An alternative evaluation of the Andrews–Burge determinant

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Dedicated to Gian-Carlo Rota

Abstract. We give a short, self-contained evaluation of the Andrews–Burge determinant (Pacific J. Math. **158** (1994), 1–14).

1. Introduction

In [9, Theorem 1], Andrews and Burge proved a determinant evaluation equivalent to

$$\det_{0\le i,j\le n-1}\left(\binom{x+i+j}{2i-j}+\binom{y+i+j}{2i-j}\right) \tag{1.1}$$
$$=(-1)^{\chi(n\equiv 3 \mod 4)}2^{\binom{n}{2}+1}$$
$$\times\prod_{j=1}^{n-1}\frac{\left(\frac{x+y}{2}+j+1\right)_{\lfloor (j+1)/2\rfloor}\left(-\frac{x+y}{2}-3n+j+\frac{3}{2}\right)_{\lfloor j/2\rfloor}}{(j)_j},$$

where the shifted factorial $(a)_k$ is given by $(a)_k := a(a+1)\cdots(a+k-1)$, $k\ge 1$, $(a)_0 := 1$, and where $\chi(\mathcal{A})$=1 if $\mathcal{A}$ is true and $\chi(\mathcal{A})$=0 otherwise. This determinant identity arose in connection with the enumeration of symmetry classes of plane partitions. The known proofs [9, 10] of (1.1) require that one knows (1.1) to hold for $x=y$. Indeed, the latter was first established by Mills, Robbins and Rumsey [15, p. 53], in turn using another determinant evaluation, due to Andrews [3], whose proof is rather complicated. In the

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meantime, simpler proofs of the $x = y$ special case of (1.1) were found by Andrews [7], Andrews and Stanton [8], and Petkovšek and Wilf [16]. In this note we describe a new, concise, and self-contained proof of (1.1), see section 3.

In fact, the main purpose of this note is to popularize the method that I use to prove (1.1) (see section 2 for a description). This method is simple but powerful. Aside from this note, evidence for this claim can be found e.g. in [11, 12, 13, 14]. Thus, the method enlarges the not at all abundant collection of methods for evaluating determinants. In fact, aside from elementary manipulations by row and column operations, we are just aware of one other method, namely Andrews' "favourite" method of evaluating determinants (cf. [1, 2, 3, 4, 5, 6, 8]), which basically consists of guessing and then proving the LU-factorization of the matrix in question (i.e., the factorization of the matrix into a product of a lower triangular times an upper triangular matrix).

We should, however, also point out a limitation of our method. Namely, in order to be able to apply our method, we need a free parameter occuring in the determinant. (In (1.1) there are even two, x and y.) Andrews' method, on the other hand, might still be applicable if there is no free parameter present. Still, it is safe to speculate that many more applications of our method are going to be found in the future.

2. The method

The method that I use to prove (1.1) is as follows. Suppose we have a matrix $\big(f_{ij}(x)\big)_{0\le i,j\le n-1}$ with entries $f_{ij}(x)$ which are polynomials in x, and we want to prove the explicit factorization of $\det\big(f_{ij}(x)\big)$ as a polynomial in x,

$$\det_{0\le i,j\le n-1}\big(f_{ij}(x)\big) = C(n)\prod_l (x-a_l(n))^{m_l(n)}, \tag{2.1}$$

where $C(n), a_l(n), m_l(n)$ are independent of x, and the $a_l(n)$'s are pairwise different for fixed n.

Then, in the first step, for each l we find $m_l(n)$ linearly independent linear combinations of the columns, or of the rows, which vanish for $x = a_l(n)$. In different, but equivalent terms, we find $m_l(n)$ linearly independent vectors in the kernel of our matrix evaluated at $x = a_l(n)$, i.e., in the kernel of $\big(f_{ij}(a_l(n))\big)$, or of its transpose. That this really guarantees that $(x-a_l(n))^{m_l(n)}$ is a factor of the determinant is a fact that might not be well-known enough. Therefore, for the sake of completeness, we state it as a lemma at the end of this section and provide a proof for it.

The finding of $a_l(n)$ linearly independent linear combinations of columns or rows which vanish for $x = a_l(n)$ (equivalently, linearly independent vectors in the kernel of the matrix, or its transpose, evaluated at $x = a_l(n)$) can be done with some skill (and primarily patience) by setting $x = a_l(n)$ in the matrix $\big(f_{ij}(x)\big)$, computing tables for the coefficients of the linear combinations for $n = 1, 2, \dots$ (by solving the respective systems of linear equations on the computer), and finally guessing what the general pattern of the coefficients could be. To prove that the guess is correct, in case of binomial determinants (such as the one in (1.1)) one has to verify certain binomial identities. But this is pure routine today, by means of Zeilberger's algorithm [18, 19].

Next, in the second step, one checks the degrees of both sides of (2.1) as polynomials in x. If it should happen that the degree of $\det\big(f_{ij}(x)\big)$ is not larger than $\sum_l m_l(n)$, the degree of the right-hand side, then it follows by what we did in the first step that the determinant $\det\big(f_{ij}(x)\big)$ has indeed the form of the right-hand side of (2.1), where $C(n)$ is some unknown constant. This constant can then be determined in the third step by comparison of coefficients of a suitable power of x.

Finally, here is the promised lemma, and its proof.

Lemma. *Let $A = A(x)$ be a matrix whose entries are polynomials in x, and u a number. If $\dim \operatorname{Ker} A(u) \ge k$, then u is a root of $\det A(x)$ of multiplicity at least k.*

Proof. Let $v_1, v_2, \dots, v_k$ be k linearly independent (column) vectors in the kernel of $A(u)$. Without loss of generality, we may assume that $v_1, v_2, \dots, v_k$ are such that the matrix $[v_1, v_2, \dots, v_k]$, formed by gluing the columns $v_1, v_2, \dots, v_k$ to a matrix, is in column-echelon form. In addition, again without loss of generality, we may assume that for any $i = 1, 2, \dots, k$ the vector v_i is of the form $v_i = (0, \dots, 0, 1, \dots)^t$, i.e., the first $i - 1$ entries are 0, the i-th entry is 1, and the remaining entries could be anything. (To justify "without loss of generality" one would possibly have to permute the columns of A.)

Now we consider the matrix $\tilde{A} = \tilde{A}(x)$, formed by replacing for $i = 1, 2, \dots, k$ the i-th column of A by the column Av_i. It is an easy observation that $\tilde{A}$ and A are related by elementary column operations. Therefore, their determinants are the same. On the other hand, the i-th column of $\tilde{A}(u)$ is $A(u)v_i = 0$, for any $i = 1, 2, \dots, k$. Hence, each entry of Av_i, being a polynomial in x, must be divisible by $(x - u)$. Therefore, in the determinant $\det \tilde{A}$, we may take $(x - u)$ out of the i-th column, $i = 1, 2, \dots, k$, with the entries in the remaining determinant still being polynomials in x. This proves that $(x - u)^k$ divides $\det \tilde{A} = \det A$, and, thus, the lemma. ∎

3. Proof of (1.1)

As announced in the previous section, the proof consists of three steps. For convenience, let us denote the determinant in (1.1) by $AB(x, y; n)$, or sometimes just $AB(n)$ for short.

Step 1. Identification of the factors. We show that the product on the right-hand side of (1.1),

$$\prod_{j=1}^{n-1} \left(\tfrac{x+y}{2}+j+1\right)_{\lfloor (j+1)/2\rfloor} \left(-\tfrac{x+y}{2}-3n+j+\tfrac{3}{2}\right)_{\lfloor j/2\rfloor},$$

is indeed a factor of $AB(n)$, in the way that was described in section 2. Let us first consider just one part of this product,

$$\prod_{j=1}^{n-1} ((x+y)/2+j+1)_{\lfloor (j+1)/2\rfloor}.$$

Let us concentrate on a typical factor $(x+y+2j+2l)$, $1\le j\le n-1$, $1\le l\le (j+1)/2$. We claim that for each such factor there is a linear combination of the columns that vanishes if the factor vanishes. More precisely, we claim that for any j,l with $1\le j\le n-1$, $1\le l\le (j+1)/2$ there holds

$$\begin{aligned}\sum_{s=2l-1}^{\lfloor (j+2l-1)/2\rfloor} &\frac{(j-2l+1)}{(j-s)}\frac{(j+2l-2s)_{s-2l+1}}{(s-2l+1)!}\\ &\cdot(\text{column } s \text{ of } AB(-y-2j-2l,y;n))\\ &\qquad+(\text{column } j \text{ of } AB(-y-2j-2l,y;n))=0.\end{aligned}\tag{3.1}$$

To avoid confusion, for $j=2l-1$ it is understood by convention that the sum in the first line of (3.1) vanishes.

To establish the claim, we have to check

$$\begin{aligned}\sum_{s=2l-1}^{\lfloor (j+2l-1)/2\rfloor} &\frac{(j-2l+1)}{(j-s)}\frac{(j+2l-2s)_{s-2l+1}}{(s-2l+1)!}\\ &\cdot\left(\binom{-y-2j-2l+i+s}{2i-s}+\binom{y+i+s}{2i-s}\right)\\ &\qquad+\left(\binom{-y-j-2l+i}{2i-j}+\binom{y+i+j}{2i-j}\right)=0.\end{aligned}\tag{3.2}$$

The exceptional case $j=2l-1$ can be treated immediately. By assumption the sum in the first line of (3.2) vanishes for $j=2l-1$, and, by inspection, also the last line in (3.2) vanishes for $j=2l-1$. So we are left with establishing (3.2) for $l\le j/2$. In terms of the usual hypergeometric notation

$${}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_r)_k}{k!\,(b_1)_k\cdots(b_s)_k}z^k\ ,$$

this means to check

$$(3.3)\qquad \frac{(-1-i-2j+2l-y)_{1+2i-2l}}{(1+2i-2l)!}\\ \times {}_4F_3\left[\begin{matrix}-\frac{1}{2}-\frac{j}{2}+l,-\frac{j}{2}+l,-1-2i+2l,i-2j-y\\ -j+2l,-\frac{1}{2}-\frac{i}{2}-j+l-\frac{y}{2},-\frac{i}{2}-j+l-\frac{y}{2}\end{matrix};1\right]\\ +\frac{(-1-i+4l+y)_{1+2i-2l}}{(1+2i-2l)!}{}_4F_3\left[\begin{matrix}-\frac{1}{2}-\frac{j}{2}+l,-\frac{j}{2}+l,-1-2i+2l,i+2l+y\\ -j+2l,-\frac{1}{2}-\frac{i}{2}+2l+\frac{y}{2},-\frac{i}{2}+2l+\frac{y}{2}\end{matrix};1\right]\\ +\left(\binom{-y-j-2l+i}{2i-j}+\binom{y+i+j}{2i-j}\right)=0,$$

for $1\le j\le n-1$, $1\le l\le j/2$. This identity can be proved routinely by means of Zeilberger's algorithm [18, 19] and Salvy and Zimmermann's Maple package GFUN [17]. However, it happens that a ${}_4F_3$-summation is already known that applies to both ${}_4F_3$-series in (3.3), namely Lemma 1 in [9]. (Here we need the assumption $l\le j/2$.) Little simplification then establishes (3.3) and hence the claim. Thus, $\prod_{j=1}^{n-1}\left((x+y)/2+j+1\right)_{\lfloor (j+1)/2\rfloor}$ is a factor of $AB(n)$.

Now we prove that

$$\prod_{j=2}^{n-1}\left(-(x+y)/2-3n+j+\frac{3}{2}\right)_{\lfloor j/2\rfloor}$$

is a factor of $AB(n)$. Also here, let us concentrate on a typical factor $(x+y+6n-2j-2l-1)$, $2\le j\le n-1$, $1\le l\le j/2$. This time we claim that for each such factor there is a linear combination of the columns that vanishes if the factor vanishes. More precisely, we claim that for any j,l with $2\le j\le n-1$, $1\le l\le j/2$ there holds

$$\sum_{s=1}^{n-l}\frac{(2n-j-s)_{j-2l}}{(n+l-j)_{j-2l}}\frac{(s)_{n-l-s}\,(3n+l-2j-2)_{n-l-s}}{4^{n-l-s}\,(n-l-s)!\,(2n-j-\frac{1}{2})_{n-l-s}}\\ \cdot(\text{column } s \text{ of } AB(-y-6n+2j+2l+1,y;n))=0.$$

This means to check

$$\sum_{s=1}^{n-l}\frac{(2n-j-s)_{j-2l}}{(n+l-j)_{j-2l}}\frac{(s)_{n-l-s}\,(3n+l-2j-2)_{n-l-s}}{4^{n-l-s}\,(n-l-s)!\,(2n-j-\frac{1}{2})_{n-l-s}}\\ \cdot\left(\binom{-y-6n+2j+2l+1+i+s}{2i-s}+\binom{y+i+s}{2i-s}\right)=0$$

Converting this into hypergeometric notation and cancelling some factors, we see that we have to check

$$(4-i+2j+2l-6n-y)_{2i-1} \times {}_5F_4\left[\begin{matrix}1-2i,\frac{5}{2}+j+l-3n,2+j-2n,\\ 4+2j-4n,2+2l-2n,2-\frac{i}{2}+j+l-3n-\frac{y}{2},\end{matrix}\right. \left.\begin{matrix}1+l-n,3+i+2j+2l-6n-y\\ \frac{5}{2}-\frac{i}{2}+j+l-3n-\frac{y}{2}\end{matrix};1\right]$$
$$=-(3-i+y)_{2i-1} \times {}_5F_4\left[\begin{matrix}1-2i,\frac{5}{2}+j+l-3n,2+j-2n,1+l-n,2+i+y\\ 4+2j-4n,2+2l-2n,\frac{3}{2}-\frac{i}{2}+\frac{y}{2},2-\frac{i}{2}+\frac{y}{2}\end{matrix};1\right] \tag{3.4}$$

Again, in view of Zeilberger's algorithm, this is pure routine. However, also this identity happens to be already in the literature. It is exactly identity (5.7) in [8], with $a=\frac{3}{2}-i+j+l-3n-y$, $x=\frac{3}{2}+j+l-3n$, $z=1+j-2n$, $p=2i-1$. Thus, $\prod_{j=2}^{n-1}\left(-(x+y)/2-3n+j+\frac{3}{2}\right)_{\lfloor j/2\rfloor}$ is a factor of $AB(n)$.

Step 2. Bounding the polynomial degrees. The degree of $AB(n)$ as a polynomial in x is obviously at most $\binom{n}{2}$. But the degree of the product on the right-hand side of (1.1) is exactly $\binom{n}{2}$. Therefore it follows that

$$\det_{0\le i,j\le n-1}\left(\binom{x+i+j}{2i-j}+\binom{y+i+j}{2i-j}\right) = C(n)\prod_{j=1}^{n-1}\left(\tfrac{x+y}{2}+j+1\right)_{\lfloor (j+1)/2\rfloor}\left(-\tfrac{x+y}{2}-3n+j+\tfrac{3}{2}\right)_{\lfloor j/2\rfloor}, \tag{3.5}$$

with some $C(n)$ independent of x, and, by symmetry, also independent of y.

Step 3. Determining the constant. To compute $C(n)$, on both sides of (3.5) set $y=x$ and then compare coefficients of $x^{\binom{n}{2}}$. On the right-hand side the coefficient is $(-1)^{\chi(n\equiv 3 \mod 4)}C(n)$, whereas on the left-hand side the coefficient is

$$\begin{aligned}\det_{0\le i,j\le n-1}\left(2\frac{1}{(2i-j)!}\right) &= 2^n\prod_{i=0}^{n-1}\frac{1}{(2i)!}\det_{0\le i,j\le n-1}\left((2i-j+1)_j\right)\\ &= 2^n\prod_{i=0}^{n-1}\frac{1}{(2i)!}\det_{0\le i,j\le n-1}\left((2i)^j\right)\\ &= 2^n\prod_{i=0}^{n-1}\frac{1}{(2i)!}\prod_{0\le i<j\le n-1}(2j-2i)\\ &= 2^{n+\binom{n}{2}}\prod_{i=0}^{n-1}\frac{i!}{(2i)!}\\ &= 2^{1+\binom{n}{2}}\prod_{i=1}^{n-1}\frac{1}{(i)_i},\end{aligned}$$

where in the step from the first to the second line we used elementary column operations, and the subsequent step is just the Vandermonde determinant evaluation.

This completes the proof of (1.1). ■

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