From Matrix Models to the theory of fundamental interactions

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Motivation and background

- aim: quantum theory of fundamental interactions incl. gravity
- present state:

  working "standard models" (el.part., cosmology)

  big mysteries: dark matter, dark energy;
- theoretical/conceptual challenges (hard): reconcile quantum mechanics & gravity

  classical geometry breaks down at Planck scale

  "naturalness" problems (separation of scales): Planck scale vs. electroweak scale, cosm. constant

... more radical approach needed?!

matrix models: simple yet far-reaching, pre-geometric

quantized geometry?

combine quantum mechanics & general relativity:

- superposition of massive objects $\rightarrow$ superposition of geometries
- measure short length $\Delta x \leq L_{\rm Planck}$:
  $\Delta x \Delta E \geq \hbar$ & $\Delta x \geq R_{\rm Schwarzschild} \sim \frac{M}{L_{\rm Planck}}$
  $\Rightarrow (\Delta x)^2 \geq L^2_{\rm Planck}$
- more rigorous: Fredenhagen Doplicher Roberts 1994

$\Rightarrow$ space-time must be quantized in some way

(cf. string thy, LQG)

Matrix Models as fundamental theory

1996: BFSS model, IKTT model proposed as
non-perturbative definition of M-theory / IIB string theory
focus on IKKT: Ishibashi, Kawai, Kitazawa, Tsuchiya 1996

$$S[X,\Psi] = Tr\Big([X^a, X^b][X^{a'}, X^{b'}]\eta_{aa'}\eta_{bb'} + \bar\Psi\Gamma_a[X^a,\Psi]\Big)$$

$X^a = X^{a\dagger} \in Mat(N,\mathbb{C})$, $a = 0, ..., 9$
η_{ab} ... $SO(9,1)$, Γ_a ... Clifford alg.
$\Psi \in \mathrm{Mat}(N,\mathbb{C}) \otimes \mathbb{C}^{32}$ ... Majorana-Weyl spinor

gauge symmetry $X^a \to U^{-1}X^aU$, $ISO(9,1)$, SUSY

1) nonpert. def. of IIB string theory (on $\mathbb{R}^{10}$) (*IKKT*)
2) $\mathcal{N}=4$ SUSY Yang-Mills gauge thy. on "*noncommutative*" $\mathbb{R}^4_\theta$

dynamical NC branes $\mathcal{M} \subset \mathbb{R}^{10}$, relation w/ string theory

→ brane-world scenarios (→ 4D gravity ? H.S. 2007 ff) quantization is understood:

$$Z = \int dX^a d\Psi \, e^{-S[X]-S[\Psi]}$$

$$\langle Tr([X,X]...) Tr(...)\rangle = \frac{1}{Z}\int dX^a d\Psi \, Tr([X,X]...) Tr(...) e^{-S[X]-S[\Psi]}$$

...suitable for non-perturbative approach

- integral exists (in Euclidean setting) Krauth, Staudacher 1998
- proposal for regularization in Minkowski setting
  ”Monte-Carlo“ studies: Nishimura etal 2012 arXiv:1108.1540 [hep-th]
  hints for ”expanding universe” behavior, 3+1 dimensions
- simplified models:
  eigenvalue distribution $\leftrightarrow$ (mass) renormalization, phase trans.
  H.S. hep-th/0501174, A. Polychronakos arXiv:1306.6645 [hep-th] etc.
  RG analysis (Grosse-Wulkenhaar), multiscale analysis (Rivasseau, Vignes-Tourneret, Gurau, ...), ...

Here: perturbative approach:

- choose background solution (e.g. $\mathbb{R}^4_\theta$), study fluctuations
  look for "effective action" (=generating function)
  in suitable (semi-classical?) limit
- expansion around $\mathbb{R}^4_\theta$:
  noncommutative gauge theory, Filk rules, (non-)planar diagrams
  ($\rightarrow$ combinatorics ...!)
- almost all models (probably) pathological (UV/IR mixing)
- ONE model expected to be well-behaved (perturbatively finite):
  $\mathcal{N} = 4$ NC SYM on $\mathbb{R}^4_\theta$ $\Leftrightarrow$ (IKKT) model, in 9+1 dimensions
- includes integral over geometries !!

basic solutions: branes

e.o.m.: $\delta S = 0 \;\Rightarrow\; [X_a, [X^a, X^b]] = 0, \qquad D\!\!\!/\Psi \equiv \Gamma^a [X_a, \Psi] = 0$

basic solutions: (allow $N \to \infty$)

- flat "branes" $\mathbb{R}^{2n}_\theta$ embedded in $\mathbb{R}^{10}$

$$X^a = \begin{pmatrix} X^\mu \\ 0 \end{pmatrix}, \qquad \mu = 1, ..., 2n$$

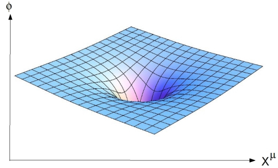
$[X^\mu, X^\nu] = i\theta^{\mu\nu}\, \mathbb{1}$ "Moyal-Weyl quantum plane"

- generic (curved) branes $\mathcal{M}^{2n}$

$$X^a \sim x^a : \; \mathcal{M}^{2n} \hookrightarrow \mathbb{R}^{10}$$

... quantized embedding map

$(\mathcal{M}^{2n}, \omega)$ ... symplectic manifold

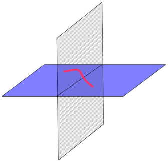


non-commutative, $\omega = \frac{1}{2}\theta^{-1}_{\mu\nu}(x) dx^\mu dx^\nu = B$ -field

stacks of branes in M.M.

- assume $X^a_{(i)}$ ... solutions of e.o.m.

  $\rightarrow$ new solution: $X^a = \begin{pmatrix} X^a_{(1)} & 0 \\ 0 & X^a_{(2)} \end{pmatrix}$



- stacks of n_1 & n_2 coincident branes $X^a = \begin{pmatrix} X^a_{(1)} \mathbf{1}_{n_1} & 0 \\ 0 & X^a_{(2)} \mathbf{1}_{n_2} \end{pmatrix}$

  breaks $U(N)$ to $U(n_1) \times U(n_2)$

- fermions may connect different branes

  $$\Psi = \begin{pmatrix} 0 & \psi_{(12)} \\ \psi_{(21)} & 0 \end{pmatrix},$$

  $\psi_{(12)}$ transforms in bifundamental $(n_1) \otimes (\overline{n}_2)$

branes as quantized Poisson (symplectic) manifolds

$(\mathcal{M}, \theta^{\mu\nu}(x))$ ... $2n$-dimensional manifold with Poisson structure

Its quantization $\mathcal{M}_\theta$ is NC algebra such that

$$\mathcal{Q}: \ \mathcal{C}(\mathcal{M}) \ \to \ \mathcal{A} \subset \mathcal{L}(\mathcal{H})$$

such that

$$\begin{aligned} \mathcal{Q}(f)\,\mathcal{Q}(g) &= \mathcal{Q}(fg) \ + O(\theta) \\ [\mathcal{Q}(f), \mathcal{Q}(g)] &= \mathcal{Q}(i\{f,g\}) \ + O(\theta^2) \end{aligned}$$

("nice") $\Phi \in Mat(\infty, \mathbb{C}) \quad \leftrightarrow \quad$ quantized function on $\mathcal{M}$

furthermore:

$$(2\pi)^n \mathrm{Tr}\, \mathcal{Q}(\phi) \quad \sim \quad \int \omega^n \, \phi(x)$$

Example: the fuzzy sphere S^2_N

classical S^2 :
$$\left.\begin{array}{rcl} x^a : S^2 & \hookrightarrow & \mathbb{R}^3 \\ x^a x^a & = & 1 \end{array}\right\} \Rightarrow \mathcal{A} = \mathcal{C}^\infty(S^2)$$

fuzzy sphere S^2_N : (Madore, Hoppe)

algebra $\mathcal{A} = \mathrm{Mat}(N, \mathbb{C})$ ... alg. of functions on S^2_N

$SO(3)$ action:

$$\begin{array}{rcl} \mathfrak{su}(2) \times \mathcal{A} & \to & \mathcal{A} \\ (J^a, \phi) & \mapsto & [J^a, \phi] \end{array}$$

decompose $\mathcal{A} = \mathrm{Mat}(N, \mathbb{C})$ into irreps of $SO(3)$:

$$\begin{array}{rcl} \mathcal{A} = \mathrm{Mat}(N, \mathbb{C}) \cong (N) \otimes (\bar{N}) & = & (1) \oplus (3) \oplus \ldots \oplus (2N-1) \\ & = & \{\hat{Y}^0_0\} \oplus \{\hat{Y}^1_m\} \oplus \ldots \oplus \{\hat{Y}^{N-1}_m\}. \end{array}$$

... fuzzy spherical harmonics; UV cutoff quantization map:

$$\mathcal{Q}: \quad \mathcal{C}(S^2) \;\to\; \mathcal{A} \;=\; \mathrm{Mat}(N,\mathbb{C})$$
$$Y^l_m \;\mapsto\; \begin{cases} \hat{Y}^l_m, & l < N \\ 0, & l \geq N \end{cases}$$

in particular $X^a := \mathcal{Q}(x^a) = \frac{1}{\sqrt{C_N}} J^a$ ... $N-$ dim irrep of $\mathfrak{su}(2)$ on $\mathbb{C}^N$

satisfies

$$\begin{aligned} \mathcal{Q}(fg) &= \mathcal{Q}(f)\mathcal{Q}(g) \; + O(\tfrac{1}{N}), \\ \mathcal{Q}(i\{f,g\}) &= [\mathcal{Q}(f),\mathcal{Q}(g)] \; + O(\tfrac{1}{N^2}) \end{aligned}$$

Poisson structure $\{x^a, x^b\} = \frac{2}{N}\,\varepsilon^{abc}\,x^c$

$$\begin{aligned} [X^a, X^b] &= \frac{i}{\sqrt{C_N}}\varepsilon^{abc}\,X^c\,, \qquad C_N = \tfrac{1}{4}(N^2-1) \\ X^a X^a &= \mathbf{1}, \end{aligned}$$

S^2_N ... quantization of $(S^2, N\omega_0)$, $\qquad \omega_0 = \varepsilon^{abc}\,x^a dx^b dx^c$

metric structure on branes:

metric encoded in NC Laplace operator

$$\Box : \mathcal{A} \to \mathcal{A},$$
$$\Box\phi = [X^a,[X^b,\phi]]\delta_{ab} = \frac{1}{C_N} J^a J^a \phi$$

$SO(3)$ invariant $\Rightarrow$ $\boxed{\Box \hat{Y}^l_m = \frac{1}{C_N} l(l+1) \hat{Y}^l_m}$

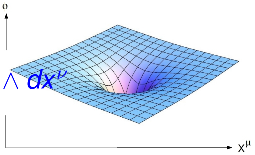
spectrum identical with classical case $\Delta_g\phi = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi)$

$\Rightarrow$ effective metric of $\Box$ = round metric on S^2

generic branes = quantized symplectic (immersed) submanifolds in $\mathbb{R}^{10}$:

1. take some nice manifold $x^a: \quad \mathcal{M}^{2n} \hookrightarrow \mathbb{R}^{10}$

   induced metric $g_{\mu\nu}$ on $\mathcal{M}$



2. equip $\mathcal{M}$ with symplectic form $\omega = \theta^{-1}_{\mu\nu} dx^\mu \wedge dx^\nu$

   $\rightarrow$ quantization of $(\mathcal{M}, \omega)$:

$$\mathcal{Q}: \ \mathcal{C}(\mathcal{M}) \to \mathcal{A} \cong Mat(\infty, \mathbb{C})$$

   in particular: $\quad X^a := \mathcal{Q}(x^a) \ \sim \ x^a$

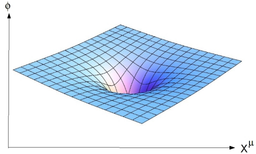
> $\rightarrow$ "noncommutative brane“, matrix geometry
>
> ... class of NC spaces under consideration.

3. $\rightarrow$ effective metric $G^{\mu\nu}$, encoded in matrix Laplacian

$$\Box = [X^a, [X^b, .]]\, \delta_{ab} \ \sim \ -\{x^a, \{x^b, .\}\}\, \delta_{ab}$$

$g_{\mu\nu} \sim G_{\mu\nu}$ if $\mathcal{M}$ almost-Kähler, $g_{\mu\nu} \neq G_{\mu\nu}$ for Minkowski sign.

extracting the geometry of matrices: (H.S. Nucl.Phys. B810 (2009))



$$X^a \sim x^a: \quad \mathcal{M} \hookrightarrow \mathbb{R}^{10}$$

Lemma: assume $\dim \mathcal{M} > 2$. Then

$$\Box f(X) \sim -\eta_{ab}\{x^a, \{x^b, f(x)\}\} = -e^{\sigma}\Box_G f(x)$$

... Matrix Laplace- operator, effective metric

$$G^{\mu\nu}(x) = e^{-\sigma}\theta^{\mu\mu'}(x)\theta^{\nu\nu'}(x)\, g_{\mu'\nu'}(x)$$ effective metric (cf. open string m.)

$$g_{\mu\nu}(x) = \partial_\mu x^a \partial_\nu x^b \eta_{ab}$$ induced metric on $\mathcal{M}^4_\theta$ (cf. closed string m.)

$$e^{-2\sigma} = \frac{|\theta^{-1}_{\mu\nu}|}{|g_{\mu\nu}|}$$

follows by coupling to scalar field φ:

$$S[\varphi] = Tr\,[X^a, \varphi][X^b, \varphi]\, g_{ab}$$

$$\sim \int d^{2n}x \sqrt{|G|}\, G^{\mu\nu}(x)\, \partial_\mu\varphi\partial_\nu\varphi = \int d\varphi \wedge \star_G d\varphi$$

fluctuations on branes $\rightarrow$ noncommutative gauge fields the Moyal-Weyl quantum plane $\mathbb{R}^4_\theta$:

$$[\bar{X}^\mu, \bar{X}^\nu] \;=\; i\theta^{\mu\nu}\,\mathbb{1}, \qquad \mu,\nu = 1,...,4$$

... Heisenberg algebra, interpreted as space of functions on $\mathbb{R}^4_\theta$

uncertainty relations $\Delta\bar{X}^\mu \Delta\bar{X}^\nu \geq |\theta^{\mu\nu}|$

relation with classical $\mathbb{R}^4$: Weyl quantization

$$\mathcal{Q}: \quad L^2(\mathbb{R}^4) \overset{\cong}{\to} HS(\mathcal{H})$$

$$\phi(x) = \int d^4k\, e^{ik_\mu x^\mu}\,\hat{\phi}(k) \;\leftrightarrow\; \int d^4k\, e^{ik_\mu \bar{X}^\mu}\,\hat{\phi}(k) =: \Phi(\bar{X})$$

note:

$\bar{X}^\mu \in Mat(\infty,\mathbb{C})$ ... quantized coordinate functions on $\mathbb{R}^4_\theta$

$\Phi(\bar{X}^\mu) \in Mat(\infty,\mathbb{C})$ ... general function on $\mathbb{R}^4_\theta$

$(2\pi)^2\, Tr(\mathcal{Q}(f)) = \int \omega^2 f \qquad \omega = \frac{1}{2}\theta^{-1}_{\mu\nu} dx^\mu dx^\nu$

claim

M.M. fluctuations on a stack of n coincident $\mathbb{R}^4_\theta$ branes
$\to$ noncommutative $U(n)$ $\mathcal{N}=4$ super-Yang-Mills on $\mathbb{R}^4_\theta$

<u>sketch:</u>

- background solution: stack of n coinciding $\mathbb{R}^4_\theta$ branes

$$X^a = \begin{pmatrix} X^\mu \\ \phi^i \end{pmatrix} = \begin{pmatrix} \bar{X}^\mu \otimes \mathbf{1}_n \\ 0 \end{pmatrix}, \qquad \begin{matrix} \mu = 0, ..., 3 \\ i = 4, 5, .., 9 \end{matrix}$$

$[\bar{X}^\mu, \bar{X}^\nu] = i\theta^{\mu\nu}$ ... Heisenberg algebra, generate $\mathcal{A}_\theta \approx End(\mathcal{H})$

- add fluctuations:

$$X^a = \begin{pmatrix} \bar{X}^\mu \otimes \mathbf{1}_n + \theta^{\mu\nu} A_\nu \\ \phi^i \end{pmatrix} \qquad \in \mathcal{A}_\theta \otimes Mat(n, \mathbb{C})$$

$$A_\mu = A_\mu(\bar{X}) = A_{\mu,\alpha}(\bar{X})\lambda_\alpha \;\; \in End(\mathcal{H}^n) \cong \mathcal{A}_\theta \otimes Mat(n, \mathbb{C})$$

formally $A_\mu = \int d^4k\, e^{ik_\mu \bar{X}^\mu} A_{\mu,\alpha}(k)\lambda_\alpha, \qquad \lambda_\alpha \in \mathfrak{su}(n)$

$$\phi = \int d^4k\, e^{ik_\mu \bar{X}^\mu} \phi_\alpha(k)\lambda_\alpha$$

define derivatives as inner derivations:

$$[\bar{X}^\mu, \phi(X)] =: i\theta^{\mu\nu}\partial_\nu\phi(X), \qquad [\partial_\mu, \partial_\nu] = 0$$

thus

$$\begin{aligned} [X^\mu, \phi(X)] &= i\theta^{\mu\nu} D_\nu \phi(X), \qquad D_\mu = \partial_\mu + i[A_\mu, .] \\ [X^\mu, X^\nu] &= i\theta^{\mu\nu} + i\theta^{\mu\mu'}\theta^{\nu\nu'}(\partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'} + [A_{\mu'}, A_{\nu'}]) \\ &= i\theta^{\mu\nu} + i\theta^{\mu\mu'}\theta^{\nu\nu'} F_{\mu'\nu'} \end{aligned}$$

$F_{\mu'\nu'}$ ... Yang-Mills field strength

$S = Tr([X^a, X^b][X_a, X_b])$ is gauge-invariant: $X^a \to U^{-1}X^a U$

$\to$ tangential fluctuations $X^\mu = \bar{X}^\mu + \theta^{\mu\nu}A_\nu$ transform as
$A_\mu \to U^{-1}A_\mu U + iU^{-1}\partial_\mu U$ $\ldots \mathfrak{u}(n)$ gauge fields!

$\to$ transversal fluctuations $\phi^i \to U^{-1}\phi^i U$ $\ldots \mathfrak{u}(n)$ scalar fields!

$\mathcal{N}=4$ Super-Yang-Mills

$\Rightarrow$ effective action on $\mathbb{R}^4_\theta$:

$$\begin{aligned} S &= \Lambda_0^4\, Tr\Big([X^a,X^b][X_a,X_b] + \bar\Psi\Gamma_a[X^a,\Psi]\Big) \\ &= \int d^4x\,\sqrt{G}\, tr_n\Big(\tfrac{1}{4g^2}(\mathcal{F}\mathcal{F})_G + \tfrac12 G^{\mu\nu}D_\mu\Phi^i D_\nu\Phi_i - \tfrac14 g^2[\Phi^i,\Phi^j][\Phi_i,\Phi_j] \\ &\quad + \bar\psi\tilde\gamma^\mu(i\partial_\mu + [\mathcal{A}_\mu,.])\psi + g\bar\psi\Gamma^i[\Phi_i,\psi]\Big) \end{aligned}$$

where

$$\begin{aligned} G^{\mu\nu} &= \rho\theta^{\mu\nu'}\theta^{\nu\nu'}g_{\mu'\nu'}, \qquad \rho=\sqrt{|\theta^{-1}|} \\ \tilde\gamma^\mu &= \rho^{1/2}\,\theta^{\nu\mu}\gamma_\nu, \\ \frac{1}{4g^2} &= \frac{\Lambda_0^4}{(2\pi)^2}\rho^{-1} \end{aligned}$$

IKKT on stack of branes $\rightarrow$ $U(n)$ $\mathcal{N}=4$ SYM coupled to $G^{\mu\nu}$

better: $U(1)_{\rm tr} \rightarrow$ dynamical $G^{\mu\nu}$, $SU(n)$ SYM coupled to $G^{\mu\nu}(x)$

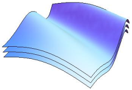
H.S., JHEP 0712:049 (2007), JHEP 0902:044,(2009)

analogous for finite matrix geometries, $\mathcal{A} = Mat(N,\mathbb{C})$

stack of coincident curved branes $\to$ $\mathfrak{su}(n)$ gauge thy

generic background branes

$$X^a = \begin{pmatrix} \bar{X}^\mu \otimes \mathbf{1}_n \\ \bar{\phi}^i \otimes \mathbf{1}_n \end{pmatrix}$$



general CR $[\bar{X}^\mu, \bar{X}^\nu] = i\theta^{\mu\nu}(\bar{X})$

fluctuations:

$$X^a = \begin{pmatrix} \bar{X}^\mu \otimes \mathbf{1}_N + \mathcal{A}^\mu \\ \bar{\phi}^i \otimes \mathbf{1}_N + \Phi^i \end{pmatrix}$$

$\mathcal{A}^\mu, \Phi^i \sim \mathbf{1}_n$ d.o.f. change background $\bar{X}^a$, geometrical d.o.f. $\theta^{\mu\nu}, g_{\mu\nu}$

write $\mathcal{A}^\mu = \theta^{\mu\nu} A_\nu$, note $[\bar{X}^\mu, f] \sim i\theta^{\mu\nu}\partial_\nu f$

$$\begin{aligned} [X^\mu, X^\nu] &= i\theta^{\mu\nu} + i\theta^{\mu\mu'}\theta^{\nu\nu'}(\partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'} + [A_{\mu'}, A_{\nu'}]) \\ &= i\theta^{\mu\nu} + i\theta^{\mu\mu'}\theta^{\nu\nu'} F_{\mu'\nu'} \qquad \text{field strength} \end{aligned}$$

$\Rightarrow$ effective action on $\mathcal{M}^4_\theta$:

$$
\begin{aligned}
S &= \Lambda_0^4\, Tr\Big([X^a, X^b][X_a, X_b] + \bar\Psi\Gamma_a[X^a, \Psi]\Big) \\
&\sim \int d^4x \sqrt{G}\, tr_n\Big(\frac{1}{4g^2}(\mathcal{F}\mathcal{F})_G + \frac{1}{2}(D\Phi^i D\Phi_i)_G - \frac{1}{4}g^2[\Phi^i, \Phi^j][\Phi_i, \Phi_j] \\
&\qquad + \bar\psi\tilde\gamma^\mu(i\partial_\mu + [\mathcal{A}_\mu, .])\psi + g\bar\psi\Gamma^i[\Phi_i, \psi]\Big) + \int 2\eta(\theta\wedge\theta + tr_n F\wedge F)
\end{aligned}
$$

where

$$
\begin{aligned}
G^{\mu\nu}(x) &= \rho\theta^{\mu\nu'}(x)\theta^{\nu\nu'}(x)g_{\mu'\nu'}(x), \qquad & \rho = \sqrt{|\theta^{-1}|} \\
\tilde\gamma^\mu(x) &= \rho^{1/2}\,\theta^{\nu\mu}(x)\gamma_\nu, & \eta = Gg \\
\frac{1}{4g^2} &= \frac{\Lambda_0^4}{(2\pi)^2}\rho^{-1}
\end{aligned}
$$

IKKT on stack of branes $\rightarrow$ $SU(n)$ $\mathcal{N} = 4$ SYM coupled to $G^{\mu\nu}$

dynamical $G^{\mu\nu}(x)$! ($\rightarrow$ gravity ?!)

H.S., JHEP 0712:049 (2007), JHEP 0902:044,(2009), Class.Quant.Grav. 27 (2010)

fermions

Ψ ... $\mathcal{A}$ - valued Majorana-Weyl spinor of $SO(9,1)$
action

$$S[\Psi] = \mathrm{Tr}\,\overline{\Psi}\Gamma_a[X^a,\Psi] \equiv \mathrm{Tr}\,\overline{\Psi}\not{D}\Psi$$
$$\sim \int d^4x\,\sqrt{\theta^{-1}}\,\overline{\Psi}i\gamma^\mu(\partial_\mu + [A_\mu,.])\Psi,$$

$$\tilde{\gamma}^\mu = \rho^{1/2}\,\Gamma_a\theta^{\nu\mu}\partial_\nu x^a$$

note

$$\{\tilde{\gamma}^\mu,\tilde{\gamma}^\nu\} = \rho\,\{\Gamma_a,\Gamma_b\}\theta^{\mu'\mu}\partial_{\mu'}x^a\theta^{\nu'\nu}\partial_{\nu'}x^b$$
$$= 2\rho\,\theta^{\mu'\mu}\theta^{\nu'\nu}\eta_{\mu'\nu'}$$
$$= 2\,G^{\mu\nu}$$

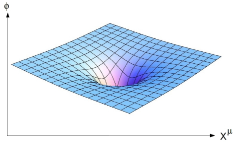
Ψ decomposes into 4 Weyl fermions $\rightarrow$ $\mathcal{N}=4$ SYM result:

- trace-$U(1)$ sector defines geometry $\mathcal{M}^{2n} \subset \mathbb{R}^{10}$
- $SU(n)$ fluctuations of matrices X^a, Ψ
  $\to$ gauge fields, scalar fields, fermions on $\mathcal{M}^{2n}$ (NOT 10 dim!)

all fields couple to metric $G^{\mu\nu}(x)$
determined by $\theta^{\mu\nu}(x)$, embedding
dynamical $\Rightarrow$ ("emergent") gravity

matrix e.o.m $[X^a, [X^{a'}, X^b]]\eta_{aa'} = 0 \quad \Longleftrightarrow$

$$\begin{aligned} \Box_G x^a &= 0, \qquad \text{"minimal surface"} \\ \nabla^\mu(e^\sigma \theta^{-1}_{\mu\nu}) &= e^{-\sigma}\, G_{\rho\nu}\theta^{\rho\mu}\partial_\mu \eta \\ & \qquad \eta \sim G^{\mu\nu} g_{\mu\nu} \end{aligned}$$



covariant formulation in semi-classical limit (H.S. Nucl.Phys. B810 (2009))

2 interpretations for quantization:

$$Z = \int dX^a d\Psi \, e^{-S[X]-S[\Psi]}$$

1. on $\mathbb{R}^4_\theta$: $X^\mu = \bar{X}^\mu + \bar{\theta}^{\mu\nu} A_\nu,$ $\bar{X}^\mu$ ...Moyal-Weyl
   $\rightarrow$ NC gauge theory on $\mathbb{R}^4_\theta$, UV/IR mixing in $U(1)$ sector
   IKKT model: $\mathcal{N} = 4$ SYM, perturb. finite !(?)
2. on $\mathcal{M}^4 \subset \mathbb{R}^{10}$: $U(1)$ absorbed in $\theta^{\mu\nu}(x)$, $g_{\mu\nu}$
   $\rightarrow$ quantized gravity, induced E-H. action

$$S_{eff} \sim \int d^4x \sqrt{|G|} \, (\Lambda^4 + c\Lambda_4^2 R[G] + \ldots)$$

- explanation for UV/IR mixing & $U(1)$ entanglement
- good quantization for theory with “gravity”! (maximal SUSY)
- emergence of Einstein equations not established, not clear

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further prospects:

- well suited for quantization
- can be put on computer (Monte Carlo; Lorentzian) !

  measure effective dimensions Kim, Nishimura, Tsuchiya PRL 108 (2012)

  result:

  3 out of 9 spatial directions start to expand at some 'critical time',

  3+1 dims at late times
- intersecting branes $\rightarrow$ chiral fermions

  A. Chatzistavrakidis, H.S., G. Zoupanos (2011)

  study compactifications $\mathcal{M}^4 \times \mathcal{K}_N$

  may get close to standard model (for low/intermediate energies)

  J. Zahn, H.S. (2014)

towards particle physics

internal Dirac operator $\leftrightarrow$ fermion masses:

$$\int d^4x \sqrt{G}\, tr_N\, g\bar{\psi}\Gamma^i[\Phi_i,\psi] =: \int d^4x \sqrt{G}\, tr_N\, g\bar{\psi}\not{D}_{\rm int}\psi$$

$$\not{D}_{\rm int} = \Gamma^i[\Phi_i, .]$$

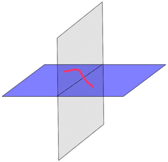
- $\not{D}_{\rm int}\psi = 0$ ... massless fermion;
  4D chirality = internal chirality (Weyl constraint!)
- $\not{D}_{\rm int}\psi = m\psi$ ... massive (Dirac) fermion
  (combining two spinors with opposite internal chirality);

mass m $\sim$ EV of $\not{D}_{\rm int}$

chiral fermions on intersecting NC branes

A. Chatzistavrakidis, H.S., G. Zoupanos (2011)

$$\Phi^i = \begin{pmatrix} \Phi^i_{(1)} & \\ & \Phi^i_{(2)} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} & \Psi_{(12)} \\ \Psi_{(21)} & \end{pmatrix}$$



M.M. Dirac operator on $\mathbb{R}^2_\theta \cap \mathbb{R}^2_\theta$

$$\begin{aligned} \not{D}_{\rm int}\Psi_{(12)} &= \Gamma_i[\Phi^i, \Psi_{(12)}] = \Gamma_i(\Phi^i_{(1)}\Psi_{(12)} - \Psi_{(12)}\Phi^i_{(2)}) \\ &= \not{D}_{(1)}\Psi_{(12)} - \not{D}_{(2)}\Psi_{(12)} \end{aligned}$$

use oscillator basis for (noncommutative!) branes

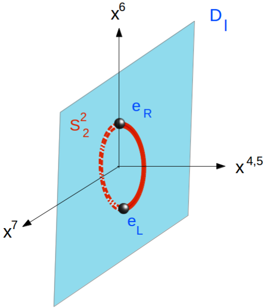
$$\begin{aligned} a &= \Phi^4 - i\Phi^5, \qquad b = \Phi^6 - i\Phi^7, \\ \alpha &= \tfrac{1}{2}(\Gamma^4 + i\Gamma^5), \quad \beta = \tfrac{1}{2}(\Gamma^6 + i\Gamma^7) \\ \not{D}_{(1)}\Psi &= (\alpha a^\dagger + \alpha^\dagger a)\Psi \\ \not{D}_{(2)}\Psi &= \beta\Psi b^\dagger + \beta^\dagger \Psi b \end{aligned}$$

$$\not{D}_{\rm int}\Psi_{(12)} = 0 \quad \Leftrightarrow \quad \Psi_{(12)} = |0,\downarrow\rangle_{(1)}\langle 0,\uparrow|_{(2)}$$

chiral zero mode in $\mathbb{R}^2 \times \mathbb{R}^2$
localized at intersection (coherent state)

fermions on intersection $S^2_2 \cap \mathbb{R}^2_\theta$:

$$\Phi^i = \begin{pmatrix} \Phi^i_{(1)} & \\ & \Phi^i_{(2)} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} & \Psi_{(12)} \\ \Psi_{(21)} & \end{pmatrix}$$



with $\quad \Phi^i_{(1)} = \begin{pmatrix} \phi\sigma_1 \\ \phi\sigma_2 \\ r\sigma_3 \\ 0 \end{pmatrix}, \quad \Phi^i_{(2)} = \begin{pmatrix} 0 \\ 0 \\ y^6 \\ y^7 \end{pmatrix}$

1. no Higgs $\phi = 0$: 2 points at $x^6 = \pm r$
   pair of zero modes, both chiralities, at each location
2. switch on Higgs $\phi \neq 0$:
   one chiral zero mode localized at each intersection $x^6 = \pm r$
   (coherent states)
   $$e_R = |+,\downarrow\rangle_{(1)}\langle +r,\uparrow|_{(2)}, \quad e_L = |-,\uparrow\rangle_{(1)}\langle -r,\uparrow|_{(2)}$$
   massive mirror fermion at each intersection, mass $m \sim \phi$
   $$\tilde{e}_L = |+,\uparrow\rangle_{(1)}\langle +r,\uparrow|_{(2)}, \quad \tilde{e}_R = |-,\downarrow\rangle_{(1)}\langle -r,\uparrow|_{(2)}$$

on $S^2 \times \mathbb{R}^2 \cap \mathbb{R}^2$:

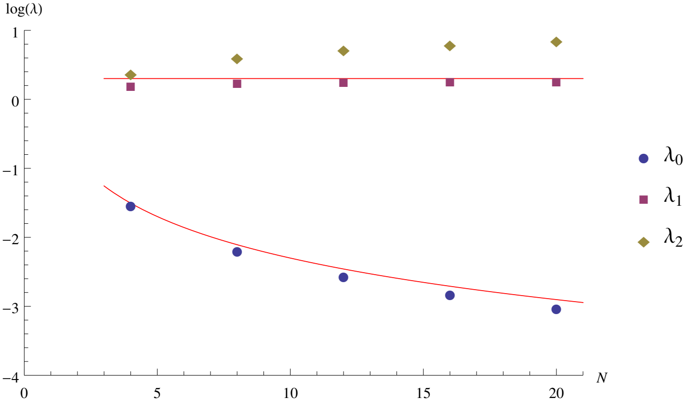
- exact chiral zero modes, e.g. $\Psi_{(12)} = |+0, \uparrow\downarrow\rangle_{(1)} \langle +r, \downarrow|_{(2)}$
- massive mirror fermions, e.g. $\tilde{\Psi}_{(12)} = |+0, \downarrow\downarrow\rangle_{(1)} \langle +r, \downarrow|_{(2)}$,
  mass $m \sim \phi$
  (opposite chirality on S^2_2, same localization)

on $\mathcal{K}^4_N \cap S^2_N$:

expect pairs of near-zero eigenmodes of $D\!\!\!/_{\rm int}$
consisting of nearly-localized chiral states

can estimate lowest eigenvalues of $\not{D}_{\rm int}$, depend on local geometry near intersections for "projected" $S^2_N \tilde{\times} S^2_{N'} \cap S^2_N$

numerical results: lowest eigenvalues of $\not{D}_{\rm int}$ (= Yukawas) for $N_i = N, R_i = 1, r = \phi = 1$:



towards the standard model (of particle physics)

- consider intersecting branes $\mathbb{R}^4 \times \mathcal{K}_i \subset \mathbb{R}^{10}$

  $\mathcal{K}_i$... fuzzy spaces (=quantized compact spaces)
  e.g. $S^2_N,\ S^2_N \times S^2_N, \mathbb{C}P^2_N$, ...

  $\rightarrow$ chiral fermions localized at $\mathcal{K}_i \cap \mathcal{K}_j$, propagate on $\mathbb{R}^4$

- stacks of n_i branes $\rightarrow$ $SU(n_i)$ gauge fields
  fermions $\Psi_{(12)}$ in $(n_1) \otimes (\bar{n}_2)$
- find explicit brane solution which breaks
  $SU(N) \rightsquigarrow SU(3)_c \times U(1)_Q \times U(1)_B$
- correct matter content of S.M. + ν_R at brane intersections

standard model fields embedded in adjoint of $SU(N)$:

A. Chatzistavrakidis, H.S., G. Zoupanos (2011)

$$\Psi = \begin{pmatrix} 0_2 & 0 & 0 & l_L & Q_L \\ & 0 & \begin{pmatrix} 0 & e_R \\ 0 & \nu_R \end{pmatrix} & & Q_R \\ & & & 0 & 0 \\ & & & & 0_3 \end{pmatrix},$$

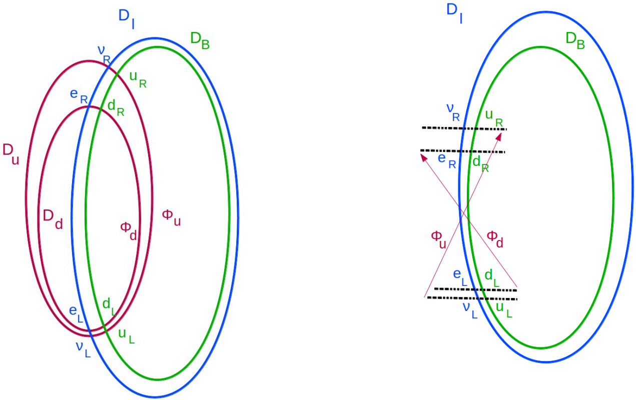
where

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \qquad Q_R = \begin{pmatrix} d_R \\ u_R \end{pmatrix}$$

Higgs:

$$\Phi^a_{(H)} = \begin{pmatrix} 0_2 & H_d & H_u & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & S & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \phi_u & 0 & 0 \\ 0 & 0 & \phi_d & 0 & 0 & 0 \\ 0 & \phi_d^\dagger & 0 & 0 & 0 & 0 \\ {\phi_u}^\dagger & 0 & 0 & 0 & S & 0 \\ 0 & 0 & 0 & S^\dagger & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

S ... sterile Higgs



precisely chiral matter content of standard model + ν_R
Higgs ... intrinsic part of fuzzy internal geometry

intersecting brane solutions

need to stabilize compact branes:

- rotating branes
- deform model by $SO(6)$-invariant potential

$$S \rightarrow S - V_{\rm def},$$

$$V_{\rm def} = f(tr_N \sum_{i=4}^{9} X_i X^i) \overset{e.g.}{=} -m^2 tr(X_i X^i) + \lambda(tr X_i X^i)^2$$

e.o.m.

$$\Box X^i = -(2\pi g \rho^{-\frac{1}{2}} f')\, X^i, \qquad \Box = [X^j, [X_j, .]]$$

- better: add cubic potential terms (”soft SUSY breaking, flux“)

$$V_{\rm def} = X^a X^b X^c f_{abc}$$

f_{abc} tot. antisymm.

- branes interact (1-loop $\rightarrow \approx$ SUGRA, typically attraction)

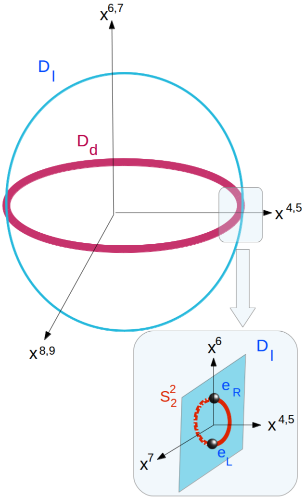
intersecting brane solutions

$\mathcal{D}_u \sim \mathcal{D}_d$:

$$\Phi^a_{(u)} = \begin{pmatrix} R'_u L_3 + \phi_u \sigma'_1 \\ \phi_u \sigma'_2 \\ r_u \sigma'_3 \\ 0 \\ R_u L_1 \\ R_u L_2 \end{pmatrix} \cong S^2_{N_u} \tilde{\times} S^2_2$$

$\mathcal{D}_l \sim \mathcal{D}_B$:

$$\Phi^a_{(l)} = \begin{pmatrix} R'_l K_3 \\ 0 \\ R_l K_1 \\ R_l K_2 \\ 0 \\ 0 \end{pmatrix} \cong S^2_{N_l}$$



solution of eom if

$$R_u = R'_u = R_l = R'_l = r_u = \phi_u = \sqrt{-\pi g \rho^{-1/2} f'}$$

J. Zahn, H.S. (2014)

results:

- background branes lead to correct symmetry breaking
  $SU(N) \rightsquigarrow SU(3)_c \times U(1)_Q \times U(1)_B$ (assume appropriate S)
- resembles S.M. at low energies:
  - correct matter content of S.M. (2 generations ...) + ν_R
    coupled to $SU(3)_c \times SU(2)_L \times U(1)_Y$
  - electroweak SSB $SU(2)_L \times U(1)_Y \rightsquigarrow U(1)_Q$ via
    two Higgs doublets,
    intrinsic part of geometry (minimal fuzzy spheres),
    essential for chiral nature of fermions
- mirror fermions at intermediate energies (above m_W),
- gauginos, towers of massive KK modes,
  ultimately completing $\mathcal{N} = 4$ SUSY

summary, conclusion

- matrix-models $Tr[X^a, X^b][X^{a'}, X^{b'}]\,\eta_{aa'}\eta_{bb'} + $ fermions

  dynamical NC branes $\leftrightarrow$ "emergent" gravity

- fluctuations of matrices $\rightarrow$ gauge theory on brane
  all ingredients for physics
- rich solutions of IKKT model with $\mathbb{R}^4 \times \mathcal{K}$ (with extra V_{def})

  building blocks for intersecting branes $\rightarrow$ standard model ?
- nonperturbative insights very desirable:
  eigenvalue distribution, ... !?
  new, adapted methods ??
- ... very rich model, more to be discovered

quantum numbers: adjoint action $Q\Psi_{(12)} = [t_Q, \Psi_{(12)}]$ etc., e.g.

$$t_Q = \frac{1}{2}\begin{pmatrix} \sigma_3 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -\frac{1}{3} \end{pmatrix}, \qquad t_Y = \begin{pmatrix} 0_2 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -\frac{1}{3} \end{pmatrix}.$$