results with the question of finding the planar set of points in general position that has the minimum possible number of triangulations — a well-known open problem from computational geometry.

1. Introduction

Let k and r be two natural numbers, $k \geq 3$, $r \geq 1$. Let $\mathrm{SC}(k,r)$ denote a convex k-gon in the plane each of whose sides is subdivided by $r-1$ points. (Thus, the whole configuration consists of kr points.) In what follows, the exact measures are not essential: without loss of generality, we may consider a regular k-gon with sides subdivided by evenly spaced points. The k vertices of the original ("basic") k-gon will be called *corners*, and they will be denoted (say, clockwise) by $P_{0,0}, P_{1,0}, \dots, P_{k-1,0}$ (with arithmetic modulo k in the first index, so that $P_{k,0} = P_{0,0}$). The $r-1$ points that subdivide the segment $P_{i,0}P_{i+1,0}$ (oriented from $P_{i,0}$ to $P_{i+1,0}$) will be denoted by $P_{i,1}, P_{i,2}, \dots, P_{i,r-1}$ (we shall also occasionally write $P_{i,r}$ for $P_{i+1,0}$). The subdivided segments $P_{i,0}P_{i+1,0}$ — that is, the point sequences of the form $P_{i,0}, P_{i,1}, P_{i,2}, \dots, P_{i,r-1}, P_{i+1,0}$ — will be referred to as *strings*. Thus, the boundary of $\mathrm{SC}(k,r)$ consists of k strings, and each corner belongs to two strings. The reader is referred to Figure 1 for an illustration. For brevity, a convex polygon with subdivided edges (not all of them necessarily subdivided by the same number of points) will be referred to as a *subdivided convex polygon*. A subdivided convex polygon is *balanced* if (as described above) all its sides are subdivided by the same number of points.

A *triangulation* of a finite planar point set S is a dissection of its convex hull by non-crossing diagonals[1] into triangles. We emphasize that maximal triangulations are meant; in particular, no triangle can have another point of the set in the interior of one of its sides. The set of triangulations of a point set S will be denoted by $\mathsf{TR}(\mathrm{S})$.

Key words and phrases. Geometric graphs, triangulations, generating functions, asymptotic analysis, Chebyshev polynomials, saddle-point method.

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[1] By a "diagonal" we mean a straight-line segment connecting two points of the set S.

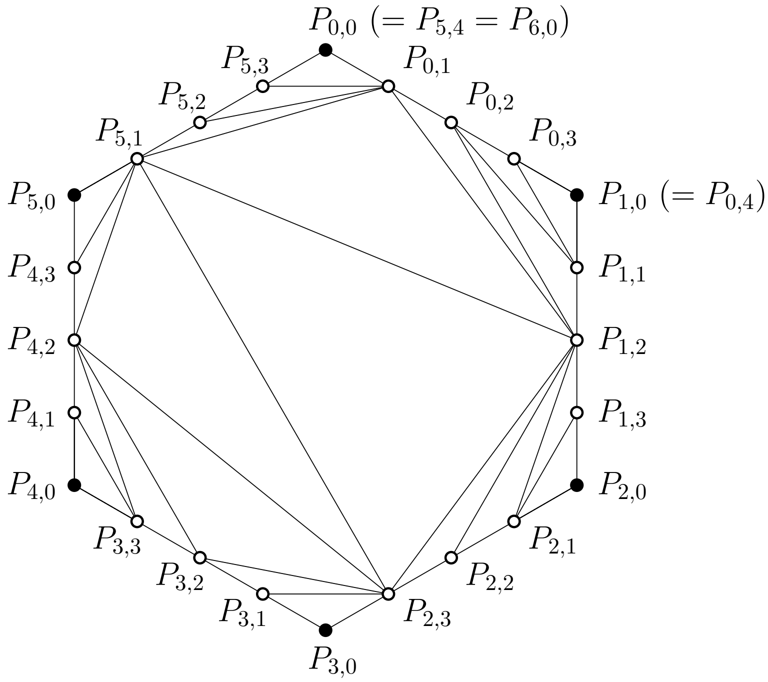

FIGURE 1. The subdivided convex polygon $\mathrm{SC}(6,4)$ and one of its triangulations.

Triangulations of (structures equivalent or related to) subdivided convex polygons have appeared in earlier work. Hurtado and Noy [11] considered triangulations of *almost convex polygons*, which turn out to be equivalent to subdivided convex polygons according to our terminology. They dealt with the non-balanced case — that is, k-gons whose sides are subdivided, but not necessarily into the same number of points. In particular, Hurtado and Noy derived an inclusion-exclusion formula for the number of triangulations of a subdivided convex k-gon whose sides are subdivided by $a_1, a_2, \dots, a_k$ points, and they showed that this number is independent of the specific distribution of the subdivisions among the sides of the basic k-gon. On the other hand, Bacher and Mouton [6, 7] considered triangulations of more general *nearly convex polygons* defined as infinitesimal perturbations of subdivided convex polygons. They derived a formula for the number of triangulations of such polygons in terms of certain polynomials that depend on the shape of chains.

The main purpose of our paper is to present enumeration formulas and precise asymptotic results for the number of triangulations of a subdivided convex polygon in the balanced case, that is, where each side of the polygon is subdivided into the same number of points. Our enumeration formulas are more compact than those of Hurtado and Noy or of Bacher and Mouton when specialised to the balanced case. We shall as well provide formulas for some non-balanced cases.

Let us denote the number of triangulations of $\mathrm{SC}(k,r)$ by $\mathsf{tr}(k,r)$. For $r = 1$ our configuration is just a convex k-gon, and, thus, $\mathsf{tr}(k,1) = C_{k-2}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. It is easy to find $\mathsf{tr}(k,r)$ for small values of k and r by

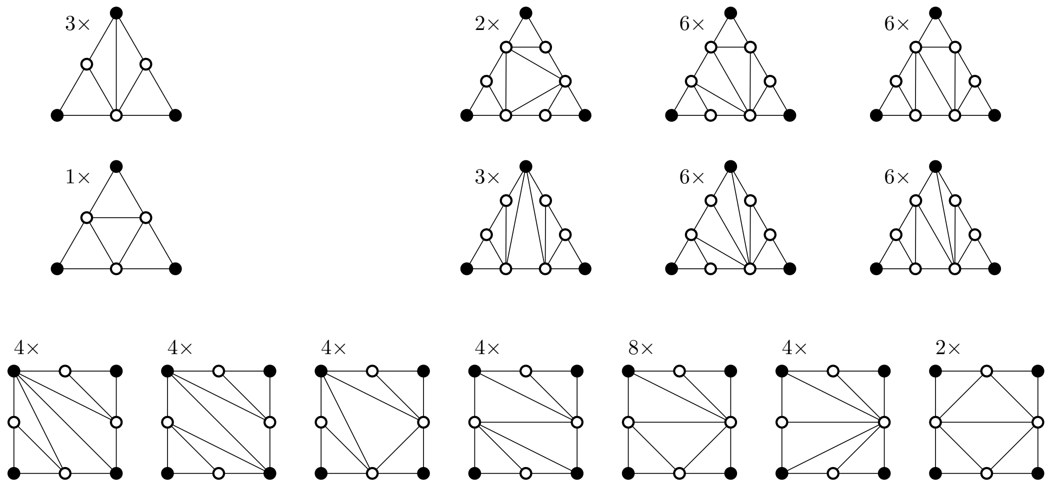

FIGURE 2. All triangulations of SC(3, 2), SC(3, 3) and SC(4, 2).

inspection. For example, we have $\mathsf{tr}(3,2) = 4$, $\mathsf{tr}(3,3) = 29$ and $\mathsf{tr}(4,2) = 30$; see Figure 2 (there, symmetries must also be taken into account; for each triangulation it is shown how many different triangulations can be obtained from it under symmetries). Values of $\mathsf{tr}(k,r)$ for $1 \le k \le 7$, $1 \le r \le 6$ are shown in Table 1; the meaning of these values for $k = 2$ — the central binomial coefficients — will be explained in Section 2 (see the remark after the proof of Theorem 4). The sequence $(\mathsf{tr}(k,2))_{k\ge3}$ is OEIS/A086452, while the sequence $(\mathsf{tr}(3,r))_{r\ge1}$ is OEIS/A087809 [14].

In the next section, we derive our formulas for the numbers $\mathsf{tr}(k,r)$. They are given in the form of double sums, see Theorem 4, thus answering an open question posed in [11]. These formulas come from a representation of $\mathsf{tr}(k,r)$ in terms of a complex contour integral (see Proposition 3), when interpreted as a coefficient extraction formula. We use this integral representation to prove in Section 3 that the "vertical" generating functions $\sum_{k\ge2} \mathsf{tr}(k,r)x^k$ as well as the "horizontal" generating functions $\sum_{r\ge1} \mathsf{tr}(k,r)x^r$ are all algebraic. More precisely, we find explicit expressions for these generating functions in terms of roots of certain (explicit) polynomials. We devote a separate section, Section 4, to the special case $k = 3$, since in that case several alternative formulas that are more attractive than the formulas in Theorem 4 are available. Moreover, in Section 5 we also consider the *non-balanced* case of $k = 3$: we count triangulations of a triangle whose sides are subdivided by a, b, and c points, respectively. The resulting compact formulas are presented in Propositions 8 and 9. Then, in Section 6, we determine the asymptotic behaviour of $\mathsf{tr}(k,r)$ as r and/or k tend to infinity, see Theorems 11 and 12. This is achieved by transforming the contour integral into a complex integral along a line in the complex plane parallel to the imaginary axis that passes through the saddle point of the integrand. In the final Section 7, we connect our results with a well-known open problem from computational geometry: the problem of determining a planar set of n points in general position with the minimum number of triangulations. We show that

	$r=1$	2	3	4	5	6
$k=2$	1	1	2	6	20	70
3	1	4	29	229	1847	14974
4	2	30	604	12168	238848	4569624
5	5	250	13740	699310	33138675	1484701075
6	14	2236	332842	42660740	4872907670	510909185422
7	42	20979	8419334	2711857491	745727424435	182814912101920

TABLE 1. Values of $\mathsf{tr}(k,r)$ for $2 \le k \le 7$, $1 \le r \le 6$. (The meaning of the values for $k=2$ is explained after the proof of Theorem 4.)

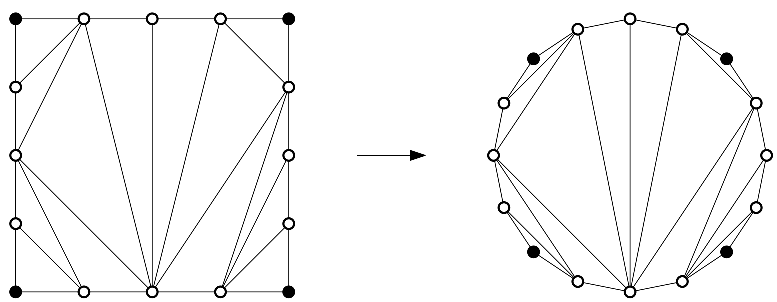

FIGURE 3. Injection $\varphi_{k,r}$ from $\mathsf{TR}(\mathrm{SC}(k,r))$ to $\mathsf{TR}(\mathrm{C}(k\cdot r))$

our results support a conjecture of Aichholzer, Hurtado and Noy [3] that this minimum is attained by the so-called *double circle*.

2. A formula for $\mathsf{tr}(k,r)$

In this section we derive two — very similar — double sum formulas for $\mathsf{tr}(k,r)$, given in (2.7) and (2.8). Starting point for finding these double sum expressions is the inclusion-exclusion formula (2.2), which is equivalent to that found in [11] and in [6, 7]. We include its derivation for the sake of completeness.

We start by "inflating" $\mathrm{SC}(k,r)$. That is, we replace its strings by slightly curved circular arcs so that a set of kr points in convex position is obtained. We keep the labels for these points. Denote this point set by $\mathrm{C}(k\cdot r)$. It is easy to see that each triangulation of $\mathrm{SC}(k,r)$ is transformed into a triangulation of $\mathrm{C}(k\cdot r)$, see Figure 3. More formally, this "inflation" defines a natural injection $\varphi=\varphi_{k,r}$ from $\mathsf{TR}(\mathrm{SC}(k,r))$ to $\mathsf{TR}(\mathrm{C}(k\cdot r))$: for each $D\in\mathsf{TR}(\mathrm{SC}(k,r))$, triangulation $\varphi(D)\in\mathsf{TR}(\mathrm{C}(k\cdot r))$ uses the diagonals with the same labels as D. Thus $\mathsf{tr}(k,r)$ is the size of the image of φ. We say that a triangulation of $\mathrm{C}(k\cdot r)$ is *legal* if it belongs to the image of φ — that is, corresponds to a (unique) triangulation of $\mathrm{SC}(k,r)$. It is easy to see the following.

Observation 1. *Let T be a triangulation of $\mathrm{C}(k\cdot r)$. T is legal if and only if it uses no diagonal whose endpoints belong to the same string (that is, to the set $\{P_{i,0},P_{i,1},\dots,P_{i,r-1},P_{i+1,0}\}$ for some i).*

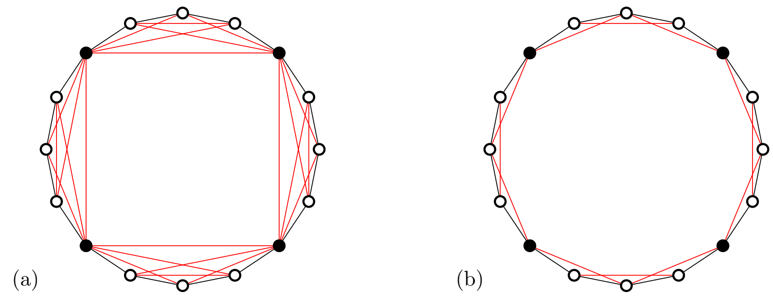

FIGURE 4. Forbidden (a) and essentially forbidden (b) diagonals of $\mathrm{C}(4\cdot 4)$.

We call the diagonals mentioned in Observation 1 *forbidden*, and we need to exclude triangulations that contain them from the set of all the triangulations of $\mathrm{C}(k\cdot r)$. Notice, however, that, if a triangulation of $\mathrm{C}(k\cdot r)$ uses some forbidden diagonal, then it necessarily (also) uses a forbidden diagonal that connects two points at distance 2 along the boundary of $\mathrm{C}(k\cdot r)$. Therefore, the characterization of legal triangulations from Observation 1 can be simplified as follows.

Observation 2. *Let T be a triangulation of $\mathrm{C}(k\cdot r)$. T is legal if and only if it uses no diagonal of the form $P_{i,j}P_{i,j+2}$ with $0\le i\le k-1$ and $0\le j\le r-2$.*

We call the diagonals mentioned in Observation 2 *essentially forbidden*. Figure 4 shows (a) forbidden and (b) essentially forbidden diagonals of $\mathrm{C}(4\cdot 4)$.

Thus, we need to exclude triangulations of $\mathrm{C}(k\cdot r)$ that use essentially forbidden diagonals. The total number of essentially forbidden diagonals is $k(r-1)$, but the neighbouring essentially forbidden diagonals (that is, $P_{i,j}P_{i,j+2}$ and $P_{i,j+1}P_{i,j+3}$ for some i and j with $0\le i\le k-1$ and $0\le j\le r-3$) cannot coexist in the same triangulation of $\mathrm{C}(k\cdot r)$. Thus, the number of possible choices of ℓ essentially forbidden diagonals from the same string, where $0\le\ell\le\lfloor r/2\rfloor$, equals the number of ℓ-subsets of $\{1,2,\dots,r-1\}$ that do not contain adjacent numbers. This is a simple exercise in elementary combinatorics, and the answer is $\binom{r-\ell}{\ell}$. Therefore, the number of ways to choose m pairwise non-crossing essentially forbidden diagonals in $\mathrm{C}(k\cdot r)$ is

$$a_{k,r,m} := [x^m]\left(\sum_{\ell=0}^{\lfloor r/2\rfloor}\binom{r-\ell}{\ell}x^\ell\right)^k,$$

where $[x^m]f(x)$ denotes the coefficient of x^m in the polynomial or formal power series $f(x)$.

Once m essentially forbidden diagonals of $\mathrm{C}(k\cdot r)$ are chosen, we are left with a convex $(kr-m)$-gon to be triangulated. Therefore, the number of illegal triangulations that use at least m essentially forbidden diagonals is $a_{k,r,m}C_{kr-m-2}$. At this point we can apply the inclusion-exclusion principle and obtain

$$\mathsf{tr}(k,r) = \sum_{m=0}^{\lfloor r/2\rfloor k}(-1)^m\,a_{k,r,m}\,C_{kr-m-2}. \tag{2.1}$$

Next, we observe that

$$\sum_{\ell=0}^{\lfloor r/2\rfloor}\binom{r-\ell}{\ell}(-x)^\ell = x^{r/2}\,U_r\left(\frac{1}{2\sqrt{x}}\right),$$

where $U_r(x)$ is the rth Chebyshev polynomial of the second kind. Thus,

$$(-1)^m\,a_{k,r,m} = [x^m]\left(x^{r/2}\,U_r\left(\frac{1}{2\sqrt{x}}\right)\right)^k,$$

and (2.1) can be rewritten as

$$\mathsf{tr}(k,r) = [x^{rk-2}]\left(\left(x^{r/2}U_r\left(\frac{1}{2\sqrt{x}}\right)\right)^k C(x)\right), \tag{2.2}$$

where

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

is the generating function for Catalan numbers. Since an explicit form of $U_r(x)$ is

$$U_r(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{r+1} - \left(x-\sqrt{x^2-1}\right)^{r+1}}{2\sqrt{x^2-1}},$$

it follows that

$$\mathsf{tr}(k,r) = [x^{rk-2}]\Bigg(\frac{1}{2^{(r+1)k}(1-4x)^{k/2}} \cdot\left(\left(1+\sqrt{1-4x}\right)^{r+1} - \left(1-\sqrt{1-4x}\right)^{r+1}\right)^k\,\frac{1-\sqrt{1-4x}}{2x}\Bigg).$$

Using Cauchy's integral formula, we may write this expression in terms of a complex contour integral, namely as

$$\mathsf{tr}(k,r) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{dx}{2^{(r+1)k+1}x^{rk}(1-4x)^{k/2}} \cdot\left(\left(1+\sqrt{1-4x}\right)^{r+1} - \left(1-\sqrt{1-4x}\right)^{r+1}\right)^k\left(1-\sqrt{1-4x}\right), \tag{2.3}$$

where $\mathcal{C}$ is a small contour encircling the origin once in positive direction. Next we perform the substitution $x = t(1-t)$, in which case $dx = (1-2t)\,dt$. This leads us to the following integral representation of our numbers $\mathsf{tr}(k,r)$.

Proposition 3. *For all positive integers k and r with $rk \ge 3$, we have*

$$\mathsf{tr}(k,r) = -\frac{1}{4\pi i}\int_{\mathcal{C}}\frac{dt}{t^{rk}(1-t)^{rk}(1-2t)^{k-2}}\left((1-t)^{r+1} - t^{r+1}\right)^k, \tag{2.4}$$

where $\mathcal{C}$ is a contour close to 0 which encircles 0 once in positive direction.

Proof. Carrying out the above described substitution in (2.3), we arrive at

$$\mathsf{tr}(k,r) = \frac{1}{2\pi i}\int_{\mathcal{C}'} \frac{(1-2t)\,dt}{t^{rk-1}(1-t)^{rk}(1-2t)^k}\left((1-t)^{r+1}-t^{r+1}\right)^k, \tag{2.5}$$

where $\mathcal{C}'$ is a(nother) contour close to the origin encircling the origin once in positive direction. In order to obtain the more symmetric form (with respect to the substitution $t \to 1-t$) in (2.4), we blow up the contour $\mathcal{C}'$ so that it is sent to infinity. While doing this, we must pass over the pole $t=1$ of the integrand. (The point $t=1/2$ is a removable singularity of the integrand.) This must be compensated by taking the residue at $t=1$ into account. The integrand is of the order $O(t^{-rk+2})$ as $|t|\to\infty$, and even of the order $O(t^{-rk+1})$ if r is odd. Together, this means that the integrand is of the order $O(t^{-2})$ as $|t|\to\infty$ for $rk\ge 3$. Hence, the integral along the contour near infinity vanishes. Thus, we obtain

$$\begin{aligned}\mathsf{tr}(k,r) &= -\operatorname{Res}_{t=1}\frac{1}{t^{rk-1}(1-t)^{rk}(1-2t)^{k-1}}\left((1-t)^{r+1}-t^{r+1}\right)^k\\ &= -\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{dt}{(1+t)^{rk-1}(-t)^{rk}(-1-2t)^{k-1}}\left((-t)^{r+1}-(1+t)^{r+1}\right)^k, \end{aligned}\tag{2.6}$$

where $\mathcal{C}$ is a contour close to 0, which encircles 0 once in positive direction. We have thus obtained two (slightly) different expressions for $\mathsf{tr}(k,r)$, namely (2.5) and (2.6). Thus, $\mathsf{tr}(k,r)$ is also equal to their arithmetic mean. If this is worked out, after having substituted $-t$ for t in (2.6), one arrives at (2.4). □

We are now in the position to derive explicit formulas for $\mathsf{tr}(k,r)$ in terms of binomial double sums.

Theorem 4. *For all positive integers k and r with $rk\ge 3$, we have*

$$\mathsf{tr}(k,r) = \sum_{j=0}^{k}\sum_{\ell=0}^{rk-(r+1)j-2}(-1)^j\,2^\ell\binom{k}{j}\binom{k-2+\ell}{\ell}\binom{(r-1)k-\ell-3}{rk-(r+1)j-\ell-2} \tag{2.7}$$

$$= \sum_{j=0}^{k}\sum_{\ell=0}^{rk-(r+1)j-1}(-1)^{j+1}\,2^{\ell-1}\binom{k}{j}\binom{k-3+\ell}{\ell}\binom{(r-1)k-\ell-2}{rk-(r+1)j-\ell-1}. \tag{2.8}$$

Proof. By Cauchy's integral formula, Equation (2.5) can also be read as

$$\mathsf{tr}(k,r) = [t^{rk-2}]\frac{1}{(1-t)^{rk}(1-2t)^{k-1}}\left((1-t)^{r+1}-t^{r+1}\right)^k.$$

If we now expand $\left((1-t)^{r+1}-t^{r+1}\right)^k$ using the binomial theorem, and subsequently do the same for powers of $1-t$ and of $1-2t$, then we are led to (2.7).

If the same is done starting from (2.4), then the formula in (2.8) is obtained. □

Remark. If we choose $k=2$ in (2.8), then the only term which does not vanish is the one with $j=1$ and $\ell=0$. This term is $\binom{2r-4}{r-2}$, a central binomial coefficient. If we interpret $\mathsf{tr}(2,r)$ (consistently with the case $k\ge 3$) as the number of triangulations of

C$(2\cdot r)$ that do not use (essentially) forbidden diagonals, then it is easy to see that this number is indeed $\binom{2r-4}{r-2}$. Indeed, there exists a well-known[2] bijective encoding of such triangulations in terms of balanced sequences over $\{0,1\}$, see Figure 5 which illustrates this encoding for $r=4$. We shall use the same idea in the proof of Theorem 8(1) below.

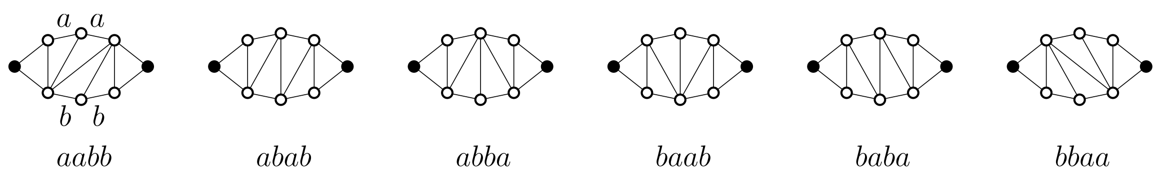

FIGURE 5. Illustration of the fact $\mathsf{tr}(2,r)=\binom{2r-4}{r-2}$.

3. GENERATING FUNCTIONS

Starting from the integral representation (2.4), we now show that "horizontal" and "vertical" generating functions for the numbers $\mathsf{tr}(k,r)$ are algebraic.

Theorem 5. *For fixed $r\ge 2$, we have*

$$\sum_{k\ge1}\mathsf{tr}(k,r)x^k=-\frac{1}{2}\sum_{i=1}^{r}\frac{t_i(x)^r(1-t_i(x))^r(1-2t_i(x))^2}{(\frac{d}{dt}P_r)(x;t_i(x))},\tag{3.1}$$

where the $t_i(x)$, $i=1,2,\dots,r$, are the "small" zeroes of the polynomial[3]

$$P_r(x;t)=t^r(1-t)^r-x\left((1-t)^{r+1}-t^{r+1}\right)(1-2t)^{-1},$$

that is, those zeroes $t(x)$ of $P_r(x;t)$ for which $\lim_{x\to0}t(x)=0$.

Proof. It should be noted that the right-hand side of (2.4) vanishes for $k=0$. Hence, multiplication of both sides of (2.4) by x^k and subsequent summation of both sides over $k=0,1,\dots$ by means of the summation formula for geometric series yield

$$\begin{aligned}\sum_{k\ge1}\mathsf{tr}(k,r)x^k&=-\frac{1}{4\pi i}\int_{\mathcal{C}}\frac{(1-2t)^2\,dt}{1-x\left((1-t)^{r+1}-t^{r+1}\right)t^{-r}(1-t)^{-r}(1-2t)^{-1}}\\&=-\frac{1}{4\pi i}\int_{\mathcal{C}}\frac{t^r(1-t)^r(1-2t)^2}{t^r(1-t)^r-x\left((1-t)^{r+1}-t^{r+1}\right)(1-2t)^{-1}}\,dt,\end{aligned}\tag{3.2}$$

provided

$$|x|<\left|\frac{t^r(1-t)^r(1-2t)}{(1-t)^{r+1}-t^{r+1}}\right|$$

for all t along the contour $\mathcal{C}$. By the residue theorem, this integral equals the sum of the residues at poles of the integrand inside $\mathcal{C}$. The poles are the "small" zeroes of

[2] For example, an encoding of this type was used by Hurtado, Noy, and Urrutia [12] for proving a lower bound on the flip distance between triangulations of polygons.

[3] $P_r(x;t)$ is indeed a polynomial in t since $1-2t$ is a polynomial divisor of $(1-t)^{r+1}-t^{r+1}$.

the denominator polynomial $P_r(x;t)$. By general theory of algebraic curves, the zeroes $t_i(x)$ of $P_r(x;t)$, $i=1,2,\dots,2r$, can be written in terms of Puiseux series in x. In order to identify the "small" zeroes, we write the equation $P_r(x;t)=0$ in the form

$$\frac{t^r(1-t)^r(1-2t)}{(1-t)^{r+1}-t^{r+1}}=x.$$

Taking the rth root, we obtain

$$\frac{t(1-t)(1-2t)^{1/r}}{\left((1-t)^{r+1}-t^{r+1}\right)^{1/r}}=\omega_r^i x^{1/r},\qquad i=1,2,\dots,r,$$

where $\omega_r=e^{2i\pi/r}$ is a primitive rth root of unity. It is easy to see that there exists a unique power series solution $t(X)$ to the equation

$$\frac{t(1-t)(1-2t)^{1/r}}{\left((1-t)^{r+1}-t^{r+1}\right)^{1/r}}=X.$$

We thus obtain the "small" zeroes of $P_r(x;t)$ as $t_i(x)=t(\omega_r^i x^{1/r})$, $i=1,2,\dots,r$. Because of the relation $P_r(x;1-t)=P_r(x;t)$, the other zeroes of $P_r(x;t)$ are $1-t_i(x)$, $i=1,2,\dots,r$, which are not "small". The $t_i(x)$ for $i=1,2,\dots,r$ are hence all "small" zeroes.

In view of the above considerations, from (3.2) we get

$$\begin{aligned}\sum_{k\ge1}\mathsf{tr}(k,r)x^k&=-\frac{1}{4\pi i}\int_{\mathcal{C}}\frac{t^r(1-t)^r(1-2t)^2}{P_r(x;t)}\,dt\\&=-\frac12\sum_{i=1}^r\operatorname{Res}_{t=t_i(x)}\frac{t^r(1-t)^r(1-2t)^2}{P_r(x;t)}\\&=-\frac12\sum_{i=1}^r\frac{t_i(x)^r(1-t_i(x))^r(1-2t_i(x))^2}{(\frac{d}{dt}P_r)(x;t_i(x))},\end{aligned}$$

as desired. □

We illustrate this theorem by considering the case where $r=2$. In this case, the polynomial $P_r(x;t)$ becomes

$$P_2(x;t)=t^2(1-t)^2-x(t^2-t+1).$$

The zeroes of this polynomial are

$$t_i(x)=\frac12\left(1\pm\sqrt{1+2x\pm2\sqrt{x+4}\sqrt{x}}\right),\qquad i=1,2,3,4.$$

The small zeroes are

$$t_1(x)=\frac12\left(1-\sqrt{1+2x-2\sqrt{x+4}\sqrt{x}}\right)\quad\text{and}\quad t_2(x)=\frac12\left(1-\sqrt{1+2x+2\sqrt{x+4}\sqrt{x}}\right).$$

If all this is used in (3.1), then we obtain

$$\sum_{k\ge1}\mathsf{tr}(k,2)x^k=\frac{1}{8}\sqrt{\frac{x}{x+4}}\left(\sqrt{1+2x+2\sqrt{x(x+4)}}\left(\sqrt{x}+\sqrt{x+4}\right)^2 -\sqrt{1+2x-2\sqrt{x(x+4)}}\left(\sqrt{x}-\sqrt{x+4}\right)^2\right)$$

after some simplification.

Theorem 6. *For fixed $k\ge2$, we have*

$$\sum_{r\ge1}\mathsf{tr}(k,r)x^r=\frac{1}{2}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{i=1}^{k-j}\frac{t_{i,j}^{j+1}(x)(1-t_{i,j}(x))^{k-j+1}}{(1-2t_{i,j}(x))^{k-2}\left(k-j-kt_{i,j}(x)\right)},\tag{3.3}$$

where the $t_{i,j}(x)$, $i=1,2,\dots,k-j$, are the "small" zeroes of the polynomial

$$Q_{j,k}(x;t)=t^{k-j}(1-t)^j-x,$$

$j=1,2,\dots,k$, that is, those zeroes $t(x)$ for which $\lim_{x\to0}t(x)=0$.

Proof. We multiply both sides of (2.4) by x^r and then sum both sides over $r=0,1,\dots$. Subsequently, we use the binomial theorem to expand $\left((1-t)^{r+1}-t^{r+1}\right)^k$ and evaluate the resulting sums over r by means of the summation formula for geometric series. Taking into account that the right-hand side of (2.4) vanishes also for $r=0$, this leads us to

$$\begin{aligned}\sum_{r=1}^{\infty}\mathsf{tr}(k,r)x^r&=-\frac{1}{4\pi i}\int_C\frac{dt}{(1-2t)^{k-2}}\sum_{j=0}^{k}(-1)^j\binom{k}{j}t^j(1-t)^{k-j}\frac{1}{1-xt^{-(k-j)}(1-t)^{-j}}\\&=-\frac{1}{4\pi i}\int_C\frac{t^k(1-t)^k\,dt}{(1-2t)^{k-2}}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{1}{t^{k-j}(1-t)^j-x}.\end{aligned}\tag{3.4}$$

The remaining arguments are completely analogous to those of the proof of Theorem 5 and are therefore left to the reader. □

4. The case $k=3$

The case of triangulations of a subdivided triangle, that is, the case where $k=3$, is particularly interesting from the point of view of exact enumeration formulas. In this section we found several such formulas; see Table 4 for the summary.

By (2.8), we know that

$$\mathsf{tr}(3,r)=-\sum_{\ell=0}^{3r-1}2^{\ell-1}\binom{3r-\ell-5}{3r-\ell-1}+3\sum_{\ell=0}^{2r-2}2^{\ell-1}\binom{3r-\ell-5}{2r-\ell-2}-3\sum_{\ell=0}^{r-3}2^{\ell-1}\binom{3r-\ell-5}{r-\ell-3}.\tag{4.1}$$

A simpler formula can be obtained if one reads coefficients from the right-hand side of (2.4) in a way that differs from the one done in the proof of Theorem 4. Namely, we write

$$\mathsf{tr}(k,r)=-\frac{1}{4\pi i}\int_C\frac{dt}{(1-2t)}\left(t^{-3r}(1-t)^3-3t^{-2r+1}(1-t)^{-r+2}+3t^{-r+2}(1-t)^{-2r+1}\right)$$

$$\mathsf{tr}(3,r) = 2^{3r-4} - 3\sum_{j=0}^{r-3}\binom{3r-4}{j}. \tag{4.3}$$

$$\mathsf{tr}(3,r) = \sum_{i,j,k\geq 0}\binom{r-1}{i+j}\binom{r-1}{j+k}\binom{r-1}{i+k}. \tag{4.4}$$

$$\mathsf{tr}(3,r+2) = 3\binom{3r+2}{r} + \sum_{j=0}^{r}\frac{5j+1}{2j+1}\binom{3j}{j}8^{r-j}. \tag{4.5}$$

$$\mathsf{tr}(3,r+1) = [x^r]\frac{1-7g(x)+17g^2(x)-10g^3(x)}{(1-3g(x))(1-8x)}, \quad \text{where } g(x)(1-g(x))^2 = x. \tag{4.8}$$

TABLE 2. Summary of Section 4: formulas for the number of triangulations of a subdivided triangle.

$$= -\frac{1}{4\pi i}\int_{\mathcal{C}}\frac{dt}{(1-2t)}t^{-3r}(1-t)^3 + \frac{3}{4\pi i}\int_{\mathcal{C}}\frac{dt}{(1-2t)}\left(t^{-2r+1}(1-t)^{-r+2} - t^{-r+2}(1-t)^{-2r+1}\right)$$
$$= -\frac{1}{4\pi i}\int_{\mathcal{C}}\frac{dt}{(1-2t)}t^{-3r}(1-t)^3 + \frac{3}{4\pi i}\int_{\mathcal{C}}\sum_{j=0}^{r}t^{-2r+1+j}(1-t)^{-r+1-j}\,dt.$$

The second integral can again be interpreted as a coefficient extraction formula. In the first integral, we blow up $\mathcal{C}$ so that it tends to the circle at infinity. While doing this, we pass over the pole at $t = 1/2$. Hence, the residue at this point must be taken into account. The integral along the circle at infinity vanishes since the integrand is of the order $O(t^{-2})$ as $|t| \to \infty$. If this is taken into account, then we obtain the alternative formula

$$\mathsf{tr}(3,r) = -2^{3r-5} + \frac{3}{2}\sum_{j=0}^{r}\binom{3r-4}{2r-2-j} = -2^{3r-5} + \frac{3}{2}\sum_{j=0}^{r}\binom{3r-4}{r-2+j}. \tag{4.2}$$

Making use of the symmetry of binomial coefficients and of the binomial theorem, it is a simple matter to verify that the above is equivalent to

$$\mathsf{tr}(3,r) = 2^{3r-4} - 3\sum_{j=0}^{r-3}\binom{3r-4}{j}. \tag{4.3}$$

We entered the sequence $(\mathsf{tr}(3,r))_{r\geq 1}$ into the On-line Encyclopedia of Integer Sequences [14]. This produced the hit OEIS/A087809, which in particular said that (according to [14] a conjecture of Benoit Cloitre) another (elegant) formula must be

$$\mathsf{tr}(3,r) = \sum_{i,j,k\geq 0}\binom{r-1}{i+j}\binom{r-1}{j+k}\binom{r-1}{i+k}. \tag{4.4}$$

We prove this conjecture, in a more general context, in the next section; see Theorem 9.

There is yet another (substantially) different formula for $\mathsf{tr}(3,r)$. By computer experiments, utilizing the guessing features of *Maple*, we were led to conjecture that

$$\mathsf{tr}(3,r+2)=3\binom{3r+2}{r}+\sum_{j=0}^{r}\frac{5j+1}{2j+1}\binom{3j}{j}8^{r-j}. \tag{4.5}$$

This formula can be established in the following way. The (already established) formula (4.3) for $\mathsf{tr}(3,r)$ satisfies the recurrence

$$\mathsf{tr}(3,r+1)-8\mathsf{tr}(3,r)=\frac{3\,(5r^2-19r+6)\,(3r-4)!}{(r-2)!\,(2r)!}. \tag{4.6}$$

This is easy to see by applying the relation

$$\binom{3r-1}{j}=\binom{3r-4}{j}+3\binom{3r-4}{j-1}+3\binom{3r-4}{j-2}+\binom{3r-4}{j-3}$$

to the binomial coefficient appearing in the definition of $\mathsf{tr}(3,r+1)$ (or by entering the sum in (4.3) into the Gosper–Zeilberger algorithm; cf. [15]). On the other hand, it is routine to verify that the expression in (4.5) (with r replaced by $r-2$) satisfies the same recurrence. Comparison of an initial value then completes the proof of (4.5).

Finally, our results also enable us to establish another conjecture reported in Entry OEIS/A087809 of [14], namely an expression for the generating function of the numbers $\mathsf{tr}(3,r)$ that is more compact than the expression produced by Theorem 6 for $k=3$. According to [14], this expression was found by Mark van Hoeij (presumably) by using his computer algebra tools. It reads

$$\sum_{r\ge1}\mathsf{tr}(3,r+1)x^r=\frac{10g^3(x)-17g^2(x)+7g(x)-1}{(1-3g(x))(2g(x)-1)(4g^2(x)-6g(x)+1)}, \tag{4.7}$$

where $g(x)(1-g(x))^2=x$. Indeed, to see this, we first observe that

$$(2g(x)-1)(4g^2(x)-6g(x)+1)=8g(x)(1-g(x))^2-1=8x-1.$$

If we use this in (4.7), then we see that van Hoeij's claim is

$$\begin{aligned}\mathsf{tr}(3,r+1)&=[x^r]\frac{1-7g(x)+17g^2(x)-10g^3(x)}{(1-3g(x))(1-8x)}\\&=\sum_{j=0}^{\infty}[x^{r-j}]8^j\frac{1-7g(x)+17g^2(x)-10g^3(x)}{(1-3g(x))}.\end{aligned} \tag{4.8}$$

The coefficient of x^{r-j} on the right-hand side is conveniently computed using the second form of Lagrange inversion (see [13, Eq. (1.2)]). We obtain

$$\begin{aligned}&[x^n]\frac{1-7g(x)+17g^2(x)-10g^3(x)}{(1-3g(x))}\\&=[x^{-1}]\frac{1-7x+17x^2-10x^3}{(1-3x)}\big(x(1-x)^2\big)^{-n-1}\frac{d}{dx}\big(x(1-x)^2\big)\\&=[x^n](1-7x+17x^2-10x^3)\,(1-x)^{-2n-1}\end{aligned}$$

This is now substituted on the right-hand side of (4.8). It yields

$$\sum_{j=0}^{\infty} 8^j \binom{3(r-j)}{r-j} - 7\sum_{j=0}^{\infty} 8^j \binom{3(r-j)-1}{r-j-1} + 17\sum_{j=0}^{\infty} 8^j \binom{3(r-j)-2}{r-j-2} - 10\sum_{j=0}^{\infty} 8^j \binom{3(r-j)-3}{r-j-3}$$
$$= \sum_{j=0}^{r} 8^{r-j} \binom{3j}{j} - 7\sum_{j=0}^{r} 8^{r-j} \binom{3j-1}{j-1} + 17\sum_{j=0}^{r} 8^{r-j} \binom{3j-2}{j-2} - 10\sum_{j=0}^{r} 8^{r-j} \binom{3j-3}{j-3}.$$

In the first sum, we shift the index by replacing j by $j-1$. Thus, we obtain

$$\binom{3r}{r} + \sum_{j=0}^{r} 8^{r-j} \left(8\binom{3j-3}{j-1} - 7\binom{3j-1}{j-1} + 17\binom{3j-2}{j-2} - 10\binom{3j-3}{j-3} \right)$$
$$= \binom{3r}{r} + \sum_{j=1}^{r} 8^{r-j} \frac{5j-4}{2j-1} \binom{3j-3}{j-1}$$
$$= \binom{3r}{r} + \sum_{j=0}^{r-1} 8^{r-1-j} \frac{5j+1}{2j+1} \binom{3j}{j}.$$

By (4.5), this expression equals $\mathsf{tr}(3, r+1)$, which establishes van Hoeij's guess.

5. The case $k = 3$, non-balanced version

In this section, we generalize two formulas for $\mathsf{tr}(3, r)$ that we obtained in Section 4 to the non-balanced case. The proofs use quite elementary tools and shed more light on the structure of subdivided triangles. More precisely, we prove a generalization of (4.4) by considering a trivariate generating function and subsequently performing coefficient extraction, and a generalization of (4.3) by partitioning a triangulation of a subdivided triangle into structural blocks.

First we introduce some notation. Let $\Delta(a, b, c)$ be the triangle ABC whose sides are subdivided as follows: the side BC is subdivided by a points, the side CA by b points, and the side AB by c points.

Let T be a triangulation of $\Delta(a, b, c)$. An *ear* is a triangle of T that contains a corner of ABC. For example, the triangulation in Figure 6(a) has ears in all three corners (marked in grey colour), while the triangulation in Figure 6(b) has ears in the corners A and B (again marked in grey colour), but none in C. An *ear diagonal* is the side of an ear that lies in the interior of ABC. A *central triangle* is a triangle of T whose vertices are interior points of different sides of ABC. For example, the triangulation in Figure 6(a) contains a central triangle (namely the green triangle), while the triangulation in Figure 6(b) is one without central triangle. A *regular triangle* is a triangle of T which is neither an ear nor a central triangle. A *corner-side diagonal* is a diagonal of T one of whose endpoints is a corner of ABC and the other an interior point of the opposite side. Examples of corner-side diagonals are the red diagonals in the triangulation in Figure 6(b). On the other hand, the triangulation in Figure 6(a) does not contain any corner-side diagonal.

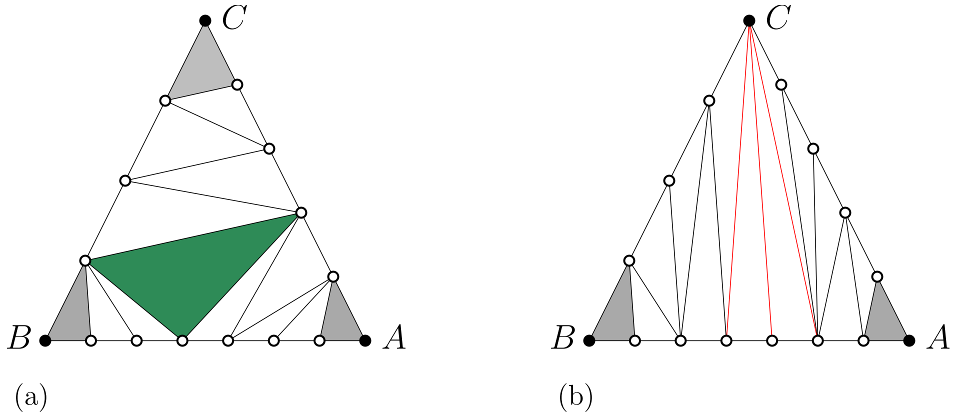

FIGURE 6. Two triangulations of $\Delta(3,4,6)$: (a) a T-triangulation; (b) a D_C-triangulation.

It is easy to observe the following facts.

Observation 7. *Triangulations of $\Delta(a,b,c)$ have the following properties:*

(1) *Each regular triangle shares exactly one edge with a side of ABC.*
(2) *Any triangulation of $\Delta(a,b,c)$ has corner-side diagonals emanating from at most one corner.*
(3) *Any triangulation of $\Delta(a,b,c)$ has at most one central triangle.*

More precisely: assume $(a,b,c) \neq (0,0,0)$, and let T be a triangulation of $\Delta(a,b,c)$. Then *either* T has one central triangle, three ears, and no corner-side diagonal, *or* T has no central triangle, two ears, and at least one corner-side diagonal emanating from the remaining corner. Triangulations of the former kind will be called T-*triangulations* (see Figure 6(a) for an example), and triangulations of the latter kind will be called D-*triangulations* (see Figure 6(b) for an example). Moreover, a D_A-*triangulation* is a (D-)triangulation that contains a corner-side diagonal one of whose endpoints is A, and D_B- and D_C-triangulations are similarly defined. The triangulation in Figure 6(b) is a D_C-triangulation.

We denote the numbers of T-, D-, D_A-, D_B-, and D_C-triangulations of $\Delta(a,b,c)$ by $\mathsf{tr}(\Delta(a,b,c))$ with appropriate specification: $\mathsf{tr}_{\mathrm{T}}(\Delta(a,b,c))$, $\mathsf{tr}_{\mathrm{D}}(\Delta(a,b,c))$, etc.

The theorem below summarizes our counting formulas for the various classes of triangulations that we just defined. In particular, it provides the promised generalization of (4.3) in (5.3).

Theorem 8. *For any non-negative integers a,b,c not all equal to zero,*

(1) *the number of* D-*triangulations of $\Delta(a,b,c)$ is*

$$\mathsf{tr}_{\mathrm{D}}(\Delta(a,b,c)) = \binom{a+b+c-1}{a-1} + \binom{a+b+c-1}{b-1} + \binom{a+b+c-1}{c-1}; \tag{5.1}$$

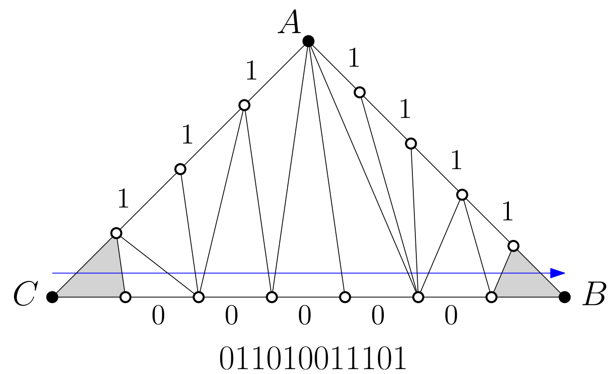

FIGURE 7. Illustration for the proof of Theorem 8.1.

(2) *the number of* T*-triangulations of* $\Delta(a,b,c)$ *is*

$$\mathsf{tr}_{\mathrm{T}}(\Delta(a,b,c)) = 2^{a+b+c-1} - \sum_{\ell=0}^{a-1}\binom{a+b+c-1}{\ell} - \sum_{\ell=0}^{b-1}\binom{a+b+c-1}{\ell} - \sum_{\ell=0}^{c-1}\binom{a+b+c-1}{\ell}; \quad (5.2)$$

(3) *the total number of triangulations of* $(\Delta(a,b,c)$ *is*

$$\mathsf{tr}(\Delta(a,b,c)) = 2^{a+b+c-1} - \sum_{\ell=0}^{a-2}\binom{a+b+c-1}{\ell} - \sum_{\ell=0}^{b-2}\binom{a+b+c-1}{\ell} - \sum_{\ell=0}^{c-2}\binom{a+b+c-1}{\ell}. \quad (5.3)$$

Proof. (1) We first show that

$$\mathsf{tr}_{\mathrm{D}_A}(\Delta(a,b,c)) = \binom{a+b+c-1}{a-1}. \quad (5.4)$$

In order to see that, consider T, a D_A-triangulation of $\Delta(a,b,c)$. The triangles of T can be linearly ordered as follows. Consider the directed segment CB, and shift it slightly ("infinitesimally") into the interior of ABC. The segment obtained in this way intersects all the triangles of T and, thus, induces a linear order on them.

By Observation 7(1), each regular triangle of T shares exactly one edge with one of the sides of ABC. We encode the regular triangles that share an edge with CB by 0, and those that share an edge with CA or with AB by 1. Using the linear order that was described above, we obtain a $\{0,1\}$-sequence of length $a+b+c-1$, in which 0 occurs $a-1$ times and 1 occurs $b+c$ times. See Figure 7 for an illustration. It is easy to see that this correspondence between D_A-triangulations of $\Delta(a,b,c)$ and $\{0,1\}$-sequences with $a-1$ occurrences of 0 and $b+c$ occurrences of 1 is bijective. (In particular, since b and c are fixed, it is determined uniquely whether a triangle encoded by 1 shares an edge with CA or with AB.) Since the number of such sequences is $\binom{a+b+c-1}{a-1}$, we obtain (5.4). Finally, due to symmetry, we get (5.1).

Remark. A special case of (5.4), the formula $\mathsf{tr}(\Delta(a,b,0)) = \binom{a+b}{a}$, was already mentioned in [11].

(2) Now we derive the formula (5.2) for the number of T-triangulations of $\Delta(a,b,c)$. By definition and by Observation 7(3), any T-triangulation T of $\Delta(a,b,c)$ has a unique

central triangle. If we remove the central triangle from T, then T decomposes into three triangulations: a triangulation of $\Delta(a_2,b_1,0)$, a triangulation of $\Delta(b_2,c_1,0)$, and a triangulation of $\Delta(c_2,a_1,0)$, where $a_1+a_2=a-1$, $b_1+b_2=b-1$, $c_1+c_2=c-1$. Conversely, each (appropriately combined) triple of such triangulations generates a T-triangulation of $\Delta(a,b,c)$. Since, as mentioned above, we have $\Delta(a,b,0)=\binom{a+b}{a}$, and since $\frac{1}{1-x-y}$ is the bivariate generating function for the array $\left(\binom{a+b}{a}\right)_{a,b\ge0}$, we conclude that $\frac{xyz}{(1-x-y)(1-y-z)(1-z-x)}$ is the trivariate generating function for $(\mathsf{tr}_{\mathrm{T}}(\Delta(a,b,c)))_{a,b,c\ge0}$. To be precise, for each fixed triple (a,b,c), we have

$$\mathsf{tr}_{\mathrm{T}}(\Delta(a,b,c))=[x^ay^bz^c]\,\frac{xyz}{(1-x-y)(1-y-z)(1-z-x)}. \tag{5.5}$$

In order to extract the coefficients, we ignore the factor xyz in the numerator for a while. We have

$$\begin{aligned}[x^ay^bz^c]\,\frac{1}{(1-x-y)(1-y-z)(1-z-x)}&=\sum_{i=0}^{a}\sum_{j=0}^{b}\left(\binom{i+j}{i}\cdot\sum_{k=0}^{c}\binom{b-j+k}{b-j}\binom{a-i+c-k}{a-i}\right)\\&=\sum_{i=0}^{a}\sum_{j=0}^{b}\binom{i+j}{i}\binom{a+b+c+1-i-j}{a+b+1-i-j}\\&=\sum_{i=0}^{a}\sum_{j=0}^{b}\binom{i+j}{i}\binom{a+b+c+1-i-j}{c}.\end{aligned} \tag{5.6}$$

For the second equality we used the standard combinatorial identity

$$\sum_{i=0}^{\ell}\binom{m+i}{m}\binom{n+\ell-i}{n}=\binom{m+n+\ell+1}{m+n+1},$$

which is a special instance of Chu–Vandermonde summation. We may use it again in order to evaluate the inner sum of the remaining double sum, for $0\le j\le a+b+1-i$ rather than $0\le j\le b$:

$$\sum_{j=0}^{a+b+1-i}\binom{i+j}{i}\binom{a+b+c+1-i-j}{c}=\binom{a+b+c+2}{c+1+i}. \tag{5.7}$$

Now we continue simplifying (5.6). We use (5.7) and subtract the extra terms which also have this form (up to an interchange of the summations over i and j). Writing $s=a+b+c+2$, we have

$$\begin{aligned}&\sum_{i=0}^{a}\sum_{j=0}^{b}\binom{i+j}{i}\binom{a+b+c+1-i-j}{c}\\&\quad=\sum_{i=0}^{a}\sum_{j=0}^{a+b+1-i}\binom{i+j}{i}\binom{a+b+c+1-i-j}{c}-\sum_{j=b+1}^{a+b+1}\sum_{i=0}^{a+b+1-j}\binom{i+j}{i}\binom{a+b+c+1-i-j}{c}\\&\quad=\sum_{i=0}^{a}\binom{s}{c+1+i}-\sum_{j=b+1}^{a+b+1}\binom{s}{c+1+j}=\sum_{\ell=c+1}^{a+c+1}\binom{s}{\ell}-\sum_{\ell=b+c+2}^{a+b+c+2}\binom{s}{\ell}=\sum_{\ell=c+1}^{a+c+1}\binom{s}{\ell}-\sum_{\ell=0}^{a}\binom{s}{\ell}\end{aligned}$$

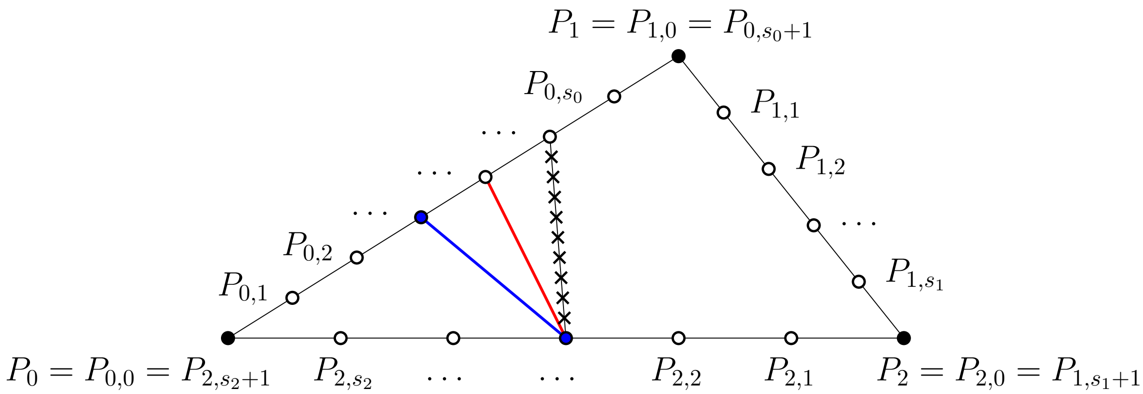

FIGURE 8. Illustration for the proof of Theorem 9: notation and definition of F_T. The diagonals shown in blue and red belong to T; the diagonal shown by crosses does not belong to T. Hence, the blue diagonal belongs to F_T.

$$= \sum_{\ell=0}^{s} \binom{s}{\ell} - \sum_{\ell=0}^{a} \binom{s}{\ell} - \sum_{\ell=0}^{c} \binom{s}{\ell} - \sum_{\ell=a+c+2}^{s} \binom{s}{\ell} = 2^s - \sum_{\ell=0}^{a} \binom{s}{\ell} - \sum_{\ell=0}^{b} \binom{s}{\ell} - \sum_{\ell=0}^{c} \binom{s}{\ell}.$$

Taking into account the factor xyz in (5.5), we obtain (5.2).

(3) Finally, we obtain (5.3) by adding (5.1) and (5.2). □

Remarks. (1) For certain specific choices of parameters, formulas that can be further simplified can be obtained. For example, we have $\mathsf{tr}_{\mathrm{T}}(\Delta(a,b,1)) = \binom{a+b}{a} - 1$. Recall that $\mathsf{tr}(\Delta(a,b,0)) = \binom{a+b}{a}$. We leave it as an exercise for the reader to find a (simple) "almost bijection" between the set of T-triangulations of $\Delta(a,b,1)$ and the set of all triangulations $\Delta(a,b,0)$.

(2) Item (1) of Theorem 8 can also be proven in a way similar to our proof of Item (2) — by considering a trivariate generating function and extracting coefficients. Doing this, we obtain $\mathsf{tr}_{\mathrm{D}_A}(\Delta(a,b,c)) = [x^a y^b z^c] \frac{xyz}{(1-x)(1-x-y)(1-x-z)}$, and similarly for $\mathsf{tr}_{\mathrm{D}_B}(\Delta(a,b,c))$ and $\mathsf{tr}_{\mathrm{D}_C}(\Delta(a,b,c))$.

Next we prove the announced generalization of Formula (4.4) to the non-balanced case.

Theorem 9. *For any non-negative integers a, b, c, we have*

$$\mathsf{tr}(\Delta(a,b,c)) = \sum_{\alpha,\beta,\gamma \ge 0} \binom{a}{\alpha+\beta} \binom{b}{\beta+\gamma} \binom{c}{\gamma+\alpha}. \tag{5.8}$$

Proof. We use a uniform notation similarly to the notation that we used for the balanced case (see Figure 8). We denote the corners of the triangle by $P_0 = P_{0,0}$, $P_1 = P_{1,0}$, $P_2 = P_{2,0}$ (say, clockwise), with arithmetic mod 3 in the first index. For each $i \in \{0,1,2\}$, the side $P_i P_{i+1}$ is subdivided by s_i points $P_{i,1}, P_{i,2}, \ldots, P_{i,s_i}$ (in the direction from P_i to

P_{i+1}). Moreover, we set $P_{i,s_i+1} = P_{i+1}$. In this notation, Formula (5.8) reads

$$\mathsf{tr}(\Delta(s_0,s_1,s_2)) = \sum_{\alpha_1,\alpha_2,\alpha_3\ge 0} \binom{s_0}{\alpha_0+\alpha_1}\binom{s_1}{\alpha_1+\alpha_2}\binom{s_2}{\alpha_2+\alpha_3}. \tag{5.9}$$

Let F be some (possibly empty) set of diagonals of $\Delta(s_0,s_1,s_2)$ which connect **interior** points of two sides of the basic triangle (that is, F does not contain corner-side diagonals), and which are pairwise disjoint (that is, they are not only non-crossing but also do not share endpoints). Such sets will be called *fundamental sets* (of diagonals of $\Delta(s_0,s_1,s_2)$). Each diagonal in a fundamental set F can be uniquely represented as $P_{i-1,\ell}P_{i,m}$ for some $i \in \{0,1,2\}$, $1 \le \ell \le s_{i-1}$, $1 \le m \le s_i$. We say that this diagonal *separates* the corner P_i.

We say that a fundamental set F has *type* $(\alpha_0,\alpha_1,\alpha_2)$ if, for $i \in \{0,1,2\}$, the number of elements of F that separate the corner P_i is exactly α_i. Notice that F is uniquely determined by the set of the endpoints of its elements. Indeed, if, for $i \in \{0,1,2\}$, exactly β_i endpoints of the elements of F lie on P_iP_{i+1}, then the type of F is $(\alpha_0,\alpha_1,\alpha_2)$, where $\alpha_i = (\beta_{i-1}+\beta_i-\beta_{i+1})/2$. Once we know the set of endpoints of the elements of F and its type, the elements of F themselves can be identified at once. It follows that the number of fundamental sets of type $(\alpha_0,\alpha_1,\alpha_2)$ is $\binom{s_0}{\alpha_0+\alpha_1}\binom{s_1}{\alpha_1+\alpha_2}\binom{s_2}{\alpha_2+\alpha_3}$, and the total number of fundamental sets is precisely the right-hand side of (5.9). Thus, in order to prove the claim, it suffices to find a bijection between the set of triangulations of $\Delta(s_0,s_1,s_2)$ and the set of its fundamental sets.

Let T be a triangulation of $\Delta(s_0,s_1,s_2)$. We define

$$F_T := \left\{ \begin{array}{ll} P_{i-1,\ell}P_{i,m}\colon & i \in \{0,1,2\},\ 1 \le \ell \le s_{i-1},\ 1 \le m \le s_i; \\ & P_{i-1,\ell}P_{i,m} \in T,\ P_{i-1,\ell}P_{i,m+1} \in T,\ P_{i-1,\ell}P_{i,m+2} \notin T \end{array} \right\}.$$

(Notice that, if $m = s_i$, then $P_{i-1,\ell}P_{i,m+1}$ is a corner-side diagonal, and the last condition, $P_{i-1,\ell}P_{i,m+2} \notin T$, is satisfied automatically.) Figure 8 illustrates this definition: the diagonal coloured blue satisfies the just described condition and, therefore, is an element of T_F.

It is easy to verify that F_T is a fundamental set. Moreover, next we show that, given a fundamental set F, there is a unique triangulation T such that $F_T = F$. This triangulation T can be reconstructed from F by applying the following procedure.

Given F, we define another set of diagonals (a *modified fundamental set*), by

$$F' = \{P_{i-1,\ell}P_{i,m+1}\colon\ \ P_{i-1,\ell}P_{i,m} \in F\}.$$

In addition, for each corner P_i such that F' contains no corner-side diagonal one of whose endpoints is P_i, we add the ear diagonal $P_{i-1,s_{i-1}}P_{i,1}$ to F'. See Figure 9(a): a "generic" element of F is coloured blue, the corresponding element of F' is coloured red; another diagonal is coloured red because it is an ear diagonal.

The elements of F' are not necessarily disjoint — they can share endpoints, — but still they are non-crossing. Therefore they partition $\Delta(s_0,s_1,s_2)$ into several parts that we call *blocks*. The boundary of each block contains at most three elements of F' (in fact, we have two or three ears whose boundaries contain exactly one element of F',

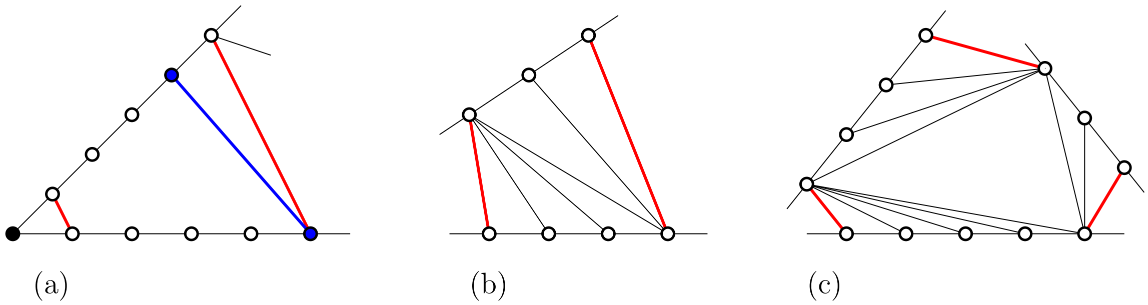

FIGURE 9. Rules for reconstructing T from $F = F_T$. Blue diagonals are the elements of F. Red diagonals are the elements of F'. (a) Definition of F'. (b) Triangulation of a block bounded by two elements of F'. (c) Triangulation of a block bounded by three elements of F'.

at most one block whose boundary contains three elements of F', and all other blocks whose boundaries contain exactly two elements of F').

Then we complete F' to a triangulation of $\Delta(s_0, s_1, s_2)$ by triangulating the blocks according to the following rules:

- Suppose B is a block whose boundary contains exactly two elements of F': $P_{i-1,\ell'}P_{i,m}$ and $P_{i-1,\ell}P_{i,m'}$, where $i \in \{0,1,2\}$, $0 \le \ell \le \ell' \le s_{i-1}$, $1 \le m \le m' \le s_i + 1$. Then we add the diagonal $P_{i-1,\ell}P_{i,m}$ (unless it belongs to F', which would happen if we have $\ell = \ell'$ or $m = m'$). At this point there is only one way to complete the triangulation of B. See Figure 9(b).
- Suppose B is a block whose boundary contains three elements of F': $P_{i-1,\ell'}P_{i,m}$, $P_{i,m'}P_{i+1,p}$, and $P_{i+1,p'}P_{i+1,\ell}$, where $i \in \{0,1,2\}$, $1 \le \ell \le \ell' \le s_{i-1}$, $1 \le m \le m' \le s_i$, $1 \le p \le p' \le s_{i+1}$. Then we add three diagonals (or, more precisely: those of them that do not belong to F') that form the triangle $P_{i-1,\ell}P_{i,m}P_{i+1,p}$. At this point there is only one way to complete the triangulation of B. See Figure 9(c).

Once this is done for all blocks, we have a triangulation T of $\Delta(s_0, s_1, s_2)$. It is routine to verify that T contains all the elements of F, and that T is the unique triangulation of $\Delta(s_0, s_1, s_2)$ such that $F_T = F$. See Figure 10 for some examples.

We established a bijection between the set of triangulations of $\Delta(s_0, s_1, s_2)$ and the set of its fundamental sets. As explained above, this completes the proof of the claim.

To summarize: while *fundamental sets* are clearly enumerated by the right-hand side of (5.8), it is *modified fundamental sets* that describe a very natural structural decomposition of triangulations into blocks. □

6. ASYMPTOTICS

Here, we determine the asymptotic behaviour of $\mathsf{tr}(k, r)$. Our starting point is another integral representation of $\mathsf{tr}(k, r)$. It is motivated by the fact that the integrand in (2.4), $I_{r,k}(t)$ say, has one saddle point at $t = 1/2$ for large k and/or r, which is easily verified

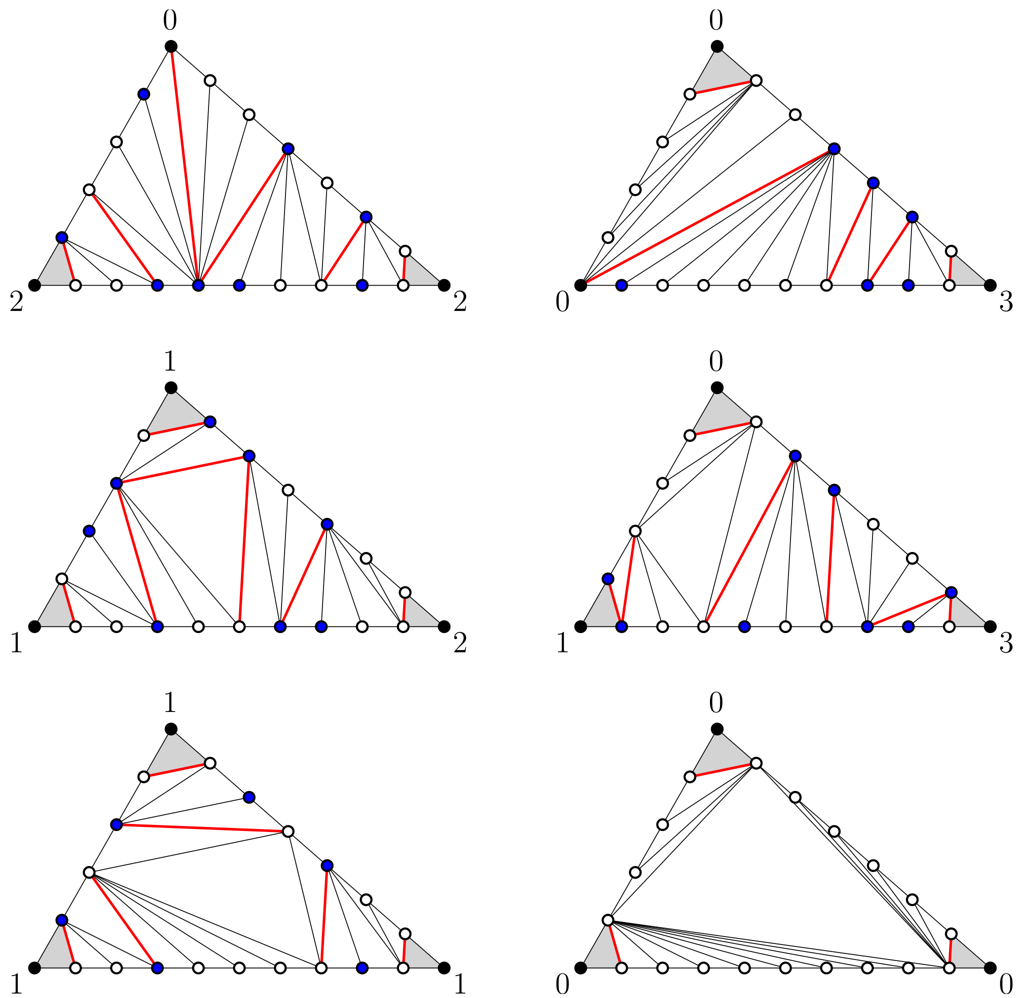

FIGURE 10. Reconstructing T from $F = F_T$. Blue points are the endpoints of the elements of F. Red diagonals are the elements of F'. The numbers at the corners are α_0, α_1 and α_2.

by solving the saddle point equation $\frac{d}{dt} I_{r,k}(t) = 0$ for large k and/or r.[4] (The subsequent arguments can however be followed without that observation.)

Proposition 10. *For all positive integers k and r with $rk \ge 3$, we have*

$$\mathsf{tr}(k,r) = -\frac{2^{(r-2)k}}{\pi} \int_{-\infty}^{\infty} \frac{du}{(1+4u^2)^{rk}(iu)^{k-2}} \left(\Big(1+2iu\Big)^{r+1} - \Big(1-2iu\Big)^{r+1} \right)^k . \tag{6.1}$$

---

[4]Strictly speaking, the point $t = 1/2$ is not a saddle point of the function $t \to |I_{r,k}(t)|$, since its value at $t = 1/2$ vanishes, that is, $I_{r,k}(1/2) = 0$. However, this is "just" caused by the factor $(1-2t)^2$ in the numerator (the factor $(1-2t)^k$ in the denominator cancels with $\big((1-t)^{r+1} - t^{r+1}\big)^k$ in the numerator). If we would ignore the factor $(1-2t)^2$, that is, if we would instead consider $I_{r,k}(t)/(1-2t)^2$, then $t = 1/2$ is a true saddle point. So, "morally," the point $t = 1/2$ *is* a saddle point of $t \to |I_{r,k}(t)|$, in the sense that the main contribution to the integral comes from a small environment around $t = 1/2$. The "only" effect of the factor $(1-2t)^2$ is to lower the polynomial factor in the asymptotic approximation, while the exponential growth is not affected.

Proof. We start with the integral representation (2.4). We deform the contour $\mathcal{C}$ so that it passes through the point $t = 1/2$. More precisely, we consider the family of contours

$$\left\{t : \Re(t) = \tfrac{1}{2} \text{ and } |\Im(t)| \le \rho\right\} \cup \left\{t : |t - \tfrac{1}{2}| = \rho \text{ and } \Re(t) \le \tfrac{1}{2}\right\}, \qquad (6.2)$$

parametrized by positive real numbers $\rho \ge 1$, which are supposed to be oriented in positive direction. In other words, these contours consist of a vertical straight line segment of length 2ρ whose midpoint is $1/2$, and the left half-circle whose diameter is this very segment. The integral over these contours still equals $\mathsf{tr}(k,r)$ since $t = 1/2$ is a removable singularity of the integrand.

Now we let $\rho \to \infty$. As we already observed in the proof of Proposition 3, the integrand is of the order $O(t^{-2})$ as $|t| \to \infty$ under our assumptions. Consequently, the integral over the circle segment of the contour (6.2) will tend to zero as $\rho \to \infty$. Thus, the number $\mathsf{tr}(k,r)$ equals the integral over the straight line $\{t : \Re(t) = 1/2\}$. If we set $t = \frac{1}{2} + iu$ in (2.4), then we obtain (6.1) after little rearrangement. ☐

The integral representation in Proposition 10 now allows for a convenient asymptotic analysis of $\mathsf{tr}(k,r)$. We distinguish between two scenarios: (1) the number k of corners is fixed, while the number of subdivisions r tends to infinity; (2) k tends to infinity, leaving it open whether r remains fixed or not.

Theorem 11. *For fixed $k \ge 3$, we have*

$$\mathsf{tr}(k,r) = \frac{2^{(r-1)k} r^{k-3}}{\pi} \left(\int_{-\infty}^{\infty} \frac{du}{u^{k-2}} \sin^k(2u)\right)\Big(1 + o(1)\Big), \qquad \textit{as } r \to \infty. \qquad (6.3)$$

Proof. We start with the integral representation (6.1), in which we make the substitution $u \to u/r$. This leads to

$$\mathsf{tr}(k,r) = -\frac{2^{(r-2)k}\, r^{k-3}}{\pi} \int_{-\infty}^{\infty} \frac{du}{\left(1 + \frac{4u^2}{r^2}\right)^{rk} (iu)^{k-2}} \left(\left(1 + \tfrac{2iu}{r}\right)^{r+1} - \left(1 - \tfrac{2iu}{r}\right)^{r+1}\right)^k.$$

Making use of dominated convergence, we may now compute the limit of the above integral as $r \to \infty$,

$$\begin{aligned}\lim_{r\to\infty} \int_{-\infty}^{\infty} \frac{du}{\left(1 + \frac{4u^2}{r^2}\right)^{rk} (iu)^{k-2}} \left(\left(1 + \tfrac{2iu}{r}\right)^{r+1} - \left(1 - \tfrac{2iu}{r}\right)^{r+1}\right)^k &= \int_{-\infty}^{\infty} \frac{du}{(iu)^{k-2}} \left(e^{2iu} - e^{-2iu}\right)^k \\ &= -2^k \int_{-\infty}^{\infty} \frac{du}{u^{k-2}} \sin^k(2u).\end{aligned}$$

The assertion of the theorem follows immediately. ☐

Remark. It is well-known that the integral in (6.3) can be evaluated for any specific k, and it equals some rational multiple of π. More precisely (cf. [10, 333.17] or [9, 3.821.12]), the relations

$$\int_0^\infty \frac{\sin^\lambda(x)}{x^k}\,dx = \frac{\lambda}{k-1}\int_0^\infty \frac{\sin^{\lambda-1}(x)\,\cos(x)}{x^{k-1}}\,dx, \qquad \text{for } \lambda > k-1 > 0, \tag{6.4}$$

$$= \frac{\lambda(\lambda-1)}{(k-1)(k-2)}\int_0^\infty \frac{\sin^{\lambda-2}(x)}{x^{k-2}}\,dx - \frac{\lambda^2}{(k-1)(k-2)}\int_0^\infty \frac{\sin^\lambda(x)}{x^{k-2}}\,dx, \quad \text{for } \lambda > k-1 > 1, \tag{6.5}$$

together with the "initial conditions" (cf. [10, 333.14, 333.15] or [9, 3.821.7, 3.832.15])

$$\int_{-\infty}^\infty \frac{\sin^{2k-1}(x)}{x}\,dx = \frac{\sqrt{\pi}\,\Gamma(k-\frac{1}{2})}{\Gamma(k)}. \tag{6.6}$$

and

$$\int_{-\infty}^\infty \frac{\sin^{2k-1}(x)\,\cos(x)}{x}\,dx = \frac{\sqrt{\pi}\,\Gamma(k-\frac{1}{2})}{2\,\Gamma(k+1)}, \tag{6.7}$$

allow for the recursive computation of the integral in (6.3) for any specific k. (*Maple* and *Mathematica* know about this.)

Theorem 12. *We have*

$$\mathsf{tr}(k,r) = \frac{\big(2^r(r+1)\big)^k}{16\sqrt{\pi}(r(r+5)/6)^{3/2}k^{3/2}}\big(1+o(1)\big), \qquad \text{as } k\to\infty, \tag{6.8}$$

where r may or may not stay fixed.

Proof. We start again with the integral representation (6.1). Here we do the substitution $u \to u/\sqrt{kR}$, where R is short for $r(r+5)/6$. Thereby we obtain

$$\mathsf{tr}(k,r) = \frac{2^{2rk-(r+1)k}}{(kR)^{3/2}}\frac{1}{\pi}\int_{-\infty}^\infty \frac{u^2\,du}{(1+\frac{4u^2}{kR})^{rk}(2iu/(kR)^{1/2})^k} \cdot\left(\left(1+\tfrac{2iu}{(kR)^{1/2}}\right)^{r+1} - \left(1-\tfrac{2iu}{(kR)^{1/2}}\right)^{r+1}\right)^k. \tag{6.9}$$

Once again, by dominated convergence, we may approximate the above integral as $k\to\infty$,

$$\int_{-\infty}^\infty \frac{u^2\,du}{(1+\frac{4u^2}{kR})^{rk}(2iu/(kR)^{1/2})^k}\left(\left(1+\tfrac{2iu}{(kR)^{1/2}}\right)^{r+1} - \left(1-\tfrac{2iu}{(kR)^{1/2}}\right)^{r+1}\right)^k$$
$$= 2^k\,(r+1)^k\left(\int_{-\infty}^\infty \frac{u^2\,du}{\exp(4u^2r/R)}\exp\left(\frac{r(r-1)}{6}\frac{(2iu)^2}{R}\right)\right)\big(1+o(1)\big)$$
$$= 2^k\,(r+1)^k\left(\int_{-\infty}^\infty u^2\,e^{-4u^2}\,du\right)\big(1+o(1)\big)$$
$$= 2^k\,(r+1)^k\frac{\sqrt{\pi}}{16}\big(1+o(1)\big),$$

as $k\to\infty$. If this is substituted back in (6.9), one obtains (6.8). □

7. Generalizations of the double circle and their triangulations

The present research was initially motivated by the following open problem from computational geometry: what is the minimum number of triangulations that a planar set of n points in general position[5] can have, and for which set(s) is this minimum attained?

This is one instance of the research direction concerning the minimum and the maximum number of plane geometric non-crossing graphs of various kinds, with respect to the number of points. One typically fixes some naturally defined class $\mathcal{C}$ of such geometric graphs (for example, triangulations, spanning trees, perfect matchings, etc.), and asks for the minimum or the maximum number of graphs from $\mathcal{C}$ that a planar set of n points in general position (playing the role of the vertex set) can have, and for a characterization of point set(s) on which these extremal values are attained. To our knowledge, in all such cases no exact results concerning **maximum** were found (except for trivialities), but rather lower and upper bounds, usually with substantial gaps (see [17] for a summary of some results of this type). In contrast, for many natural families of plane graphs, the **minimum** is attained for sets in convex position: Aichholzer et al. [2] proved that this is the case for any class of acyclic graphs (thus, for spanning trees, forests, perfect matchings, etc.[6]), as well as for the family of all plane graphs, and that of all connected plane graphs. However, this is not the case for triangulations: in [3], Aichholzer, Hurtado and Noy presented a configuration, which they called *double circle*, and which has less triangulations than sets of the same size (that is, with the same number of points) in convex position. Indeed, as was shown by Santos and Seidel in [16], the double circle of size n has $\Theta^*(\sqrt{12}^n)$ triangulations[7]. It was proven by exhaustive computations [4, 1] that, for $n \leq 15$, (only) the double circle of size n has the minimal number of triangulations over all point sets of size n in general position. Therefore it was conjectured in [3] that (only) the double circle minimizes the number of triangulations for any n. As for the lower bound, Aichholzer et al. [1] recently proved that, for all point sets of size n in general position, the number of triangulations is $\Omega(2.63^n)$ (the first result of this kind, $\Omega(2.33^n)$, was proven in [3]).

Next we recall the definition of the double circle of size n, which we denote by DC_n. For the sake of simplicity, we restrict ourselves to even n. In this case, DC_n consists of $n/2$ points, denoted by $P_1, P_2, \ldots, P_{n/2}$, in convex position; and $n/2$ points, $Q_1, Q_2, \ldots, Q_{n/2}$, such that for each i, $1 \leq i \leq n/2$, Q_i lies in the interior of the convex hull of $\{P_1, P_2, \ldots, P_{n/2}\}$, very ("infinitesimally") close to the midpoint of P_iP_{i+1}[8]. Figure 11(a) shows DC_{12} and one of its triangulations.

[5] *General position* means that no three points lie on the same line.

[6] For some of these families it was proven earlier by other authors, but Aichholzer et al. gave a unified proof.

[7] The notation $\Theta^*(\ldots)$ corresponds to the usual $\Theta(\ldots)$ notation, but with polynomial and sub-polynomial factors omitted.

[8] By convention, $P_{n/2+1} = P_1$.

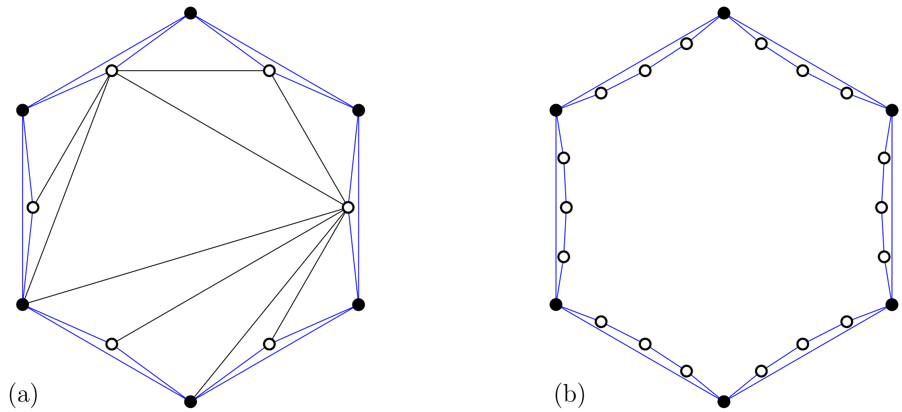

FIGURE 11. (a) Double Circle of size 12 and one of its triangulations. (b) A generalized configuration. Unavoidable edges are shown in blue colour.

Notice that each triangulation of DC_n necessarily uses the edges Q_iP_i and Q_iP_{i+1} for each i, $1 \le i \le n/2$, and, of course, all the edges that form the boundary of its convex hull. Therefore we refer to them as *unavoidable edges*. (In Figure 11, unavoidable edges are shown in blue colour.) This observation leads to a simple bijection between $\mathsf{TR}(\mathrm{DC}_n)$ and $\mathsf{TR}(\mathrm{SC}(n/2,2))$: given a triangulation of DC_n, move all the points Q_i "outwards", until they lie on the segments P_iP_{i+1}. Thus, from this point of view, triangulations of DC_n are equivalent to triangulations of $\mathrm{SC}(n/2,2)$, and the above cited bound $\mathsf{tr}(\mathrm{DC}_n) = \Theta^*(\sqrt{12}^{\,n})$ is a special case of our Theorem 12 for $r = 2$, $k = n/2 \to \infty$.

Our goal was to investigate whether the number of triangulations can decrease if one inserts more points between the corners. A similar idea, applied to the so-called *double chain*, led to an improvement of the lower bound on the *maximum* number of triangulations [8] and of perfect matchings [5].

Let us define our construction precisely. For fixed k and r, we take $\mathrm{SC}(k,r)$ and slightly pull the inner points of the strings into the convex hull so that, after this transformation, they lie on circular arcs of sufficiently big radius. This radius is chosen so that the orientation of triples of points which do not belong to the same string is not changed. See Figure 11(b) for an illustration. We denote this construction by $\mathrm{ISC}(k,r)$. Notice that for $r = 2$ we have the double circle: $\mathrm{ISC}(k,2) = \mathrm{DC}(2k)$. Observe that the segments that connect consecutive points of a string of $\mathrm{ISC}(k,r)$ are unavoidable for triangulations. Together with the segments that form the boundary of the convex hull, they split the convex hull into $k+1$ regions: k convex regions, each being spanned by $r+1$ points in convex position, and one non-convex region whose triangulations are in an obvious bijection with triangulations of $\mathrm{SC}(k,r)$. Due to this fact, the analysis of the number of triangulations of $\mathrm{ISC}(k,r)$ is now easy: we have $\mathsf{tr}(\mathrm{ISC}(k,r)) = \mathsf{tr}(\mathrm{SC}(k,r)) \cdot C_{r-1}^k$. By our asymptotic result in Theorem 12, we see that the exponential growth factor of the number of triangulations of $\mathrm{SC}(k,r)$ as $k \to \infty$ —

and thus the total number $n = kr$ of points tends to infinity — is $2(r+1)^{1/r}$.[9] Hence the growth factor for the number of triangulations of $\mathrm{ISC}(k,r)$ equals $2(r+1)^{1/r}C_{r-1}^{1/r}$. This expression is minimal for $r = 2$, that is, for the double circle. If, on the other hand, we keep k fixed and let r tend to infinity — so that again the total number $n = kr$ of points tends to infinity — then similar reasoning using our asymptotic result in Theorem 11 leads to the conclusion that the exponential growth factor of the number of triangulations of $\mathrm{ISC}(k,r)$ is 8. Thus, somewhat disappointingly, the asymptotic count of $\Theta^*(\sqrt{12}^n)$ attained by $\mathrm{DC}(n)$ cannot be improved by using balanced generalizations of the double circle, in whatever way $n \to \infty$.

Let us return to the case of fixed r and $k \to \infty$. As stated above, the exponential growth factor in this case is $g_r := 2(r+1)^{1/r}C_{r-1}^{1/r}$. As $r \to \infty$, we have $(r+1)^{1/r} \searrow 1$ and $C_{r-1}^{1/r} \nearrow 4$, in both cases monotonically for $r \geq 1$. Thus, the fact $g_2 < g_1$ can be interpreted intuitively as follows: when we pass from $r = 1$ to $r = 2$, the former expression decreases, while the k regions in convex position are just triangles with the unique (trivial) triangulation, and so there is no extra factor. On the other hand, for $r = 3$ these k regions are convex quadrilaterals with two triangulations, and, as calculations above show, their "positive" contribution to the total number of triangulations already dominates over the "negative" contribution of the central region. For $r \geq 3$, this tendency holds monotonically, and, thus, g_r has its minimum at $r = 2$.

However, if one extends the expression g_r for *real* values of r by using the Gamma function in the definition of Catalan numbers (namely, $C_n = \frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(n+2)}$), one can observe that g_r has its minimum not at $r = 2$ but rather at $r \approx 1.4957$. This may lead to the idea that, perhaps, we may get less triangulations if we "mix" sides subdivided by one point (corresponding to $r = 2$) and non-subdivided sides (corresponding to $r = 1$). More precisely, let $n \geq 2s$, and let us consider a subdivided convex $(n-s)$-gon in which s sides are subdivided by one point, and all other sides are not subdivided. (Thus, the total number of points is n.) We denote this partially subdivided polygon by $\mathrm{MC}(n-s,s)$, and its number of triangulations by $\mathsf{tr}^*(n-s,s)$. Recall from the introduction that, by [11], this number does not depend on the specific distribution of the subdivisions among the sides of the polygon. Therefore we can assume that the subdivided sides of $\mathrm{MC}(n-s,s)$ appear consecutively.

This conclusion can be also confirmed by calculations similar to those from Section 2. Proceeding in analogy with the inclusion-exclusion argument there, we observe that the number of ways to choose m pairwise non-crossing essentially forbidden diagonals in $\mathrm{MC}(n-s,s)$ is $\binom{s}{m}$. Once m essentially forbidden diagonals of $\mathrm{MC}(n-s,s)$ are chosen, we are left with a convex $(n-m)$-gon to be triangulated. Therefore, the number of illegal triangulations that use at least m essentially forbidden diagonals is $a_{n,s,m}C_{n-m-2}$.

[9]This result is also stated in [8]; however, the argument given there is non-rigorous since it relies on [11, Theorem 3] which holds for *fixed* k rather than for $k \to \infty$.

We apply the inclusion-exclusion principle to get

$$\mathsf{tr}^*(n-s,s) = \sum_{m=0}^{s} (-1)^m \, a_{n,s,m} \, C_{n-m-2} = \sum_{m=0}^{s} (-1)^m \binom{s}{m} C_{n-m-2}.$$

Thus, the analogue of (2.3) in the current context reads

$$\mathsf{tr}^*(n-s,s) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{dx}{2x^n} (1-x)^s \left(1-\sqrt{1-4x}\right), \tag{7.1}$$

where $\mathcal{C}$ is a small contour encircling the origin once in positive direction. The substitution $x = t(1-t)$, followed by the arguments used in the proof of Proposition 3, turns this into

$$\mathsf{tr}^*(n-s,s) = -\frac{1}{4\pi i} \int_{\mathcal{C}} \frac{(1-2t)^2 \, dt}{t^n (1-t)^n} \left(1-t+t^2\right)^s. \tag{7.2}$$

Deformation of the contour as described in the proof of Proposition 10 then leads us to the following integral representation of $\mathsf{tr}^*(n,s)$.

Proposition 13. *For all positive integers n and s with $n \ge 3$ and $n \ge 2s$, we have*

$$\mathsf{tr}^*(n-s,s) = \frac{4^{n-s} \, 3^s}{\pi} \int_{-\infty}^{\infty} \frac{u^2 \, du}{(1+4u^2)^n} \left(1-\tfrac{4}{3}u^2\right)^s. \tag{7.3}$$

Finally, following the proof of Theorem 12, we obtain the following asymptotic estimate for $\mathsf{tr}^*(n-s,s)$, where both n and s tend to infinity under the condition of approaching a fixed ratio.

Theorem 14. *Let α be a real number with $0 \le \alpha \le 1/2$. Then we have*

$$\mathsf{tr}^*(n-s,s) = \frac{\left(4^{1-\alpha} 3^{\alpha}\right)^n}{16\sqrt{\pi}(1+\frac{\alpha}{3})^{3/2} n^{3/2}} \big(1+o(1)\big), \qquad \text{as } n,s \to \infty \text{ subject to } s/n \to \alpha. \tag{7.4}$$

As is obvious from this asymptotic formula, the minimal exponential growth is attained for the maximal possible α, that is, for $\alpha = 1/2$, as expected. As explained above, the equivalent (from the point of view of triangulations) point set in general position is again the double circle.

In summary, our results provide further support for the conjecture of Aichholzer, Hurtado and Noy that, asymptotically, the double circle yields the minimal number of triangulations of n points in general position.

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