no boundary:

$$G_F(x,t) = 1 + t\left(x + \frac{1}{x}\right) G_F(x,t) \quad \rightsquigarrow \quad G_F(x,t) = \frac{1}{1 - t(x+\frac{1}{x})}$$

correct for steps crossing the boundary: $\quad - t\,\frac{1}{x}\,G(0,t)$

$$\underline{\underline{\left[1 - t\left(x + \frac{1}{x}\right)\right] G(x,t) = 1 - \frac{t}{x}\,G(0,t)}}$$

$$G(x,t) = \frac{1 - \frac{t}{x}\,G(0,t)}{1 - t(x+\frac{1}{x})}$$ so $G(0,t)$ determines $G(x,t)$

- cannot simply plug in $x=0$
- factor $1 - t(x+\frac{1}{x}) = -\frac{t}{x}\left(x - x_0(t)\right)\left(x - x_1(t)\right)$

$$x^2 - \frac{1}{t}x + 1 = 0 \quad \rightsquigarrow \quad x_{0,1} = \frac{1}{2t}\left(1 \pm \sqrt{1-4t^2}\right) \qquad x_0 x_1 = 1$$

$$x_0 + x_1 = \frac{1}{t}$$

so if $x \to x_0$ or $x \to x_1$ then (hopefully)

$$\lim_{x \to x_i} \left[1 - t\left(x + \frac{1}{x}\right)\right] G(x,t) = 0 \quad \text{and} \quad 0 = 1 - \frac{t}{x_i}\,G(0,t)$$

so $G(0,t) = \dfrac{x_i}{t}$ and $x_i = x_1$: $\dfrac{x_1}{t} = \dfrac{1}{2t^2}\left(1 - \sqrt{1-4t^2}\right) = C(t^2)$

whereas $x_i = x_0$: $\dfrac{x_0}{t} = \dfrac{1}{2t^2}\left(1 + \sqrt{1-4t^2}\right) = O\left(\frac{1}{t^2}\right)$

$$\left[\; C(t) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{t^n}{n+1} \quad \text{Catalan GF solving} \quad C(t) = 1 + t\,C(t)^2 \;\right]$$

$G(0,t) = \frac{x_1}{t}$ and therefore

$$G(x,t) = \frac{1-\frac{t}{x}G(0,t)}{1-t(x+\frac{1}{x})} = \frac{1-\frac{x_1}{x}}{-\frac{t}{x}(x-x_0)(x-x_1)} = \frac{1}{t(x_0-x)}$$

GF for walks w/o boundary

$$G_F(1,t) = \frac{1}{1-2t} \quad \rightsquigarrow \quad c_N = 2^N$$

GF for walks with boundary

$$G(0,t) = C(t^2) \quad \rightsquigarrow \quad C_{2N} \sim C_1 \frac{4^N}{N^{3/2}}$$

$$G(1,t) = \frac{1-t\,C(t^2)}{1-2t} \quad \rightsquigarrow \quad C_N \sim C_1 \frac{2^N}{N^{1/2}}$$

(C_1 "generic")

Note: Walks w/o boundary ending at 0 cannot be computed by substituting $x=0$. We need

<u>Exercise:</u>

$$[x^0]\frac{1}{1-t(x+\frac{1}{x})} = CT_x \frac{1}{1-t(x+\frac{1}{x})} = \frac{1}{2\pi i}\oint \frac{1}{-\frac{t}{x}(x-x_0)(x-x_1)}\frac{dx}{x}$$

$$= \frac{1}{2\pi i}\oint \frac{1}{\sqrt{1-4t^2}}\left(\frac{1}{x-x_1}-\frac{1}{x-x_0}\right)dx = \frac{1}{\sqrt{1-4t^2}}, \quad C_{2N} \sim C\frac{4^N}{N^{1/2}}$$

An application: adsorption of directed polymers

weigh contacts with boundary with weight a

$$G(x,a,t) = 1 + t\left(x + \frac{1}{x}\right) G(x,a,t) - \frac{t}{x} G(0,a,t)$$

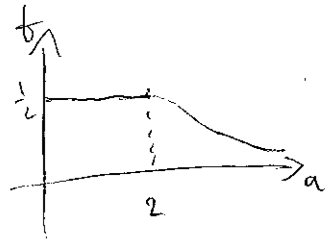
$$\underline{\underline{+\,tx(a-1)\; G(0,a,t)}}$$

$$\underbrace{\left[1 - t\left(x + \frac{1}{x}\right)\right]}_{K(x_1,t)\,=\,0} G(x,a,t) = 1 - t\left(\frac{1}{x} + x(1-a)\right) G(0,a,t)$$

$$\rightsquigarrow \quad G(0,a,t) = \frac{x_1}{t\,(1 + x_1^2(1-a))} = \frac{C(t^2)}{1 + (1-a)(C(t^2)-1)}$$

- square-root singularity at $t = \frac{1}{2}$ for $a < 2$ $\qquad Z_N \sim 2^N N^{-3/2}$
- pole at $C(t^2) = \frac{a}{a-1}$ for $a > 2$ $\qquad Z_N \sim \mu^N$
- $\frac{1}{\text{square-root}}$ singularity for $a = 2$ $\qquad Z_N \sim 2^N N^{-1/2}$

$\rightsquigarrow$ polymer adsorbs at $a_c = 2$, phase transition



3. A trick used twice becomes a method:

We've just encountered the Kernel Method for

a functional equation of the form (dropping t)

$$K(x)\; G(x) = F(x, G(0))$$

Often, one isn't interested in the variable x, but only in special values ($x = 0$, or $x = 1$, say), but varying x is essential for solving the eqn. The variable x is called catalytic. $K(x)$ is called the Kernel.

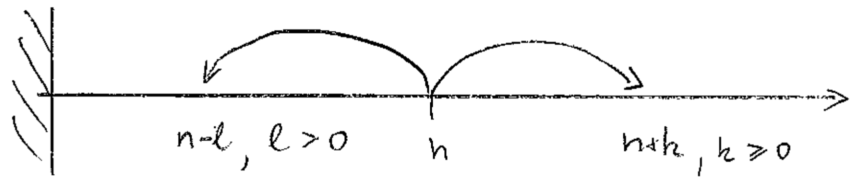
Method: (i) solve $K(x) = 0 \rightsquigarrow x = X$

(ii) solving $F(X, G(0))$ determines $G(0)$

Origin: Knuth TACP I 1968 (exercise)

Prodinger '99: "The French have a new toy. They call it the Kernel Method"

Kernel method for a larger class of walks



$l \in A$ $\qquad$ $k \in B$

allow finitely many forward and backward jumps

$$A(x) = \sum_{l \in A} x^l$$

$$B(x) = \sum_{k \in B} x^k$$

$\deg A = a$

$\deg B = b$

$$G(x,t) = 1 + t\left(A(x) + B\left(\tfrac{1}{x}\right)\right) G(x,t)$$

$\uparrow$

no walk

$$- t \left[B\left(\tfrac{1}{x}\right) G(x,t) \right]_{<0}$$

$$= 1 + t\left(A(x) + B\left(\tfrac{1}{x}\right)\right) G(x,t)$$

$$- t \sum_{k=0}^{b-1} b_k\left(\tfrac{1}{x}\right) G_k(t)$$

rewrite

$$\underbrace{\left[1 - t\left(A(x) + B\left(\tfrac{1}{x}\right)\right)\right] x^b}_{K(x,t)} G(x,t) = \underbrace{x^b \left(1 - t \sum_{k=0}^{b-1} b_k\left(\tfrac{1}{x}\right) G_k(t)\right)}_{R(x,t)}$$

<u>exercise</u>: write functional equation for $A = \{0,1\}$ and $B = \{1,2,3\}$

~~

$$K(x,t) = x^b\left(1 - t\left[A(x) + B\left(\tfrac{1}{x}\right)\right]\right)$$

has degree $a+b$ in x and admits $a+b$ solutions as algebraic functions of t.

- b "small" branches $x = u_i \sim \omega t^{1/b}$ $\quad \omega^b = 1$
- a "large" branches $x = v_i \sim \omega t^{-1/a}$ $\quad \omega^a = 1$

[Puiseaux expansion]

$$K(x,t) = -t \prod_{i=1}^{b}(x-u_i) \prod_{i=1}^{a}(x-v_i)$$

$R(x,t)$ is a polynomial of degree b in x, therefore necessarily

$$R(x,t) = \prod_{i=1}^{b}(x-u_i)$$

[no need to work with $G_k(t)$!]

Therefore we find

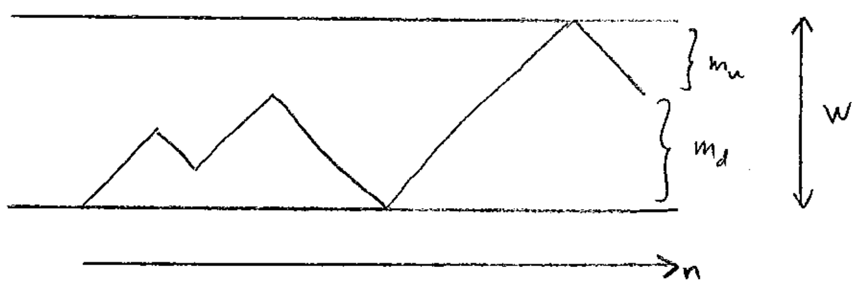
$$G(x,t) = -\frac{1}{t\prod_{i=1}^{a}(x-v_i)}$$

and

$$G(0,t) = \frac{(-1)^{a-1}}{t\prod_{i=1}^{a} v_i}, \quad G(1,t) = -\frac{1}{t\prod_{i=1}^{a}(1-v_i)}$$

[compare Dyck: $G(0,t) = \frac{1}{tx_0} = \frac{x_1}{t}$; $G(1,t) = -\frac{1}{t(1-x_0)}$]

4. Directed walks in a strip



$$G(\underset{m_d}{\underset{\uparrow}{x}},\underset{m_u}{\underset{\uparrow}{y}},\underset{n}{\underset{\uparrow}{t}}) = y^w + t\left(\frac{x}{y}+\frac{y}{x}\right) G(x,y,t)$$

$$- t\,\frac{x}{y}\, G(x,0,t) - t\,\frac{y}{x}\, G(0,y,t)$$

rewrite: $$xy\left(1-t\left(\frac{x}{y}+\frac{y}{x}\right)\right) G(x,y,t) = xy^{w+1} - tx^2 G(x,0,t) - ty^2 G(0,y,t)$$

or $$K(x,y,t)\,xy\; G(x,y,t) = xy^{w+1} - R(x,t) - S(y,t)$$

$K(x,y,t)=0 \leadsto y=y(x,t)$ relates y and x

think of <u>pairs</u> (x,y) killing the kernel. Here, $y=qx$

where $q+\frac{1}{q}=\frac{1}{t}$, so if $K(x,qx,t)=0$ then

K vanishes also for (x,qx), (q^2x,qx), (q^2x,q^3x), ...

So iterate: $$R(x,t) = x(qx)^{w+1} - S(qx,t)$$

$$S(qx,t) = q^2x(qx)^{w+1} - R(q^2x,t) \quad \text{etc}$$

$$R(x,t) = x^{w+2} q^{w+1} - x^{w+2} q^{w+3} + R(q^2 x, t)$$

$$= x^{w+2} q^{w+1} (1-q^2) + R(q^2 x, t)$$

$$= \dots = \frac{x^{w+2} q^{w+1} (1-q^2)}{1-q^{2(w+2)}} \qquad \text{(geom series)}$$

Therefore
$$G(x,0,t) = x^w \frac{q^{w+1}(1+q^2)}{t\left(1-q^{2(w+2)}\right)} = x^w \frac{q^w (1-q^4)}{1-q^{2(w+2)}}$$

<u>exercise:</u> compute
$$G(0,y,t) \left[= y^w \frac{(1+q^2)\left(1-q^{2(w+1)}\right)}{1-q^{2(w+2)}} \right]$$

and
$$G(1,1,t) \left[= \frac{(1+q^2)(1-q^{w+1})(1-q^{w+2})}{(1-q)\left(1-q^{2(w+2)}\right)} \right]$$

Generalize to contact weights a/b at bottom / top :

e.g.
$$G_{ab}(x,0,t) = x^w \frac{b\, q^w (1-q^4)}{\left(1-(a-1)q^2\right)\left(1-(b-1)q^2\right) - \left((1-a)+q^2\right)\left((1-b)+q^2\right) q^{2w}}$$

5. 2-dimensional lattice walks I: walks on the slit plane

C_{N, m_x, m_y}, # of N-step walks from $\mathcal{O}$ to (m_x, m_y)

$$G(x,y,t) = \sum_{N, m_x, m_y} C_{N, m_x, m_y}\, t^N x^{m_x} y^{m_y} \quad , \text{ step set } \Omega \text{ :}$$

- no boundaries: $\quad G_F(x,y,t) = 1 + t \sum_{\sigma \in \Omega} x^{\sigma_x} y^{\sigma_y}\, G_F(x,y,t)$

$$\rightsquigarrow G_F(x,y,t) = \left[K(x,y,t)\right]^{-1}$$

- walks in the slit plane (starting at 0 but must not return to $(-n, 0)$, $n \in \mathbb{N}_0$)

  $\Omega = \{\uparrow, \leftarrow, \downarrow, \rightarrow\}$

$$G(x,y,t) = 1 + t\left(x + \frac{1}{x} + y + \frac{1}{y}\right) G(x,y,t) - B\left(\frac{1}{x}, t\right)$$

walks starting at $\mathcal{O}$, avoiding slit but ending on it.

$$\left[1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)\right] G(x,y,t) = 1 - B\left(\frac{1}{x}, t\right)$$

careful: mindless application of the method gives nonsense:

$LHS = 0 \rightsquigarrow RHS = 0 \rightsquigarrow B\left(\frac{1}{x}\right) = 1$ does not make sense!

What's wrong? With the power of hindsight,

$$c_N \sim 4^N N^{-1/4}$$ so that $G(x,y,t)$ diverges so fast that

$$\lim_{y \to y(x,t)} \left[1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)\right] G(x,y,t) \neq 0$$

Repair this by considering instead of $G(x,y,t) = \sum_M y^M G_M(x,t)$

$$H(x,y,t) = \sum c_{N,m_x,m_y}\, t^N x^{m_x} y^{|m_y|} = \sum_M y^{|M|} G_M(x,t)$$

$$[G_M = G_{-M}]$$

$$H(x,y,t) = 1 + t\left(x + \frac{1}{x} + y + \frac{1}{y}\right) H(x,y,t) + t\left(y - \frac{1}{y}\right) G_0(x,t) - B\left(\frac{1}{x}, t\right)$$

walks ending on x-axis

so that

$$(*) \quad \underbrace{\left[1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)\right]}_{K(x,y,t)} H(x,y,t) = 1 + t\left(y - \frac{1}{y}\right) G_0(x,t) - B\left(\frac{1}{x}, t\right)$$

$$K(x,y,t) = 0 \quad \rightsquigarrow \quad y^2 - \left(\frac{1}{t} - \left(x + \frac{1}{x}\right)\right) y + 1$$

two roots $Y_0 Y_1 = 1$ and $Y_1 = Y_1(x,t) = \frac{1}{2}\left(\frac{1}{t} - \left(x + \frac{1}{x}\right)\right) - \sqrt{\frac{1}{4}\left(\frac{1}{t} - \left(x + \frac{1}{x}\right)\right)^2 - 1}$

$Y_0 \sim \frac{1}{t}$ $\quad Y_1 \sim t$ with coefficients Laurent pols in x.

Substitute $y = Y_1$ into $(*)$ to get

$$1 - B\left(\tfrac{1}{x},t\right) = t\left(\frac{1}{Y_1} - Y_1\right) G_0(x,t) \qquad \left[\text{compare } 1 - B\left(\tfrac{1}{x},t\right) = 0\right]$$

$$= 2t\sqrt{\frac{1}{4}\left(\frac{1}{t} - x - \frac{1}{x}\right)^2 - 1}\; G_0(x,t)$$

$$\underbrace{1 - B\left(\tfrac{1}{x},t\right)}_{\text{non-pos. powers in } x} = \sqrt{\left(1 - t\left(x + \tfrac{1}{x}\right)\right)^2 - 4t^2}\; \underbrace{G_0(x,t)}_{\text{non-neg powers in } x}$$

<u>Trick</u>: Factorize $\left(1 - t\left(x + \frac{1}{x}\right)\right)^2 - 4t^2 = \left(1 - t\left(x + \frac{1}{x} + 2\right)\right)\left(1 - t\left(x + \frac{1}{x} - 2\right)\right)$

$$= D(t)\, \Delta(x,t)\, \Delta\left(\tfrac{1}{x},t\right)$$

where $\Delta(x,t) = \left(1 - x\left(C(t) - 1\right)\right)\left(1 - x\left(1 - C(-t)\right)\right)$

and $D(t) = \left[C(t)\,C(-t)\right]^{-2}$

<u>exercise</u>: confirm factorization

now
$$\frac{1 - B\left(\frac{1}{x},t\right)}{\sqrt{\Delta\left(\frac{1}{x},t\right)}} = \sqrt{D(t)}\,\sqrt{\Delta(x,t)}\; G_0(x,t)$$

LHS non-pos powers in x

RHS non-neg powers in x

LHS = RHS must be constant independent of x

$$\Delta(0,t) = 1, \qquad G_0(0,t) = 1$$ [walks cannot return to 0]

so that
$$\frac{1 - B(\frac{1}{x}, t)}{\Delta(\frac{1}{x}, t)} = \sqrt{D(t)}$$
, or

$$1 - B(\tfrac{1}{x}, t) = \sqrt{D(t)\,\Delta(\tfrac{1}{x}, t)}$$

and thus

$$G(x,y,t) = \frac{\sqrt{D(t)\,\Delta(\frac{1}{x}, t)}}{1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)}$$

$$G(1,1,t) = \frac{\left(1 + \sqrt{1+4t}\right)^{1/2}\left(1 + \sqrt{1-4t}\right)^{1/2}}{2(1-4t)^{3/4}}$$

$$c_n \sim \frac{\sqrt{1+\sqrt{2}}}{2\,\Gamma(3/4)}\, 4^n\, n^{-1/4}$$