Workshop “2D-Quantum Gravity and Statistical Mechanics”
Erwin Schrödinger Institute, Wien, Juni 20

Summing Planar Graphs in 0, 1, 2, 3 and 4 Dimensions

Vladimir Kazakov (ENS,Paris)

ENS

UPMC PARIS UNIVERSITAS

European Research Council
Established by the European Commission

Unterstützt von / Supported by
Alexander von Humboldt Stiftung/Foundation

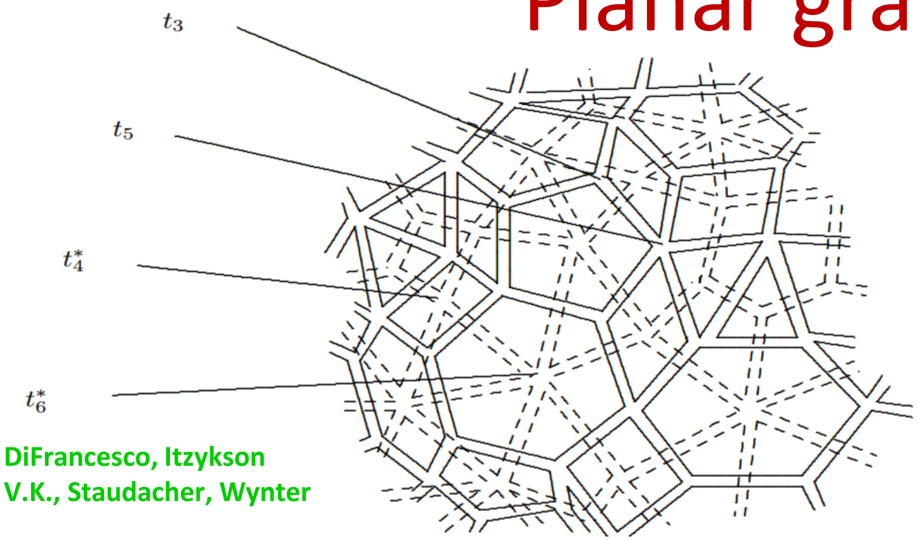
Outline

- The topological classification of ribbon (Feynman) graphs in matrix models and quantum field theories in terms of 1/N expansion w.r.t. size of NxN matrix fields was introduced by Gerard ‘t Hooft in 70’s. He hoped to solve in this interesting, planar limit ($N=\infty$) the SU(N) non-abelian gauge theory which, at N=3, describes the strong interactions. We are still not yet there but…
- ‘t Hooft’s limit was successfully used in variety of math. and physical problems: graph counting, topological characteristics of moduli space, stat. mech. models on random surfaces, non-critical string theory and 2D quantum gravity, 2D black hole, superstring theory and M-theory, AdS/CFT correspondence etc.
- In dimensions D>1 it gets tough (famous c=1 barrier). A “theorem” says that graphs (surfices degenerate into trees)
- A rare 2D matrix field theory solvable at any N is the SU(N)×SU(N) principal chiral field (PCF). I will briefly discuss its large N limit.
- Recent developments: Unique higher D examples of solvable planar QFT’s: 3D ABJM gauge theory (N=4 super-Chern-Simons) and, the most important, N=4 Super-Yang-Mills - a superconformal 4D gauge theory, exactly solvable at any ‘t Hooft coupling (fugacity of vertices of Feynman graphs). Summing genuine 4D Feynman diagrams for most important physical quantities: anomalous dimensions, correlation functions, Wilson loops, gluon scattering amplitudes…

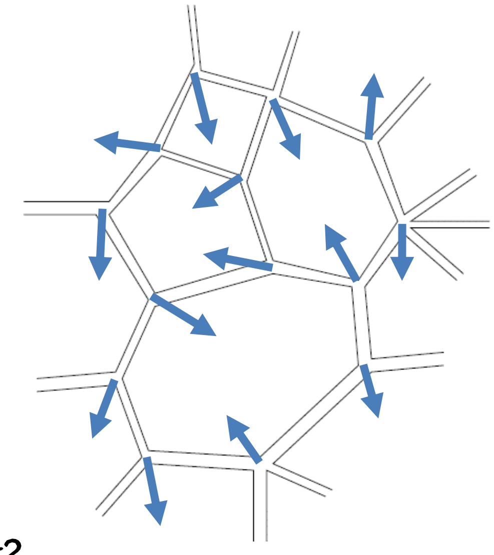
- Important common feature: Integrability. Hirota equation and its Wronskian determinant solutions in terms of Baxter-like Q-functions give a general point of view on these models.

Planar graphs in D≤1:
Matrix Models and
Matrix Quantum Mechanics

Planar graphs in D=0



't Hooft
Brezin, Itzykson, Parizi, Zuber



DiFrancesco, Itzykson
V.K., Staudacher, Wynter

$$Z = \int d^{N^2}M \, \exp[-N \mathrm{tr}\, V(M)]$$

$$Z(t, t^*) = \sum_G \prod_{v_q, v_q^* \in G} t_q^{\# v_q} \, t_q^{* \# v_q^*}$$

$$V(M) = \frac{1}{2}M^2 - \sum_n \frac{t_n}{n}(AM)^n$$

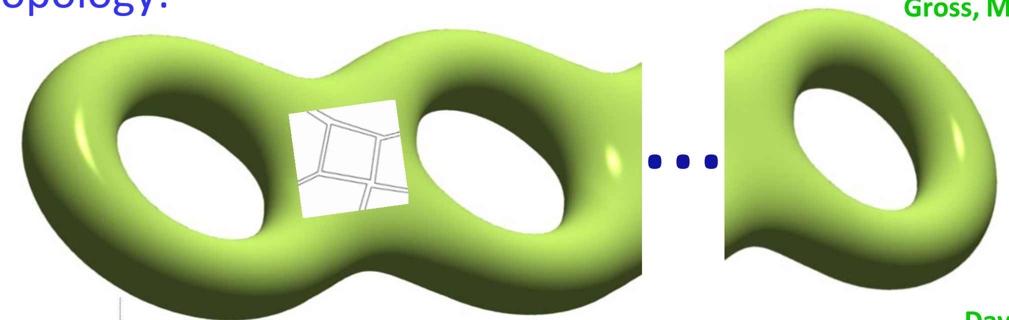
$$t_n^* = \frac{1}{N}\mathrm{tr}\, A^n$$

V.K., Kostov

- Number of planar graphs - via one-matrix model or loop equations
- Number of dually weighted graphs - via character expansion methods
- Spins on dynamical planar graphs (Ising, Potts, O(N),…) - via multi-matrix models, loop equations, orthogonal polynomials, integrability, Hirota eqs.
- Double scaling: big graphs of fixed topology:

Brezin, V.K.
Douglas, Shenker
Gross, Migdal

$$\sum_{\text{graphs genus } g}$$



David
V.K.

- Describes non-critical strings and 2d quantum gravity + CFT(c<1) etc

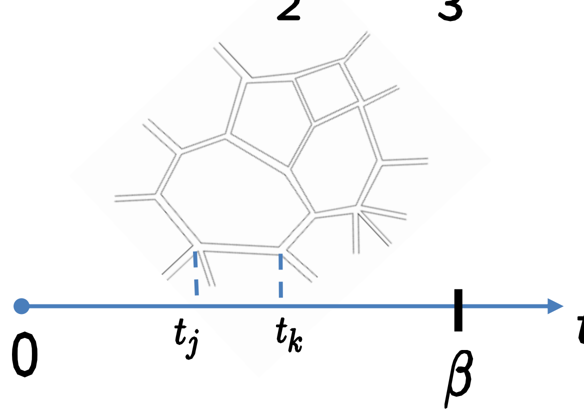
Planar graphs in D=1

- Solvable via matrix quantum mechanics.

$$Z_{MQM} = \int D^{N^2}M(t)\ \exp -\frac{N}{g^2}\int_0^\beta dt\,\mathrm{tr}\ \left[\dot{M}^2 + V(M)\right] \qquad V(M) = \frac{1}{2}M^2 - \frac{1}{3}M^3$$

$$M(\beta) = \Omega^{-1}M(0)\Omega$$

$$\log Z = \beta \sum_{h=0}^{\infty} N^{2-2h} \sum_{n=1}^{\infty} g^{2n} \sum_{G_n^{(h)}} \int dt_2 \cdots dt_n\ e^{-\sum_{<ab>_G} |t_a - t_b|}$$

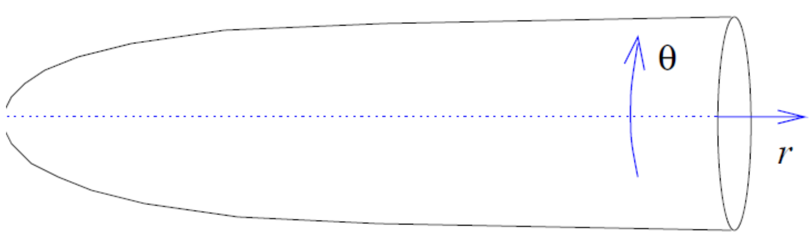


- In double scaling (critical) regime one can estimate the sum of graphs of any fixed topology

$$\frac{\log Z_{sphere}}{N^2} \sim -\frac{\Delta^2}{|\log \Delta|}, \qquad \Delta = g_{crit} - g \to 0$$

V.K., Migdal

- Describes non-critical strings in D=1. Effective string theory has 2 dimensional background, one coming from labeling of eigenvalues (Liouville field in Polyakov string formulation)
- Compactification of time leads to Berezinsky Kosterlitz-Thouless vortices coupled to 2d gravity: Sine-Liouville string theory. Dual to Witten's black hole with cigar background



V.K., Kostov, Kutasov

Summing exactly planar graphs in 2D: Principal Chiral Field Model

$SU(N)_L \times SU(N)_R$ principal chiral field

$$\mathcal{S} = \frac{N}{2g^2} \int d\sigma d\tau \; \mathrm{Tr} \left(G^{-1}\partial_\mu G\right)^2, \qquad G(\tau,\sigma) \in SU(N).$$

- 1/N-expansion → sum of planar Feynman graphs embedded in 2D:

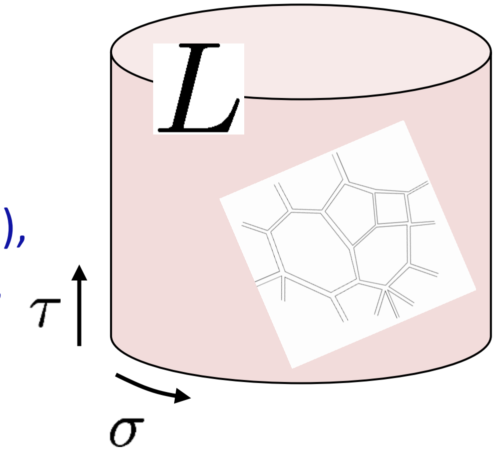
$$G = e^{iM} = I + iM - \frac{1}{2}M^2 + \cdots$$

Zamolodchikov&Zamolodchikov
Karowski
Polyakov, Wiegmann;
Wiegmann

- Integrable: S-matrix, N-1 types of particles with masses

$$m_a = m\frac{\sin\frac{\pi a}{N}}{\sin\frac{\pi}{N}}, \qquad m = \frac{\Lambda}{g}\, e^{-\frac{4\pi}{g^2}}, \qquad a = 1,\ldots,N-1$$

- An interesting solvable matrix problem: PCF on a cylinder.



- Methods: Integrability, Thermodynamical Bethe Ansatz (TBA), Analytic Y-system, T-system (Hirota bi-linear difference equation), Wronskian solution of T-system in terms of Baxter's Q-functions, Riemann-Hilbert problem, etc .

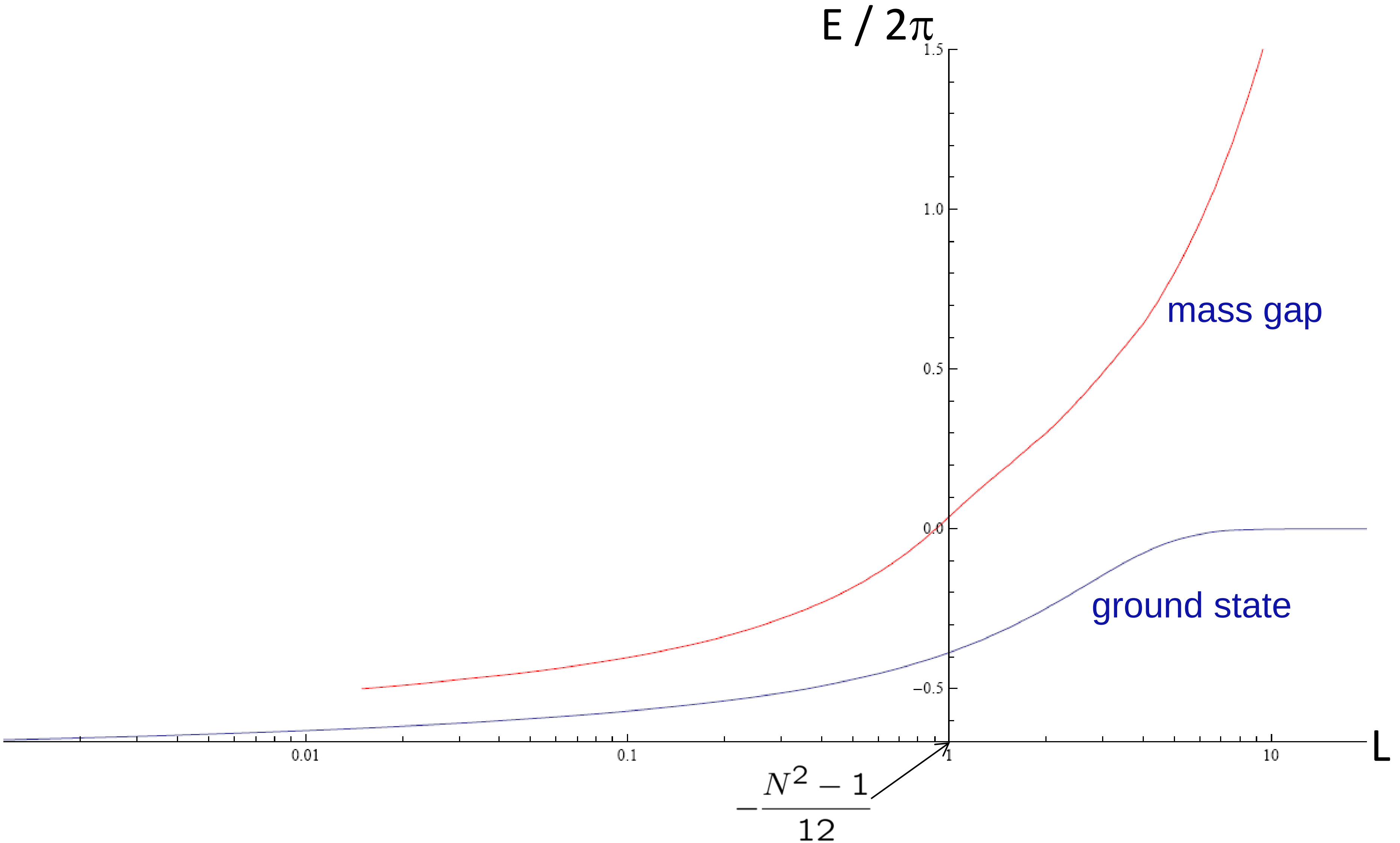
- Numerical results at any coupling available for N=2,3,4

Balos, Hegedus
Gromov, V.K. Vieira
V.K., Leurent

- Riemann-Hilbert equations are available at any N.
  To analyse their large N limit is our current research project.

SU(3) PCF numerics

V.K.,Leurent'09



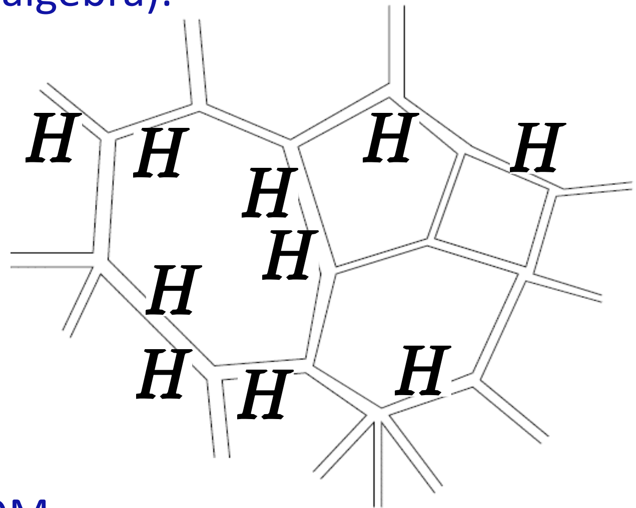
Planar PCF in external field

Fateev, V.K., Wiegmann

- Infinite volume, External magnetic field (element of Cartan algebra):

$$\partial_\mu G \to D_\mu G = \partial_\mu G - \frac{i}{2}\delta_{\mu,0}(H_L G + G H_R)$$

$$H_{R(L)} = \{h_1, h_2 - h_1, \cdots, h_{N-1} - h_{N-2}, -h_{N-1}\}$$



- Explicit result in planar limit, for the choice $h_a = h\frac{m_a}{m}$

$$\frac{\log Z}{N^2} = -\frac{hm}{8\pi} B I_1(B)\,, \qquad \frac{m}{h} = B K_1(B)$$

- Parameter B reminds the Fermi level for eigenvalues in MQM and is related to renormalized charge of PCF as $\bar{g}^2(h) = \frac{4\pi}{B}$
- Weak coupling - Feynman perturbation theory, asymptotic freedom:

$$16\pi\frac{\log Z}{N^2} = -h^2\left(\log\frac{h}{m} + \frac{1}{2}\log\log\frac{h}{m} + \frac{1}{2}\log\frac{\pi}{2} + \mathcal{O}(\log^{-1}\frac{h}{m})\right), \qquad \frac{h}{m} \gg 1$$

- Strong coupling - non-perturbative critical regime near threshold, big planar graphs:

$$16\pi\frac{\log Z}{N^2} \simeq -\frac{\Delta}{|\log\Delta|}, \qquad \Delta = \frac{h}{m} - 1 \to 0$$

- Strongly reminds D=1 MQM scaling! Another non-critical string theory, but in 2D?
- A third dimension comes from Dynkin labels

Planar graphs in D>2

Quantum integrability for
3D planar N=6 ABJM model
(super-Chern-Simons model)
and
4D planar N=4 Super-Yang Mills theory

N=4 SYM and a string in AdS5 x S^5 metric

$$S_{SYM} = \frac{N}{g^2}\int d^4x \,\mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + \bar{\Psi}\mathcal{D}\Psi + \bar{\Psi}\Phi\Psi + [\Phi,\Phi]^2\right)$$

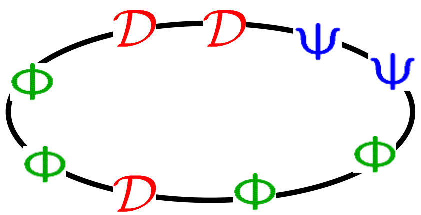
'tHooft coupling

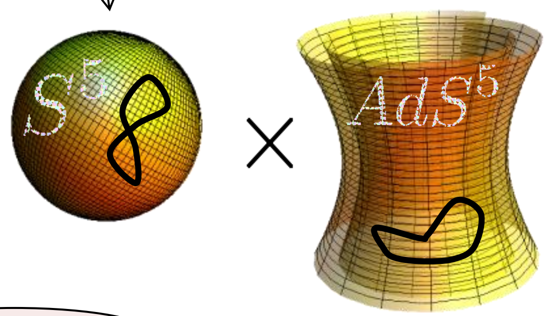
Metsaev-Tseytlin σ-model on super-coset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$

$$S_{sigma} = g\int d\tau \int_0^L d\sigma \left[\left(\partial\vec{X}\right)^2 + \left(\partial\vec{Y}\right)^2 + \text{fermions}\right]$$

CFT/AdS duality

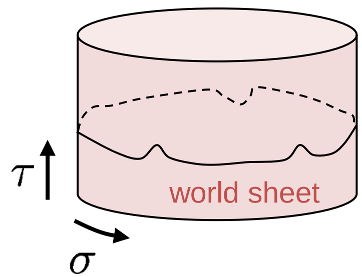
$$\mathcal{O}(x) = \mathrm{Tr}\,[\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots](x) \quad \longleftrightarrow$$





Maldacena
Gubser,Klebanov,Polyakov
Witten

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \frac{\delta_{ij}}{|x|^{2\Delta_j(g)}}$$



Dimension of a local operator = Energy of a string state

Global superconformal symmetry → psu(2,2|4) ← isometry of background

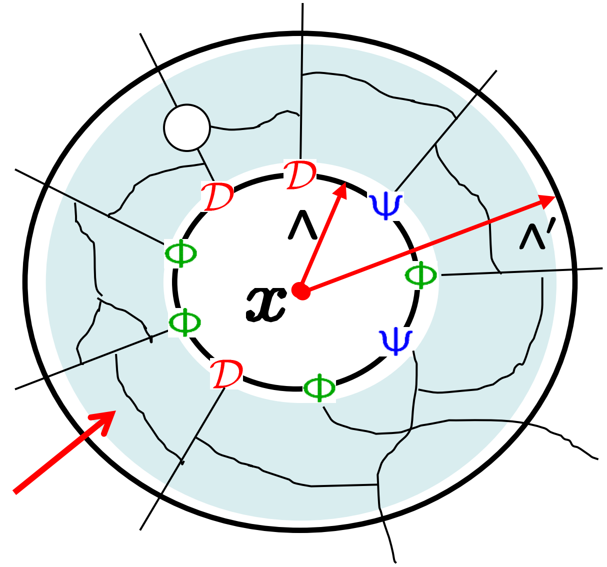
It is an integrable theory: the spectrum is defined by the AdS/CFT Y-system|

*

Dilatation operator in SYM perturbation theory

- Dilatation operator $\hat{D}$ from point-splitting and renormalization

$$\mathcal{O}_j^{\Lambda'}(x) = \left[\left(\frac{\Lambda'}{\Lambda}\right)^{\hat{D}}\right]_{jk} \mathcal{O}_k^{\Lambda}(x)$$



- Conformal dimensions are eigenvalues of dilatation operator

$$\hat{D}_{jk}\mathcal{O}_k(x) = \Delta_j \mathcal{O}_j$$

- Can be computed from perturbation theory in $g^2 \equiv N g_{YM}^2$

$$\hat{D} = \hat{D}^{(0)} + g^2 \hat{D}^{(2)} + g^4 \hat{D}^{(4)} + \ldots$$

$$\Delta = \Delta^{(0)} + g^2 \Delta^{(2)} + g^4 \Delta^{(4)} + \ldots$$

- $\tau \sim \log \Lambda$ corresponds to AdS time

Exact spectrum at one loop (su(2)-sector)

- Dilatation operator = Heisenberg Hamiltonian, integrable by Bethe ansatz!

$\mathrm{Tr} Z^L(x)$ - vacuum:

$—Z—Z—X—Z—Z—X—Z—$

$$\hat{D} = L + \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(1 - \sigma_l \cdot \sigma_{l+1}\right)$$

Minahan, Zarembo

$\lambda = 16\pi^2 g^2$

Beisert, Kristijansen,Staudacher

- One loop Bethe equations for the N=4 SYM spectrum:

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{(k\neq)j=1}^{J} \frac{u_k - u_j + i}{u_k - u_j - i}$$

momentum - rapidity parameterization:

$$e^{ip} = \frac{u + i/2}{u - i/2}$$

Bethe'31

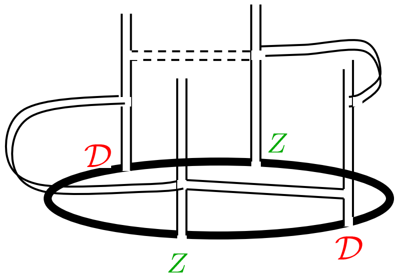
Anomalous dimension:

$$\Delta - L = \frac{\lambda}{8\pi^2} \sum_{k=1}^{J} \frac{1}{u_k^2 + 1/4} + \mathcal{O}(\lambda^2)$$

- To go to higher loops one has to use the Y-system and Thermodynamic Bethe Ansatz (TBA)

Perturbative Konishi: integrability versus Feynman graphs

$\mathcal{O}_{\text{Konishi}} = \text{Tr}\ [\mathcal{D}, Z]^2$



- Integrability allows to sum exactly enormous numbers of Feynman diagrams of N=4 SYM

$$
\begin{aligned}
\Delta = \;& 4 + 12\,g^2 - 48\,g^4 + 336\,g^6 + 96\,g^8\,(-26 + 6\,\zeta_3 - 15\,\zeta_5) \\
& -96\,g^{10}\,(-158 - 72\,\zeta_3 + 54\,\zeta_3^2 + 90\,\zeta_5 - 315\,\zeta_7) \\
& -48\,g^{12}\,(160 + 5472\,\zeta_3 - 3240\,\zeta_3\,\zeta_5 + 432\,\zeta_3^2 - 2340\,\zeta_5 - 1575\,\zeta_7 + 10206\,\zeta_9) \\
& +48\,g^{14}\,(-44480 + 108960\,\zeta_3 + 8568\,\zeta_3\,\zeta_5 - 40320\,\zeta_3\,\zeta_7 - 8784\,\zeta_3^2 + 2592\,\zeta_3^3 \\
& \qquad - 4776\,\zeta_5 - 20700\,\zeta_5^2 - 26145\,\zeta_7 - 17406\,\zeta_9 + 152460\,\zeta_{11}) \\
& +96\,g^{16}\,(566752 - 869760\,\zeta_3 - 45360\,\zeta_3\,\zeta_5 - 64890\,\zeta_3\,\zeta_7 + 241920\,\zeta_3\,\zeta_9 + 82656\,\zeta_3^2 - 33912\,\zeta_3^2\,\zeta_5 + 20736\,\zeta_3^3 \\
& \qquad - 204984\,\zeta_5 + 231840\,\zeta_5\,\zeta_7 + 24840\,\zeta_5^2 + 227799\,\zeta_7 + 97164\,\zeta_9 + 135927\,\zeta_{11} - 1104246\,\zeta_{13} \\
& \qquad + 7128\,\frac{\zeta_{11} - \zeta_3\,\zeta_{3,5} + \zeta_{3,5,3}}{5}) \\
& -96\,g^{18}\,(10568224 - 11884608\,\zeta_3 + 148896\,\zeta_3\,\zeta_5 - 177768\,\zeta_3\,\zeta_5^2 - 354384\,\zeta_3\,\zeta_7 - 1244484\,\zeta_3\,\zeta_9 + 2901096\,\zeta_{11}\,\zeta_3 \\
& \qquad + 533952\,\zeta_3^2 + 284904\,\zeta_3^2\,\zeta_5 - 229824\,\zeta_3^2\,\zeta_7 + 209952\,\zeta_3^3 - 5993280\,\zeta_5 + 963954\,\zeta_5\,\zeta_7 + 2553120\,\zeta_5\,\zeta_9 - 576000\,\zeta_5^2 \\
& \qquad + 2324196\,\zeta_7 + 1184274\,\zeta_7^2 + 2573892\,\zeta_9 + 355266\,\zeta_{11} + 2644434\,\zeta_{13} - 15810795\,\zeta_{15} \\
& \qquad + 163296\,\frac{\zeta_{11} - \zeta_3\,\zeta_{3,5} + \zeta_{3,5,3}}{5} - 13608\,(\zeta_3\,\zeta_{3,7} - \zeta_{3,7,3} + \zeta_3^2\,\zeta_5 - \zeta_5\,\zeta_{5,3} + \zeta_{5,3,5}))
\end{aligned}
$$

Bajnok,Janik
Leurent,Serban,Volin
Bajnok,Janik,Lukowski
Lukowski,Rej,
Velizhanin,Orlova
Leurent, Volin

Leurent, Volin
(8 loops from FiNLIE)

Volin
(9-loops from spectral curve)

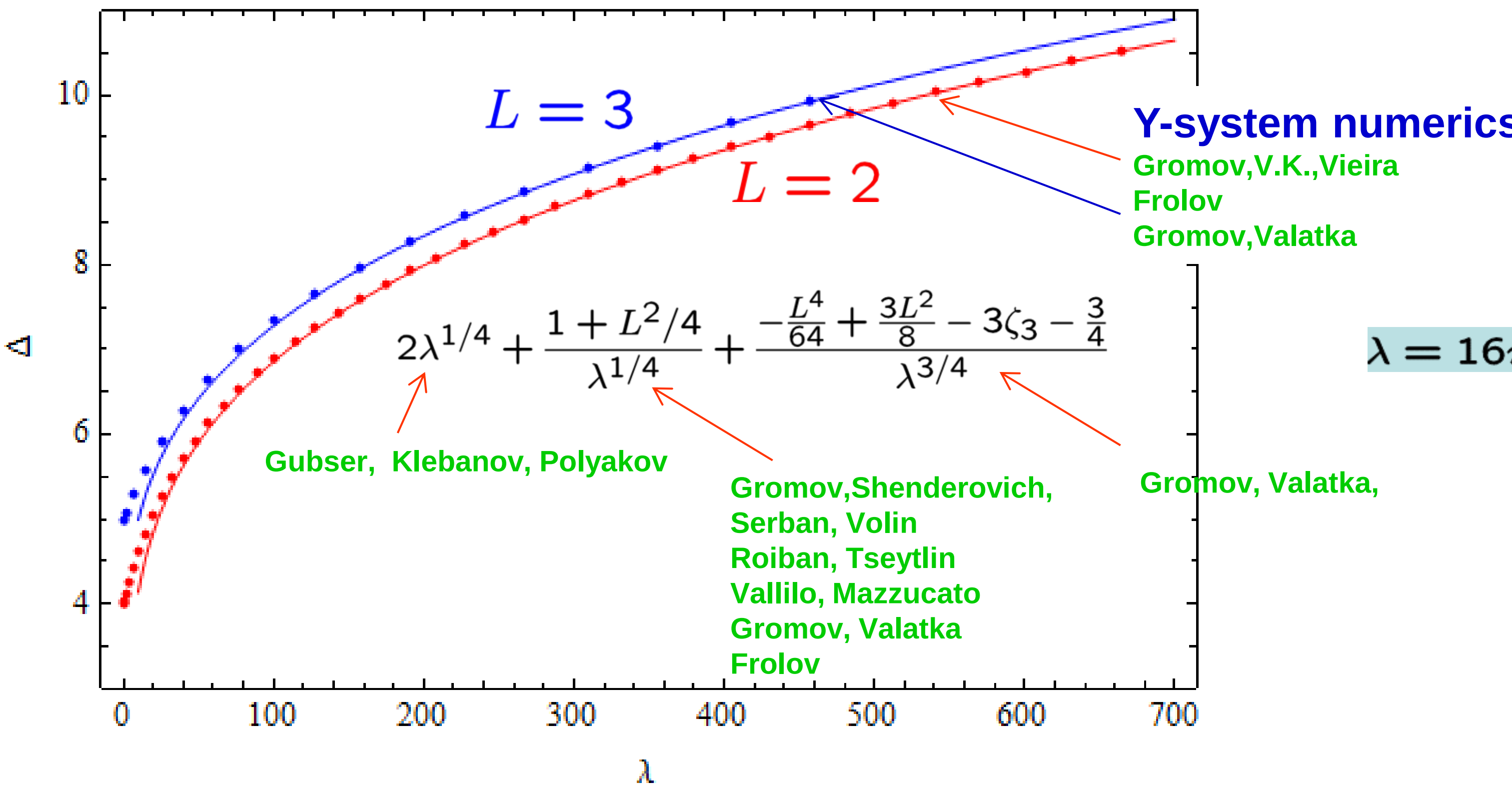
- Confirmed up to 5 loops by direct graph calculus (6 loops promised)

Fiamberti,Santambrogio,Sieg,Zanon
Velizhanin
Eden,Heslop,Korchemsky,Smirnov,Sokatchev

AdS string quasiclassics and numerics in SL(2) sector: twist-L operators of spin S $\mathrm{Tr}\,\mathcal{D}^S Z^L$

- 3 leading strong coupling terms were calculated for any S and L
- Numerics from Y-system, TBA, FiNLIE, at any coupling:
  - for Konishi operator $S=2,\quad L=2,\quad n=1$
  - and twist-3 operator $S=2,\quad L=3,\quad n=1$

  They perfectly reproduce the TBA/Y-system or FiNLIE numerics



$\lambda = 16\pi^2 g^2$

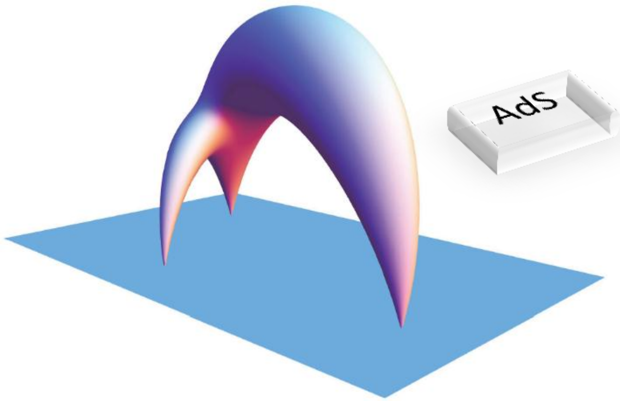
- AdS/CFT Y-system passes all known tests!

3 point function of classical operators

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}(g)}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}|x_{31}|^{\Delta_i+\Delta_k-\Delta_j}}$$

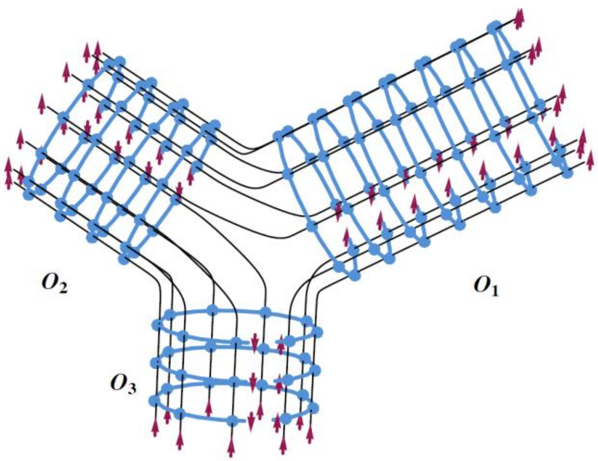
- Strong coupling limit: The problem reduces to finding the classical solution: minimal surface in AdS space

Zarembo
Janik, Wereszczynski
Kazama, Komatsu



$$ds^2 = \frac{dx^\mu dx_\mu + dz^2}{z^2}$$

- Perturbation theory: summing graphs ("spin chain" integrability helps…)



Gromov, Vieira, Sever
Kostov,
Serban
V.K., Sobko, etc

Riemann-Hilbert problem for spectrum of planar N=4 SYM

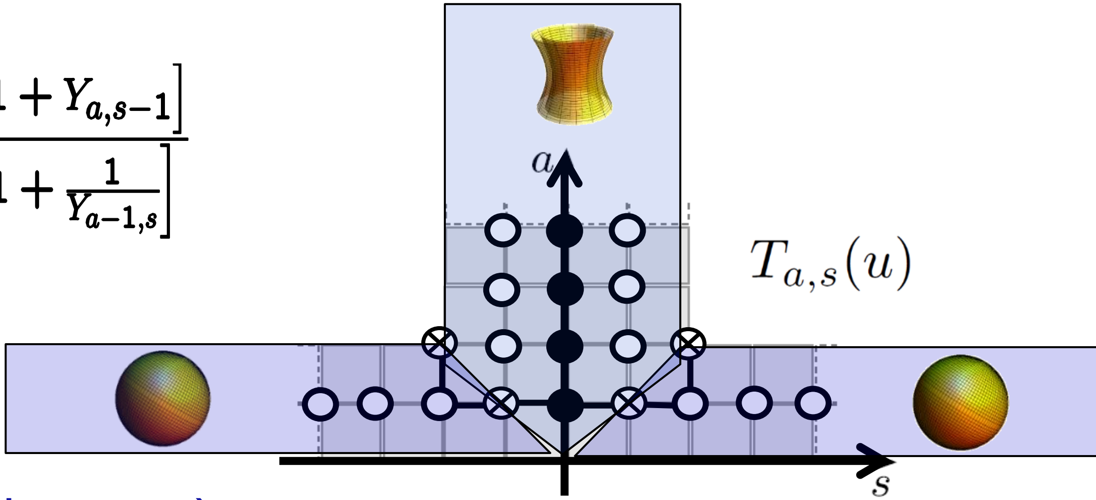
Y-system and T-system: discrete integrability

- From TBA equations we get the AdS/CFT Y-system in psu(2,2|4) T-hook:

Gromov,V.K.,Vieira

$$Y_{a,s}(u+\frac{i}{2})\,Y_{a,s}(u-\frac{i}{2}) = \frac{\left[1+Y_{a,s+1}\right]\,\left[1+Y_{a,s-1}\right]}{\left[1+\frac{1}{Y_{a+1,s}}\right]\,\left[1+\frac{1}{Y_{a-1,s}}\right]}$$

$$Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$$



- Equivalent to the T-system (Hirota eq.):

$$T_{a,s}\left(u+\frac{i}{2}\right)\,T_{a,s}\left(u-\frac{i}{2}\right) = T_{a,s-1}(u)\,T_{a,s+1}(u) + T_{a+1,s}(u)T_{a-1,s}(u)$$

- Integrable system, solvable in terms of Wronskians of Baxter's Q-functions
- Example: solution for right band via two functions:

$$T_{1,s}(u) = \mathbf{P}_1(u+\frac{is}{2})\mathbf{P}_2(u-\frac{is}{2}) - \mathbf{P}_1(u-\frac{is}{2})\mathbf{P}_2(u+\frac{is}{2})$$

- It's a quantum analogue of Weyl formula for U(2) characters: $\mathbf{P}_j(u) \to e^{iu\,\phi_j}$
- Complete solution described by Q-system – full set of 2^8 Q-functions
  All of them can be expressed through 8 basic Q-functions

Gromov, V.K., Leurent, Volin 2013

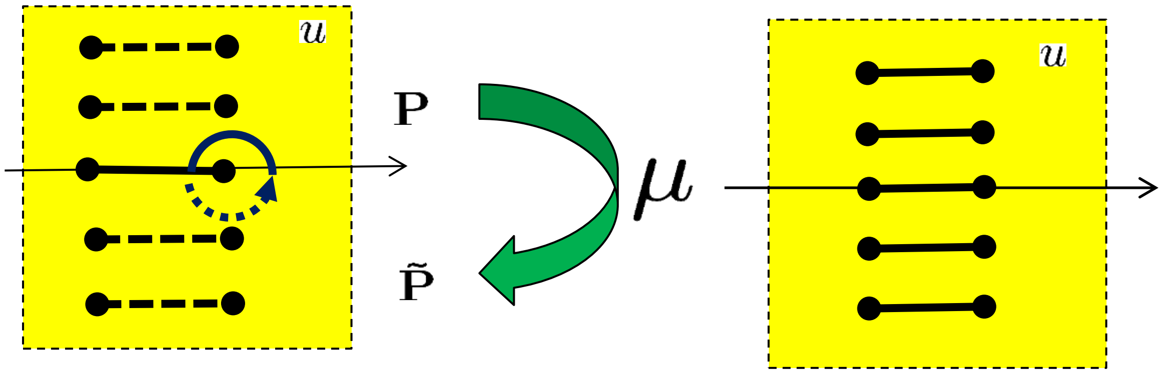
Spectral Riemann-Hilbert equations (Pμ-system)

- 4-vector of functions with cut $[-2g, 2g]$ $\quad \mathbf{P}(u) = \{\mathbf{P}_1(u), \mathbf{P}_2(u), \mathbf{P}_3(u), \mathbf{P}_4(u)\}$

$$\tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b \qquad \tilde{\mu}_{ab}(u) = \mu_{ab}(u+i)$$

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a \qquad \mu^{ab} \equiv \mu^{-1}_{ab} = -\frac{1}{2}\epsilon^{abcd}\mu_{cd}, \qquad \mathrm{Pf}(\mu) = 1$$

$\tilde{\mathbf{P}}$ is the analytic continuation of $\mathbf{P}$ through the cut:



- "Left-Right symmetric" case, e.g. twist L operators $\mathrm{Tr}(\nabla^S Z^L)$

$$\mu^{-1} = \chi\mu\chi \qquad \mathbf{P}^a = -\chi^{ab}\mathbf{P}_b \qquad \chi = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mu_{23} = \mu_{14}$$

- Cut structure on defining sheet and asymptotics at $u \to \infty$

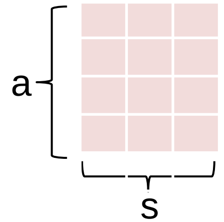
$$\begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{pmatrix} \sim \begin{pmatrix} A_1 u^{-\frac{L}{2}} \\ A_2 u^{-\frac{L+2}{2}} \\ A_3 u^{\frac{L}{2}} \\ A_4 u^{\frac{L-2}{2}} \end{pmatrix} \qquad \begin{pmatrix} \mu_{12} \\ \mu_{13} \\ \mu_{14} \\ \mu_{24} \\ \mu_{34} \end{pmatrix} \sim \begin{pmatrix} u^{\Lambda-L} \\ u^{\Lambda+1} \\ u^{\Lambda} \\ u^{\Lambda-1} \\ u^{\Lambda+L} \end{pmatrix}, \qquad \Lambda = 0,\ \pm\Delta,\ \pm(S-1)$$

Y-system, T-system
and
Integrable Hirota dynamics

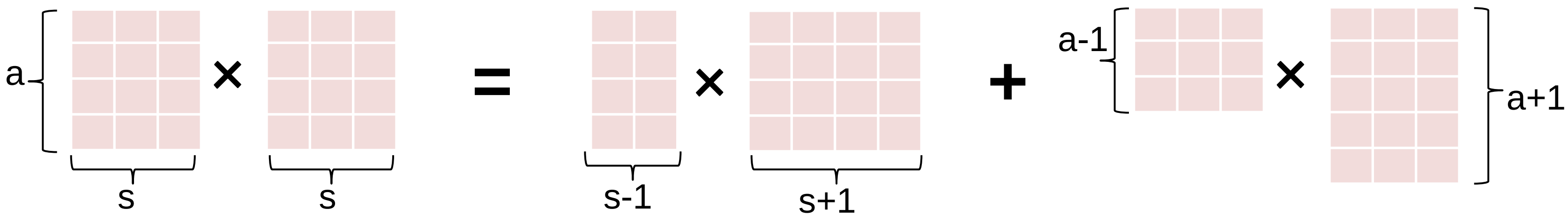
gl(K|M) (super)characters

- Character can be presented as a matrix integral, e.g. for "rectangular" irreps $\lambda = a^s$:

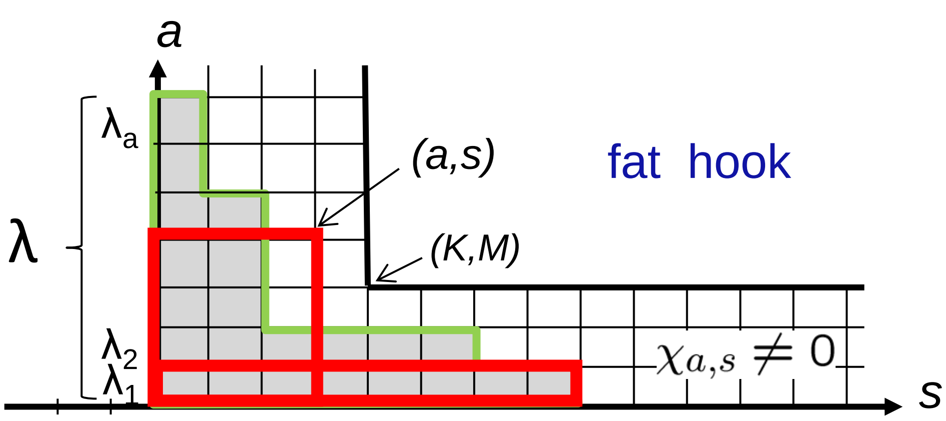
$$\chi_{a,s}(g) = \int \frac{[dh]_{U(a)}}{(\det h)^{s+1}} \mathrm{sdet}\ (1 - h \otimes g)^{-1}\ , \qquad g \in gl(K|M)$$



- A curious property of gl(K|M) representations with rectangular Young tableaux:

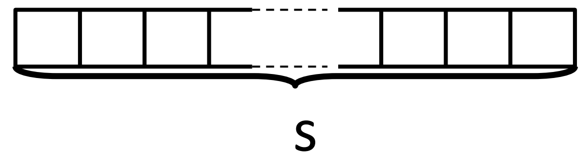


- For characters – simplified Hirota eq.: $\chi^2_{a,s} = \chi_{a+1,s}\chi_{a-1,s} + \chi_{a,s+1}\chi_{a,s-1}$
- Boundary conditions for Hirota eq.: gl(K|M) representations in "fat hook":



Compact u(K|M) versus non-compact u(K_1,K_2|M)

- Generating function for symmetric irreps:



$$w(z) \equiv \mathrm{sdet}\ (1 - zg)^{-1} \equiv \frac{\prod_{\hat{m}=1}^{M}(1 - zx_{\hat{m}})}{\prod_{n=1}^{N}(1 - zx_n)} = \sum_{s=1}^{\infty} \chi_{1,s}(g)\, z^s$$

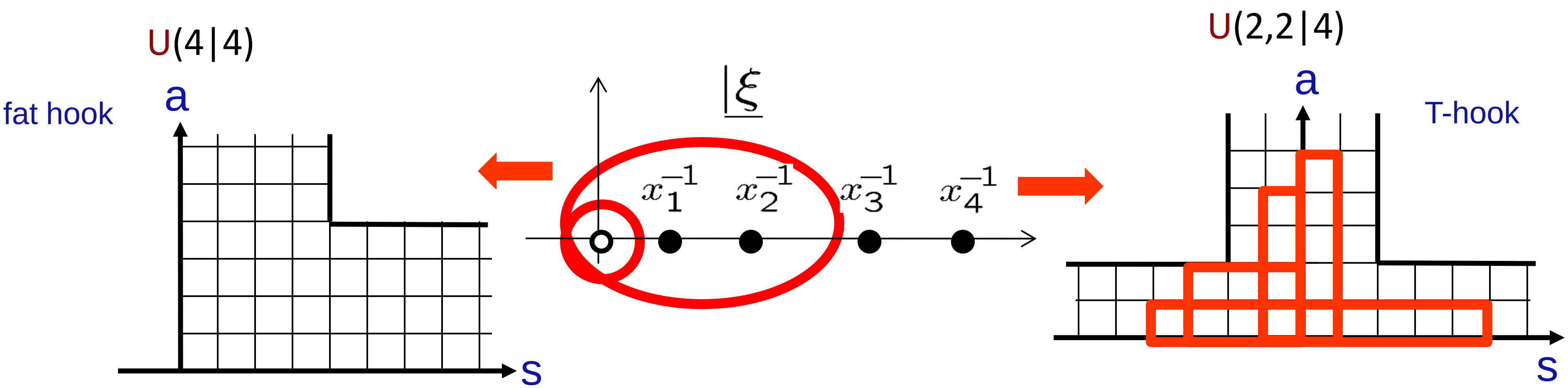
$$g = \mathrm{diag}\{x_1, \cdots, x_M | x_{\hat{1}}, \cdots, x_{\hat{N}}\}$$

- Solution of Hirota: Gambelli-Jacobi-Trudi formula for GL(K|M) characters

$$\chi_{\{\lambda\}}[g] = \det_{1 \le i,j \le a} \chi_{1,\lambda_i - i + j}[g]\,, \qquad g \in GL(K|M).$$

$$\chi_{1,s} = \oint \frac{dz}{2\pi i}\, z^{-s-1}\, w(z;g)$$

- Important example: superconformal su(2,2|4): $g = \mathrm{diag}\{x_1, x_2, x_3, x_4 | y_{\hat{1}}, y_{\hat{2}}, y_{\hat{3}}, y_{\hat{4}}\}$



∞ - dim. unitary highest weight representations of u(2,2|4) !

Kwon
Cheng,Lam,Zhang
Gromov, V.K., Tsuboi

Krichever,Lipan, Wiegmann,Zabrodin
Gromov, Vieira
V.K., Leurent, Volin.

Q-system

- One-form on N single indexed Q-functions:

$$Q_{(1)} \equiv \sum_{j=1}^{N} Q_j(u)\xi^j, \qquad \{\xi^i, \xi^j\} = 0$$

- l-form encodes all Q-functions with l indices:

$$Q_{(l)} \equiv Q_{(1)}^{[-l+1]} \wedge Q_{(1)}^{[-l+3]} \wedge \ldots \wedge Q_{(1)}^{[l-1]}$$

Notations:

$$Q^{[n]} \equiv Q(u + \frac{in}{2})$$

$$Q^{\pm} \equiv Q(u \pm \frac{i}{2})$$

- Multi-index Q-functions: coefficient of $\xi_{i_1} \wedge \xi_{i_2} \wedge \ldots \wedge \xi_{i_l}$

$$Q_{j_1,\ldots,j_k} = \det_{1 \le m,n \le k} Q_{j_m}^{[-1-k+2n]}$$

- Example for N=2: $Q_{(2)} = 2Q_{12}\,\xi_1 \wedge \xi_2, \qquad Q_{12} = Q_1^+ Q_2^- - Q_1^- Q_2^+$

- Notations in terms of sets of indices:

$$\mathrm{Q}_{j_1,\ldots,j_k} \equiv \mathrm{Q}_I\,, \qquad I = \{j_1, \ldots, j_k\} \subset \{1, 2, \ldots, N\}$$

- Plücker's QQ-relations: $Q_I Q_{I,i,j} = Q_{I,i}^+ Q_{I,j}^- - Q_{I,i}^- Q_{I,j}^+$

Tsuboi
Gromov,V.K., Leurent, Tsuboi
V.K.,Leurent,Volin

(K|M)-graded Q-system

- Split the full set of K+M indices as $\{B\}\cup\{F\}$

$$B=\{1,2,\ldots,K\},\quad F=\{K+1,K+2,\ldots,K+M\}$$

- Grading = re-labeling of F-indices (subset → complimentary subset of F)

$$Q_{I|J}\equiv \mathbb{Q}_{I,F\backslash J}\,,\qquad I\in B,\;\; J\in F$$

We impose for AdS/CFT $Q_{\emptyset|\emptyset}=Q_{1234|5678}=1$

- Examples for (4|4): $Q_{j|\emptyset}=\mathbb{Q}_{j5678}\,,\quad j=1,2,3,4,\qquad Q_{12|57}=\mathbb{Q}_{1268}$

- Graded forms:

$$Q_{(n|p)}=\sum_{\{b\}\in B}\sum_{\{f\}\in F}Q_{b_1,b_2,\ldots,b_n|f_1,f_2,\ldots,f_p}\cdot\xi^{b_1}\wedge\xi^{b_2}\wedge\cdots\wedge\xi^{b_n}\wedge\xi^{f_1}\wedge\xi^{f_2}\wedge\cdots\wedge\xi^{f_p}$$

- New type of QQ-relations involwing 2 indices of opposite grading:

$$Q_{I|J,j}Q_{I,i|J}=Q^+_{I,i|J,j}Q^-_{I|J}-Q^-_{I,i|J,j}Q^+_{I|J}$$

- Hodge duality is a simple relabeling: $Q^{I|J}\equiv Q_{B\backslash I\,|\,F\backslash J}$

- Example for (4|4): $Q^{1|134}=Q_{234|2}$

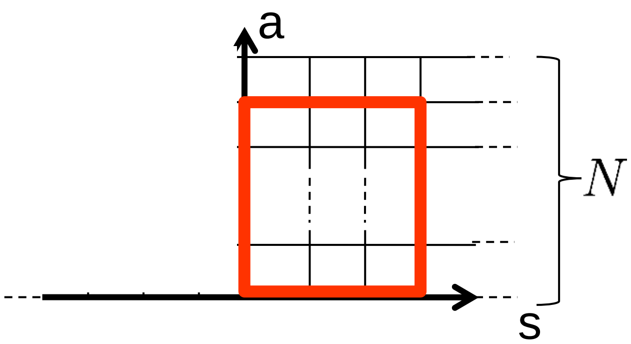
Now we can label:
$F=\{1,2,\ldots,M\}$

Wronskian solution of Hirota eq.

- Example: solution of Hirota equation in a band of width N in terms of exterior full-forms via 2N arbitrary functions $Q_j(u),\ \tilde{Q}_j(u)$

Krichever,Lipan, Wiegmann,Zabrodin

$$T_{a,s} = Q^{[s]}_{(a)} \wedge \tilde{Q}^{[-s]}_{(N-a)}$$

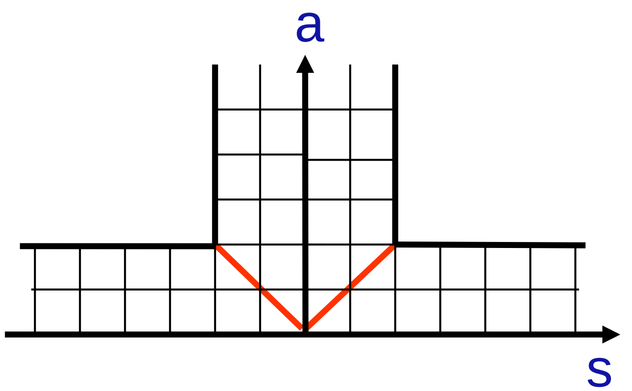


- For su(N) spin chain (half-strip) we impose:

$$\tilde{Q}(u) = Q^{[N]}, \qquad \tilde{Q}_{(0)} = Q_{(0)} = 1$$

- Solution of Hirota eq. for (2,2|4) T-hook

Tsuboi
V.K.,Leurent,Volin



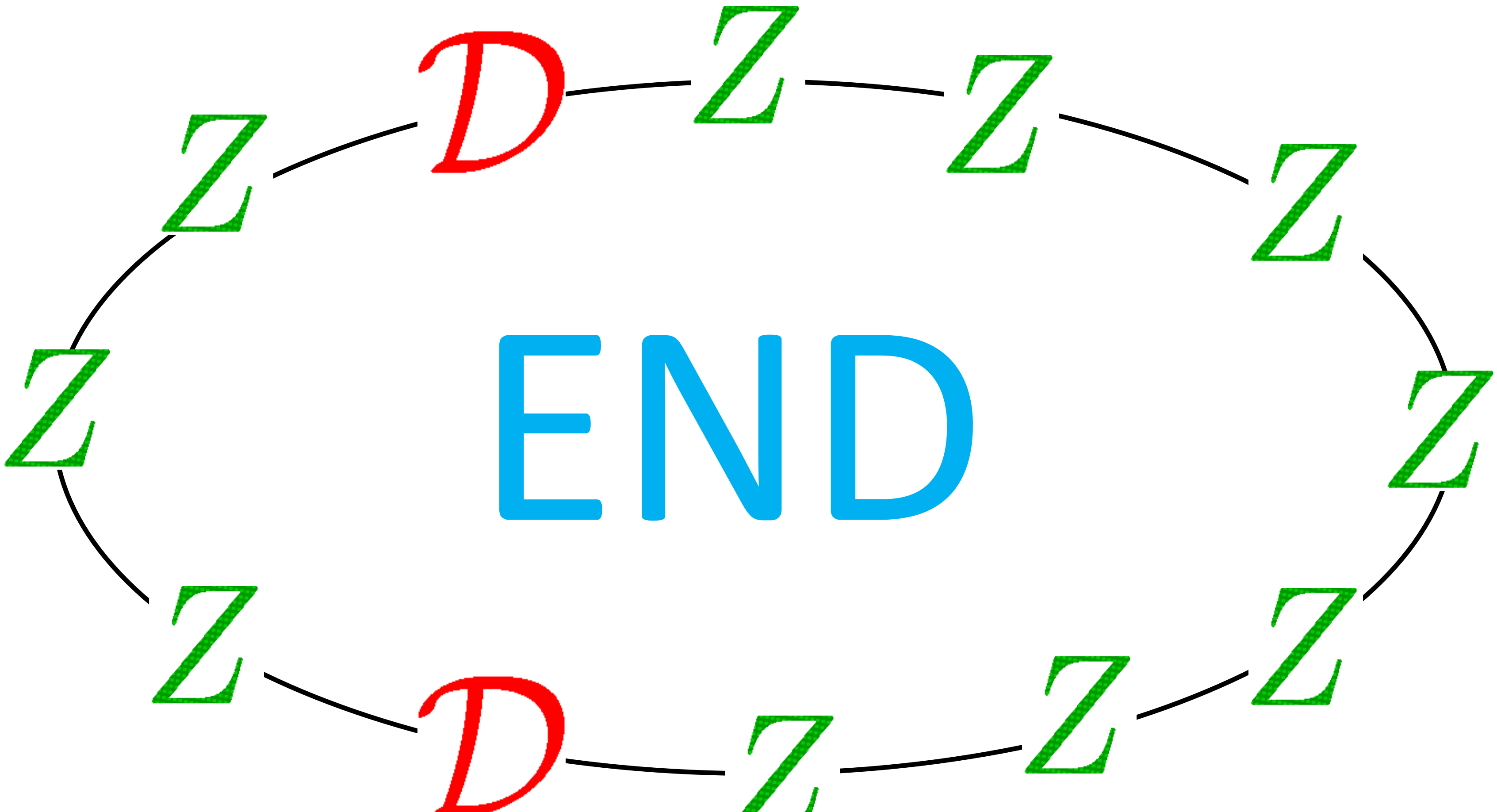
$$Q_{I_1,I_2|J} \qquad \{I_1, I_2|J\} \subset \{B_1, B_2|F\}$$

$$T_{a,s} = \begin{cases} Q^{[+s]}_{(a,0|0)} \wedge Q^{[-s]}_{(2-a,2|4)} & s \geq a \\ Q^{[+a]}_{(2,0|2-s)} \wedge Q^{[-a]}_{(0,2|2+s)} & a \geq |s| \\ Q^{[-s]}_{(2,2-a|4)} \wedge Q^{[+s]}_{(0,a|0)} & s \leq -a \end{cases}$$

Conclusions and prospects

- We have exact solutions for non-trivial physical models summing planar graphs embedded into D>1 dimensions. AdS/CFT correspondence relates them to string theory.
- Solutions are achieved using quantum integrability. Integrability (normally 2D…) is a window into D>2 physics.
- TBA and Y-system describe the Hirota integrable dynamics: T-functions can be expressed through Wronskian determinants of Baxter's Q-functions.
- N=4 SYM is a first 4D QFT with calculable spectrum of anomalous dimensions (sum of non-trivial 4D Feynman graphs!); bears some common features with QCD, in particular, in Balitsky-Fadin-Lipatov-Kuraev approximation (BFKL)
- Efficient system of Riemann-Hilbert equations – quite a step w.r.t. the original functional integral!
- Another example of solvable AdS/CFT duality: 3D ABJM gauge theory

- In my opinion, the way to self-consistent 4D quantum gravity goes through new models of strings/planar graphs embedded into higher dimensions


END
Z
D
Z
Z
Z
Z
Z
Z
Z
Z
D
Z