PROOF OF A SUMMATION FORMULA FOR AN $\tilde{A}_n$ BASIC HYPERGEOMETRIC SERIES CONJECTURED BY WARNAAR

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Abstract. A proof of an unusual summation formula for a basic hypergeometric series associated to the affine root system $\tilde{A}_n$ that was conjectured by Warnaar is given. It makes use of Milne's A_n extension of Watson's transformation, Ramanujan's ${}_1\psi_1$-summation, and a determinant evaluation of the author. In addition, a transformation formula between basic hypergeometric series associated to the affine root systems $\tilde{A}_n$ respectively $\tilde{A}_m$, which generalizes at the same time the above summation formula and an identity due to Gessel and the author, is proposed as a conjecture.

1. Introduction, statement of the result, and of the conjecture

The purpose of this note is to prove a summation formula for a basic hypergeometric series associated to the affine root system $\tilde{A}_{n-1}$ that was conjectured by Warnaar (private communication). (Another frequently used term for such series is 'basic hypergeometric series in $SU(n)$.' We follow however the terminology for multiple basic hypergeometric series associated to root systems as laid down in [4, Sec. 7] and [1, Sec. 1]. For an overview of the state of the art of this theory and of its relevance we refer the reader to [10, 1, 2, 8] and the references cited therein.)

Theorem. *Let n be a positive integer, let M_1 and M_2 be nonnegative integers, and let S be an integer with $-M_1 \le S \le M_2$. Then*

$$\sum_{k_1+\dots+k_n=S} (-1)^{(n-1)S} q^{\binom{n+1}{2}\sum_{i=1}^n k_i^2+\sum_{i=1}^n ik_i} \prod_{1\le i<j\le n} (1-q^{nk_j-nk_i+j-i})$$
$$\cdot \prod_{i=1}^n \frac{(q;q)_{M_1+M_2+i-1}}{(q;q)_{M_1+nk_i+i-1}\,(q;q)_{M_2-nk_i+n-i}} = q^{(n+1)\binom{S+1}{2}} \frac{(q;q)_{M_1+M_2}}{(q;q)_{M_1+S}\,(q;q)_{M_2-S}}, \quad (1)$$

where, as usual, the shifted q-factorial $(a;q)_n$ is defined by $(a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1})$ if $k>0$, $(a;q)_0 := 1$, and $(a;q)_k := 1/(1-a/q)(1-a/q^2)\cdots(1-aq^k)$ if $k<0$.

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This identity is remarkable, because it essentially[1] reduces to an identity originally due to Milne [9, Theorem 1.9] if we let M_2 tend to infinity. The proof of Milne's identity in [9] uses a great deal of machinery (in fact a large part of his paper [9] is devoted to the proof of this identity), which, apparently, does not allow any generalization or extension. On the other hand, an elementary, combinatorial proof of Milne's identity has been given in [4, Theorem 22]. But, again, it seems impossible to extend this combinatorial approach to a proof of the above Theorem.

I will prove the above Theorem by an unusual combination of, on the one hand, classical and, on the other hand, more recent results in classical analysis. The proof will require Milne's A_n extension of Watson's transformation [11, Theorem 6.1], Ramanujan's classical ${}_1\psi_1$-summation (see e.g. [3, Eq. (5.2.1); Appendix (II.29)]), and a determinant evaluation of the author [6, Lemma 2.2] which is ubiquitous in classcial and combinatorial analysis (cf. [7, Theorem 26 and the subsequent paragraphs] for a list of occurrences).

An independent proof of the above Theorem results from an identity for supernomial coefficients due to Schilling and Shimozono [13, Eq. (6.6)] (cf. [14, remarks preceding Eq. (6.6)]). I believe that the proof of this paper is still of interest, because variations of this approach will certainly turn out to be useful in other cases as well.

A test candidate for the above judgement may be the following conjectural generalization of the Theorem. Before I state it precisely, let me recall that in [4, Theorem 26] it is shown that Milne's identity (i.e., the $M_2 \to \infty$ case of the above Theorem) is in fact part of an infinite *hierarchy* of *transformation* formulas between multiple basic hypergeometric *of different dimension.* (Such transformations are, up to now, very rare. Except for Section 8 of [4], the only occurrence of such transformations that I am aware of is [5].) Since Milne's identity admits the generalization stated in the above Theorem, an immediate question is whether or not it is possible to also introduce an additional parameter into this infinite hierarchy of transformation formulas. On the basis of computer experiments, there is overwhelming evidence that this is indeed the case. We state the formula in the Conjecture below.

Conjecture. *Let n and m be positive integers, let M_1 and M_2 be nonnegative integers, and let S_1 and S_2 be integers with $-M_1 \le S_1 \le M_2$ and $-M_1 \le S_2 \le M_2$. Then*

$$\sum_{k_1+\dots+k_n=S_1} (-1)^{(n-1)S_1} q^{\frac{n(n+m)}{2}\sum_{i=1}^n k_i^2+m\sum_{i=1}^n ik_i-m\binom{S_1+1}{2}-nS_1(S_1+m)/2}$$

$$\cdot \prod_{1\le i<j\le n} (1-q^{nk_j-nk_i+j-i}) \prod_{i=1}^n \frac{(q;q)_{M_1+M_2+i-1}}{(q;q)_{M_1-S_1+nk_i+i-1}\,(q;q)_{M_2+S_1-nk_i+n-i}}$$

[1] In fact, Milne's identity is the $M_2 \to \infty$, $M_1 = 0$ case of (1). However, it is shown in [4, paragraph before Theorem 22], that, by what is called there the "rotation trick", Milne's identity does also imply the $M_2 \to \infty$ case of (1) (i.e., with M_1 arbitrary). The rotation trick will also be used in our proof of the Theorem.

$$= \sum_{l_1+\dots+l_m=S_2} (-1)^{(m-1)S_2} q^{\frac{m(m+n)}{2}\sum_{i=1}^m l_i^2 + n\sum_{i=1}^m il_i - n\binom{S_2+1}{2} - mS_2(S_2+n)/2}$$

$$\cdot \prod_{1\le i<j\le m} (1-q^{ml_j-ml_i+j-i}) \prod_{i=1}^m \frac{(q;q)_{M_1+M_2+i-1}}{(q;q)_{M_1-S_2+ml_i+i-1}\,(q;q)_{M_2+S_2-ml_i+m-i}}. \quad (2)$$

Clearly, our Theorem is the $m=1$ case of this conjecture. Even more evidence in favour of the conjecture comes from the fact that for $M_2 \to \infty$ it reduces to Theorem 26 in [4].

By means of the "rotation trick" (see [4, paragraph before Theorem 22] and the first paragraph of the next section), it can be seen that it suffices to prove the Conjecture for $S_1 = S_2 = 0$. However, in contrast to our proof of the Theorem, for a proof of the Conjecture it will not be sufficient to apply Milne's A_{n-1} extension of Watson's transformation. Perhaps one has to start with a higher order transformation formula, for example, with one of the A_{n-1} extensions of Bailey's very-well-poised ${}_{10}\phi_9$-transformation formula from [12].

2. Proof of the Theorem

First of all, analogously to the remark of the last paragraph of the previous section, I claim that it is enough to prove (1) for $S=0$, i.e.,

$$\sum_{k_1+\dots+k_n=0} q^{\binom{n+1}{2}\sum_{i=1}^n k_i^2 + \sum_{i=1}^n ik_i} \prod_{1\le i<j\le n} (1-q^{nk_j-nk_i+j-i})$$

$$\cdot \prod_{i=1}^n \frac{(q;q)_{M_1+M_2+i-1}}{(q;q)_{M_1+nk_i+i-1}\,(q;q)_{M_2-nk_i+n-i}} = \frac{(q;q)_{M_1+M_2}}{(q;q)_{M_1}\,(q;q)_{M_2}}. \quad (3)$$

This is seen by resorting to the "rotation trick" [4, paragraph before Theorem 22]. Let us assume that we already proved (3). Let S be some fixed integer. Division of S by n gives a unique representation $S = Qn + R$ where Q, R are integers with $0 \le R < n$. Then in (3) replace k_1 by $k_{1+R} - Q$, ... , k_{n-R} by $k_n - Q$, k_{n-R+1} by $k_1 - Q - 1$, ... , k_n by $k_R - Q - 1$. So the effect is a rotation of the summation indices, combined with a certain shift. If we rewrite (3) after these replacements and finally replace M_1 by $M_1 + S$ and M_2 by $M_2 - S$, we obtain (1) after some simplification.

Next, I claim that it is enough to prove (3) for $M_1 \equiv 0 \bmod n$. To see this, suppose that M_2 is given. Multiply both sides of (3) by $\prod_{i=1}^n (q^{M_1+M_2+i};q)_n$ and write the result in the form

$$\sum_{k_1+\dots+k_n=0} q^{\binom{n+1}{2}\sum_{i=1}^n k_i^2 + \sum_{i=1}^n ik_i} \prod_{1\le i<j\le n} (1-q^{nk_j-nk_i+j-i})$$

$$\times \prod_{i=1}^n \frac{(q^{M_1+nk_i+i};q)_{M_2-nk_i+n}}{(q;q)_{M_2-nk_i+n-i}} = \frac{(q^{M_1+1};q)_{M_2}}{(q;q)_{M_2}} \prod_{i=1}^n (q^{M_1+M_2+i};q)_n. \quad (4)$$

Both sides are most obviously polynomials in q^{M_1}, of degree at most $n^2(n+M_2)$, because, in the summation, each k_i is bounded above by $1+M_2/n$, and, hence, bounded below by $-(n-1)(1+M_2/n)$. A polynomial is uniquely determined by its evaluation at enough points, certainly at infinitely many points. Therefore, if (4) is true for all $M_1 \equiv 0$ mod n then it is true for all M_1. Since (4) and (3) are equivalent, the same applies to (3).

Now, choose some $M_1 \equiv 0$ mod n. If we want to prove (3) for this particular M_1, then an analogous argument shows that it is enough to prove it for all $M_2 \equiv 0$ mod n.

Summarizing, it is sufficient to prove (3) for $M_1 \equiv M_2 \equiv 0$ mod n. Therefore, for the rest of the proof, we assume that this congruence condition is satisfied.

To begin with, let us rewrite the left-hand side of (3) by replacing k_i by $k_i - M_1/n$, $i = 1, 2, \dots, n$, and performing some rearrangement of terms,

$$(-1)^{nM_1} q^{M_1(M_1n-M_1+2nM_2+2n^2-1)/2} \\ \times \sum_{k_1+\dots+k_n=M_1} q^{\frac{n}{2}\sum_{i=1}^n k_i^2-(n-1)\sum_{i=1}^n ik_i} \prod_{1\le i<j\le n} \frac{(1-q^{nk_j-nk_i+j-i})}{1-q^{j-i}} \prod_{i=1}^n \frac{(q^{-M_1-M_2-n+i};q)_{nk_i}}{(q^i;q)_{nk_i}}. \tag{5}$$

Next we want to apply a limiting case of Milne's A_n Watson transformation [11, Theorem 6.1],

$$\sum_{k_1,\dots,k_l\ge0} \Bigg(\prod_{1\le r<s\le l} \frac{1-\frac{x_r}{x_s}q^{k_r-k_s}}{1-\frac{x_r}{x_s}}\Bigg)\Bigg(\prod_{i=1}^l \frac{1-\frac{x_i}{x_l}aq^{k_i+(k_1+\dots+k_l)}}{1-\frac{x_i}{x_l}a}\Bigg)\Bigg(\prod_{r=1}^l\prod_{s=1}^l \frac{(\frac{x_r}{x_s}q^{-N_s};q)_{k_r}}{(q\frac{x_r}{x_s};q)_{k_r}}\Bigg) \\ \times \Bigg(\prod_{i=1}^l \frac{(\frac{x_i}{x_l}a;q)_{k_1+\dots+k_l}}{(\frac{x_i}{x_l}aq^{1+N_i};q)_{k_1+\dots+k_l}}\Bigg)\Bigg(\prod_{i=1}^l \frac{(\frac{x_i}{x_l}c;q)_{k_i}\,(\frac{x_i}{x_l}d;q)_{k_i}}{(\frac{x_i}{x_l}\frac{aq}{b};q)_{k_i}\,(\frac{x_i}{x_l}\frac{aq}{e};q)_{k_i}}\Bigg) \\ \times \frac{(b;q)_{k_1+\dots+k_l}\,(e;q)_{k_1+\dots+k_l}}{(\frac{aq}{c};q)_{k_1+\dots+k_l}\,(\frac{aq}{d};q)_{k_1+\dots+k_l}}\left(\frac{a^2q^{1+N_1+\dots+N_l}}{bcde}\right)^{k_1+\dots+k_l} q^{\sum_{i=1}^l ik_i} \\ = \frac{(aq/de;q)_{N_1+\dots+N_l}}{(aq/d;q)_{N_1+\dots+N_l}}\Bigg(\prod_{i=1}^l \frac{(\frac{x_i}{x_l}aq;q)_{N_i}}{(\frac{x_i}{x_l}aq/e;q)_{N_i}}\Bigg)\sum_{k_1,\dots,k_l\ge0} q^{\sum_{i=1}^l ik_i}\Bigg(\prod_{1\le r<s\le l}\frac{1-\frac{x_r}{x_s}q^{k_r-k_s}}{1-\frac{x_r}{x_s}}\Bigg) \\ \times \Bigg(\prod_{r=1}^l\prod_{s=1}^l \frac{(\frac{x_r}{x_s}q^{-N_s};q)_{k_r}}{(q\frac{x_r}{x_s};q)_{k_r}}\Bigg)\Bigg(\prod_{i=1}^l \frac{(\frac{x_i}{x_l}d;q)_{k_i}}{(\frac{x_i}{x_l}\frac{aq}{b};q)_{k_i}}\Bigg)\frac{(\frac{aq}{bc};q)_{k_1+\dots+k_l}\,(e;q)_{k_1+\dots+k_l}}{(\frac{aq}{c};q)_{k_1+\dots+k_l}\,(\frac{de}{a}q^{-N_1-\dots-N_l};q)_{k_1+\dots+k_l}}, \tag{6}$$

where $N_1, \dots, N_l$ are nonnegative integers. For convenience, let us set $k_{l+1} = M - k_1 - \dots - k_l$ and $a = x_l/q^M x_{l+1}$, so that (6) becomes

$$q^M\Bigg(\prod_{i=1}^l \frac{(q\frac{x_{l+1}}{x_i};q)_M}{(\frac{x_{l+1}}{x_i}q^{-N_i};q)_M}\Bigg)\frac{(q^{1-M}/b;q)_M\,(q^{1-M}/e;q)_M}{(\frac{x_{l+1}}{x_l}c;q)_M\,(\frac{x_{l+1}}{x_l}d;q)_M}\prod_{i=1}^l \frac{(1-\frac{x_i}{x_{l+1}})}{(1-\frac{x_i}{q^Mx_{l+1}})} \\ \times \sum_{k_1+\dots+k_l+k_{l+1}=M} q^{-\sum_{i=1}^{l+1} ik_i}\Bigg(\prod_{1\le r<s\le l+1}\frac{1-\frac{x_s}{x_r}q^{k_s-k_r}}{1-\frac{x_s}{x_r}}\Bigg)\Bigg(\prod_{r=1}^{l+1}\prod_{s=1}^l \frac{(\frac{x_r}{x_s}q^{-N_s};q)_{k_r}}{(q\frac{x_r}{x_s};q)_{k_r}}\Bigg) \\ \times \Bigg(\prod_{i=1}^{l+1} \frac{(\frac{x_i}{x_l}c;q)_{k_i}\,(\frac{x_i}{x_l}d;q)_{k_i}}{(\frac{x_i}{x_{l+1}}\frac{q^{1-M}}{b};q)_{k_i}\,(\frac{x_i}{x_{l+1}}\frac{q^{1-M}}{e};q)_{k_i}}\Bigg)$$

$$= \frac{(q^{1-M}x_l/x_{l+1}de;q)_{N_1+\dots+N_l}}{(q^{1-M}x_l/x_{l+1}d;q)_{N_1+\dots+N_l}} \left(\prod_{i=1}^{l} \frac{(\frac{x_i}{x_{l+1}}q^{1-M};q)_{N_i}}{(\frac{x_i}{x_{l+1}}q^{1-M}/e;q)_{N_i}}\right)$$
$$\times \sum_{k_1,\dots,k_l\ge 0} q^{\sum_{i=1}^{l} ik_i} \left(\prod_{1\le r<s\le l} \frac{1-\frac{x_r}{x_s}q^{k_r-k_s}}{1-\frac{x_r}{x_s}}\right)\left(\prod_{r=1}^{l}\prod_{s=1}^{l} \frac{(\frac{x_r}{x_s}q^{-N_s};q)_{k_r}}{(q\frac{x_r}{x_s};q)_{k_r}}\right)$$
$$\times \left(\prod_{i=1}^{l} \frac{(\frac{x_i}{x_l}d;q)_{k_i}}{(\frac{x_i}{x_{l+1}}\frac{q^{1-M}}{b};q)_{k_i}}\right) \frac{(\frac{q^{1-M}x_l}{x_{l+1}bc};q)_{k_1+\dots+k_l}\,(e;q)_{k_1+\dots+k_l}}{(\frac{q^{1-M}x_l}{x_{l+1}c};q)_{k_1+\dots+k_l}\,(\frac{deq^M x_{l+1}}{x_l}q^{-N_1-\dots-N_l};q)_{k_1+\dots+k_l}}. \tag{7}$$

In this identity we replace q by q^n. Then we set $l = n-1$, $M = M_1$, $x_i = q^i$ for $i = 1,2,\dots,l+1$, $b = q^{-nM_1}$, $d = \delta q^{-M_1-M_2-1}$, $N_i = (M_1+M_2)/n$. Next we multiply both sides by $(1-\delta)$ (this cancels one factor in the term $(x_{l+1}d/x_l;q)_M \sim (\delta q^{-M_1-M_2};q^n)_{M_1}$ in the denominator of the left-hand side of (7) and one factor in the term $(q^{1-M}x_l/x_{l+1}d;q)_{N_1+\dots+N_l} \sim (q^{n-nM_1+M_1+M_2}/\delta;q^n)_{(M_1+M_2)/n}$ in the denominator of the right-hand side of (7)). Finally, we let $\delta\to 1$, $c\to\infty$, and $e\to\infty$. This reduces (7) to the following transformation formula,

$$q^{nM_1(n-M_1/2-1)} \frac{(q;q)_{nM_1}}{(q^{-M_1-M_2};q)_{M_1+M_2}\,(q;q)_{nM_1-M_1-M_2-1}}$$
$$\times \sum_{k_1+\dots+k_n=M_1} q^{\frac{n}{2}\sum_{i=1}^{n}k_i^2-(n-1)\sum_{i=1}^{n}ik_i}\left(\prod_{1\le r<s\le n}\frac{1-q^{nk_s-nk_r+s-r}}{1-q^{s-r}}\right)$$
$$\times\left(\prod_{r=1}^{n}\frac{(q^{-M_1-M_2-n+r};q)_{nk_r}}{(q^r;q)_{nk_r}}\right)$$
$$= -\frac{(q^{-nM_1};q)_{M_1+M_2+1}}{(q^{-nM_1};q^n)_{M_1}\,(q^n;q^n)_{M_2}} \sum_{k_1,\dots,k_{n-1}\ge 0}\left(\prod_{1\le r<s\le n-1}\frac{1-q^{nk_r-nk_s+r-s}}{1-q^{r-s}}\right)$$
$$\times\left(\prod_{r=1}^{n-1}\frac{(q^{-M_1-M_2-n+r};q)_{nk_r}}{(q^r;q)_{nk_r}}\left(q^{M_2}\right)^{nk_r}\right)q^{n\sum_{i=1}^{n-1}ik_i}. \tag{8}$$

The series on the left-hand side of (8) is exactly the series in (5). What the transformation (8) does with this series is, in some sense which will become more transparent below, that it "entangles" the summation indices. Thus, we obtain the following expression for the left-hand side of (3),

$$(-1)^{M_1}q^{-\binom{M_1+1}{2}}\frac{(q;q)_{M_1+M_2}}{(q^{-nM_1};q^n)_{M_1}\,(q^n;q^n)_{M_2}}$$
$$\times\sum_{k_1,\dots,k_{n-1}\ge 0}\left(\prod_{1\le r<s\le n-1}\frac{1-q^{nk_r-nk_s+r-s}}{1-q^{r-s}}\right)$$
$$\times\left(\prod_{r=1}^{n-1}\frac{(q^{-M_1-M_2-n+r};q)_{nk_r}}{(q^r;q)_{nk_r}}\left(q^{M_2}\right)^{nk_r}\right)q^{n\sum_{i=1}^{n-1}ik_i}. \tag{9}$$

(The sign $(-1)^{M_1}$ is no misprint since our assumption $M_1\equiv 0 \bmod n$ implies $nM_1\equiv M_1 \bmod 2$.)

The next task is to split the sum in (9) into many pieces, each of which being a product of $n-1$ one-dimensional summations. This is done by replacing the product over $1 \le s < r \le n-1$ by a Vandermonde determinant. More precisely, we have

$$\begin{aligned}\prod_{1\le r<s\le n-1}(1-q^{nk_r-nk_s+r-s}) &= q^{-\sum_{i=1}^{n-1}(i-1)(nk_i+i)}\prod_{1\le r<s\le n-1}(q^{nk_s+s}-q^{nk_r+r})\\ &= q^{-\sum_{i=1}^{n-1}(i-1)(nk_i+i)}\det_{1\le i,j\le n-1}\left(\left(q^{nk_i+i}\right)^{j-1}\right)\\ &= q^{-n\sum_{i=1}^{n-1}ik_i+n\sum_{i=1}^{n-1}k_i-2\binom{n}{3}}\sum_{\sigma\in S_{n-1}}\operatorname{sgn}\sigma\prod_{i=1}^{n-1}q^{(\sigma(i)-1)(nk_i+i)}.\end{aligned}$$

Hence, the sum in (9) equals

$$\begin{aligned}(-1)^{\binom{n-1}{2}}\left(\prod_{i=1}^{n-1}\frac{1}{(q;q)_{i-1}}\right)&\sum_{\sigma\in S_{n-1}}\operatorname{sgn}\sigma\, q^{-\binom{n}{3}}q^{\sum_{i=1}^{n-1}i(\sigma(i)-1)}\\ &\times\prod_{i=1}^{n-1}\left(\sum_{k_i\ge0}\frac{(q^{-M_1-M_2-n+i};q)_{nk_i}}{(q^i;q)_{nk_i}}\left(q^{M_2+\sigma(i)}\right)^{nk_i}\right).\end{aligned}\tag{10}$$

The next ingredient is Ramanujan's ${}_1\psi_1$-summation (see [3, (5.2.1)]),

$$\sum_{k=-\infty}^{\infty}\frac{(a;q)_k}{(b;q)_k}z^k=\frac{(q;q)_\infty\,(b/a;q)_\infty\,(az;q)_\infty\,(q/az;q)_\infty}{(b;q)_\infty\,(q/a;q)_\infty\,(z;q)_\infty\,(b/az;q)_\infty}.\tag{11}$$

Each of the inner sums in (10) is an n-section of a special case of the left-hand side of (11). (To be precise, it is the special case $a=q^{-M_1-M_2-n+i}$, $b=q^i$, and $z=q^{M_2+\sigma(i)}$.) Thus, (10) simplifies to

$$\begin{aligned}(-1)^{\binom{n-1}{2}}\left(\prod_{i=1}^{n-1}\frac{1}{(q;q)_{i-1}}\right)&\sum_{\sigma\in S_{n-1}}\operatorname{sgn}\sigma\, q^{-\binom{n}{3}}q^{\sum_{i=1}^{n-1}i(\sigma(i)-1)}\\ \times\prod_{i=1}^{n-1}\left(\frac{1}{n}\sum_{\ell_i=0}^{n-1}\frac{(q;q)_\infty\,(q^{M_1+M_2+n};q)_\infty\,(q^{i+\sigma(i)-M_1-n}\omega^{\ell_i};q)_\infty\,(q^{1-i-\sigma(i)+M_1+n}\omega^{-\ell_i};q)_\infty}{(q^i;q)_\infty\,(q^{1-i+M_1+M_2+n};q)_\infty\,(q^{M_2+\sigma(i)}\omega^{\ell_i};q)_\infty\,(q^{-\sigma(i)+M_1+n}\omega^{-\ell_i};q)_\infty}\right)&,\end{aligned}\tag{12}$$

where ω denotes a primitive n-th root of unity. An immediate observation is that if any ℓ_i equals 0 then the corresponding summand vanishes, because of the term

$$(q^{i+\sigma(i)-M_1-n}\omega^{\ell_i};q)_\infty\,(q^{1-i-\sigma(i)+M_1+n}\omega^{-\ell_i};q)_\infty$$

in the numerator. Hence, we may as well sum over ℓ_i from 1 to $n-1$, $i=1,2,\dots,n-1$.

Some manipulation transforms (12) into

$$(-1)^{\binom{n-1}{2}}\frac{1}{n^{n-1}}\prod_{i=1}^{n-1}\frac{1}{(q^{1-i+M_1+M_2+n};q)_{i-1}}\sum_{\sigma\in S_{n-1}}\operatorname{sgn}\sigma \cdot\Bigg(\sum_{\ell_1,\dots,\ell_{n-1}=1}^{n-1}\Bigg(\prod_{i=1}^{n-1}\frac{(q^{-M_1}\omega^{\ell_i};q)_\infty}{(q^{M_2+1}\omega^{\ell_i};q)_\infty}\omega^{\ell_i(n-i-\sigma(i))}\Bigg) \cdot (q^{M_2+1}\omega^{\ell_i};q)_{\sigma(i)-1}\,(q^{M_1+1}\omega^{-\ell_i};q)_{n-\sigma(i)-1}\Bigg). \tag{13}$$

Now it is not difficult to see that if $\ell_r=\ell_s$, $r\ne s$, then the summand corresponding to the permutation σ cancels with the summand corresponding to the permutation $\sigma\circ(rs)$. (Here, (rs) denotes the transposition which interchanges r and s.) Therefore the only summands which survive this cancellation are those where the summation indices $\ell_1,\ell_2,\dots,\ell_{n-1}$ are a permutation of $\{1,2,\dots,n-1\}$. Thus, (13) reduces to

$$(-1)^{\binom{n-1}{2}}\frac{1}{n^{n-1}}\frac{(q^{-nM_1};q^n)_{M_1}\,(q^n;q^n)_\infty\,(q^{M_2+1};q)_\infty}{(q^{-M_1};q)_{M_1}\,(q;q)_\infty\,(q^{nM_2+n};q^n)_\infty}\prod_{i=1}^{n-1}\frac{(1-\omega^i)}{(q^{1-i+M_1+M_2+n};q)_{i-1}}$$
$$\times\sum_{\sigma,\tau\in S_{n-1}}(\operatorname{sgn}\sigma)\,\omega^{\tau(i)(n-i-\sigma(i))}(q^{M_2+1}\omega^{\tau(i)};q)_{\sigma(i)-1}\,(q^{M_1+1}\omega^{-\tau(i)};q)_{n-\sigma(i)-1}$$
$$=(-1)^{\binom{n-1}{2}}\frac{1}{n^{n-1}}\frac{(q^{-nM_1};q^n)_{M_1}\,(q^n;q^n)_\infty\,(q^{M_2+1};q)_\infty}{(q^{-M_1};q)_{M_1}\,(q;q)_\infty\,(q^{nM_2+n};q^n)_\infty}\prod_{i=1}^{n-1}\frac{(1-\omega^i)}{(q^{1-i+M_1+M_2+n};q)_{i-1}}$$
$$\times\sum_{\tau\in S_{n-1}}\omega^{\tau(i)(n-i)}\det_{1\le i,j\le n-1}\left(\omega^{-j\tau(i)}(q^{M_2+1}\omega^{\tau(i)};q)_{j-1}\,(q^{M_1+1}\omega^{-\tau(i)};q)_{n-j-1}\right). \tag{14}$$

The determinant is easily evaluated with the help of the determinant lemma [6, Lemma 2.2],

$$\det_{1\le i,j\le n}\Big((X_i+A_n)\cdots(X_i+A_{j+1})(X_i+B_j)\cdots(X_i+B_2)\Big) =\prod_{1\le i<j\le n}(X_i-X_j)\prod_{2\le i\le j\le n}(B_i-A_j), \tag{15}$$

where $X_1,\dots,X_n$, $A_2,\dots,A_n$, and $B_2,\dots B_n$ are arbitrary indeterminates. In order to apply (15), we rewrite the determinant in (14) as

$$\det_{1\le i,j\le n-1}\left(\omega^{-j\tau(i)}(q^{M_2+1}\omega^{\tau(i)};q)_{j-1}\,(q^{M_1+1}\omega^{-\tau(i)};q)_{n-j-1}\right)$$
$$=(-1)^{\binom{n-1}{2}}\prod_{i=1}^{n-1}\omega^{-\tau(i)}q^{(M_1+1)+(M_1+2)+\dots+(M_1+n-i-1)}$$
$$\det_{1\le i,j\le n-1}\Big((\omega^{-\tau(i)}-q^{M_2+1})(\omega^{-\tau(i)}-q^{M_2+2})\dots(\omega^{-\tau(i)}-q^{M_2+j-1})$$
$$\cdot(\omega^{-\tau(i)}-q^{-M_1-n+j+1})(\omega^{-\tau(i)}-q^{-M_1-n+j+2})\cdots(\omega^{-\tau(i)}-q^{-M_1-1})\Big).$$

Now the determinant evaluation (15) applies with $X_i = \omega^{-\tau(i)}$, $A_j = -q^{-M_1-n+j}$, and $B_j = -q^{M_2+j-1}$. If the resulting expression is substituted back into (14), we obtain

$$\frac{1}{n^{n-1}} \frac{(q^{-nM_1};q^n)_{M_1}\,(q^n;q^n)_\infty\,(q^{M_2+1};q)_\infty}{(q^{-M_1};q)_{M_1}\,(q;q)_\infty\,(q^{nM_2+n};q^n)_\infty} \prod_{i=1}^{n-1}(1-\omega^i)$$
$$\times \sum_{\tau\in S_{n-1}} \omega^{\tau(i)(n-i-1)} \prod_{1\le i<j\le n-1} (\omega^{-\tau(i)}-\omega^{-\tau(j)})$$
$$= \frac{1}{n^{n-1}} \frac{(q^{-nM_1};q^n)_{M_1}\,(q^n;q^n)_\infty\,(q^{M_2+1};q)_\infty}{(q^{-M_1};q)_{M_1}\,(q;q)_\infty\,(q^{nM_2+n};q^n)_\infty} \prod_{i=1}^{n-1}(1-\omega^i) \prod_{1\le i<j\le n-1}(\omega^{-i}-\omega^{-j})$$
$$\times \sum_{\tau\in S_{n-1}} (\operatorname{sgn}\tau)\,\omega^{\tau(i)(n-i-1)}. \tag{16}$$

The sum over permutations in the last line is just a Vandermonde determinant, and as such easily evaluated. If we substitute this in (16), the resulting expression for the sum in (9), we obtain

$$\frac{1}{n^{n-1}} \frac{(q;q)_{M_1+M_2}}{(q;q)_{M_1}\,(q;q)_{M_2}} \prod_{i=1}^{n-1}(1-\omega^i) \prod_{1\le i<j\le n-1} (\omega^{-i}-\omega^{-j})(\omega^i-\omega^j) \tag{17}$$

for the left-hand side of (3). Clearly, there holds

$$\prod_{i=1}^{n-1}(1-\omega^i) = n,$$

because it is the limit $\lim_{z\to 1}(1-z^n)/(1-z)$. Moreover, we have

$$\prod_{1\le i<j\le n-1} (\omega^i-\omega^j)(\omega^{-i}-\omega^{-j}) = \prod_{i=1}^{n-1}(1-\omega)\cdots(1-\omega^{i-1}) \prod_{i=1}^{n-1}(1-\omega^{-1})\cdots(1-\omega^{-i+1})$$
$$= \prod_{i=1}^{n-2}(1-\omega)\cdots(1-\omega^i) \prod_{i=1}^{n-2}(1-\omega^{n-1})\cdots(1-\omega^{i+1})$$
$$= \prod_{i=1}^{n-2}(1-\omega)\cdots(1-\omega^{n-1}) = n^{n-2},$$

in view of the previous observation. Thus, (17) does indeed reduce to the right-hand side of (3). In view of the remarks of the first paragraph of this section, the proof of the theorem is complete. □

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