Cyclic descents of standard Young tableaux

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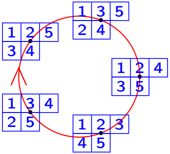
Joint work with Ron Adin and Yuval Roichman



Workshop on Algorithmic and Enumerative Combinatorics — ESI, Vienna, October 2017

Descents and cyclic descents of permutations

Let $\pi = \pi_1 \dots \pi_n \in \mathcal{S}_n$ be a permutation.

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Introduced by Cellini '95; further studied by Dilks, Petersen and Stembridge '09 among others.

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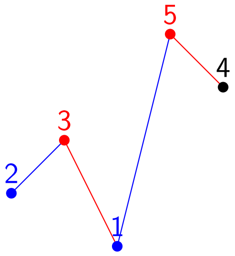

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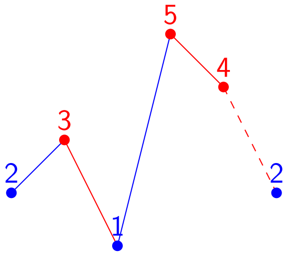

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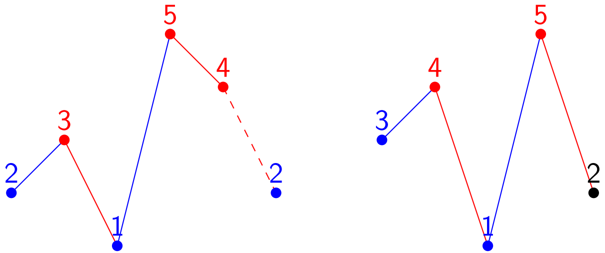

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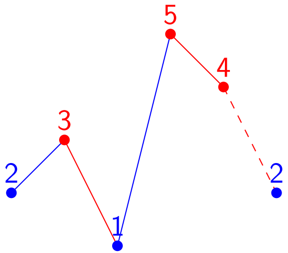

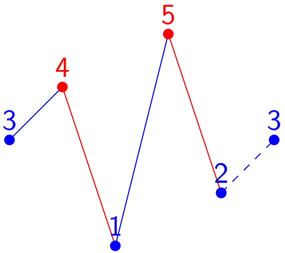

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Indeed, we can just define ϕ by

$$\pi_1\pi_2 \dots \pi_{n-1}\pi_n \quad \stackrel{\phi}{\longmapsto} \quad \pi_n\pi_1\pi_2 \dots \pi_{n-1}$$

Young diagrams

A partition of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ and $\lambda_1 + \lambda_2 + \dots = n$. We write $\lambda \vdash n$.

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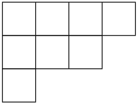

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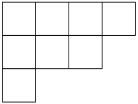

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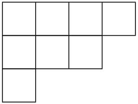
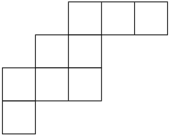


If the diagram of μ is contained in the diagram of λ, then the difference of these diagrams is a diagram of skew shape λ/μ.

Example: $\lambda/\mu = (5, 3, 3, 1)/(2, 1)$



When μ is the empty partition, λ/μ is simply λ.

Standard Young Tableaux

A standard Young tableau (SYT) of shape λ/μ is a filling of the diagram of λ/μ with the numbers $1, \ldots, n$ (where $n = \#\text{boxes}$) so that entries increase along rows and along columns.

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$$\lambda = (4, 3, 1)$$

1	2	4	8
3	5	7	
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Denote the set of all SYT of shape λ/μ by $\mathrm{SYT}(\lambda/\mu)$.

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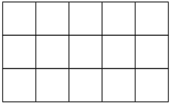
Motivating Problem:

Define a cyclic descent set for SYT of any shape λ/μ.

SYT of rectangular shapes



For $r \mid n$, let $\lambda = (r, \ldots, r) \vdash n$ be a rectangular shape.

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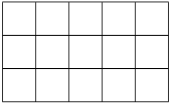

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Theorem (Rhoades '10)

For $\lambda = (r, \ldots, r)$, there exists a cyclic descent map $\mathrm{cDes} : \mathrm{SYT}(\lambda) \to 2^{[n]}$ satisfying

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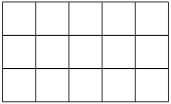

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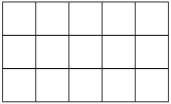

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Here, ϕ is Schützenberger's *jeu-de-taquin* promotion operator p.

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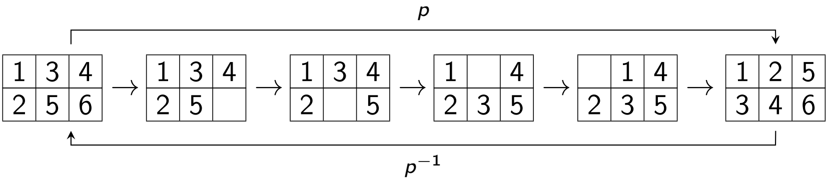

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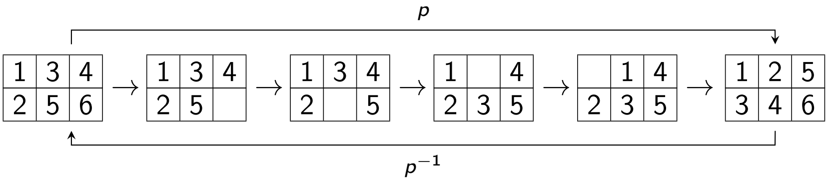

Rhoades' definition of cDes for $T \in \mathrm{SYT}(r, \ldots, r)$ declares that

$$n \in \mathrm{cDes}(T) \quad \text{iff} \quad n - 1 \in \mathrm{Des}(p^{-1}(T)).$$

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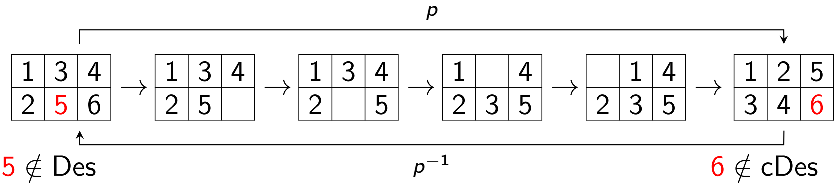

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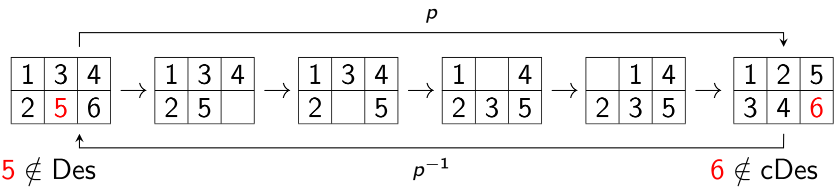

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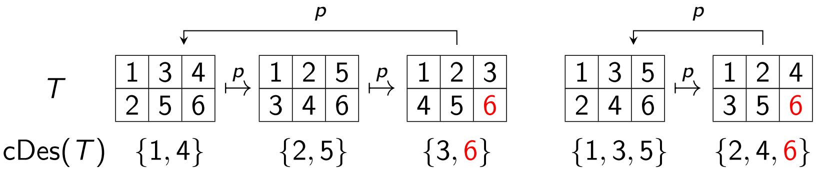
In fact, p determines a $\mathbb{Z}_n$-action. Here it is for $\lambda = (3, 3)$:



Reformulation

Definition

Given a set $\mathcal{T}$ and map $\text{Des} : \mathcal{T} \to 2^{[n-1]}$,
a cyclic descent extension is a pair (cDes, ϕ), where
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Examples

- $\mathcal{T} = \mathcal{S}_n$, with Cellini's cDes and $\phi =$ cyclic rotation.
- $\mathcal{T} = \mathrm{SYT}(r, \ldots, r)$, with Rhoades' cDes and $\phi =$ promotion.

Reformulation

Motivating Problem:

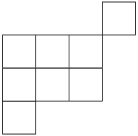
Is there a cyclic descent extension on $\mathrm{SYT}(\lambda/\mu)$?

Cyclic descents on SYT($\lambda^{\square}$)

For a partition $\lambda \vdash n-1$, let $\lambda^{\square}$ be the skew shape obtained from λ by placing a disconnected box at its upper right corner.

Example

$$(3,3,1)^{\square} =$$



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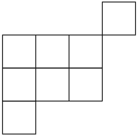
$(3,3,1)^{\Box} =$ 

Theorem (E.-Roichman '16)

For every $\lambda \vdash n-1$, there exists a cyclic descent extension on SYT($\lambda^{\Box}$).

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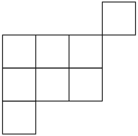

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What is the definition of cDes and ϕ in this case?

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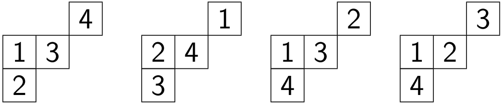

$\{1, 4\}$ $\{1, 2\}$ $\{2, 3\}$ $\{3, 4\}$

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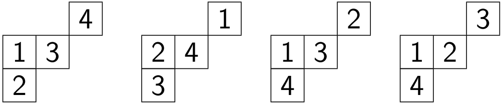

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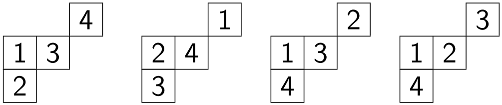

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What is $\mathrm{jdt}(T - d)$?

A *jeu-de-taquin* straightening algorithm

Given an SYT T with n boxes, let $T+k$ be obtained by adding k mod n to each entry.

$$T = \begin{array}{cccc} & & & 6 \\ 1 & 3 & 5 & \\ 2 & 4 & & \end{array} \qquad T+3 = \begin{array}{cccc} & & & 3 \\ 4 & 6 & 2 & \\ 5 & 1 & & \end{array}$$

A *jeu-de-taquin* straightening algorithm

Given an SYT T with n boxes, let $T+k$ be obtained by adding k mod n to each entry.

$$T = \begin{array}{cccc} & & & 6 \\ 1 & 3 & 5 & \\ 2 & 4 & & \end{array} \qquad T+3 = \begin{array}{cccc} & & & 3 \\ 4 & 6 & 2 & \\ 5 & 1 & & \end{array}$$

Let $\mathrm{jdt}(T+k)$ be the SYT obtained from $T+k$ by repeatedly applying the following step:

- Let i be the minimal entry for which the entry immediately above or to its left is $> i$.
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$T =$

			6
1	3	5	
2	4		

$T+3 =$

			3
4	6	2	
5	1		

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			3
4	6	2	
5	1		

$\mapsto$

			3
4	1	2	
5	6		

$\mapsto$

			3
1	4	2	
5	6		

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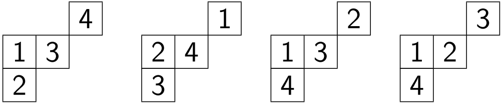
Note: promotion is just $p(T) = \mathrm{jdt}(T+1)$, $p^{-1}(T) = \mathrm{jdt}(T-1)$.

Definition of cDes on SYT($\lambda^{\square}$)



$\{1, 4\}$ $\{1, 2\}$ $\{2, 3\}$ $\{3, 4\}$

For $T \in \mathrm{SYT}(\lambda^{\square})$, define $n \in \mathrm{cDes}(T)$ iff

- n is strictly north of 1, or
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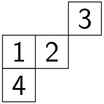
$T =$ 

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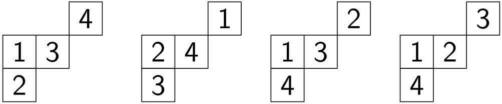

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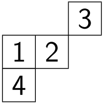
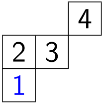
$T =$ 

$T - 3 =$ 

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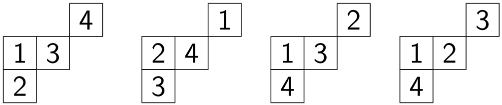

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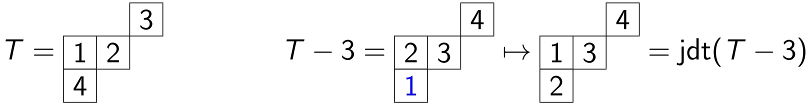
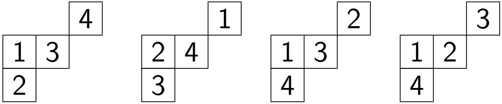


Definition of cDes on $\mathrm{SYT}(\lambda^{\square})$



$\{1, {\color{red}4}\}$ $\{1, 2\}$ $\{2, 3\}$ $\{3, {\color{red}4}\}$

For $T \in \mathrm{SYT}(\lambda^{\square})$, define $n \in \mathrm{cDes}(T)$ iff

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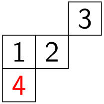
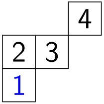
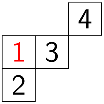
$T =$ 

$T - 3 =$  $\mapsto$  $= \mathrm{jdt}(T - 3)$

${\color{red}4} \in \mathrm{cDes}$

$4 - 3 = {\color{red}1} \in \mathrm{Des}$

The bijection ϕ that rotates cDes on SYT($\lambda^{\square}$)

The map $\phi : \mathrm{SYT}(\lambda^{\square}) \to \mathrm{SYT}(\lambda^{\square})$ given by

$$\phi(T) = \mathrm{jdt}\,(\,\mathrm{jdt}(T - d) + d + 1\,),$$

where d is the letter in the disconnected cell of T,
is a bijection such that $\mathrm{cDes}(\phi(T)) = \mathrm{cDes}(T) + 1$ for all T.

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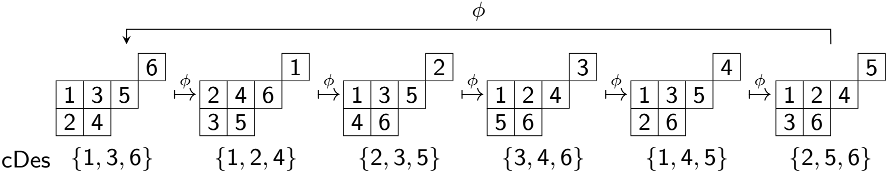
Example:



Cyclic descent extensions for other shapes

Theorem (Adin-E.-Roichman '17)

There exists a cyclic descent extension on $\mathsf{SYT}(\lambda/\mu)$ *for* λ/μ *of each of these shapes:*

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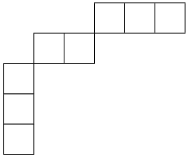

(strip)

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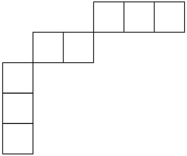
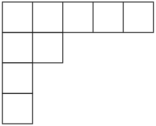


(*strip*)



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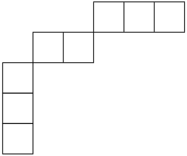

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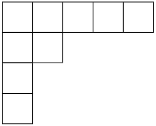

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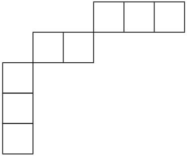

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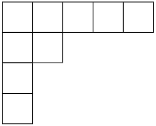

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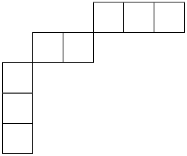
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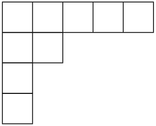
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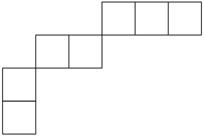
In each case we have an explicit combinatorial definition of cDes.

Definition of cDes on strips

Let λ/μ be a strip of size *n*, i.e., a shape whose components are one-row or one-column shapes.



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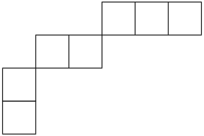

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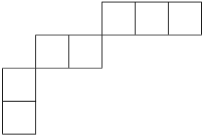

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Equivalently, $n \in \mathrm{cDes}(T)$ iff $n-1 \in \mathrm{Des}(p^{-1}(T))$.

Definition of ϕ on strips

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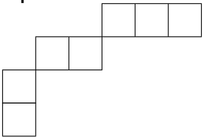
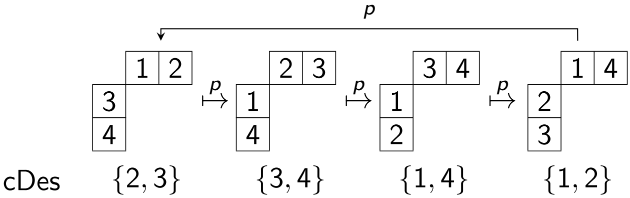
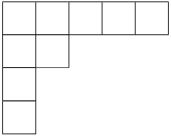


As in the case of rectangles, the promotion operator $p : T \mapsto \mathrm{jdt}(T+1)$ shifts cDes.



Definition of cDes on hooks plus a box

Let $\lambda = (n-k-2, 2, 1^k)$, where $0 \leq k \leq n-4$.



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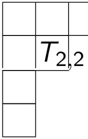

For $T \in \mathrm{SYT}(\lambda)$, let $n \in \mathrm{cDes}(T)$ iff

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Definition of cDes on hooks plus a box

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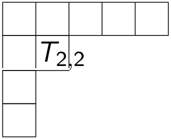

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For this shape, this definition of cDes is unique.

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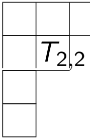

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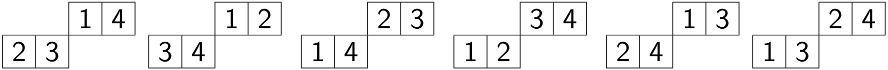
We have a complicated explicit definition of a bijection ϕ that shifts cDes. It determines a $\mathbb{Z}$-action, but not a $\mathbb{Z}_n$-action.

Non-uniqueness of cDes

For many shapes, cyclic descent completions are not unique.

Example: Let $\lambda = (4,2)/(2)$.



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		1	4
2	3		

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1	4		

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Our defintion of cDes:

$\{1\}$ $\{2\}$ $\{3\}$ $\{4\}$ $\{1,3\}$ $\{2,4\}$

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2 3	3 4	1 4	1 2	2 4	1 3

Our defintion of cDes:

$\{1\}$ $\{2\}$ $\{3\}$ $\{4\}$ $\{1,3\}$ $\{2,4\}$

Another possible definition of cDes:

$\{1\}$ $\{2,4\}$ $\{3\}$ $\{4\}$ $\{1,3\}$ $\{2\}$

Non-uniqueness of ϕ

Even for shapes where cDes in unique, different definitions of ϕ may give different orbit lengths:

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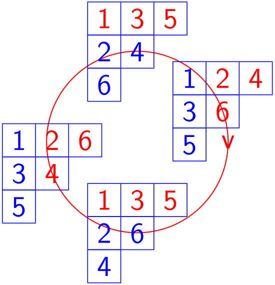

vs.

(cDes in red)

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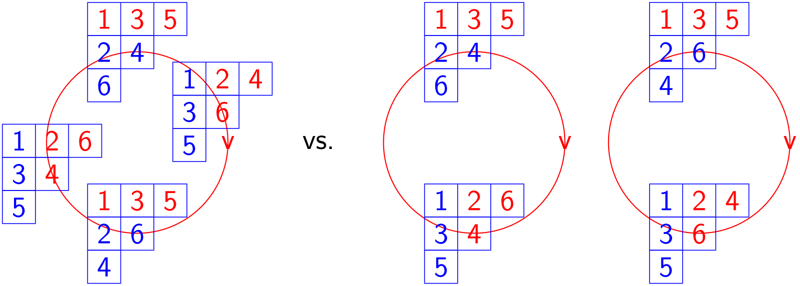
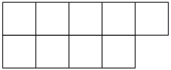


(cDes in red)

Definition of cDes on two-row straight shapes

Let $\lambda = (n-k, k)$, where $2 \leq k \leq n/2$.



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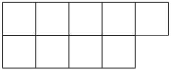

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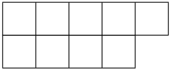

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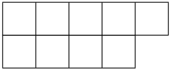

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Examples:

$$9 \in \mathrm{cDes}\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 9 \\ \hline 4 & 6 & 7 & 8 & \\ \cline{1-4}\end{array}\right)$$ because $8 = 7 + 1$, $4 > 2$ and $6 > 3$.

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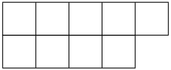

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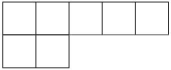
Definition of cDes on two-row straight shapes

Remarks

- When $\lambda = (n-2, 2)$, the definition of cDes viewed as a two-row shape coincides with the definition viewed as a hook plus a box.



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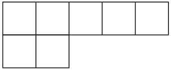

- For $\lambda = (r, r)$, the definition of cDes viewed as a two-row shape coincides with Rhoades' definition viewed as a rectangular shape.

Definition of ϕ on two-row straight shapes

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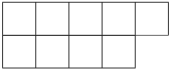

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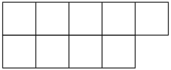

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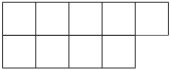

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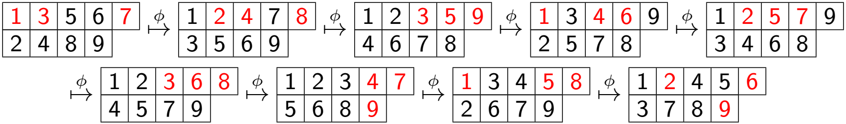
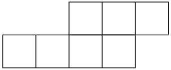


(cDes in red)

Definition of cDes on two-row skew shapes

Let $\lambda/\mu = (n-k+m, k)/(m)$ with $k \neq m+1$.



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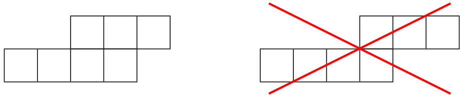

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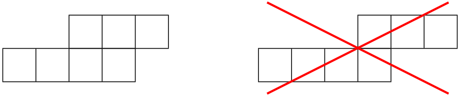

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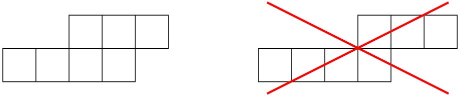

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We do not have an explicit description of ϕ in this case.

How about other shapes?

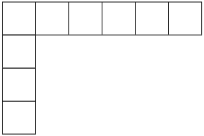
For which shapes λ/μ is there a cyclic descent extension for $\mathrm{SYT}(\lambda/\mu)$?

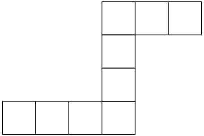
Connected ribbons

Definition

A connected skew shape λ/μ is a ribbon if it does not contain a 2×2 rectangle.

Examples:





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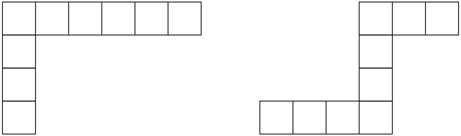

Proposition

If λ/μ is a connected ribbon, then there is no cyclic descent extension on $\mathrm{SYT}(\lambda/\mu)$.

Other shapes

After running computations for all partitions of size $n < 16$...

Conjecture (Adin-E.-Roichman '16)

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Unfortunately, it does not provide an explicit description of cDes on a given SYT.

Future work

Problem: For each non-ribbon shape λ/μ:

- Find an explicit combinatorial description of cDes on SYT(λ/μ).

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Thanks!

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Also:

Permutation Patterns
Dartmouth College
July 9-14, 2018