The Potts model on planar maps

Mireille Bousquet-Mélou, CNRS, Bordeaux, France
(joint work with Olivier Bernardi, Brandeis, Boston)

http://www.labri.fr/~bousquet
Planar maps
There are finitely many maps with $n$ edges.
Planar maps

- vertices \( V(M) \)
- edges
- and faces
Rooted planar maps

- vertices $V(M)$
- edges
- and faces
Map enumeration

In combinatorics, statistical physics

- The original question: how many maps with a given number of edges?
  - Tutte and his descendents (1960 → 2014)
  - Brézin-Itzykson-Parisi-Zuber and their descendents (1978 → 2014)

- Key object: the generating function of planar maps, counted by edges:
  \[ M(t) := \sum_{M} t^{e(M)} = 1 + 2t + O(t^2) \]
  where \( e(M) \) is the number of edges of \( M \).
A chronology of planar maps

- **Recursive approach**: Tutte, Brown, Bender, Canfield, Richmond, Goulden, Jackson, Wormald, Walsh, Lehman, Gao, Wanless...
- **Matrix integrals**: Brézin, Itzykson, Parisi, Zuber, Bessis, Ginsparg, Zinn-Justin, Boulatov, Kazakov, Mehta, Bouttier, Di Francesco, Guitter, Eynard...
- **Bijections**: Cori & Vauquelin, Schaeffer, Bouttier, Di Francesco & Guitter (BDG), Bernardi, Fusy, Poulalhon, mbm, Chapuy...
- **Geometric properties**: Chassaing & Schaeffer, BDG, Marckert & Mokkadem, Le Gall, Miermont, Curien...
A chronology of planar maps

- Recursive approach (Tutte)
- Matrix integrals
- Bijections with trees
- Geometric properties


This talk
A chronology of planar maps

- Recursive approach (Tutte)
- Matrix integrals
- Bijections with trees
- Geometric properties

This talk
A chronology of planar maps

Recursive approach (Tutte)


Matrix integrals

Bijections with trees

Geometric properties

... but for maps equipped with a $q$-colouring (or a Potts model)

This talk
Maps equipped with an additional structure

In combinatorics, but mostly in statistical physics

How many maps equipped with... | What is the expected partition function of...
--- | ---
– a spanning tree? | – the Ising model?  
[Mullin 67]  
– a spanning forest?  
[Bouttier et al., Sportiello et al., mbm-Courtiel 13]  
– a self-avoiding walk?  
[Duplantier-Kostov 88]  
– a proper $q$-colouring?  
[Tutte 74, Bouttier et al. 02]  
– the hard-particle model?  
[mbm, Schaeffer, Jehanne, Bouttier et al. 02, 07]  
– the Potts model?  
[Eynard-Bonnet 99, Baxter 01, mbm-Bernardi 09, Guionnet et al. 10, Borot et al. 12]
The partition function of the $q$-state Potts model on a planar map $M$:

$$Z_M(q, \nu) = \sum_{c:V(M) \to \{1,2,\ldots,q\}} \nu^{m(c)}$$

where $m(c)$ of the number of monochromatic edges in the colouring $c$. In fact, $Z_M(q, \nu)$ is a polynomial in $q$ (and $\nu$).

**Example:** When $M$ has one edge and two vertices, $Z_M(q, \nu) = q\nu + q(q - 1)$.
The Potts model on planar maps (cont’d)

• Generating function:

\[ M(q, \nu, t) = \sum M_Z(q, \nu) t^{e(M)} = \sum t^{e(M)} \nu^{m(c)} \]

\[ = q + (q \nu + q \nu + q(q - 1)) t + O(t^2) \]

“The Potts generating function of planar maps”

⇒ Enumeration of \( q \)-coloured planar maps, counted by edges and monochromatic edges.
I. Uncoloured planar maps: the recursive approach
Let

\[ M(t; y) \equiv M(y) = \sum_{M} t^{e(M)} y^{df(M)} = \sum_{d \geq 0} M_d(t) y^d \]

where \( e(M) \) is the number of edges and \( df(M) \) the degree of the outer face.
Planar maps: the recursive approach

Let

\[ M(t; y) \equiv M(y) = \sum_M t^{e(M)} y^{df(M)} = \sum_{d \geq 0} M_d(t) y^d \]

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where \( e(M) \) is the number of edges and \( df(M) \) the degree of the outer face.

\[
M(y) = 1 + ty^2M(y)^2 + t \sum_{d \geq 0} M_d(t) \left( y^{d+1} + y^d + \cdots + y \right) \\
= 1 + ty^2M(y)^2 + t y \frac{yM(y) - M(1)}{y - 1}
\]

[Tutte 68] A quadratic equation with one catalytic variable, \( y \)
Planar maps: the recursive approach

Let

\[ M(t; y) \equiv M(y) = \sum_{M} t^{e(M)} y^{d_f(M)} = \sum_{d \geq 0} M_d(t) y^d \]

where \( e(M) \) is the number of edges and \( df(M) \) the degree of the outer face.

Remark. Let us write the equation

\[ M(y) = 1 + ty^2 M(y)^2 + ty \frac{yM(y) - M_1}{y - 1} \]

Letting \( y \to 1 \), we see that the value of \( M_1 = M(1) \) is forced by the fact that \( M(y) \equiv M(t; y) \) is a series in \( t \) with polynomial coefficients in \( y \).
Planar maps: algebraic solution

- Form a square:

\[
(2ty^2(y - 1)M(y) + ty^2 - y + 1)^2 = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)M_1
\]

\[:= \Delta(y) \quad \text{(a polynomial in } y)\]
Planar maps: algebraic solution

• Form a square:

\[
(2ty^2(y - 1)M(y) + ty^2 - y + 1)^2 = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)M_1
\]

\[
:= \Delta(y)
\]

(a polynomial in \( y \))

• There exists a (unique) series \( Y \equiv Y(t) \) that cancels the LHS:

\[
Y = 1 + tY^2 + 2tY^2(Y - 1)M(Y).
\]

⇒ characterizes inductively the coefficient of \( t^n \)
Planar maps: algebraic solution

- Form a square:

\[
(2ty^2(y - 1)M(y) + ty^2 - y + 1)^2 = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)M_1
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\[\Rightarrow \text{characterizes inductively the coefficient of } t^n\]

- This series \( Y \) must be a root of \( \Delta(y) \), and in fact a double root.
Planar maps: algebraic solution

- Form a square:

\[
(2ty^2(y - 1)M(y) + ty^2 - y + 1)^2 = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)M_1
:= \Delta(y) \quad \text{(a polynomial in } y)\]

- There exists a (unique) series \( Y \equiv Y(t) \) that cancels the LHS:

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Y = 1 + tY^2 + 2tY^2(Y - 1)M(Y).
\Rightarrow \text{ characterizes inductively the coefficient of } t^n
\]

- This series \( Y \) must be a root of \( \Delta(y) \), and in fact a \textbf{double} root.

- \textbf{Algebraic consequence:} The discriminant of \( \Delta(y) \) w.r.t. \( y \) is zero:

\[
27t^2M_1^2 + (1 - 18t)M_1 + 16t - 1 = 0
\]

or

\[
M(t; 1) = \frac{(1 - 12t)^3/2 - 1 + 18t}{54t^2} \quad \text{(algebraic series)}
\]
Planar maps: differential solution

- The polynomial

\[ \Delta(y) = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)M_1 \]

has degree 4 in \( y \), and admits a double root \( Y(t) \):

\[ \Delta(t; y) = P(t; y)(y - Y(t))^2, \quad P \text{ of degree 2 in } y \]
Planar maps: differential solution

- The polynomial

\[
\Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)M_1
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has degree 4 in \(y\), and admits a double root \(Y(t)\):

\[
\begin{align*}
\Delta(t; y) &= P(t; y)(y - Y(t))^2, & P \text{ of degree 2 in } y \\
\Delta_y'(t; y) &= Q(t; y)(y - Y(t)), & Q \text{ of degree 2 in } y
\end{align*}
\]
Planar maps: differential solution

- The polynomial

\[ \Delta(y) = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)M_1 \]

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\[ \Delta(t; y) = P(t; y)(y - Y(t))^2, \quad P \text{ of degree 2 in } y \]
\[ \Delta_y(t; y) = Q(t; y)(y - Y(t)), \quad Q \text{ of degree 2 in } y \]
\[ \Delta_t(t; y) = R(t; y)(y - Y(t)), \quad R \text{ of degree 3 in } y \]
Planar maps: differential solution

- The polynomial

\[ \Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)M_1 \]

has degree 4 in \( y \), and admits a double root \( Y(t) \):

\[
\begin{align*}
\Delta(t; y) &= P(t; y)(y - Y(t))^2, \quad P \text{ of degree 2 in } y \\
\Delta'_y(t; y) &= Q(t; y)(y - Y(t)), \quad Q \text{ of degree 2 in } y \\
\Delta'_t(t; y) &= R(t; y)(y - Y(t)), \quad R \text{ of degree 3 in } y
\end{align*}
\]

- Elimination of \( Y \) and \( \Delta \):

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{P} \right) = \frac{1}{R} \frac{\partial}{\partial y} \left( \frac{R^2}{P} \right).
\]
Planar maps: differential solution

• The polynomial

\[ \Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)M_1 \]

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• Elimination of \( Y \) and \( \Delta \):

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{P} \right) = \frac{1}{R} \frac{\partial}{\partial y} \left( \frac{R^2}{P} \right).
\]

• Writing \( P(t; y) = P_0(t) + yP_1(t) + y^2P_2(t) \) and so on for \( Q \) and \( R \), this gives a system of differential equations in \( t \) relating the series \( P_i, Q_i \) and \( R_i \).
Planar maps: differential solution

• The polynomial

\[ \Delta(y) = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)M_1 \]

has degree 4 in \( y \), and admits a double root \( Y(t) \):

\[ \Delta(t; y) = P(t; y)(y - Y(t))^2, \quad P \text{ of degree 2 in } y \]
\[ \Delta_y(t; y) = Q(t; y)(y - Y(t)), \quad Q \text{ of degree 2 in } y \]
\[ \Delta_t(t; y) = R(t; y)(y - Y(t)), \quad R \text{ of degree 3 in } y \]

• Elimination of \( Y \) and \( \Delta \):

\[ \frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{P} \right) = \frac{1}{R} \frac{\partial}{\partial y} \left( \frac{R^2}{P} \right). \]

• Writing \( P(t; y) = P_0(t) + yP_1(t) + y^2P_2(t) \) and so on for \( Q \) and \( R \), this gives a system of differential equations in \( t \) relating the series \( P_i, Q_i \) and \( R_i \).

• Finally, the series \( M_1 \) can be expressed rationally in terms of the \( P_i, Q_i, R_i \).
II. The Potts model on planar maps

A recursive approach

- Other approaches: [Eynard-Bonnet 99], [Guionnet et al. 10], [Borot et al. 12]
An equation with two catalytic variables

- Let

\[ M(x, y) \equiv M(q, \nu, t; x, y) = \frac{1}{q} \sum_{M} Z_{M}(q, \nu) t^{e(M)} x^{dv(M)} y^{df(M)}, \]

where \( dv(M) \) (resp. \( df(M) \)) is the degree of the root-vertex (resp. root-face).

- The Potts generating function of planar maps satisfies:

\[
M(x, y) = 1 + xy t ((\nu - 1)(y - 1) + qy) M(x, y) M(1, y) \\
+ xy t (x \nu - 1) M(x, y) M(x, 1) \\
+ xy t (\nu - 1) \frac{x M(x, y) - M(1, y)}{x - 1} + xy t \frac{y M(x, y) - M(x, 1)}{y - 1}.
\]

[Tutte 68]

This equation has been sleeping for 40 years:

What is \( M(1, 1) \equiv M(q, \nu, t; 1, 1) \)?
In the footsteps of W. Tutte

• For the GF $T(q,t;x,y) \equiv T(x,y)$ of properly $q$-coloured triangulations:

$$T(x,y) = x y^2 q(q-1) + \frac{xt}{yq} T(1,y) T(x,y) + \frac{T(x,y) - y^2 T_2(x)}{y} - x^2 y t \frac{T(x,y) - T(1,y)}{x - 1}$$

where $T_2(x)$ is the coefficient of $y^2$ in $T(x,y)$.

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$
[Tutte 73] Chromatic sums for rooted planar triangulations, II: the case $\lambda = \tau + 1$
[Tutte 73] Chromatic sums for rooted planar triangulations, III: the case $\lambda = 3$
[Tutte 73] Chromatic sums for rooted planar triangulations, IV: the case $\lambda = \infty$
[Tutte 74] Chromatic sums for rooted planar triangulations, V: special equations
[Tutte 78] On a pair of functional equations of combinatorial interest
[Tutte 82] Chromatic solutions
[Tutte 82] Chromatic solutions II
[Tutte 84] Map-colourings and differential equations

[Tutte 95]: Chromatic sums revisited
In the footsteps of W. Tutte

• For the GF \( T(q, t; x, y) \equiv T(x, y) \) of properly \( q \)-coloured triangulations:

\[
T(x, y) = xy^2q(q-1) + \frac{xt}{yq} T(1, y) T(x, y) + xt \frac{T(x, y) - y^2T_2(x)}{y} - x^2yt \frac{T(x, y) - T(1, y)}{x - 1}
\]

where \( T_2(x) \) is the coefficient of \( y^2 \) in \( T(x, y) \).

**Theorem [Tutte]**

• For \( q = 2 + 2 \cos \frac{2\pi}{m} \), \( q \neq 4 \), the series \( T(1, y) \equiv T(q, t; 1, y) \) satisfies a polynomial equation with one catalytic variable \( y \).
In the footsteps of W. Tutte

• For the GF $T(q, t; x, y) \equiv T(x, y)$ of properly $q$-coloured triangulations:

$$T(x, y) = x y^2 q(q-1) + \frac{x t}{y q} T(1, y) T(x, y) + x t \frac{T(x, y) - y^2 T_2(x)}{y} - x^2 y t \frac{T(x, y) - T(1, y)}{x - 1}$$

where $T_2(x)$ is the coefficient of $y^2$ in $T(x, y)$.

**Theorem** [Tutte]

• For $q = 2 + 2 \cos \frac{2\pi}{m}$, $q \neq 4$, the series $T(1, y) \equiv T(q, t; 1, y)$ satisfies a polynomial equation with one catalytic variable $y$.

• When $q$ is generic, the generating function of properly $q$-coloured planar triangulations is differentially algebraic in $t$:

$$2q^2(1 - q)t + (qt + 10H - 6tH')H'' + q(4 - q)(20H - 18tH' + 9t^2H'') = 0$$

with $H(t) = t^2 T_2(q, \sqrt{t}; 1)/q$. 
Adapt this to other equations!

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$
[Tutte 73] Chromatic sums for rooted planar triangulations, II: the case $\lambda = \tau + 1$
[Tutte 73] Chromatic sums for rooted planar triangulations, III: the case $\lambda = 3$
[Tutte 73] Chromatic sums for rooted planar triangulations, IV: the case $\lambda = \infty$
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[Tutte 78] On a pair of functional equations of combinatorial interest
[Tutte 82] Chromatic solutions
[Tutte 82] Chromatic solutions II
[Tutte 84] Map-colourings and differential equations

[Tutte 95]: Chromatic sums revisited
Our results

• Let \( M(q, \nu, t; x, y) \) be the Potts generating function of planar maps:

\[
M(x, y) \equiv M(q, \nu, t; x, y) = \frac{1}{q} \sum_M \mathbb{Z}_M(q, \nu) t^{e(M)} x^{\text{dv}(M)} y^{\text{df}(M)},
\]
where \( \text{dv}(M) \) (resp. \( \text{df}(M) \)) is the degree of the root-vertex (resp. root-face).

Theorem

• For \( q = 2 + 2 \cos \frac{j\pi}{m} \), \( q \neq 0, 4 \), the series \( M(q, \nu, t; 1, y) \equiv M(y) \) satisfies a polynomial equation with one catalytic variable \( y \), and the complete Potts generating function \( M(q, \nu, t; x, y) \) is algebraic.

• When \( q \) is generic, \( M(q, \nu, t; 1, 1) \) is differentially algebraic in \( t \):

(\text{an explicit system of differential equations})

[mbm-Bernardi 14?] Counting coloured planar maps: differential equations
When \( q = 2 + 2 \cos(j\pi/m) \): an equation with one catalytic variable

Examples

For \( q = 1 \) and \( M(y) \equiv M(1, y) \), one obtains

\[
M(y) = 1 + y^2 t \nu M(y)^2 + \nu t y \frac{y M(y) - M(1)}{y - 1}.
\]
When $q = 2 + 2 \cos(j\pi/m)$: an equation with one catalytic variable

**Examples**

- For $q = 1$ and $M(y) \equiv M(1, y)$, one obtains
  \[ M(y) = 1 + y^2 t \nu M(y)^2 + \nu y \frac{y M(y) - M(1)}{y - 1}. \]

- For $q = 2$ (the Ising model),
  \[
  4 t^2 y^4 \nu (y - 1)^2 (\nu + 1) M(y)^3 + 4 (y - 1) y^4 \nu (\nu + 1) t^2 M(y)^2 \\
  + (y - 1)^2 y^2 \left( \nu^2 y - y - \nu^2 - 6 \nu - 1 \right) t M(y)^2 + \nu y^4 (\nu + 1) t^2 M(y) \\
  + y^2 (y - 1) \left( \nu^2 y + 3 \nu y - \nu^2 - 6 \nu - 1 \right) t M(y) - (y - 1)^2 (\nu y - y - \nu - 1) M(y) \\
  - 2 t^2 y^3 \nu (y - 1)(\nu + 1) M(1) M(y) - 2 t^2 y^2 \nu (\nu + 1)(y - 1) M(1)^2 \\
  - y^2 \nu (2 y - 1)(\nu + 1) t^2 M(1) - y(y - 1) \left( -y + \nu^2 y - 2 \nu y - \nu - \nu^2 \right) t M(1) \\
  - \nu t^2 y^2 (\nu + 1)(y - 1) M_y'(1) + (y - 1)^2 (\nu y - y - \nu - 1) = 0
  \]
When \( q = 2 + 2 \cos(j \pi/m) \): an equation with one catalytic variable

**Examples**

- For \( q = 1 \) and \( M(y) \equiv M(1, y) \), one obtains

  \[
  M(y) = 1 + y^2 t \nu M(y)^2 + \nu t y \frac{y M(y) - M(1)}{y - 1}.
  \]

- For \( q = 2 \) (the Ising model),

  \[
  4 \ t^2 \ y^4 \nu \ (y - 1)^2 (\nu + 1) M(y)^3 + 4 \ (y - 1) y^4 \nu \ (\nu + 1) t^2 M(y)^2
  \]
  \[
  + \ (y - 1)^2 \ y^2 \ (\nu^2 y - y - \nu^2 - 6 \nu - 1) \ t M(y)^2 + \nu y^4 (\nu + 1) t^2 M(y)
  \]
  \[
  + \ y^2 (y - 1) \ (\nu^2 y + 3 \nu y - \nu^2 - 6 \nu - 1) \ t M(y) - (y - 1)^2 (\nu y - y - \nu - 1) M(y)
  \]
  \[
  - 2 \ t^2 y^3 \nu \ (y - 1)(\nu + 1) M(1) M(y) - 2 \ t^2 y^2 \nu \ (\nu + 1)(y - 1) M(1)^2
  \]
  \[
  - y^2 \nu \ (2y - 1)(\nu + 1) t^2 M(1) - y(y - 1) \left(-y + \nu^2 y - 2 \nu y - \nu - \nu^2\right) t M(1)
  \]
  \[
  - \nu t^2 y^2 (\nu + 1)(y - 1) M_y'(1) + (y - 1)^2 (\nu y - y - \nu - 1) = 0
  \]

- For \( q = 3 \), one obtains a big equation involving \( M(y) \), and then \( M(1), M'(1), M''(1), M'''(1) \).
When $q = 2 + 2 \cos(2k\pi/m)$: an equation with one catalytic variable

- Let $I(y)$ be the following variant of $M(y)$:

$$I(y) = tyqM(y) + \frac{y - 1}{y} + \frac{ty}{y - 1}.$$
When $q = 2 + 2 \cos(2k\pi/m)$: an equation with one catalytic variable

- Let $I(y)$ be the following variant of $M(y)$:
  \[
  I(y) = tyqM(y) + \frac{y-1}{y} + \frac{ty}{y-1}.
  \]

- Let $N(y, x)$ and $D(t, x)$ be the following (Laurent) polynomials (with $\beta = \nu - 1$):
  \[
  N(y, x) = \beta(4 - q)(1/y - 1) + (q + 2\beta)x - q,
  \]
  \[
  D(t, x) = (q\nu + \beta^2)x^2 - q(\nu + 1)x + \beta t(q - 4)(q + \beta) + q.
  \]
When $q = 2 + 2 \cos(2k\pi/m)$: an equation with one catalytic variable

• Let $I(y)$ be the following variant of $M(y)$:

$$I(y) = tyqM(y) + \frac{y - 1}{y} + \frac{ty}{y - 1}.$$ 

• Let $N(y, x)$ and $D(t, x)$ be the following (Laurent) polynomials (with $\beta = \nu - 1$):

$$N(y, x) = \beta(4 - q)(1/y - 1) + (q + 2\beta)x - q,$$

$$D(t, x) = (q\nu + \beta^2)x^2 - q(\nu + 1)x + \beta t(q - 4)(q + \beta) + q.$$ 

• There exists $(m + 1)$ formal power series in $t$ with coefficients in $\mathbb{Q}(q, \nu)$, denoted $C_0(t), \ldots, C_m(t)$, such that

$$D(t, I(y))^{m/2} T_m \left( \frac{N(y, I(y))}{2\sqrt{D(t, I(y))}} \right) = \sum_{r=0}^{m} C_r(t)I(y)^r := C(t, I(y)),$$

where $T_m$ be the $m$th Chebyshev polynomial of the first kind, defined by

$$T_m(\cos \theta) = \cos(m\theta).$$
When \( q = 2 + 2 \cos(2k\pi/m) \): an equation with one catalytic variable

\[
D(t, I(y))^{m/2} T_m \left( \frac{N(y, I(y))}{2\sqrt{D(t, I(y))}} \right) = \sum_{r=0}^{m} C_r(t) I(y)^r := C(t, I(y)),
\]

where

\[
C(t, x) = \sum_{r=0}^{m} C_r(t) x^r.
\]

The series \( C_m, C_{m-1}, C_{m-2} \) are explicit, which leaves us with \( (m - 2) \) unknown series \( C_r(t) \) (in addition to the main bivariate series \( I(t; y) \simeq M(t; y) \)). Moreover, \( C_{m-3} \simeq M(1) \) is the series we want to determine.
When $q = 2 + 2 \cos(2k\pi/m)$: an equation with one catalytic variable

\[
D(t, I(y))^{m/2} T_m \left( \frac{N(y, I(y))}{2\sqrt{D(t, I(y))}} \right) = \sum_{r=0}^{m} C_r(t) I(y)^r := C(t, I(y)),
\]

cf.

\[
(2ty^2(y - 1)M(y) + ty^2 - y + 1)^2 = (y - 1 - yt^2)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)M_1
\]
\[:= \Delta(y) \quad \text{(a polynomial in } y)\]
When $q = 2 + 2 \cos(2k\pi/m)$: an equation with one catalytic variable

$$D(t, I(y))^{m/2} T_m \left( \frac{N(y, I(y))}{2\sqrt{D(t, I(y))}} \right) = \sum_{r=0}^{m} C_r(t) I(y)^r := C(t, I(y)),$$

**Proposition.** The polynomial (in $x$) $C(t, x)^2 - D(t, x)^m$ has degree $2m$ and admits $(m - 2)$ double roots $X_1, \ldots, X_{m-2}$:

$$C(t, x)^2 - D(t, x)^m = \widehat{P}(t, x) \prod_{i=1}^{m-2} (x - X_i)^2$$

where

$$C(t, x) = \sum_{r=0}^{m} C_r(t) x^r,$$

$$D(t, x) = (q\nu + \beta^2)x^2 - q(\nu + 1)x + \beta t(q - 4)(q + \beta) + q.$$
When \( q = 2 + 2 \cos(2k\pi/m) \): an equation with one catalytic variable

\[
D(t, I(y))^{m/2} T_m \left( \frac{N(y, I(y))}{2\sqrt{D(t, I(y))}} \right) = \sum_{r=0}^{m} C_r(t) I(y)^r := C(t, I(y)),
\]

**Proposition.** The polynomial (in \( x \)) \( C(t, x)^2 - D(t, x)^m \) has degree \( 2m \) and admits \((m - 2)\) double roots \( X_1, \ldots, X_{m-2} \):

\[
C(t, x)^2 - D(t, x)^m = \hat{P}(t, x) \prod_{i=1}^{m-2} (x - X_i)^2
\]

⇒ algebraic consequences? Only for \( q = 1, 2, 3 \)
When $q = 2 + 2\cos(2k\pi/m)$: an equation with one catalytic variable

$$D(t, I(y))^{m/2} T_m \left( \frac{N(y, I(y))}{2\sqrt{D(t, I(y))}} \right) = \sum_{r=0}^{m} C_r(t) I(y)^r := C(t, I(y)),$$

Proposition. The polynomial (in $x$) $C(t, x)^2 - D(t, x)^m$ has degree $2m$ and admits $(m - 2)$ double roots $X_1, \ldots, X_{m-2}$:

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⇒ algebraic consequences? Only for $q = 1, 2, 3$

⇒ differential consequences? YES! uniform in $q$
An explicit system of differential equations

Let \( \beta = \nu - 1 \) and \( D(t, x) = (q \nu + \beta^2)x^2 - q(\nu + 1)x + \beta t(q - 4)(q + \beta) + q. \)

- There exists a unique 8-tuple \((P_0(t), \ldots, P_3(t), Q_0(t), Q_1(t), R_0(t), R_1(t))\) of series in \( t \) with polynomial coefficients in \( q \) and \( \nu \) such that
  \[
  \frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial x} \left( \frac{R^2}{PD^2} \right),
  \]
  where
  \[
  P(t, x) = P_0(t) + P_1(t)x + P_2(t)x^2 + P_3(t)x^3 + x^4,
  
  Q(t, x) = Q_0(t) + Q_1(t)x + x^2,
  
  R(t, x) = R_0(t) + R_1(t)x + (q + \nu - 3)x^2,
  \]
  with the initial conditions (at \( t = 0 \)):
  \[
  P(0, x) = x^2(x - 1)^2 \quad \text{and} \quad Q(0, x) = x(x - 1).
  \]

+ explicit expression of \( M(1) \) (the Potts GF of planar maps) in terms of the \( P_i \)'s and \( Q_i \)'s.
An analogous system for triangulations

Let \( \beta = \nu - 1 \) and \( D(t, x) = q\nu^2 x^2 + \beta (4\beta + q) x + (q\nu\beta(q - 4)t + \beta^2) \).

- There exists a unique 7-tuple \((P_0(t), \ldots, P_2(t), Q_0(t), Q_1(t), R_0(t), R_1(t))\) of series in \( t \) with polynomial coefficients in \( q \) and \( \nu \) such that
  \[
  \frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial x} \left( \frac{R^2}{PD^2} \right),
  \]
  where

\[
\begin{align*}
P(t, x) &= P_0(t) + P_1(t)x + P_2(t)x^2 + x^3, \\
Q(t, x) &= Q_0(t) + Q_1(t)x + 2\nu x^2, \\
R(t, x) &= R_0(t) + R_1(t)x,
\end{align*}
\]

with the initial conditions (at \( t = 0 \)):

\[
P(0, x) = x^2(x + 1/4) \quad \text{and} \quad Q(0, x) = x(2\nu x + 1).
\]

- Expression of the Potts GF of triangulations in terms of the \( P_i \) and \( Q_i \)
What can one do with these systems?

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial x} \left( \frac{R^2}{PD^2} \right)
\]

+ Expression of the Potts GF \( M(1) \) in terms of the \( P_i \) and \( Q_i \)

- Differential elimination to obtain a DE for \( M(1) \)?
- Solution? Connections with elliptic functions?
III. Special values of $q$ and $\nu$

- The algebraic approach: $q = 2, q = 3$

- The differential approach: when $D$ is simple
  - $q = 0$: maps equipped with a spanning forest
  - $\nu = 0$: properly coloured maps
  - $q = 4$: four colours
Let $A$ be the series in $t$, with polynomial coefficients in $\nu$, defined by

$$A = t \left( \frac{1 + 3\nu A - 3\nu A^2 - \nu^2 A^3}{1 - 2A + 2\nu^2 A^3 - \nu^2 A^4} \right)^2.$$

Then the Ising generating function of planar maps is

$$M(2, \nu, t; 1, 1) = \frac{1 + 3 \nu A - 3 \nu A^2 - \nu^2 A^3}{(1 - 2A + 2\nu^2 A^3 - \nu^2 A^4)^2} P(\nu, A)$$

where

$$P(\nu, A) = \nu^3 A^6 + 2\nu^2 (1 - \nu) A^5 + \nu (1 - 6\nu) A^4$$

$$- \nu (1 - 5\nu) A^3 + (1 + 2\nu) A^2 - (3 + \nu) A + 1.$$

~ Asymptotics: Phase transition at $\nu_c = \frac{3 + \sqrt{5}}{2}$, critical exponents...
Let $A$ be the quartic series in $t$ defined by

$$A = t \frac{(1 + 2A)^3}{(1 - 2A^3)}.$$ 

Then the generating function of properly 3-coloured planar maps is

$$M(3,0,t;1,1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}$$
$q = 0$: Triangulations equipped with a spanning forest

$D(x) = 4x + 1$ has degree 1 only and is independent of $t$

$$\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial x} \left( \frac{R^2}{PD^2} \right)$$

$\Rightarrow$ A DE of order 2 for $T(0, \nu, t; 1, 1)$, which counts planar triangulations equipped with a spanning forest, by edges and number of trees in the forest.

Combinatorial solution: [Bouttier, Di Francesco & Guitter 07], [mbm-Courtiel 13]
\( \nu = 0: \) Properly coloured triangulations
(\text{Tutte's problem})

\[ D(x) = 1 + x(4 - q) \] has degree 1 only and is independent of \( t \)

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial x} \left( \frac{R^2}{PD^2} \right)
\]

\( \Rightarrow \) A DE of order 2 for \( T(q, 0, t; 1, 1) \), which counts properly coloured planar triangulations.
$q = 4$: Four coloured triangulations

$D(x) = (2x\nu + \nu - 1)^2$ is a square and is independent of $t$

$$\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial x} \left( \frac{R^2}{PD^2} \right),$$

$\Rightarrow$ A DE of order 2 for $T(4, \nu, t; 1, 1)$, which counts 4-coloured planar triangulations (the 4-state Potts model).
Many questions are left...

A. The differential system
• Elimination in the systems of differential equations
• Connections with elliptic functions
• Connections with [Eynard et al. 99], [Guionnet et al. 10] and [Borot et al. 12]

B. More combinatorics
• Understand algebraic series, e.g., for 3-coloured planar maps:
  \[ M(3, 0, t; 1, 1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2} \quad \text{with} \quad A = t \frac{(1 + 2A)^3}{(1 - 2A^3)} \]
• Understand differential equations, e.g., for properly \(q\)-coloured triangulations:
  \[ 2q^2(1 - q)t + (qt + 10H - 6tH')H'' + q(4 - q)(20H - 18tH' + 9t^2H'') = 0 \]

C. Asymptotics
• Asymptotic number of properly \(q\)-coloured maps?
  (done for triangulations \(q \in (28/11, 4) \cup [5, \infty)\) [Odlyzko-Richmond 83])
• More generally, phase transitions and critical exponents of the Potts model