trick to method: <u>The factorisation lemma</u>

Let $\delta(x,t)$ be a polynomial in t with coefficients in $\mathbb{R}[x,\bar{x}]$ and assume $\delta(x,0)=1$. $\quad \left(\bar{x}=\frac{1}{x}\right)$

There exist a unique triple $\left(D(t),\ \Delta(x,t),\ \bar{\Delta}(\bar{x},t)\right)$ of FPS in t satisfying

- $\delta(x,t) = D(t)\cdot\Delta(x,t)\cdot\bar{\Delta}(\bar{x},t)$
- coeffs of $D(t)$ belong to $\mathbb{R}$
- " $\Delta(x)$ " $\mathbb{R}[x]$
- " $\bar{\Delta}(\bar{x})$ " $\mathbb{R}[\bar{x}]$
- $D(0) = \Delta(0,t) = \bar{\Delta}(0,t) = \Delta(x,0) = \bar{\Delta}(\bar{x},0) = 1$

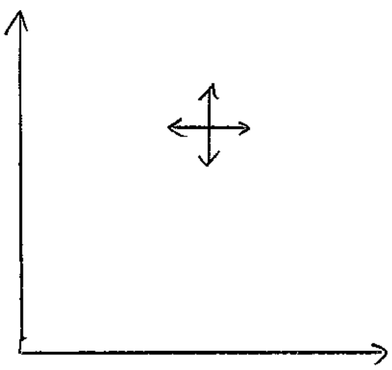
Moreover, these three series are algebraic, and $\Delta(x)$ is a pol in x $\left(\bar{\Delta}(\bar{x})\right)$ $\quad (\bar{x})$

Change steps: 

$$\left[1-t\left(\frac{x}{y}+\frac{y}{x}+xy+\frac{1}{xy}\right)\right]H(x,y,t) = 1+t\left(xy+\frac{y}{x}-\frac{x}{y}-\frac{1}{xy}\right)G_0(x,t) - B\left(\frac{1}{x},t\right)$$

<u>Exercise</u>: find $\delta(x,t)$, compute factorisation

6. 2-dimensional lattice walks II : walks on the quarter plane



$$G(x,y;t) = 1 + t\left(x+y+\frac{1}{x}+\frac{1}{y}\right) G(x,y,t) - t\frac{1}{x}\,G(0,y,t) - t\frac{1}{y}\,G(x,0,t)$$

$$\rightarrow K(x,y,t)\,xy\,G(x,y,t) = xy - t\,x\,G(x,0,t) - t\,y\,G(0,y,t)$$

Note that $K(x,y,t) = 1 - t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)$ has symmetries $x \leftrightarrow \frac{1}{x}$, $y \leftrightarrow \frac{1}{y}$, $x \leftrightarrow y$

(1) $K(x,y,t) = 0 \quad \rightsquigarrow \quad Y_1 = f_t\left(x+\frac{1}{x}\right) \sim t \qquad Y_0 = \frac{1}{Y_1} \sim \frac{1}{t}$

iterate : $(x, Y_1) \rightarrow \left(\frac{1}{x}, Y_1\right) \rightarrow \left(\frac{1}{x}, \frac{1}{Y_1}\right) \rightarrow \left(x, \frac{1}{Y_1}\right) \rightarrow (x, Y_1)$

$\checkmark \qquad \checkmark \qquad \uparrow \qquad \uparrow$

no powerseries int. not admissible

$K(x,Y_1) = 0 \quad \rightsquigarrow \quad x\,Y_1 = t\,x\,G(x,0,t) - t\,Y_1\,G(0,Y_1,t)$

$K\left(\frac{1}{x},Y_1\right) = 0 \quad \rightsquigarrow \quad \frac{1}{x}\,Y_1 = t\,\frac{1}{x}\,G\left(\frac{1}{x},0,t\right) - t\,Y_1\,G(0,Y_1,t)$

so that $\quad t\left[x\,G(x,0,t) - \frac{1}{x}\,G\left(\frac{1}{x},0,t\right)\right] = \left(x-\frac{1}{x}\right) Y_1$

and therefore

$$G(x,0,t) = \frac{1}{tx}\left[\left(x-\frac{1}{x}\right) Y_1\right]_{\geq x} \qquad \text{positive part (in } x\text{)}$$

$$= \frac{1}{t}\,\frac{1}{2\pi i}\oint_{|z|=1}\left(z-\frac{1}{z}\right) f_t\left(z+\frac{1}{z}\right)\frac{dz}{z(z-x)}$$

(2) "algebraic" kernel method: only use kernel symmetries

(don't set $K(x,y,t)=0$, just eliminate bdry terms)

exercise?

$$K(x,y,t)\,xy\,G(x,y,t) = xy - tx\,G(x,0,t) - ty\,G(0,y,t) \qquad +$$

$$x\to\tfrac{1}{x}:\; K(x,y,t)\,\tfrac{1}{x}y\,G(\tfrac{1}{x},y,t) = \tfrac{1}{x}y - t\tfrac{1}{x}\,G(\tfrac{1}{x},0,t) - ty\,G(0,y,t) \qquad -$$

$$y\to\tfrac{1}{y}:\; K(x,y,t)\,x\tfrac{1}{y}\,G(x,\tfrac{1}{y},t) = x\tfrac{1}{y} - tx\,G(x,0,t) - t\tfrac{1}{y}\,G(0,\tfrac{1}{y},t) \qquad -$$

$$K(x,y,t)\,\frac{1}{xy}\,G(\tfrac{1}{x},\tfrac{1}{y},t) = \frac{1}{xy} - t\frac{1}{x}\,G(\tfrac{1}{x},0,t) - t\frac{1}{y}\,G(0,\tfrac{1}{y},t) \qquad +$$

$$K(x,y,t)\left[xy\,G(x,y,t) - \tfrac{1}{x}y\,G(\tfrac{1}{x},y,t) - x\tfrac{1}{y}\,G(x,\tfrac{1}{y},t) + \frac{1}{xy}\,G(\tfrac{1}{x},\tfrac{1}{y},t)\right]$$

$$= xy - \tfrac{1}{x}y - y\tfrac{1}{x} + \frac{1}{xy} = \left(x-\tfrac{1}{x}\right)\left(y-\tfrac{1}{y}\right)$$

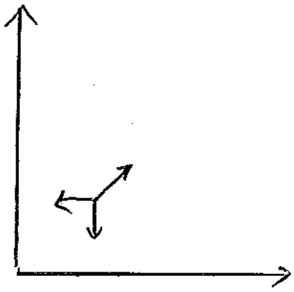
so that
$$G(x,y,t) = \frac{1}{xy}\left[\frac{(x-\frac{1}{x})(y-\frac{1}{y})}{K(x,y,t)}\right]_{>(x,y)}$$

$$= \left(\frac{1}{2\pi i}\right)^2 \oint\oint \frac{(z-\frac{1}{z})(w-\frac{1}{w})}{K(z,w,t)}\,\frac{dz}{z(z-x)}\,\frac{dw}{w(w-y)}$$

exercise: show that

$$G(x,0,t) = \left(\frac{1}{2\pi i}\right)^2 \oint\oint \frac{(z-\frac{1}{z})(w-\frac{1}{w})}{K(z,w,t)}\,\frac{dz}{z(z-x)}\,\frac{dw}{w^2}$$

reduces to the previous result



$$\underbrace{\left[1-t\left(\frac{1}{x}+\frac{1}{y}+xy\right)\right]}_{K(x,y,t)} xy\,G(x,y,t) = xy - tx\,G(x,0,t) - ty\,G(0,y,t)$$

$$= xy - R(x) - R(y)$$

[$x \leftrightarrow y$ symmetry]

Symmetry: $K(x,y,t) = K\left(\frac{1}{xy},y,t\right) = K\left(\frac{1}{xy},x,t\right) = \dots$

$(x,y) \to \left(\frac{1}{xy},y\right) \to \left(\frac{1}{xy},x\right) \to (y,x) \to \left(y,\frac{1}{xy}\right) \to \left(x,\frac{1}{xy}\right) \to (x,y)$

C_6 cycle

$$y^2 - \frac{1}{x}\left(\frac{1}{t}-\frac{1}{x}\right)y + \frac{1}{x} = 0 \qquad Y_0+Y_1 = \frac{1}{x}\left(\frac{1}{t}-\frac{1}{x}\right),\ Y_0Y_1 = \frac{1}{x}$$

$$\rightsquigarrow\ Y_0(x,t) = \frac{1}{2t}\left[\frac{1}{x}\left(1-\frac{t}{x}\right) \mp \sqrt{\frac{1}{x}\left(1-\frac{t}{x}\right)^2 - \frac{4}{x}}\right]$$

$$Y_0(x,t) = t + \frac{1}{x}t^2 + O(t^3), \qquad Y_1(x,t) = \frac{1}{xt} - \frac{1}{x^2} - t - \frac{1}{x}t^2 + O(t^3)$$

Y.

in iteration cycle, pick $y = Y_0(x,t)$:

$(x, Y_0) \to (Y_1, Y_0) \to (Y_1, x) \to (Y_0, x) \to (Y_0, Y_1) \to (x, Y_1) \to (x, Y_0)$

$$\rightsquigarrow \quad R(x) + R(Y_0) = xY_0 \sim xt$$

$$R(Y_1) + R(Y_0) = Y_0 Y_1 = \frac{1}{x}$$

$$\left[R(Y_1) + R(x) = Y_1 x = \frac{1}{Y_0} \sim \frac{1}{t} \qquad \text{no good} \right]$$

rewrite:

$$\underline{R(Y_0) - xY_0 = -R(x)}$$

$$\underline{R(Y_1) - xY_1} = Y_0 Y_1 - \underbrace{R(Y_0)} - xY_1 \qquad \swarrow \; Y_0 + Y_1 = \frac{1}{x}\left(\frac{1}{t} - \frac{1}{x}\right)$$

$$= \frac{1}{x} + R(x) - xY_0 - xY_1$$

$$= \frac{1}{x} + R(x) - \left(\frac{1}{t} - \frac{1}{x}\right)$$

$$= \underline{R(x) + \frac{2}{x} - \frac{1}{t}}$$

so that

$$\frac{R(Y_0) - R(Y_1)}{Y_0 - Y_1} - x = tx \, \frac{2R(x) + \frac{2}{x} - \frac{1}{t}}{\sqrt{\delta(x,t)}}$$

where $\delta(x,t) = \left(1 - t\frac{1}{x}\right)^2 - 4t^2 x$. Use the factorisation lemma to

argue that

$$\delta(x,t) = D(t)\, \Delta(x,t)\, \bar{\Delta}\left(\tfrac{1}{x}, t\right)$$

$$\left[\begin{array}{ll} \text{three roots } X_0, X_1, X_2 : & D(t) = 4t^2 X_2, \quad \Delta(x,t) = 1 - \frac{x}{X_2} \\ X_{0,1} \sim t \quad X_2 \sim \frac{1}{t^2} & \bar{\Delta}\left(\frac{1}{x}, t\right) = \left(1 - \frac{X_0}{x}\right)\left(1 - \frac{X_1}{x}\right) \end{array} \right]$$

therefore

$$\sqrt{\bar{\Delta}\left(\frac{1}{x},t\right)}\left[\frac{R(y_0)-R(y_1)}{y_0-y_1}-x\right] = t\,\frac{2xR(x)+2-\frac{x}{t}}{\sqrt{D(t)\,\Delta(x,t)}}$$

extracting the positive part gives

$$-x = \frac{t}{\sqrt{D(t)}}\left[\frac{2xR(x)+2-\frac{x}{t}}{\sqrt{\Delta(x,t)}}-2\right]$$

rewriting (exercise) gives

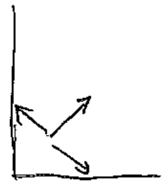
Theorem

$$G(x,0,t) = \frac{1}{tx}\left(\frac{1}{2t}-\frac{1}{x}-\left(\frac{1}{W}-\frac{1}{x}\right)\sqrt{1-xW^2}\right)$$

where $W = t(2+W^3)$ defines the power series $W = W(t)$

some further work gives for walks returning to the origin

$$c_{3n} = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$$



$$\left[1 - t\left(\frac{x}{y} + \frac{y}{x} + xy\right)\right] xy\, G(x,y,t) = xy - t x^2 G(x,0,t) - t y^2 G(0,y,t)$$

$K(x,y,t)=0 \rightsquigarrow \frac{1}{y^2} - \frac{1}{tx}\frac{1}{y} + \left(\frac{1}{x^2}+1\right) = 0$

$\rightsquigarrow$ $y_{0,1}$ satisfy $\frac{1}{y_0} + \frac{1}{y_1} = \frac{1}{t}\frac{1}{x}$ $\rightsquigarrow$ 3-term recurrence

iteration: $K(x_n, x_{n+1}, t) = 0$ gives

$\ldots, x_{-1} = y_0, \quad x_0 = x, \quad x_1 = y_1, \quad x_2 = \ldots$ with x_n given by

$$\frac{1}{x_n} = \alpha \lambda^n + \beta \lambda^{-n}, \quad \lambda + \frac{1}{\lambda} = \frac{1}{t}, \quad \alpha + \beta = \frac{1}{x}, \quad \alpha\lambda + \frac{\beta}{\lambda} = \frac{1}{y_1}$$

<u>exercise</u>: check that $x_n = x t^n + O(t^{n+1})$

$K(x_n, x_{n+1}, t) = 0 \rightsquigarrow t x_n^2 G(x_n, 0, t) = x_n x_{n+1} - t x_{n+1}^2 G(x_{n+1}, 0, t)$

leads to
$$G(x,0,t) = \frac{1}{x^2 t} \sum_{k=0}^{\infty} (-1)^k x_k x_{k+1}$$

$$G(1,1,t) = \frac{1 - 2t\, G(1,0,t)}{1-3t}$$

exercise: $c_n \sim \left(1 - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{F_{2k-1} F_{2k+1}}\right) 3^n$

[Note: $G(x,y,t)$ is not differentiably finite (poles accumulate)]