A unifying combinatorial approach to refined little Göllnitz and Capparelli's companion identities

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Plan of the talk

Background and Motivation

k-strict partition

$k = 3$: Capparelli's new companion

$k = 2$: New little Göllnitz

Final remarks

Background and Motivation

- Partition Identities
- Little Göllnitz family
- Capparelli family
- Boulet's four-variable generating function

Partition Identities

Original: Partitions of n with parts satisfying condition A are equinumerous with partitions of n with parts satisfying condition B. Euler's Distinct–Odd, Rogers-Ramanujan, etc. The prototype for condition A is "Gap condition", and for B "Modular condition".

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Companion: With the same or similar Modular condition, but with different Gap condition.

Little Göllnitz family

- Göllnitz (1967)
  G1: parts differing by at least 2 and no consecutive odd parts;
  M1: parts congruent to $1, 5, 6 \pmod 8$.
  G2: parts ≥ 2 differing by at least 2 and no consecutive odd parts;
  M2: parts congruent to $2, 3, 7 \pmod 8$.

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- Savage-Sills (2011)
  G1: distinct parts in which even-indexed parts are even;
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- Berkovich-Uncu (2016)
  G: distinct parts with i odd-indexed odd parts and j even-indexed odd parts;
  M: distinct parts with i parts congruent to $1 \pmod{4}$ and j parts congruent to $3 \pmod{4}$.

Capparelli family

- Capparelli (1988), related to representations of twisted affine Lie algebras.

  G1: parts $\neq 1$ with $\lambda_i - \lambda_{i+1} \begin{cases} \geq 2 & \text{if } 3 \mid \lambda_i + \lambda_{i+1}, \\ \geq 4 & \text{otherwise.} \end{cases}$

  M1: distinct parts congruent to $0, 2, 3, 4 \pmod 6$.

  G2: parts $\neq 2$ with $\lambda_i - \lambda_{i+1} \begin{cases} \geq 2 & \text{if } 3 \mid \lambda_i + \lambda_{i+1}, \\ \geq 4 & \text{otherwise.} \end{cases}$

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- Alladi-Andrews-Gordon (1995)

  G: as previous G1 or G2 with exactly i parts congruent to $1 \pmod 3$ and j parts congruent to $2 \pmod 3$;

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  G1: distinct parts with odd-indexed parts $\not\equiv 1 \pmod 3$, even-indexed parts $\not\equiv 2 \pmod 3$, and no $(3l+2, 3l+1)$ as consecutive parts;
  M1: the same as previous M1.
  G2: distinct parts with odd-indexed parts $\not\equiv 2 \pmod 3$, even-indexed parts $\not\equiv 1 \pmod 3$, and no $(3l+2, 3l+1)$ as consecutive parts;
  M2: the same as previous M2.

Boulet's four-variable generating function

a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	d
a	b	a	b	a	b	a			
c	d	c	d	c					
a	b								

$$\omega_\pi^2(a,b,c,d) = a^{10}b^9c^8d^7$$

a	b	c	a	b	c	a	b	c	a
d	e	f	d	e	f	d	e	f	d
a	b	c	a	b	c	a			
d	e	f	d	e					
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$$\omega_\pi^3(a,b,c,d,e,f) = a^8b^6c^5d^6e^5f^4$$

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Boulet (2006):

$$\Phi(a,b,c,d) := \sum_{\pi\in\mathcal{P}} \omega_\pi^2(a,b,c,d) = \frac{(-a,-abc;Q)_\infty}{(Q,ab,ac;Q)_\infty}, \quad Q := abcd, \qquad (1)$$

$$\Psi(a,b,c,d) := \sum_{\pi\in\mathcal{D}} \omega_\pi^2(a,b,c,d) = \frac{(-a,-abc;Q)_\infty}{(ab;Q)_\infty}, \quad Q := abcd. \qquad (2)$$

Where $\mathcal{P}$ (resp. $\mathcal{D}$) denotes the set of ordinary (resp. strict) partitions.

$$(a;q)_0 := 1, \quad (a;q)_k := \prod_{i=1}^{k}(1-aq^{i-1}), \quad k\in\mathbb{N}^*\cup\{\infty\}$$

$$(a_1,a_2,\ldots,a_m;q)_s := (a_1;q)_s(a_2;q)_s\ldots(a_m;q)_s.$$

Boulet's four-variable generating function

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$$\Psi(q,q,q,q) = (-q;q)_\infty,$$

$$\Psi(xt, x/t, yz, y/z) = \frac{(-xt,-x^2yz;x^2y^2)_\infty}{(x^2;x^2y^2)_\infty},$$

Savage-Sills: $x = y = q$, $t = 0$ or $z = 0$,

Berkovich-Uncu: $x = y = q$.

k-strict partition

- definition
- ω^k-weight and a key decomposition
- weighted generating function

Definition

For $k \geq 1$, we call a partition π "k-strict" if for any integers r_1, r_2, with $1 \leq r_1 \leq r_2 \leq k-1$,

$$mk + r_1 \quad \text{and} \quad mk + r_2 \quad \text{do not appear together as parts in } \pi. \qquad (\star)$$

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- 1-strict: ordinary partition.
- 2-strict: partitions with odd parts distinct

$$\sum_{n=0}^{\infty} \mathrm{pod}(n) q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} = \Big(\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+n} \Big)^{-1}.$$

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For example, there are nine 3-strict partitions of 10:

$$(10), (9,1), (8,2), (7,3), (6,4), (6,3,1), (5,3,2), (4,3,3), (3,3,3,1).$$

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Note that $\mathcal{D} \cap \mathcal{S}^1 = \mathcal{D} \cap \mathcal{S}^2 = \mathcal{D}$, but $\mathcal{D} \cap \mathcal{S}^k \neq \mathcal{D}$ for $k \geq 3$. Denote $\mathcal{DS}^k = \mathcal{D} \cap \mathcal{S}^k$ and $\mathcal{E}^k$ as the set of partitions into parts as mk each appearing an even number of times.

Definition (ω^k-weight)

Given a partition π and $k \geq 1$, we label the cells in the odd-indexed (resp. even-indexed) rows of π's diagram cyclically from left to right with $a_1, a_2, \ldots, a_k$ (resp. $b_1, b_2, \ldots, b_k$) and define the product of all the labels on the diagram as its ω^k-weight, denoted by $\omega_\pi^k((a_i),(b_i))$.

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Previous figures are two examples when $k = 2, \omega^2(a, b, c, d)$ and $k = 3, \omega^3(a, b, c, d, e, f)$.

Theorem

For any $k \geq 1$, the map $\psi_k : \pi \longmapsto (\pi^1, \pi^2)$ is a weight-preserving bijection from $\mathcal{S}^k$ to $\mathcal{DS}^k \times \mathcal{E}^k$ such that $\ell(\pi) = \ell(\pi^1) + \ell(\pi^2)$ and

$$\omega^k_\pi((a_i), (b_i)) = \omega^k_{\pi^1}((a_i), (b_i))\, \omega^k_{\pi^2}((a_i), (b_i)), \tag{3}$$

where $\ell(\pi)$ stands for the number of parts of π.

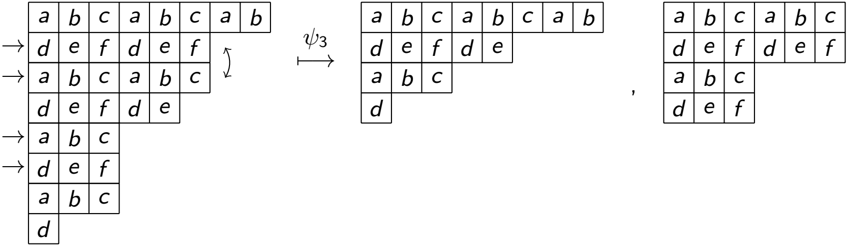

Fig.: Decomposition of $\pi = (8,6,6,5,3,3,3,1)$ into (π^1, π^2) with ω^3-labels

Weighted generating functions

Theorem

For any integer $k \geq 1$, *let* $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$ *be* $2k$ *commutable variables, and let*

$$z_k = a_1 \ldots a_k, \quad w_k = a_1 b_1 \ldots a_k b_k,$$
$$x_k = a_1 + a_1 a_2 + \cdots + a_1 \ldots a_{k-1},$$
$$y_k = z_k (b_1 + b_1 b_2 + \cdots + b_1 \ldots b_{k-1}).$$

Then we have

$$\sum_{\pi \in \mathcal{E}^k} \omega_\pi^k((a_i),(b_i)) = \frac{1}{(w_k; w_k)_\infty}, \quad (4)$$

$$\sum_{\pi \in \mathcal{S}^k} \omega_\pi^k((a_i),(b_i)) = \frac{(-x_k, -y_k; w_k)_\infty}{(z_k, w_k; w_k)_\infty}, \quad (5)$$

$$\sum_{\pi \in \mathcal{DS}^k} \omega_\pi^k((a_i),(b_i)) = \frac{(-x_k, -y_k; w_k)_\infty}{(z_k; w_k)_\infty}. \quad (6)$$

Four types of blocks when read by columns

$$\begin{array}{ccc} a_1 \dots & a_\ell & \dots a_k \\ b_1 \dots & b_\ell & \dots b_k \\ \vdots & \vdots & \vdots \\ b_1 \dots & b_\ell & \dots b_k \\ a_1 \dots & a_\ell & \end{array} \qquad \begin{array}{ccc} a_1 \dots & a_\ell & \dots a_k \\ b_1 \dots & b_\ell & \dots b_k \\ \vdots & \vdots & \vdots \\ a_1 \dots & a_\ell & \dots a_k \\ b_1 \dots & b_\ell & \end{array}$$

I II

$$\begin{array}{ccc} a_1 \dots & a_\ell & \dots a_k \\ b_1 \dots & b_\ell & \dots b_k \\ \vdots & \vdots & \vdots \\ b_1 \dots & b_\ell & \dots b_k \\ a_1 \dots & a_\ell & \dots a_k \end{array} \qquad \begin{array}{ccc} a_1 \dots & a_\ell & \dots a_k \\ b_1 \dots & b_\ell & \dots b_k \\ \vdots & \vdots & \vdots \\ a_1 \dots & a_\ell & \dots a_k \\ b_1 \dots & b_\ell & \dots b_k \end{array}$$

III IV

Fig.: Four possible types of vertical blocks where $1 \leq \ell \leq k - 1$.

odd-indexed/even-indexed

We use $o_l(\pi)$ (resp. $e_l(\pi)$) to denote the number of odd-indexed (resp. even-indexed) parts that are $\equiv l \pmod{k}$. Denote by π_o (resp. π_e) the partition consisting of the odd-indexed (resp. even-indexed) parts of π.

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Theorem

For any integer $k \geq 1$, *we have*

$$\sum_{\pi \in \mathcal{DS}^k} x^{|\pi_o|} y^{|\pi_e|} \prod_{l=1}^{k-1} u_l^{o_l(\pi)} v_l^{e_l(\pi)} = \frac{\left(-\sum_{l=1}^{k-1} u_l x^l, -x^k \sum_{l=1}^{k-1} v_l y^l; x^k y^k\right)_\infty}{(x^k; x^k y^k)_\infty}. \quad (7)$$

Proof.

In (6), simply take $a_l = u_l x / u_{l-1}, b_l = v_l y / v_{l-1}$, for $l = 1, \ldots, k$, where $u_0 = u_k = v_0 = v_k = 1$. □

A new companion

$$\sum_{\pi\in\mathcal{DS}^3} x^{|\pi_o|} y^{|\pi_e|} s^{o_1(\pi)} t^{o_2(\pi)} u^{e_1(\pi)} v^{e_2(\pi)} = \frac{(-sx - tx^2, -ux^3y - vx^3y^2; x^3y^3)_\infty}{(x^3; x^3y^3)_\infty}.$$

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$(\quad - tx^2, -ux^3y \quad)$ and $(-sx \quad , \quad - vx^3y^2)$

Theorem (Berkovich-Uncu, refined)

For integers $n, i, j \geq 0, m \in \{1, 2\}$, *the number of partitions enumerated by* $A_m(n)$ *that have exactly* i *parts* $\equiv 2 \pmod 3$ *and* j *parts* $\equiv 1 \pmod 3$ *equals the number of partitions enumerated by* $C_m(n)$ *that have exactly* i *parts* $\equiv 3m - 1 \pmod 6$ *and* j *parts* $\equiv 3m + 1 \pmod 6$.

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$(-sx \quad , -ux^3y \quad)$ and $(\quad - tx^2, \quad - vx^3y^2)$

Theorem (F.-Zeng)

For integers $n, i, j \geq 0, m \in \{1, 2\}$, let $D^I_m(i, j, n)$ be the number of partitions of n into distinct parts $\not\equiv -m \pmod 3$ that have exactly i odd-indexed parts $\equiv m \pmod 3$ and j even-indexed parts $\equiv m \pmod 3$, and $D^{II}_m(i, j, n)$ the number of partitions of n into distinct parts $\not\equiv -m \pmod 3$ that have exactly i parts $\equiv m \pmod 6$ and j parts $\equiv m + 3 \pmod 6$. Then

$$D^I_m(i, j, n) = D^{II}_m(i, j, n).$$

Bounded case

$$\sum_{\pi\in\mathcal{E}^3_{N,\infty}} \omega^3_\pi = \frac{1}{(R;R)_{\lfloor N/3\rfloor}}, \quad R = abcdef,$$

$$S^3_{3N+\mu} := S^3_{3N+\mu}(a,b,c,d,e,f) := \sum_{\pi\in\mathcal{S}^3_{3N+\mu,\infty}} \omega^3_\pi,$$

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$$S^3_{3N} = \sum_T R^{\binom{t_1}{2}+\binom{t_2}{2}} F(T),$$

$$S^3_{3N+1} = S^3_{3N}(a,b,c,d,e,f) + a(abc)^N S^3_{3N}(d,e,f,a,b,c),$$

$$S^3_{3N+2} = (1+a+ab)\sum_T R^{\binom{t_1+1}{2}+\binom{t_2}{2}} F(T),$$

$$DS^3_{3N+\mu} = (R;R)_N S^3_{3N+\mu} \quad \text{for} \quad \mu\in\{0,1,2\},$$

where the summation $\sum_T$ is over all quadruples $T := (t_1,t_2,t_3,t_4)\in\mathbb{N}^4$ such that

$$\sum_{j=1}^4 t_j = N, \text{ and } F(T) := \frac{(a+ab)^{t_1}(abcd+abcde)^{t_2}(abc)^{t_3}}{(R;R)_{t_1}(R;R)_{t_2}(R;R)_{t_3}(R;R)_{t_4}}.$$

Idea of the proof

The largest part $\leq 3N$, when viewed vertically, means the total number of blocks with four types is bounded by N. Alternatively, we can think of this number is exactly N by filling in empty blocks (of type IV) if necessary. Hence the constraint on the summation $t_1 + t_2 + t_3 + t_4 = N$.

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Next we invoke the following identity due to Euler, for dealing with type I and II blocks.

$$\sum_{t_1=0}^{\infty} \frac{(a+ab)^{t_1} R^{\binom{t_1}{2}}}{(R;R)_{t_1}} = (-a-ab;R)_{\infty}.$$

$a = sx,\ b = tx/s,\ c = x/t,\ d = uy,\ e = vy/u,\ f = y/v,\ x = y = q.$

$$P_{3N+\mu}(s,t,u,v;q) := \sum_{\pi \in \mathcal{DS}^3_{3N+\mu}} s^{o_1(\pi)} t^{o_2(\pi)} u^{e_1(\pi)} v^{e_2(\pi)} q^{|\pi|},$$

For $N \geq 0\,,\ \mu \in \{0,2\}$ we have:

$$P_{3N+\mu}(0,t,u,0;q) = \left(1 + \frac{\mu}{2} tq^2\right) \sum \begin{bmatrix} N \\ i,j,k,l \end{bmatrix}_{q^6} t^i u^j q^{3i^2+(3\mu-1)i+3j^2+j+3k}, \quad (8)$$

$$P_{3N+\mu}(s,0,0,v;q) = \left(1 + \frac{\mu}{2} sq\right) \sum \begin{bmatrix} N \\ i,j,k,l \end{bmatrix}_{q^6} s^i v^j q^{3i^2+(3\mu-2)i+3j^2+2j+3k}, \quad (9)$$

$$P_{3N+\mu}(s,0,u,0;q) = \left(1 + \frac{\mu}{2} sq\right) \sum \begin{bmatrix} N \\ i,j,k,l \end{bmatrix}_{q^6} s^i u^j q^{3i^2+(3\mu-2)i+3j^2+j+3k}, \quad (10)$$

$$P_{3N+\mu}(0,t,0,v;q) = \left(1 + \frac{\mu}{2} tq^2\right) \sum \begin{bmatrix} N \\ i,j,k,l \end{bmatrix}_{q^6} t^i v^j q^{3i^2+(3\mu-1)i+3j^2+2j+3k}, \quad (11)$$

and for $\mu = 1$,

$$P_{3N+1}(0,t,u,0;q) = \sum \begin{bmatrix} N \\ i,j,k,l \end{bmatrix}_{q^6} t^i u^j q^{3i^2-i+3j^2+j+3k}, \tag{12}$$

$$P_{3N+1}(s,0,0,v;q) = \sum \begin{bmatrix} N \\ i,j,k,l \end{bmatrix}_{q^6} q^{3i^2-2i+3j^2+2j+3k} \left(s^i v^j + s^{j+1} v^i q^{i-j+3N+1}\right), \tag{13}$$

$$P_{3N+1}(s,0,u,0;q) = \sum \begin{bmatrix} N \\ i,j,k,l \end{bmatrix}_{q^6} q^{3i^2-2i+3j^2+j+3k} \left(s^i u^j + s^{j+1} u^i q^{3N+1}\right), \tag{14}$$

$$P_{3N+1}(0,t,0,v;q) = \sum \begin{bmatrix} N \\ i,j,k,l \end{bmatrix}_{q^6} t^i v^j q^{3i^2-i+3j^2+2j+3k}. \tag{15}$$

Definition

For integers $N, n, i, j \geq 0, m \in \{1, 2\}$ *, let* $D^{I}_{m,3N}(i, j, n)$ *be the number of partitions of* n *into distinct parts such that*

i. *each part* $\not\equiv -m \pmod 3$ *;*
ii. *each part* $\leq 3N$ *;*
iii. *there are exactly* i *odd-indexed parts* $\equiv m \pmod 3$ *;*
iv. *there are exactly* j *even-indexed parts* $\equiv m \pmod 3$ *.*

Let $D^{II}_{m,3N}(i, j, n)$ *be the number of partitions of* n *into distinct parts such that*

i. *each part* $\not\equiv -m \pmod 3$ *;*
ii. *there are exactly* i *parts* $\equiv m \pmod 6$ *and these parts are all* $\leq 6N + m - 6$ *;*
iii. *there are exactly* j *parts* $\equiv m + 3 \pmod 6$ *and these parts are all* $\leq 6(N - i) + m - 3$ *;*
iv. *all parts* $\equiv 0 \pmod 3$ *are* $\leq 3(N - i - j)$ *.*

Theorem

For integers $N, n, i, j \geq 0, m \in \{1, 2\}$ *,*

$$D^{I}_{m,3N}(i, j, n) = D^{II}_{m,3N}(i, j, n).$$

The role of conjugation

Proposition

For N and M being any positive integers or ∞, the operation of conjugation, denoted as τ, is a bijection from $\mathcal{P}_{N,M}$ to $\mathcal{P}_{M,N}$, such that for any $\pi \in \mathcal{P}_{N,M}$, we have

$$\omega_\pi^2(a,b,c,d) = \omega_{\tau(\pi)}^2(a,c,b,d).$$

In terms of generating function, we have

$$\Phi_{N,M}(a,b,c,d) = \Phi_{M,N}(a,c,b,d). \tag{16}$$

Two expressions for bounded case of $k = 2$

$$\Psi_{2N+\nu,\infty}(a,b,c,d) = \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_Q (-a;Q)_{N-i+\nu}(-c;Q)_i(ab)^i, \tag{17}$$

$$\Phi_{2N+\nu,\infty}(a,b,c,d) = \frac{1}{(ac;Q)_{N+\nu}} \sum_{i=0}^{N} \frac{(-a;Q)_{N-i+\nu}(-c;Q)_i(ab)^i}{(Q;Q)_{N-i}(Q;Q)_i}, \tag{18}$$

$$\Psi_{2N+\nu,\infty}(a,b,c,d) = \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_Q (-a;Q)_{i+\nu}(-abc;Q)_i \frac{(ac;Q)_{N+\nu}}{(ac;Q)_{i+\nu}}(ab)^{N-i}, \tag{19}$$

$$\Phi_{2N+\nu,\infty}(a,b,c,d) = \sum_{i=0}^{N} \frac{(-a;Q)_{i+\nu}(-abc;Q)_i(ab)^{N-i}}{(Q;Q)_i(ac;Q)_{i+\nu}(Q;Q)_{N-i}}. \tag{20}$$

Doubly-bounded case

Theorem

For N, M being non-negative integers, $\nu, \mu = 0$ or 1 such that $N + \nu \geq 1$, we have the following expansions:

$$\Phi_{2N+\nu,2M+\mu}(a,b,c,d) \tag{21}$$

$$= \delta_{0\mu}(ac)^M \begin{bmatrix} N+M+\nu-1 \\ M \end{bmatrix}_Q + \sum_{k=0}^{M+\mu-1} (ac)^k \begin{bmatrix} N+k+\nu-1 \\ k \end{bmatrix}_Q$$

$$\times \sum_{\substack{m_1,m_2,m_3,m_4 \geq 0 \\ m_1+m_2+m_3+m_4=N}}^{N} \begin{bmatrix} M-k+m_4 \\ m_4 \end{bmatrix}_Q \begin{bmatrix} M-k+\mu-\nu \\ m_1 \end{bmatrix}_Q (1+a\nu)a^{m_1} Q^{\binom{m_1+\nu}{2}}$$

$$\times \begin{bmatrix} M-k+\mu-1+m_2 \\ m_2 \end{bmatrix}_Q (ab)^{m_2} \begin{bmatrix} M-k \\ m_3 \end{bmatrix}_Q (abc)^{m_3} Q^{\binom{m_3}{2}},$$

$$\Psi_{N,M}(a,b,c,d) \tag{22}$$

$$= \sum_{m=0}^{\lfloor M/2 \rfloor} (-1)^m \sum_{k=0}^{m} \begin{bmatrix} \lfloor N/2 \rfloor \\ k \end{bmatrix}_Q \begin{bmatrix} \lceil N/2 \rceil \\ m-k \end{bmatrix}_Q (ac)^{m-k} Q^{k(k+1-m)+\binom{m}{2}} \Phi_{N,M-2m}(a,b,c,d).$$

Recap and final remarks

- New little Göllnitz and new companion of Capparelli fit nicely into the framework of k-strict partitions.
- Further companions present themselves naturally.
- This combinatorial approach is amenable to bounded cases as well.
- Shed some lights on the connection between two different expansions for the same weighted generating functions.

Recap and final remarks

- New little Göllnitz and new companion of Capparelli fit nicely into the framework of k-strict partitions.
- Further companions present themselves naturally.
- This combinatorial approach is amenable to bounded cases as well.
- Shed some lights on the connection between two different expansions for the same weighted generating functions.

- Do the new companion identities possess Lie theoretical implications as the original Capparelli's identities?
- Study k-strict partitions ($k \geq 3$) for their own sake. For instance, it appears the sequence enumerating 3-strict partitions, $(1, 1, 1, 1, 2, 3, 4, 5, \ldots)$ is not registered on OEIS.

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Thank You! Any comments/questions?