Schur Function Factorizations, with Applications to Alternating Sign Matrices and Plane Partitions

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Plan

1. Obtain, with full proof, a curious factorization identity for the Schur function $s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1})$.

2. Present, without proof, some further similar identities.

3. Apply the first identity to rectangular and double-staircase partitions, which will lead to relations for numbers of certain plane partitions & alternating sign matrices.

1. Obtaining the Main Schur Function Factorization

Consider:

- a partition $(\lambda_1, \ldots, \lambda_n)$ & integer $k \geq \lambda_1$,
  or a half-partition $(\lambda_1, \ldots, \lambda_n)$ & half-integer $k \geq \lambda_1$
- indeterminates $x_1, \ldots, x_n$
- the Schur function $s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1})$

The determinant formula $s_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n) = \dfrac{\det_{1\leq i,j\leq n}\left(x_i^{\lambda_j+n-j}\right)}{\prod_{1\leq i<j\leq n}(x_i-x_j)}$ gives

$$s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) =$$

$$\frac{\det\left(\begin{array}{c|c} \left(x_i^{k+\lambda_j+2n-j}\right)_{1\leq i,j\leq n} & \left(x_i^{k-\lambda_{n+1-j}+n-j}\right)_{1\leq i,j\leq n} \\ \hline \left(x_i^{-k-\lambda_j-2n+j}\right)_{1\leq i,j\leq n} & \left(x_i^{-k+\lambda_{n+1-j}-n+j}\right)_{1\leq i,j\leq n} \end{array}\right)}{\prod_{i=1}^{n}(x_i-x_i^{-1})\prod_{1\leq i<j\leq n}(x_i-x_j)(x_i^{-1}-x_j^{-1})(x_i-x_j^{-1})(x_j-x_i^{-1})}$$

- We have $s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) =$ numerator/denominator

  with

$$\text{numerator} = \det\left(\begin{array}{c|c} \left(x_i^{k+\lambda_j+2n-j}\right)_{1\le i,j\le n} & \left(x_i^{k-\lambda_{n+1-j}+n-j}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-k-\lambda_j-2n+j}\right)_{1\le i,j\le n} & \left(x_i^{-k+\lambda_{n+1-j}-n+j}\right)_{1\le i,j\le n} \end{array}\right)$$

$$\text{denominator} = (-1)^{n(n-1)/2}\prod_{i=1}^{n}(x_i - x_i^{-1})\prod_{1\le i<j\le n}(x_i + x_i^{-1} - x_j - x_j^{-1})^2$$

- We have $s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) = \text{numerator}/\text{denominator}$

  with
$$\text{numerator} = \det\left(\begin{array}{c|c} \left(x_i^{k+\lambda_j+2n-j}\right)_{1\le i,j\le n} & \left(x_i^{k-\lambda_{n+1-j}+n-j}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-k-\lambda_j-2n+j}\right)_{1\le i,j\le n} & \left(x_i^{-k+\lambda_{n+1-j}-n+j}\right)_{1\le i,j\le n} \end{array}\right)$$
$$\text{denominator} = (-1)^{n(n-1)/2}\prod_{i=1}^{n}(x_i - x_i^{-1})\prod_{1\le i<j\le n}(x_i + x_i^{-1} - x_j - x_j^{-1})^2$$

- Multiplying each row i in top blocks by $x_i^{-k-n+1/2}$ & each row i in bottom blocks by $x_i^{k+n-1/2}$

  $$\Rightarrow \quad \text{numerator} = \det\left(\begin{array}{c|c} \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} & \left(x_i^{-\lambda_{n+1-j}-j+1/2}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} & \left(x_i^{\lambda_{n+1-j}+j-1/2}\right)_{1\le i,j\le n} \end{array}\right)$$

- We have $s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) =$ numerator/denominator

with

$$\text{numerator} = \det\left(\begin{array}{c|c} \left(x_i^{k+\lambda_j+2n-j}\right)_{1\le i,j\le n} & \left(x_i^{k-\lambda_{n+1-j}+n-j}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-k-\lambda_j-2n+j}\right)_{1\le i,j\le n} & \left(x_i^{-k+\lambda_{n+1-j}-n+j}\right)_{1\le i,j\le n} \end{array}\right)$$

$$\text{denominator} = (-1)^{n(n-1)/2}\prod_{i=1}^{n}(x_i - x_i^{-1})\prod_{1\le i<j\le n}(x_i + x_i^{-1} - x_j - x_j^{-1})^2$$

- Multiplying each row i in top blocks by $x_i^{-k-n+1/2}$ & each row i in bottom blocks by $x_i^{k+n-1/2}$

$$\Rightarrow\ \text{numerator} = \det\left(\begin{array}{c|c} \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} & \left(x_i^{-\lambda_{n+1-j}-j+1/2}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} & \left(x_i^{\lambda_{n+1-j}+j-1/2}\right)_{1\le i,j\le n} \end{array}\right)$$

- Reversing order of columns in right blocks

$$\Rightarrow\ \text{numerator} = (-1)^{n(n-1)/2}\det\left(\begin{array}{c|c} \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} & \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} & \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} \end{array}\right)$$

- We have $s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) =$ numerator/denominator

  with numerator $= \det \left(\begin{array}{c|c} \left(x_i^{k+\lambda_j+2n-j}\right)_{1\le i,j\le n} & \left(x_i^{k-\lambda_{n+1-j}+n-j}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-k-\lambda_j-2n+j}\right)_{1\le i,j\le n} & \left(x_i^{-k+\lambda_{n+1-j}-n+j}\right)_{1\le i,j\le n} \end{array} \right)$

  denominator $= (-1)^{n(n-1)/2} \prod_{i=1}^n (x_i - x_i^{-1}) \prod_{1\le i<j\le n} (x_i + x_i^{-1} - x_j - x_j^{-1})^2$

- Multiplying each row i in top blocks by $x_i^{-k-n+1/2}$ & each row i in bottom blocks by $x_i^{k+n-1/2}$

  $\Rightarrow$ numerator $= \det \left(\begin{array}{c|c} \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} & \left(x_i^{-\lambda_{n+1-j}-j+1/2}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} & \left(x_i^{\lambda_{n+1-j}+j-1/2}\right)_{1\le i,j\le n} \end{array} \right)$

- Reversing order of columns in right blocks

  $\Rightarrow$ numerator $= (-1)^{n(n-1)/2} \det \left(\begin{array}{c|c} \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} & \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} & \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} \end{array} \right)$

- Setting $A = \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n}$ & $B = \left(x_i^{-(\lambda_j+n-j+1/2)}\right)_{1\le i,j\le n}$ $\Rightarrow$ num. $= (-1)^{n(n-1)/2} \det\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right)$

- We have $s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) = \text{numerator}/\text{denominator}$

  with
$$\text{numerator} = \det\left(\begin{array}{c|c} \left(x_i^{k+\lambda_j+2n-j}\right)_{1\le i,j\le n} & \left(x_i^{k-\lambda_{n+1-j}+n-j}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-k-\lambda_j-2n+j}\right)_{1\le i,j\le n} & \left(x_i^{-k+\lambda_{n+1-j}-n+j}\right)_{1\le i,j\le n} \end{array}\right)$$
$$\text{denominator} = (-1)^{n(n-1)/2}\prod_{i=1}^n(x_i - x_i^{-1})\prod_{1\le i<j\le n}(x_i + x_i^{-1} - x_j - x_j^{-1})^2$$

- Multiplying each row i in top blocks by $x_i^{-k-n+1/2}$ & each row i in bottom blocks by $x_i^{k+n-1/2}$
$$\Rightarrow\ \text{numerator} = \det\left(\begin{array}{c|c} \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} & \left(x_i^{-\lambda_{n+1-j}-j+1/2}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} & \left(x_i^{\lambda_{n+1-j}+j-1/2}\right)_{1\le i,j\le n} \end{array}\right)$$

- Reversing order of columns in right blocks
$$\Rightarrow\ \text{numerator} = (-1)^{n(n-1)/2}\det\left(\begin{array}{c|c} \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} & \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} & \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} \end{array}\right)$$

- Setting $A = \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n}$ & $B = \left(x_i^{-(\lambda_j+n-j+1/2)}\right)_{1\le i,j\le n}$ $\Rightarrow$ num. $= (-1)^{n(n-1)/2}\det\left(\begin{array}{c|c} A & B \\ \hline B & A\end{array}\right)$

- Subtracting right blocks from left blocks, adding top blocks to bottom blocks & using "det of block triangular matrix = product of dets of diagonal blocks"
$$\Rightarrow\ \det\left(\begin{array}{c|c} A & B \\ \hline B & A\end{array}\right) = \det\left(\begin{array}{c|c} A-B & B \\ \hline B-A & A\end{array}\right) = \det\left(\begin{array}{c|c} A-B & B \\ \hline 0 & A+B\end{array}\right) = \det(A-B)\det(A+B)$$

- We have $s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) = \text{numerator}/\text{denominator}$

  with
$$\text{numerator} = \det\left(\begin{array}{c|c} \left(x_i^{k+\lambda_j+2n-j}\right)_{1\le i,j\le n} & \left(x_i^{k-\lambda_{n+1-j}+n-j}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-k-\lambda_j-2n+j}\right)_{1\le i,j\le n} & \left(x_i^{-k+\lambda_{n+1-j}-n+j}\right)_{1\le i,j\le n}\end{array}\right)$$
$$\text{denominator} = (-1)^{n(n-1)/2}\prod_{i=1}^n(x_i-x_i^{-1})\prod_{1\le i<j\le n}(x_i+x_i^{-1}-x_j-x_j^{-1})^2$$

- Multiplying each row i in top blocks by $x_i^{-k-n+1/2}$ & each row i in bottom blocks by $x_i^{k+n-1/2}$
$$\Rightarrow \quad \text{numerator} = \det\left(\begin{array}{c|c} \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} & \left(x_i^{-\lambda_{n+1-j}-j+1/2}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} & \left(x_i^{\lambda_{n+1-j}+j-1/2}\right)_{1\le i,j\le n}\end{array}\right)$$

- Reversing order of columns in right blocks
$$\Rightarrow \quad \text{numerator} = (-1)^{n(n-1)/2}\det\left(\begin{array}{c|c} \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n} & \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} \\ \hline \left(x_i^{-\lambda_j-n+j-1/2}\right)_{1\le i,j\le n} & \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n}\end{array}\right)$$

- Setting $A = \left(x_i^{\lambda_j+n-j+1/2}\right)_{1\le i,j\le n}$ & $B = \left(x_i^{-(\lambda_j+n-j+1/2)}\right)_{1\le i,j\le n}$ $\Rightarrow$ num. $= (-1)^{n(n-1)/2}\det\left(\begin{array}{c|c} A & B \\ \hline B & A\end{array}\right)$

- Subtracting right blocks from left blocks, adding top blocks to bottom blocks & using "det of block triangular matrix = product of dets of diagonal blocks"
$$\Rightarrow \quad \det\left(\begin{array}{c|c} A & B \\ \hline B & A\end{array}\right) = \det\left(\begin{array}{c|c} A-B & B \\ \hline B-A & A\end{array}\right) = \det\left(\begin{array}{c|c} A-B & B \\ \hline 0 & A+B\end{array}\right) = \det(A-B)\det(A+B)$$

- Therefore $s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1})$
$$= \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j+1/2}-x_i^{-(\lambda_j+n-j+1/2)}\right)\ \det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j+1/2}+x_i^{-(\lambda_j+n-j+1/2)}\right)}{\prod_{i=1}^n(x_i-x_i^{-1})\prod_{1\le i<j\le n}(x_i+x_i^{-1}-x_j-x_j^{-1})^2}$$

- Using determinant formulae for odd orthogonal ($SO(2n+1,\mathbb{C})$) character

$$so^{\text{odd}}_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n) = \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j+1/2} - x_i^{-(\lambda_j+n-j+1/2)}\right)}{\prod_{i=1}^n (x_i^{1/2} - x_i^{-1/2}) \prod_{1\le i<j\le n}(x_i + x_i^{-1} - x_j - x_j^{-1})}$$

& even orthogonal ($O(2n,\mathbb{C})$) character

$$o^{\text{even}}_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n) = \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)}\right)}{\prod_{1\le i<j\le n}(x_i + x_i^{-1} - x_j - x_j^{-1})}$$

(strictly speaking, RHS should be divided by 2 if $\lambda_n = 0$)

the factorization

$$s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1})$$
$$= \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j+1/2} - x_i^{-(\lambda_j+n-j+1/2)}\right)\ \det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j+1/2} + x_i^{-(\lambda_j+n-j+1/2)}\right)}{\prod_{i=1}^n (x_i - x_i^{-1}) \prod_{1\le i<j\le n}(x_i + x_i^{-1} - x_j - x_j^{-1})^2}$$

can be written as

$$\prod_{i=1}^{n}\left(x_i^{1/2} + x_i^{-1/2}\right) s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1})$$
$$= so^{\text{odd}}_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n)\ o^{\text{even}}_{(\lambda_1+1/2,\ldots,\lambda_n+1/2)}(x_1,\ldots,x_n).$$

- This is the main Schur function factorization.

(Independence on k is due to simple property $s_{(k+\lambda_1,\ldots,k+\lambda_n)}(x_1,\ldots,x_n) = (x_1\ldots x_n)^k\, s_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n)$.)

2. Further Schur Function Factorizations

- Similar methods give the following results.

- For any partition $(\lambda_0,\ldots,\lambda_n)$ & integers $k_1,k_2 \ge \lambda_0$,
  or half-partition $(\lambda_0,\ldots,\lambda_n)$ & half-integers $k_1,k_2 \ge \lambda_0$:

$$\prod_{i=1}^{n}(x_i^{1/2}+x_i^{-1/2})\,\big(s_{(k_1+\lambda_1,\ldots,k_1+\lambda_n,k_1-\lambda_{n-1},\ldots,k_1-\lambda_0)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) + s_{(k_2+\lambda_1,\ldots,k_2+\lambda_{n-1},k_2-\lambda_n,\ldots,k_2-\lambda_0)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1})\big) = so^{\text{odd}}_{(\lambda_0+1/2,\ldots,\lambda_{n-1}+1/2)}(x_1,\ldots,x_n)\; o^{\text{even}}_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n)$$

- For any partition $(\lambda_0,\ldots,\lambda_n)$ & integer $k \ge \lambda_0$,
  or half-partition $(\lambda_0,\ldots,\lambda_n)$ & half-integer $k \ge \lambda_0$:

$$\prod_{i=1}^{n}(x_i^{1/2}+x_i^{-1/2})(x_i^{1/2}-x_i^{-1/2})^2\, s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_0)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1},1) = so^{\text{odd}}_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n)\sum_{i=0}^{n}(-1)^{n+i}\, o^{\text{even}}_{(\lambda_0+3/2,\ldots,\lambda_{i-1}+3/2,\lambda_{i+1}+1/2,\ldots,\lambda_n+1/2)}(x_1,\ldots,x_n)$$

- Consider $(\lambda_0,\dots,\lambda_n) = (nb+a,\dots,2b+a,b+a,a)$ & $k \ge \lambda_0$, where a & k are both nonnegative integers or both positive half-integers, and b is a nonnegative integer. Then:

$$\prod_{i=1}^{n}(x_i^{1/2}+x_i^{-1/2})\; s_{(k+\lambda_1,\dots,k+\lambda_n,k-\lambda_n,\dots,k-\lambda_0)}(x_1,\dots,x_n,x_1^{-1},\dots,x_n^{-1},1)$$
$$= \prod_{i=1}^{n}\Big(\sum_{j=-b/2}^{b/2} x_i^j\Big)\; so^{\text{odd}}_{(\lambda_1,\dots,\lambda_n)}(x_1,\dots,x_n)\; so^{\text{odd}}_{(\lambda_1+(b+1)/2,\dots,\lambda_n+(b+1)/2)}(x_1,\dots,x_n)$$

- Consider $(\lambda_0,\dots,\lambda_n) = (\lfloor\frac{n+1}{2}\rfloor b+(-1)^n a,\dots,2b+a,2b-a,b+a,b-a,a)$ & $k \ge \lambda_0$, where a & k are both nonnegative integers or both positive half-integers, and b is a nonnegative integer with $b \ge 2a$. Then:

$$\prod_{i=1}^{n}(x_i^{1/2}+x_i^{-1/2})\; s_{(k+\lambda_1,\dots,k+\lambda_n,k-\lambda_n,\dots,k-\lambda_0)}(x_1,\dots,x_n,x_1^{-1},\dots,x_n^{-1},1)$$
$$= \prod_{i=1}^{n}\Big(\sum_{j=-(b+1)/2}^{(b+1)/2} x_i^j\Big)\; so^{\text{odd}}_{(\lambda_1,\dots,\lambda_n)}(x_1,\dots,x_n)\; so^{\text{odd}}_{(\lambda_0-b/2,\dots,\lambda_{n-1}-b/2)}(x_1,\dots,x_n)$$

Remarks on the Factorizations

- Further factorizations could presumably be obtained.
- These factorizations generalize all of the cases obtained for rectangular partitions by Ciucu & Krattenthaler (2009).
- The proofs involve elementary operations on determinants.
- We do not currently have representation theoretic interpretations in terms of irreducible representations of classical Lie groups.
- We do not currently have combinatorial proofs which use expressions for characters as generating functions for tableaux.

3. Applications

For any partition $(\lambda_1,\ldots,\lambda_n)$ & integer $k\geq\lambda_1$

(i) $$s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) = (-1)^{|\lambda|}\, so^{\text{odd}}_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n)\; so^{\text{odd}}_{(\lambda_1,\ldots,\lambda_n)}(-x_1,\ldots,-x_n)$$

(ii) $$s_{(k+\lambda_1+1,\ldots,k+\lambda_n+1,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) = sp_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n)\; o^{\text{even}}_{(\lambda_1+1,\ldots,\lambda_n+1)}(x_1,\ldots,x_n)$$

where $|\lambda|=\sum_{i=1}^n\lambda_i$ & $sp_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n)$ is a symplectic ($Sp(2n,\mathbb{C})$) character.

- These factorizations follow immediately from the main factorization
$$\prod_{i=1}^n(x_i^{1/2}+x_i^{-1/2})\; s_{(k+\lambda_1,\ldots,k+\lambda_n,k-\lambda_n,\ldots,k-\lambda_1)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) = so^{\text{odd}}_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n)\; o^{\text{even}}_{(\lambda_1+1/2,\ldots,\lambda_n+1/2)}(x_1,\ldots,x_n). \qquad (*)$$

- To obtain (i), apply the identity
$$o^{\text{even}}_{(\lambda_1+1/2,\ldots,\lambda_n+1/2)}(x_1,\ldots,x_n) = (-1)^{|\lambda|}\prod_{i=1}^n(x_i^{1/2}+x_i^{-1/2})\; so^{\text{odd}}_{(\lambda_1,\ldots,\lambda_n)}(-x_1,\ldots,-x_n)$$
to $(*)$.

- To obtain (ii), replace $(\lambda_1,\ldots,\lambda_n)$ by $(\lambda_1+\frac{1}{2},\ldots,\lambda_n+\frac{1}{2})$ & k by $k+\frac{1}{2}$ in $(*)$. Then apply the identity
$$so^{\text{odd}}_{(\lambda_1+1/2,\ldots,\lambda_n+1/2)}(x_1,\ldots,x_n) = \prod_{i=1}^n(x_i^{1/2}+x_i^{-1/2})\; sp_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n).$$

Rectangular Partitions

- Taking $(\lambda_1,\ldots,\lambda_n)=(\underbrace{m,\ldots,m}_{n})$ (denoted (m^n)) & $k=m$ in previous factorizations (i) & (ii) gives

$$s_{((2m)^n,0^n)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1})=(-1)^{mn}\,so^{\text{odd}}_{(m^n)}(x_1,\ldots,x_n)\;so^{\text{odd}}_{(m^n)}(-x_1,\ldots,-x_n)$$

$$s_{((2m+1)^n,0^n)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1})=sp_{(m^n)}(x_1,\ldots,x_n)\;o^{\text{even}}_{((m+1)^n)}(x_1,\ldots,x_n)$$

These identities were first obtained by Ciucu & Krattenthaler (2009).

- It can be shown that for $x_1=\ldots=x_n=1$ (denoted 1^n), the terms are:

$$s_{((2m)^n,0^n)}(1^{2n})=\#\text{ plane partitions (PPs) in }(2m)\times n\times n\text{ box}$$

$$so^{\text{odd}}_{(m^n)}(1^n)=\#\text{ symmetric PPs in }(2m)\times n\times n\text{ box}$$

$$(-1)^{mn}so^{\text{odd}}_{(m^n)}((-1)^n)=\#\text{ transpose complementary PP's in }(2m)\times n\times n\text{ box}$$

$$s_{((2m+1)^n,0^n)}(1^{2n})=\#\text{ PPs in }(2m+1)\times n\times n\text{ box}$$

$$sp_{(m^n)}(1^n)=\#\text{ transpose complementary PPs in }2m\times(n+1)\times(n+1)\text{ box}$$

$$o^{\text{even}}_{((m+1)^n)}(1^n)=\#\text{ certain restricted symmetric PPs in }(2m+2)\times n\times n\text{ box}$$

All cases can also be expressed in terms of certain rhombus tilings of hexagons.

Double-Staircase Partitions

- Taking $n=2m+1$, $(\lambda_1,\ldots,\lambda_{2m+1})=(m,m,m-1,m-1,\ldots,2,2,1,1,0)$ & $k=m$, or $n=2m$, $(\lambda_1,\ldots,\lambda_{2m})=(m-1,m-1,m-2,m-2,\ldots,1,1,0,0)$ & $k=m-1$ in previous factorizations (i) or (ii) respectively gives

$$s_{(2m,2m,\ldots,1,1,0,0)}(x_1,\ldots,x_{2m+1},x_1^{-1},\ldots,x_{2m+1}^{-1}) = so^{\text{odd}}_{(m,m,\ldots,2,2,1,1,0)}(x_1,\ldots,x_{2m+1})\; so^{\text{odd}}_{(m,m,\ldots,2,2,1,1,0)}(-x_1,\ldots,-x_{2m+1})$$

$$s_{(2m-1,2m-1,\ldots,1,1,0,0)}(x_1,\ldots,x_{2m},x_1^{-1},\ldots,x_{2m}^{-1}) = sp_{(m-1,m-1,\ldots,1,1,0,0)}(x_1,\ldots,x_{2m})\; o^{\text{even}}_{(m,m,\ldots,2,2,1,1)}(x_1,\ldots,x_{2m})$$

- It can be shown that for $x_1=x_2=\ldots=1$, the terms satisfy:

$3^{-n(n-1)/2}\, s_{(n-1,n-1,\ldots,1,1,0,0)}(1^{2n})=\#$ order n descending PPs
$=\#$ totally symmetric self-complementary PPs (TSSCPPs) in $(2n)\times(2n)\times(2n)$ box
$=\#$ $n\times n$ alternating sign matrices (ASMs)
$=\#$ alternating sign triangles (ASTs) with n rows

$3^{-(n-1)^2} so^{\text{odd}}_{(n-1,n-1,\ldots,2,2,1,1,0)}(1^{2n-1}) = 2^{-1}\,3^{-n^2} o^{\text{even}}_{(n,n,\ldots,2,2,1,1)}(1^{2n}) = \#$ certain rhombus tilings

$3^{-n(n+1)}\, so^{\text{odd}}_{(n,n,\ldots,2,2,1,1,0)}((-1)^{2n+1}) = 3^{-n(n-1)}\, sp_{(n-1,n-1,\ldots,2,2,1,1,0,0)}(1^{2n})$
$=\#$ order $2n+1$ descending PPs invariant under certain involution
$=\#$ TSSCPPs in $(4n+2)\times(4n+2)\times(4n+2)$ box invariant under certain involution
$=\#$ $(2n+1)\times(2n+1)$ vertically symm. ASMs $=\#$ $2n\times 2n$ off-diagonally symm. ASMs
$\stackrel{?}{=}\#$ vertically symmetric ASTs with $2n+1$ rows

Further Double-Staircase Partitions

- Taking $n=2m$, $(\lambda_1,\ldots,\lambda_{2m})=(m,m-1,m-1,\ldots,2,2,1,1,0)$ & $k=m$, or $n=2m+1$, $(\lambda_1,\ldots,\lambda_{2m+1})=(m,m-1,m-1,\ldots,1,1,0,0)$ & $k=m$ in previous factorizations (i) or (ii) respectively gives

$$s_{(2m,2m-1,2m-1,\ldots,2,2,1,1,0)}(x_1,\ldots,x_{2m},x_1^{-1},\ldots,x_{2m}^{-1})$$
$$= (-1)^m\, so^{\text{odd}}_{(m,m-1,m-1,\ldots,2,2,1,1,0)}(x_1,\ldots,x_{2m})\; so^{\text{odd}}_{(m,m-1,m-1,\ldots,2,2,1,1,0)}(-x_1,\ldots,-x_{2m})$$

$$s_{(2m+1,2m,2m,\ldots,2,2,1,1,0)}(x_1,\ldots,x_{2m+1},x_1^{-1},\ldots,x_{2m+1}^{-1})$$
$$= sp_{(m,m-1,m-1,\ldots,1,1,0,0)}(x_1,\ldots,x_{2m+1})\; o^{\text{even}}_{(m+1,m,m,\ldots,2,2,1,1)}(x_1,\ldots,x_{2m+1})$$

- It can be shown that for $x_1=x_2=\ldots=1$, the terms satisfy:

$$3^{-n(n-1)/2}\, s_{(n,n-1,n-1,\ldots,2,2,1,1,0)}(1^{2n}) = \#\text{ cyclically symmetric PPs in } n\times n\times n \text{ box}$$
$$= \#\text{ quasi ASTs with } n \text{ rows}$$

$$3^{-n(n-1)}\, so^{\text{odd}}_{(n,n-1,n-1,\ldots,2,2,1,1,0)}(1^{2n}) = 2^{-1}\, 3^{-n(n+1)}\, o^{\text{even}}_{(n+1,n,n,\ldots,2,2,1,1)}(1^{2n+1})$$
$$= \#\text{ totally symmetric PPs in } (2n)\times(2n)\times(2n) \text{ box}$$

$$(-1)^n\, 3^{-n^2}\, so^{\text{odd}}_{(n,n-1,n-1,\ldots,2,2,1,1,0)}((-1)^{2n}) = 3^{-(n-1)^2}\, sp_{(n-1,n-2,n-2,\ldots,2,2,1,1,0,0)}(1^{2n-1})$$
$$= \#\text{ cyclically symmetric transpose complementary PPs in } (2n)\times(2n)\times(2n) \text{ box}$$
$$\stackrel{?}{=} \#\text{ vertically symmetric quasi ASTs with } 2n \text{ rows}$$

Remarks on the Applications

- The identification of numbers of certain PPs or ASMs/ASTs with characters at $x_1 = x_2 = \ldots = 1$ corresponds to a wide range of results, with proofs of varying degrees of difficulty.

  For example, it can be shown straightforwardly that
  $$\begin{aligned}&\#\text{ PPs in } a\times b\times c \text{ box}\\ &= \#\text{ semistandard Young tableaux of shape } a^b \text{ with entries from } \{1,\ldots,b+c\}\\ &= s_{(a^b,0^c)}(1^{b+c}).\end{aligned}$$
  It is harder to show that
  $$\#\ n\times n \text{ ASMs} = 3^{-n(n-1)/2}\, s_{(n-1,n-1,\ldots,1,1,0,0)}(1^{2n}).$$

- Some of the factorizations for numbers of PPs can be explained combinatorially using Ciucu's matchings factorization theorem.

- Some of the characters (at arbitrary $x_1, x_2, \ldots$) are essentially partition functions of certain cases of the six-vertex model.