How Not to Define Desingularization

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Joint work with Moulay A. Barkatou (University of Limoges, XLIM)

Formal Methods
in Systems Engineering

"Pólya said: 'First guess, then prove.'" (A. Bostan, 2017)

Recurrences

$F(z+2) - F(z+1) - F(z) = 0$ $\quad F(0) = 0, F(1) = 1$

$S(z+1) - (z+1)S(z) = 0$ $\quad S(0) = 1$

$(z+2)H(z+2) - (2z+3)H(z+1) + (z+1)H(z) = 0$ $\quad H(0) = 1, H(1) = \frac{3}{2}, H(2) = \frac{11}{6}$

Holonomic Sequences

These are holonomic:

- Fibonacci numbers
- Factorials
- Harmonic numbers
- Catalan numbers
- Sequences given by polynomial / rational functions
- Sums, products, (certain) subsequences of these

These are not:

- Sequence of prime numbers
- Bernoulli numbers
- Partition numbers

From Single Equations to Systems

$$
\begin{aligned}
zA(z+1) - 3(z+2)A(z) - (z+6)B(z) &= 0\\
zB(z+1) - (-2(z+2)+1)A(z) + 3B(z) &= 0
\end{aligned}
$$

$$
\begin{pmatrix} A(z+1) \\ B(z+1) \end{pmatrix} = \begin{pmatrix} \frac{3(z+2)}{z} & \frac{z+6}{z} \\ \frac{-2(z+2)+1}{z} & \frac{-3}{z} \end{pmatrix} \begin{pmatrix} A(z) \\ B(z) \end{pmatrix}
$$

From Single Equations to Systems

$$(z+2)H(z+2) - (2z+3)H(z+1) + (z+1)H(z) = 0$$

$$\begin{pmatrix} H(z+1) \\ H(z+2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-(z+1)}{z+2} & \frac{2z+3}{z+2} \end{pmatrix} \begin{pmatrix} H(z) \\ H(z+1) \end{pmatrix}$$

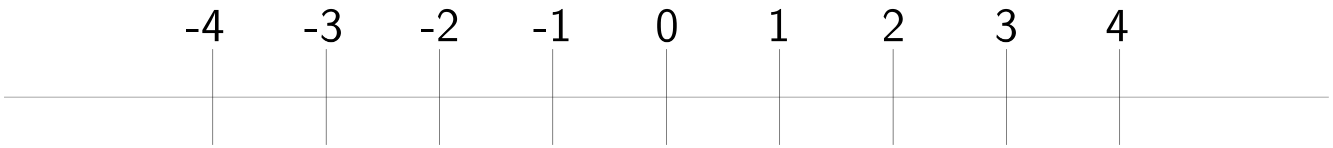
What is a Difference System

Definition: Linear Difference System

$$Y(z+1) = A(z)Y(z)$$

Y : d-dimensional column vector

A : invertible matrix of size d$\times$d with entries in $\mathbb{K}(z), \mathbb{K} \leq \mathbb{C}$



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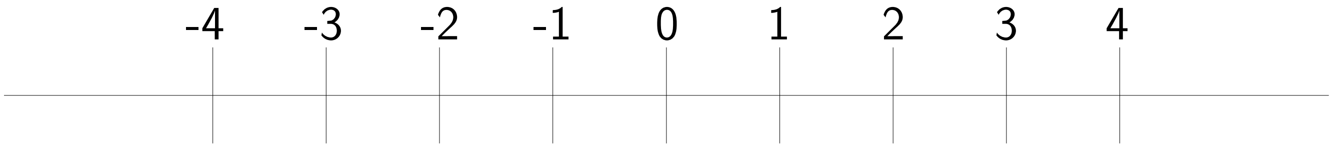

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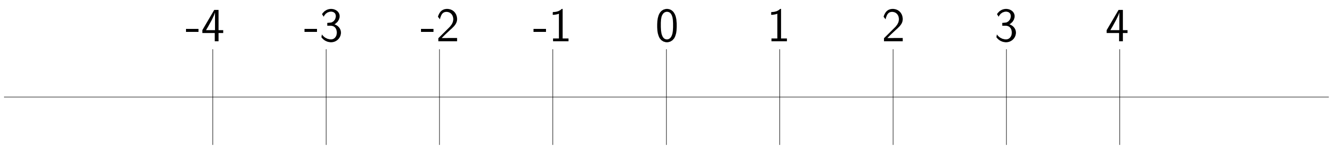

What is a Solution?

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Meromorphic Functions

$f : \mathbb{C} \setminus S \to \mathbb{C}^d$, where S is a set of isolated points.

Number Sequences

$s_z : \mathbb{Z} \to \mathbb{C}^d$, $s_{z+1} = A(z)s_z$ for all z where $A(z)$ is defined.

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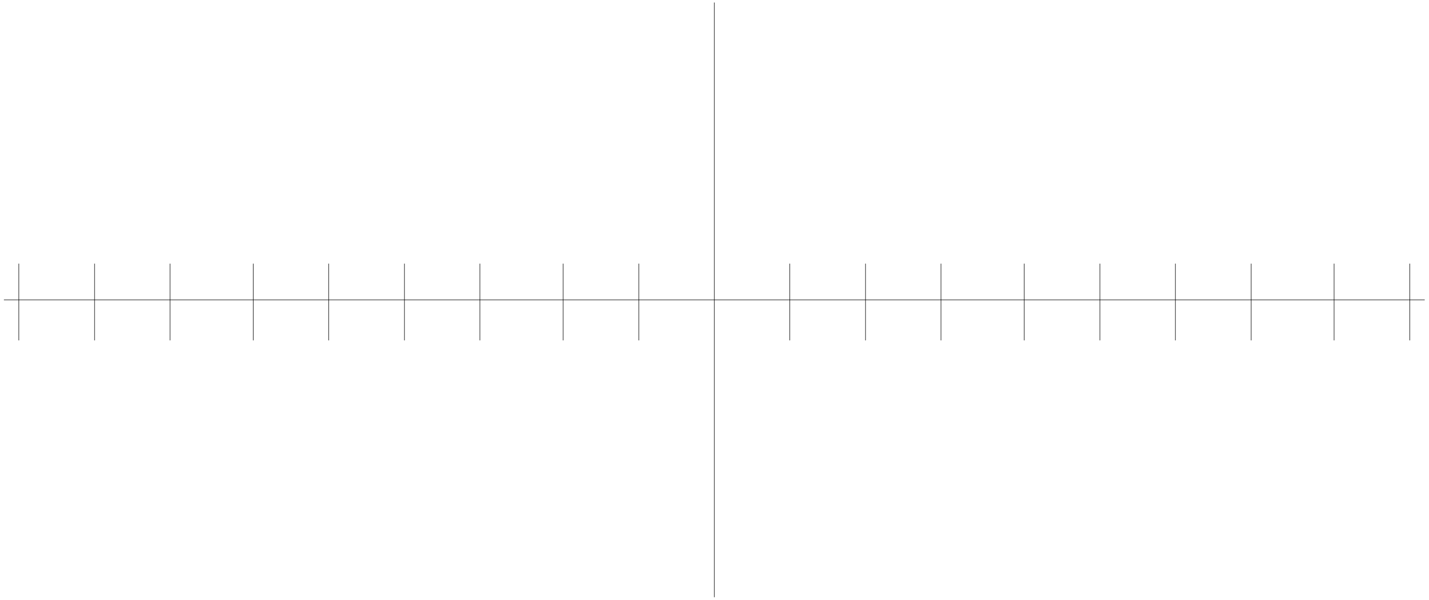
Theorem

For any complex number q with $-\mathrm{Re}(q)$ large enough, there exist d linearly independent meromorphic solutions which are holomorphic for $-\mathrm{Re}(z)$ large enough and the associated fundamental matrix F satisfies $F(q) = I_d$. (Ramis 1987, Barkatou 1989, Immink 1999)

Singularities in Meromorphic Solutions



$$A(z+1) = \frac{1}{z+(3-2i)} A(z)$$

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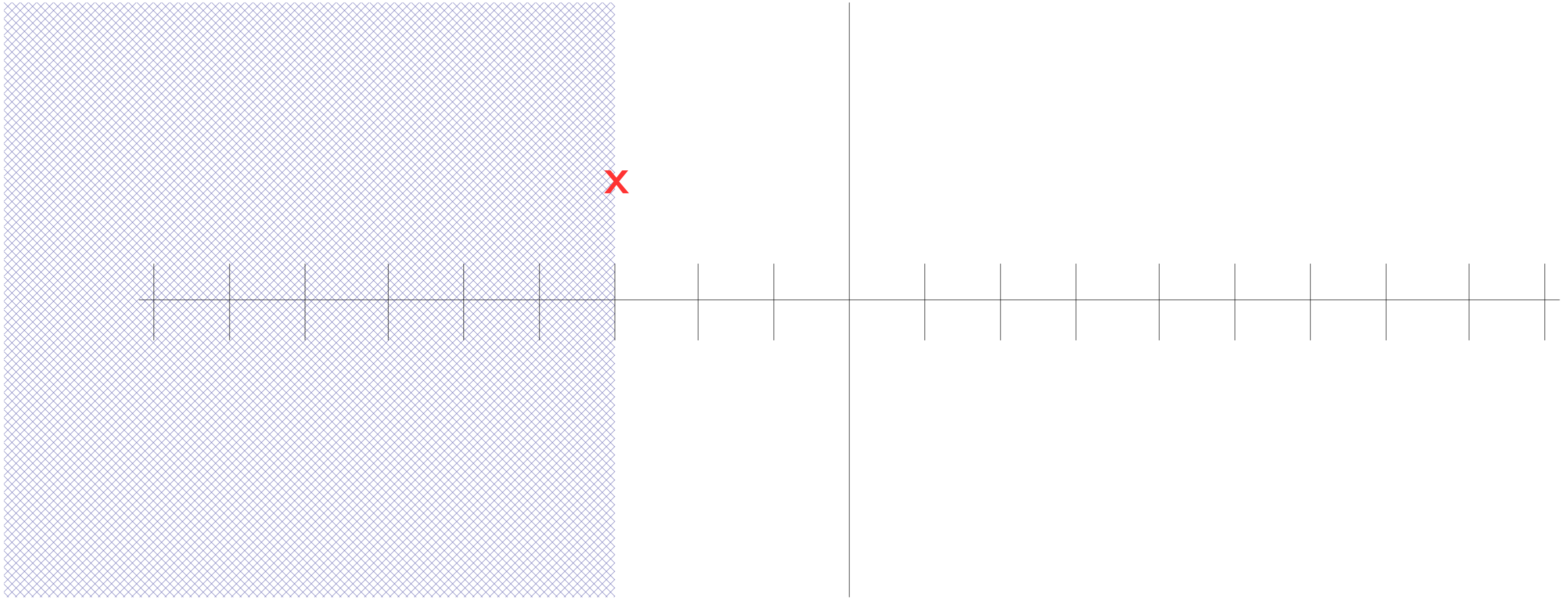

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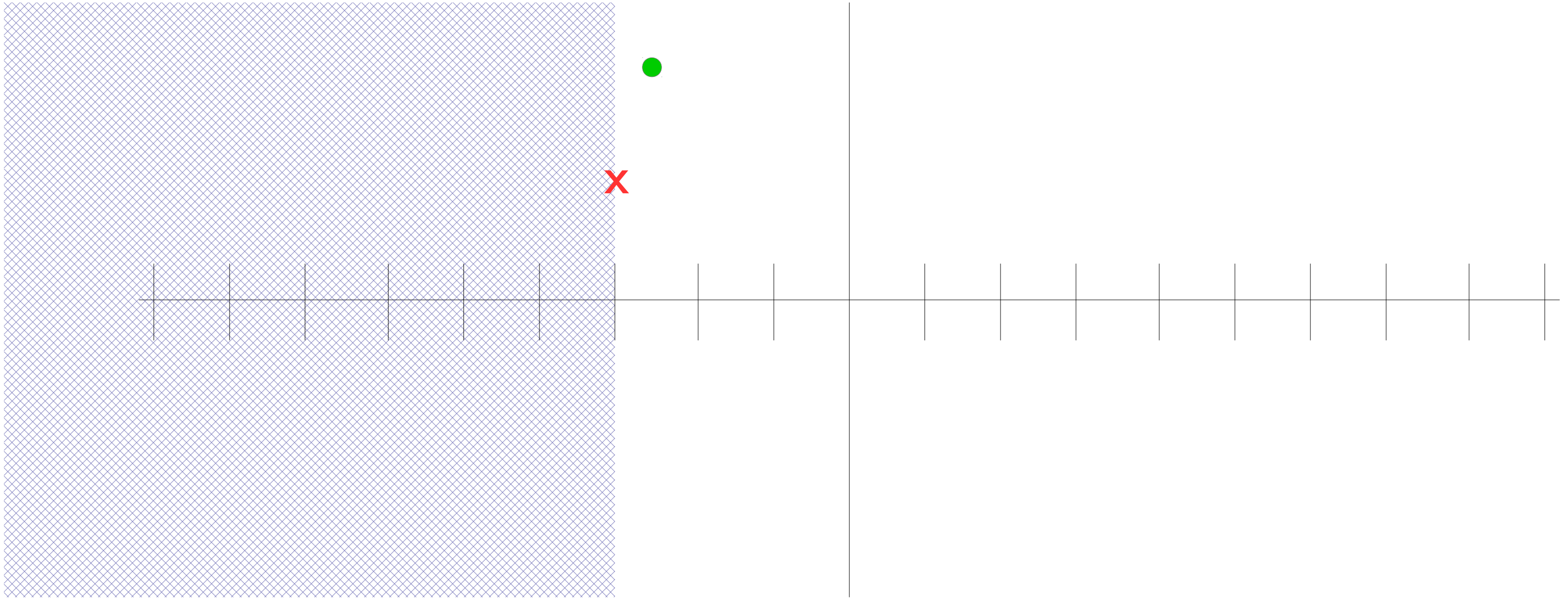

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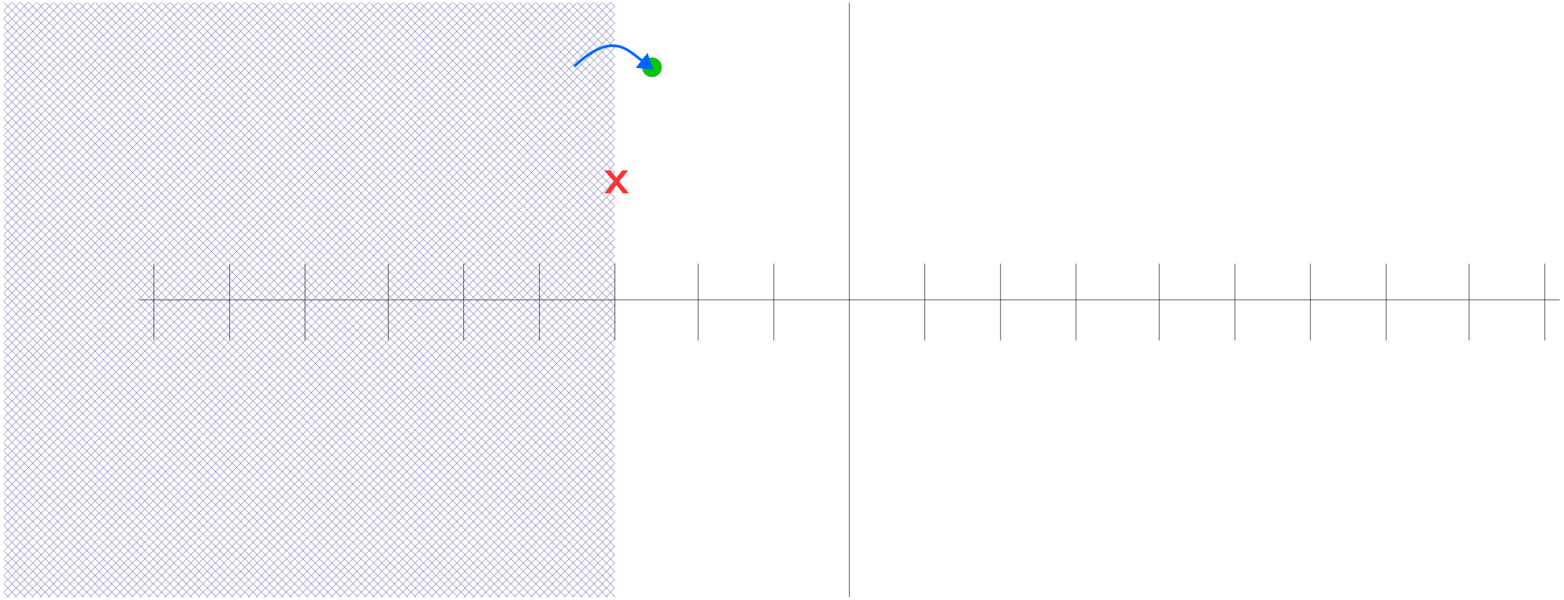

$$\color{red}{A(z+1)} = \frac{1}{z+(3-2i)}\color{red}{A(z)}$$

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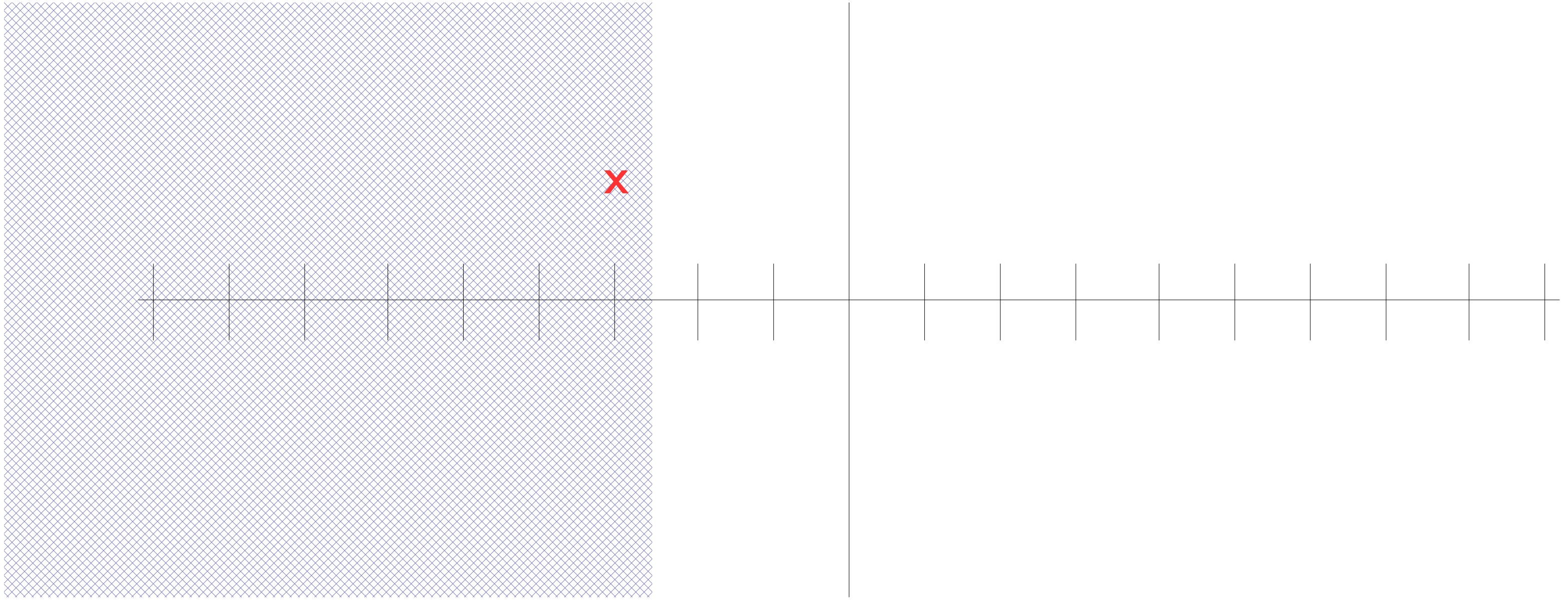

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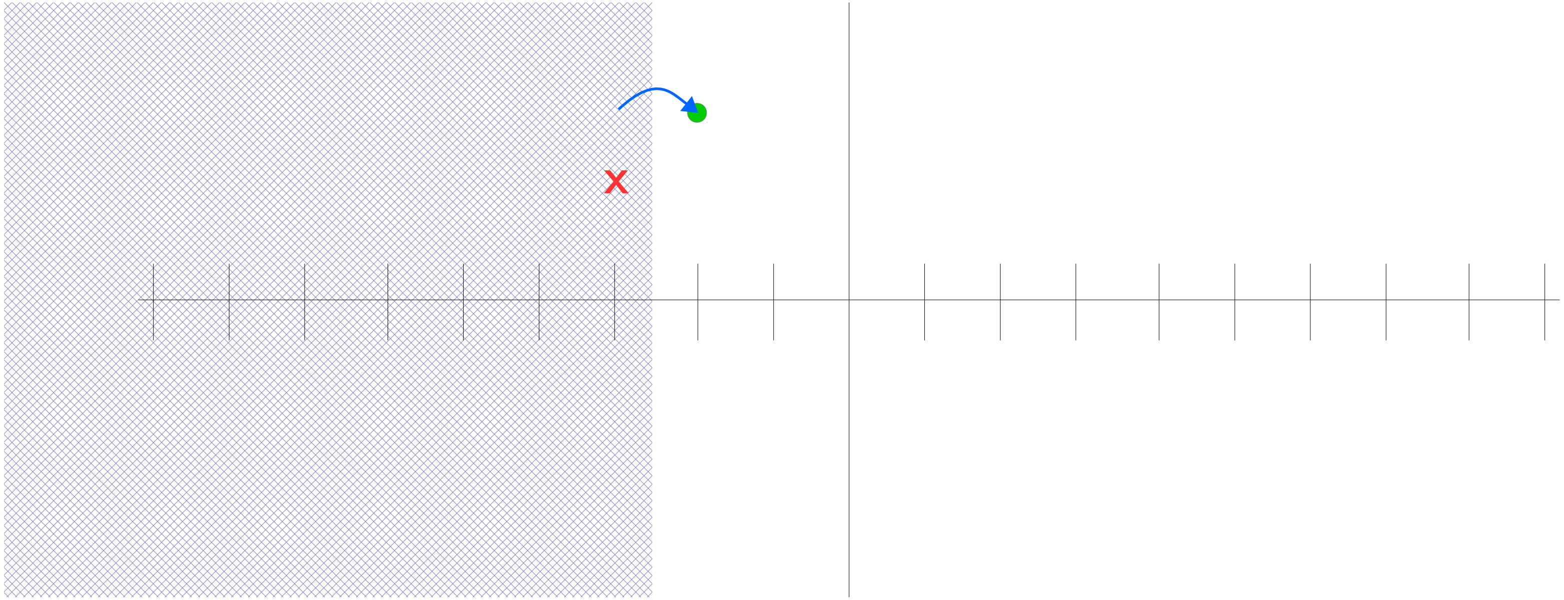

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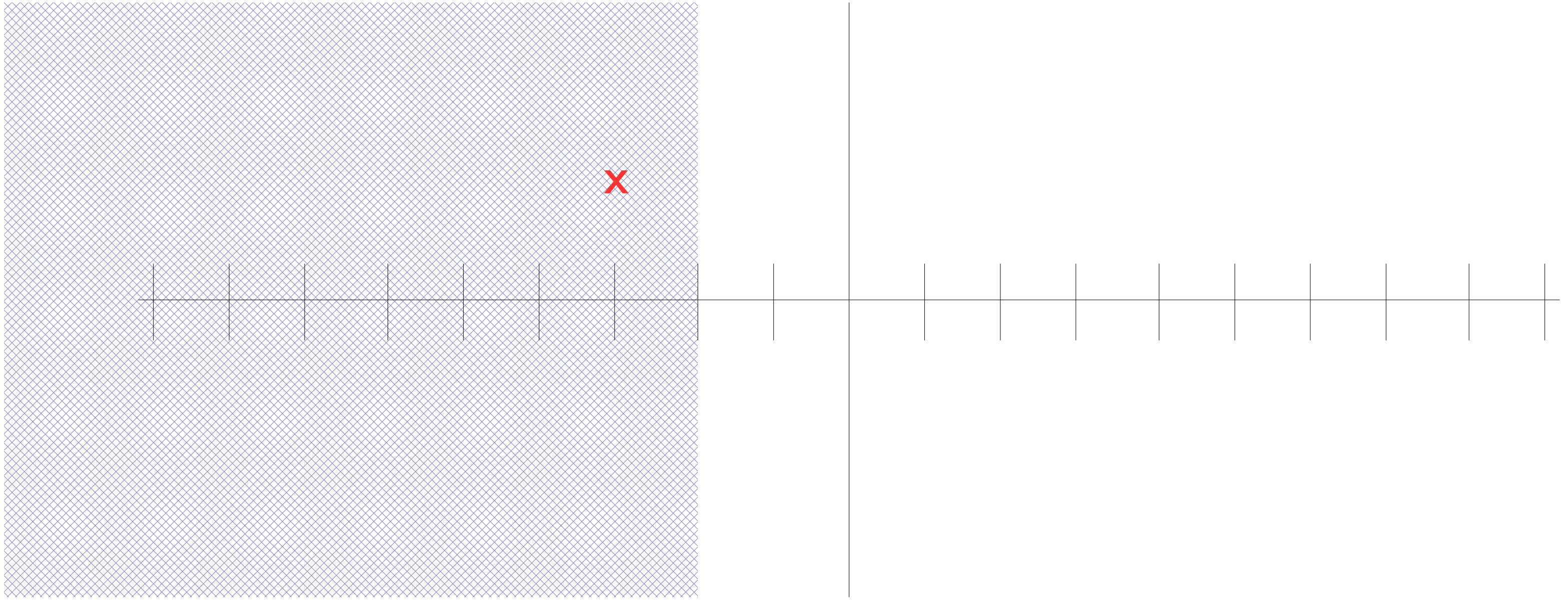

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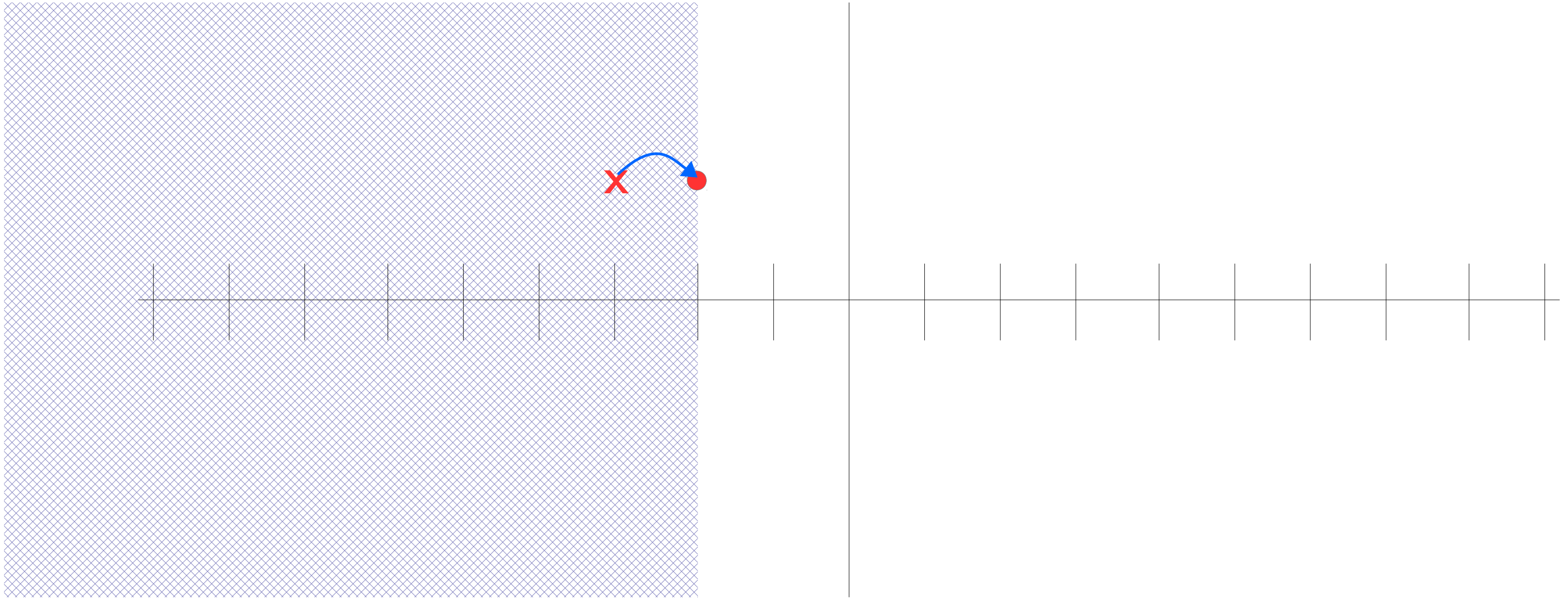

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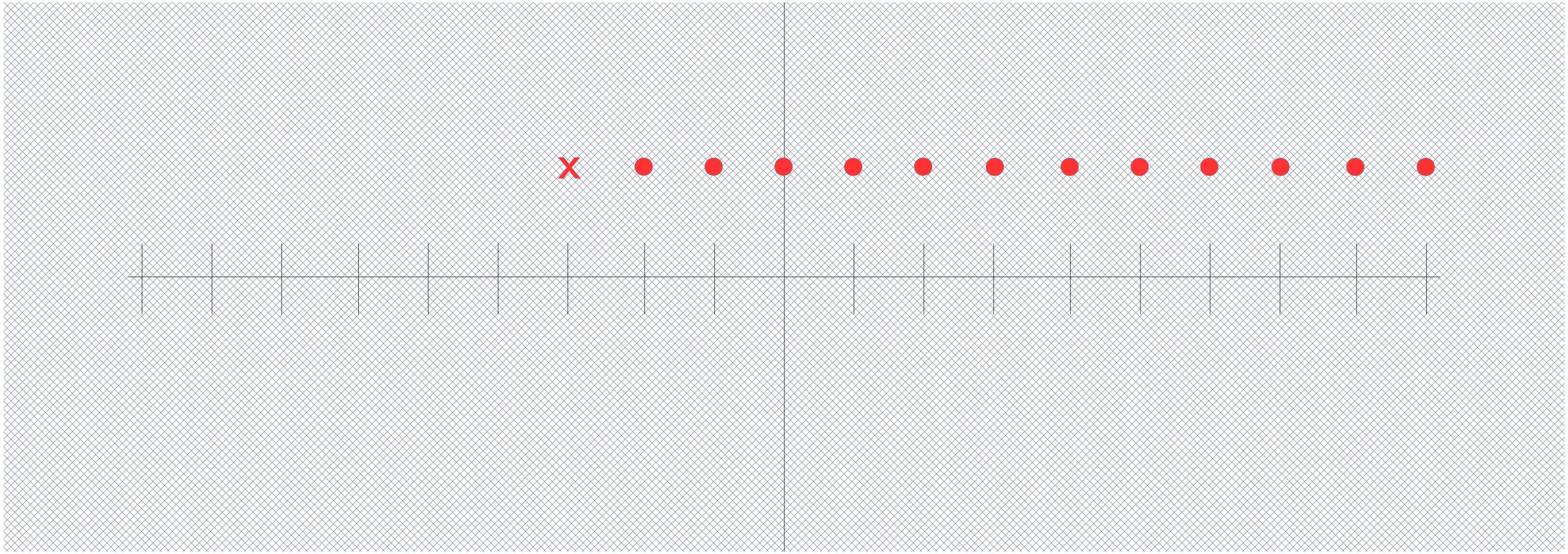

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holomorphic

Question: Which poles in A correspond to poles in solutions?

Apparent Singularities

Definition

A pole of $A(z)$ is called an apparent singularity, if any solution of $[A]$ which is holomorphic in some left half-plane can be analytically continued to a meromorphic solution which is holomorphic at each point of $\zeta + \mathbb{N}^*$.

Previous and Related Work

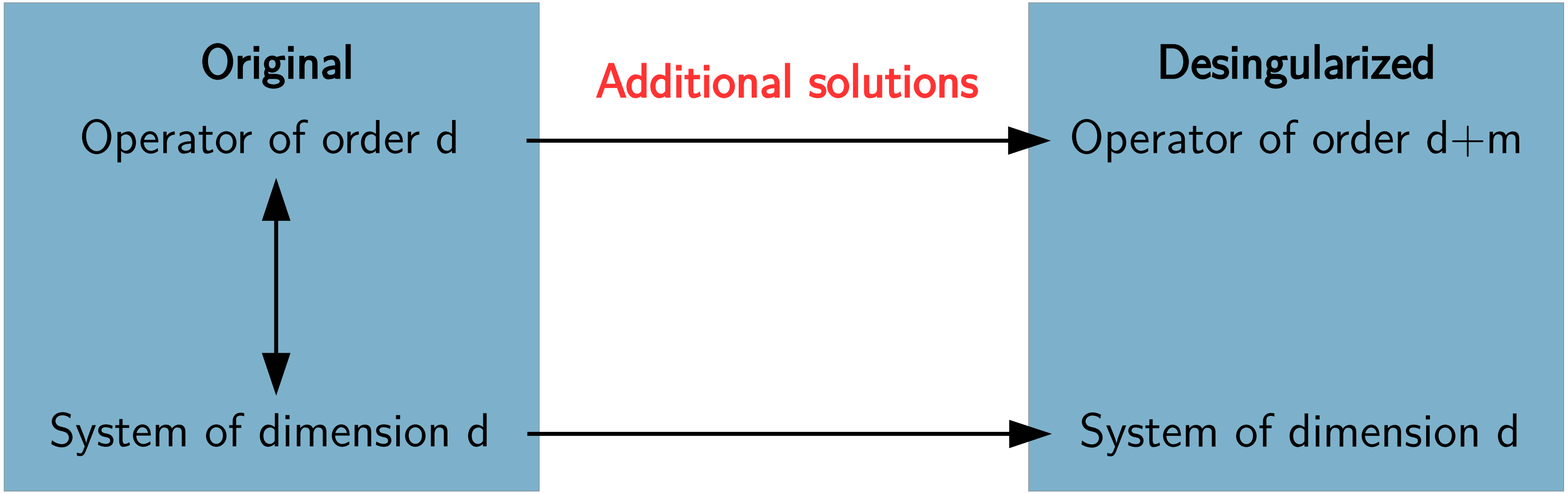
Desingularization of Ore operators:

- Abramov, van Hoeij 1999
- Tsai 2000
- Abramov, Barkatou, van Hoeij 2006
- Chen, J., Kauers, Singer 2013
- Chen, Kauers, Singer 2015
- Zhang 2016

Desingularization of linear differential systems:

- Barkatou 2010
- Barkatou, Maddah 2015

Desingularization of Operators vs. Desingularization of Systems



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Humble Beginnings

$$Y(z+1) = \begin{pmatrix} \frac{-2z+\frac{1}{2}}{\frac{1}{2}z} & \frac{\frac{1}{4}z^2}{-\frac{1}{2}z-19} & 0 & \frac{2z-3}{-\frac{1}{5}z-3} \\ \frac{-\frac{1}{4}z^2-\frac{1}{4}z+1}{20z^2-27} & \frac{\frac{3}{4}z^2-\frac{470}{3}z+4}{-\frac{1}{2}z^2-\frac{4}{3}z} & \frac{19}{2}z-\frac{1}{2} & \frac{2z^2-z-\frac{3}{2}}{-z^2-5z} \\ \frac{\frac{16}{11}z^2-z-\frac{1}{2}}{-z^2+6z-2} & \frac{-5z^2-3z-1}{-12z^2-4z-5} & \frac{25z^2+5z-2}{51z^2+2z-1} & \frac{-\frac{1}{2}z^2-\frac{1}{8}z-1}{-\frac{2}{3}z-5} \\ \frac{\frac{8}{3}z^2+z+1}{-z^2-z} & \frac{-2z^2-25z+\frac{2}{3}}{-3z^2+\frac{1}{49}z+1} & \frac{-12z^2+\frac{1}{3}z-1}{-2z^2-\frac{1}{2}z-7} & \frac{\frac{1}{5}z+1}{-\frac{2}{5}z^2-\frac{9}{2}z-7} \end{pmatrix} Y(z)$$

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$$A(z) \in \mathbb{K}(z), \quad q \cdot A(z) \in \mathrm{Mat}_d(\mathbb{K}[z]), \quad q \in \mathbb{K}[z]$$

The Algebraic Approach – Problem Statement

Problem Statement

Let A be a $d \times d$ matrix with coefficients in $\frac{1}{q}\mathbb{K}[z]$ where q is an irreducible polynomial in z.

Find a polynomial transformation T such that $T[A] \in \mathbb{K}[z]$ or show that no such T exists.

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Strategy

- Construct T as a composition of easy to understand transformations.
- Things that work for differential equations might work for difference equations.

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Dispersion Reduction

$$\begin{pmatrix} \frac{1}{z+1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{z+2}{z} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{z+2}{z+1} & 0 \\ 0 & 1 \end{pmatrix}$$

Observation: Entries in the intersection were multiplied with $\frac{z}{z+1} \rightarrow$ Reduced dispersion.

Idea: Reduce dispersion to zero $\rightarrow$ Singularity removed or not removal not possible.

Column Reduced Form

Lemma

Let r be the rank of the residue matrix of A with respect to q. There exists a unimodular polynomial transformation S such that S[A] is of the form

$$\begin{pmatrix} \frac{1}{q}A_1 & A_2 \end{pmatrix}$$

where A_1, A_2 are polynomial matrices of size $d \times r$ and $d \times d - r$ respectively.

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where A_1, A_2 are polynomial matrices of size $d \times r$ and $d \times d - r$ respectively.

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where A_1, A_2 are polynomial matrices of size $d \times r$ and $d \times d - r$ respectively.

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Shearing Transformation

Lemma

Let A be desingularizable at q and of the form

$$\begin{pmatrix} \underbrace{\tfrac{1}{q}A_1}_{r \text{ columns}} & A_2 \end{pmatrix}$$

Any desingularizing transformation T for A can be written as $T = D\tilde{T}$, where

$$D = \mathrm{diag}(\underbrace{q, \ldots, q}_{r \text{ times}}, 1, \ldots, 1) \quad \text{and} \quad \tilde{T} \in \mathrm{GL}_d(\mathbb{K}[z]).$$

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$$\begin{pmatrix} \frac{z+3}{z} & 0 \\ \frac{3}{2} & 2 \end{pmatrix} \;\text{—}\; D = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{z+3}{z+1} & 0 \\ \frac{3}{2}z & 2 \end{pmatrix}$$

Assembling the Transformation

Lemma

If A is desingularizable at q with dispersion ℓ, then there exist polynomial transformations D, S such that $(SD)[A]$ is either desingularized or desingularizable at $q(z+1)$ with dispersion $\ell - 1$.

Assembling the Transformation

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$$\begin{tikzcd} \begin{pmatrix}\text{dispersion } \ell \\ \end{pmatrix} \arrow[r, "S"] \arrow[rr, bend right, "SD"'] & \begin{pmatrix}\text{dispersion } \ell \\ \text{column reduced}\end{pmatrix} \arrow[r, "D"] & \begin{pmatrix}\text{dispersion } \ell-1 \\ \text{desingularizable}\end{pmatrix}\end{tikzcd}$$

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Repeat $\leq \ell$ times

Desingularization Theorem

Theorem

Let A be desingularizable at q. Then there exists an integer m, unimodular polynomial matrices $S_1, \dots, S_m$ and diagonal polynomial matrices $D_1, \dots, D_m$ such that

$$T = S_1 D_1 \cdots S_m D_m$$

is a desingularizing transformation for A at q. Furthermore, any other desingularizing transformation T_0 for A at p can be written as

$$T_0 = T\tilde{T} \quad \text{with } \tilde{T} \in GL_d(\mathbb{K}[z]).$$

Our Example

$$\begin{pmatrix} \frac{3(z+2)}{z} & \frac{z+6}{z} \\ \frac{-2(z+2)+1}{z} & \frac{-3}{z} \end{pmatrix} \quad S_1 = \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}$$

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$$\begin{pmatrix} \frac{3(z+2)}{z} & \frac{z+6}{z} \\ \frac{-2(z+2)+1}{z} & \frac{-3}{z} \end{pmatrix} \xrightarrow{S_1 = \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}} \begin{pmatrix} \frac{z+3}{z} & 0 \\ \frac{3}{2} & 2 \end{pmatrix}$$

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$$\begin{pmatrix} \frac{3(z+2)}{z} & \frac{z+6}{z} \\ \frac{-2(z+2)+1}{z} & \frac{-3}{z} \end{pmatrix} \xrightarrow{S_1 = \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}} \begin{pmatrix} \frac{z+3}{z} & 0 \\ \frac{3}{2} & 2 \end{pmatrix} \xrightarrow{D_1 = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} \frac{z+3}{z+1} & 0 \\ \frac{3}{2}z & 2 \end{pmatrix}$$

Our Example

$$\begin{tikzcd}[column sep=huge]
\begin{pmatrix} \frac{3(z+2)}{z} & \frac{z+6}{z} \\ \frac{-2(z+2)+1}{z} & \frac{-3}{z} \end{pmatrix} \arrow[r, "S_1 = \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}"] & \begin{pmatrix} \frac{z+3}{z} & 0 \\ \frac{3}{2} & 2 \end{pmatrix} \arrow[r, "D_1 = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}"] & \begin{pmatrix} \frac{z+3}{z+1} & 0 \\ \frac{3}{2}z & 2 \end{pmatrix} \arrow[d, "D_2 = \begin{pmatrix} z+1 & 0 \\ 0 & 1 \end{pmatrix}"'] \\
 & & \begin{pmatrix} \frac{z+3}{z+2} & 0 \\ \frac{3}{2}z(z+1) & 2 \end{pmatrix}
\end{tikzcd}$$

Our Example

$$\begin{tikzcd}[column sep=huge, row sep=huge]
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\begin{pmatrix} 1 & 0 \\ \frac{3}{2}z(z+1)(z+2) & 2 \end{pmatrix} & & \begin{pmatrix} \frac{z+3}{z+2} & 0 \\ \frac{3}{2}z(z+1) & 2 \end{pmatrix} \arrow[ll, "D_3 = \begin{pmatrix} z+2 & 0 \\ 0 & 1 \end{pmatrix}"]
\end{tikzcd}$$

Our Example

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\begin{pmatrix} \frac{3(z+2)}{z} & \frac{z+6}{z} \\ \frac{-2(z+2)+1}{z} & \frac{-3}{z} \end{pmatrix} \arrow[r, "S_1 = \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}"] \arrow[d] & \begin{pmatrix} \frac{z+3}{z} & 0 \\ \frac{3}{2} & 2 \end{pmatrix} \arrow[r, "D_1 = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}"] & \begin{pmatrix} \frac{z+3}{z+1} & 0 \\ \frac{3}{2}z & 2 \end{pmatrix} \arrow[d, "D_2 = \begin{pmatrix} z+1 & 0 \\ 0 & 1 \end{pmatrix}"'] \\
\begin{pmatrix} 1 & 0 \\ \frac{3}{2}z(z+1)(z+2) & 2 \end{pmatrix} & & \begin{pmatrix} \frac{z+3}{z+2} & 0 \\ \frac{3}{2}z(z+1) & 2 \end{pmatrix} \arrow[ll, "D_3 = \begin{pmatrix} z+2 & 0 \\ 0 & 1 \end{pmatrix}"]
\end{tikzcd}
$$

$$T = S_1 D_1 D_2 D_3 = \begin{pmatrix} x^3 + 3x^2 + 2x & 1 \\ -\frac{1}{2}x^3 - \frac{3}{2}x^2 - x & -1 \end{pmatrix}$$

The Story So Far

What we did:

- Given: Difference system with irreducible polynomial as denominator.
- Can we find a polynomial basis transformation that removes the denominator?
- We know how to construct such a transformation.
- We know the dispersion $\rightarrow$ We know an upper bound for #loop iterations.

Question: How can we generalize this to several different poles or poles with higher multiplicity?

Desingularization – Take One

Definition: Desingularization

We call $[A]$ desingularizable at q if $\frac{\operatorname{den}(A)}{q}A$ is desingularizable at q.

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$$\left(\frac{z+1}{z^2}\right) \xrightarrow{T\,=\,(z)} \left(\frac{1}{z}\right)$$

Desingularization – Take Two

$$A = q^k(A_{0,q} + qA_{1,q} + q^2A_{2,q} \cdots)$$

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$$\begin{pmatrix} \frac{1}{z} \end{pmatrix} \xrightarrow{T = (z)} \begin{pmatrix} \frac{1}{z+1} \end{pmatrix}$$

Desingularization – Take Three

$$A = q^k(A_{0,q} + qA_{1,q} + q^2A_{2,q} \cdots)$$

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We call $[A]$ desingularizable at q if there exists a polynomial transformation T s.t.

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$$\begin{pmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{(z+1)^2} \end{pmatrix} \xrightarrow{T = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} \frac{1}{z+1} & 0 \\ 0 & \frac{1}{(z+1)^2} \end{pmatrix}$$

Phi Minimality

$$\left(\frac{z+2}{z(z+1)} \right)$$

Phi Minimality

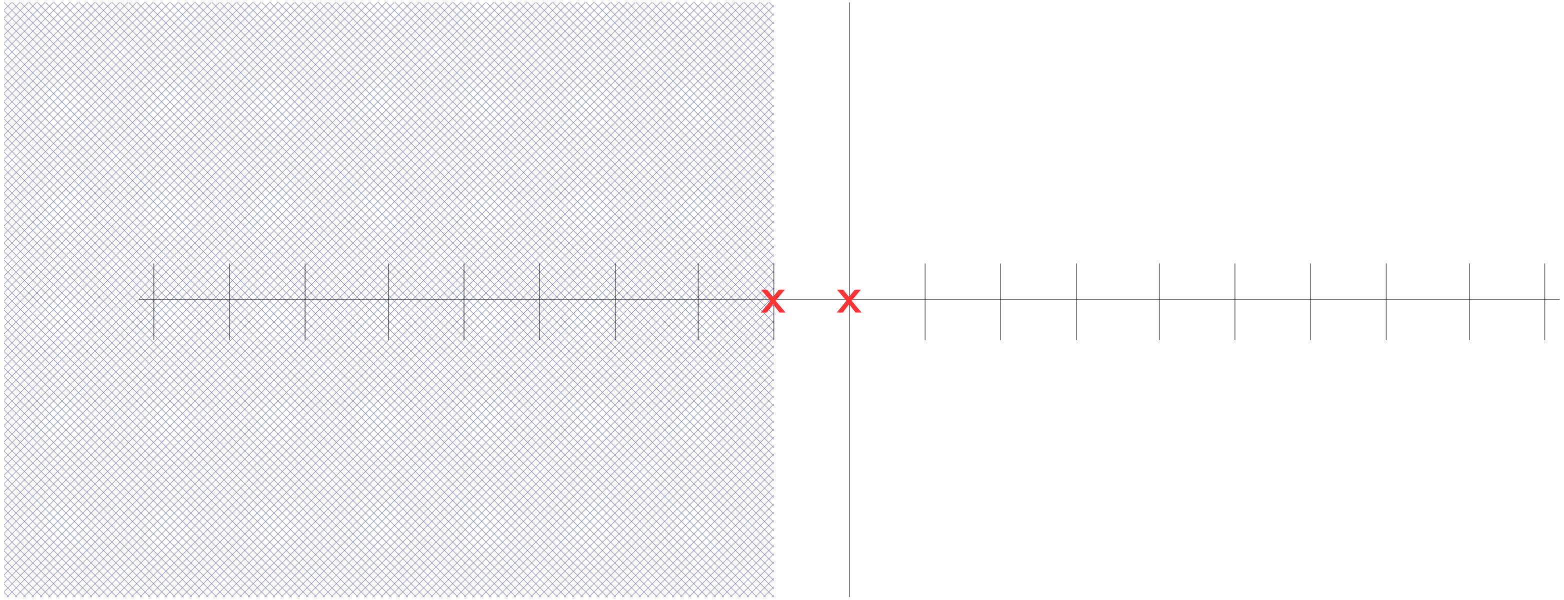
$$\begin{pmatrix} \frac{z+2}{z(z+1)} \end{pmatrix} \xrightarrow{T_1 = \left(z(z+1)\right)} \begin{pmatrix} \frac{1}{z+1} \end{pmatrix}$$

Phi Minimality

$$\left(\frac{1}{z}\right) \xleftarrow{T_2 = \left(z+1\right)} \left(\frac{z+2}{z(z+1)}\right) \xrightarrow{T_1 = \left(z(z+1)\right)} \left(\frac{1}{z+1}\right)$$

Phi Minimality

$$\begin{pmatrix}\frac{1}{z}\end{pmatrix} \xleftarrow{T_2 = (z+1)} \begin{pmatrix}\frac{z+2}{z(z+1)}\end{pmatrix} \xrightarrow{T_1 = (z(z+1))} \begin{pmatrix}\frac{1}{z+1}\end{pmatrix}$$



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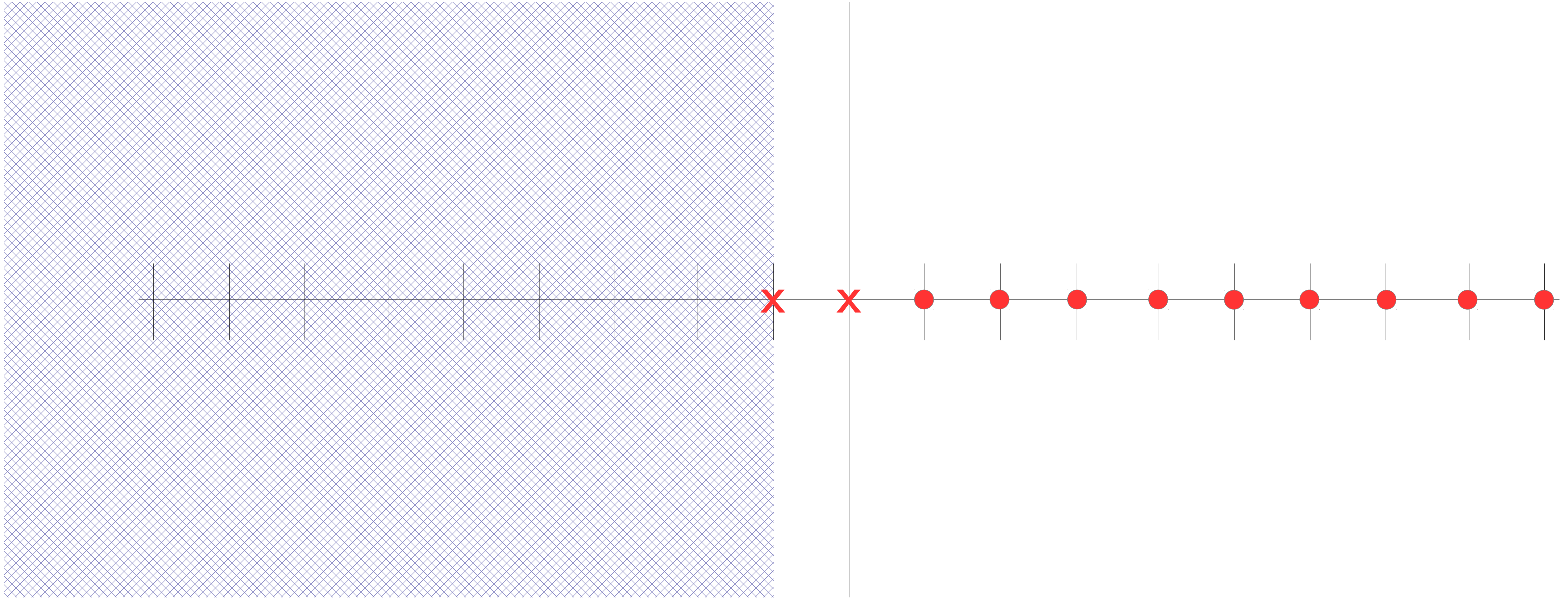

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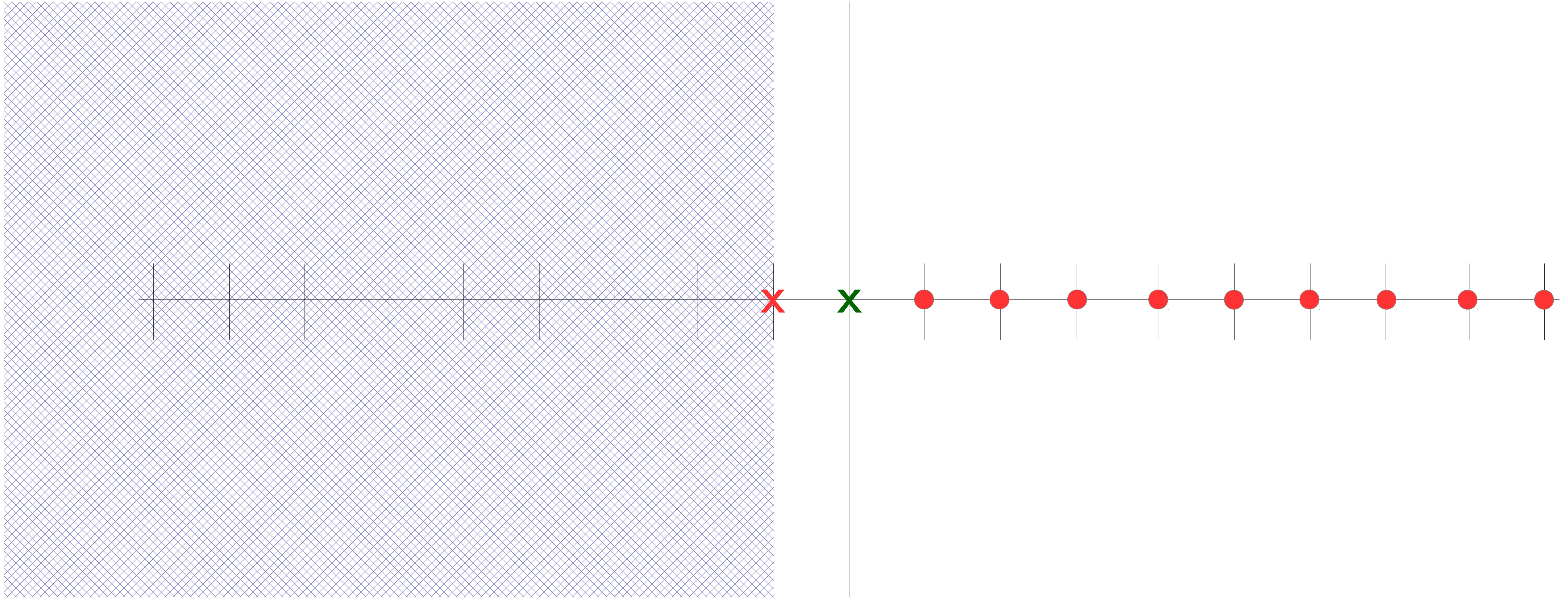

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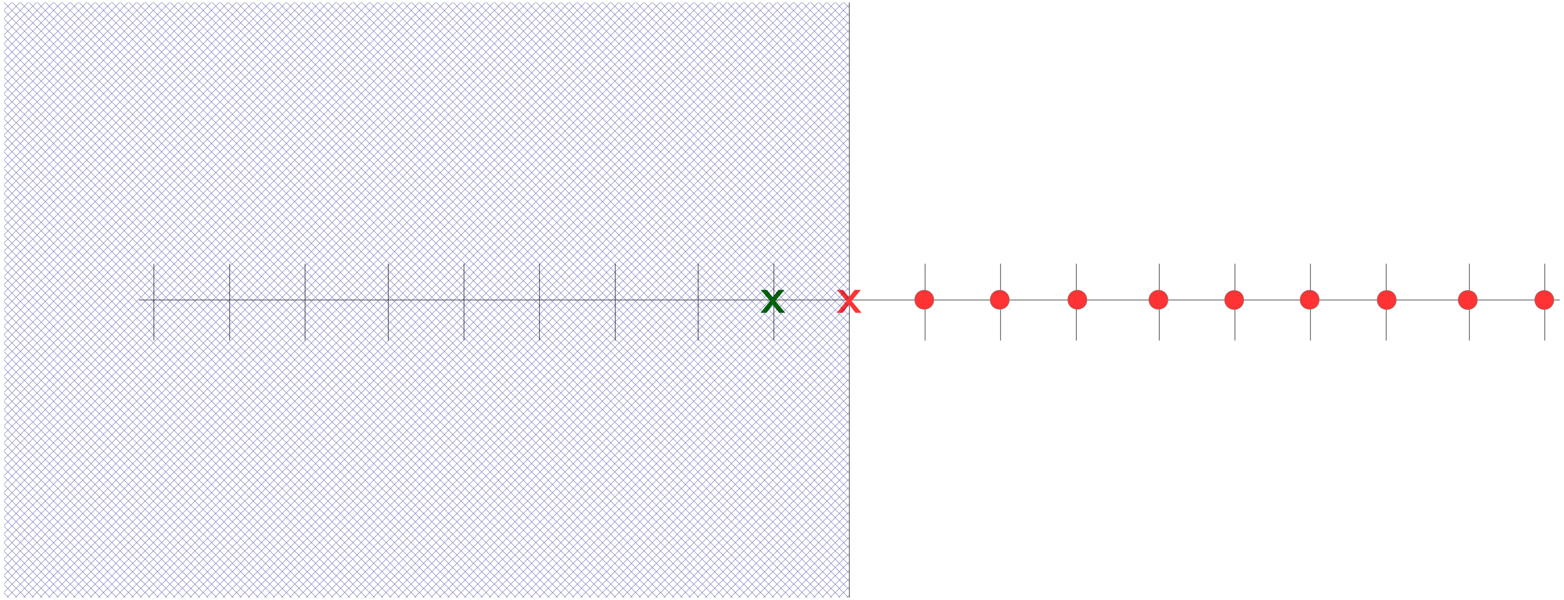

Desingularization – Take Four

Definition: Desingularization

We call [A] desingularizable at a phi-minimal q if there exists a polynomial transformation T s.t.

$$\mathrm{ord}_q(T[A]) > \mathrm{ord}_q(A), \quad \mathrm{ord}_p(T[A]) \geq \mathrm{ord}_p(A), \quad p \in \mathbb{K}[z].$$

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$$\begin{pmatrix} \frac{z+1}{z} & 0 \\ 0 & \frac{1}{(z+1)^2} \end{pmatrix} \xrightarrow{T = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{(z+1)^2} \end{pmatrix}$$

Desingularization – Take Five

Definition: Desingularization

?

Desingularization and Rank Reduction

$$\begin{pmatrix} \frac{z+1}{z} & 0 \\ 0 & \frac{1}{z} \end{pmatrix} \xrightarrow{T = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z} \end{pmatrix}$$

From Nilpotency to the Factorial Relation

Proposition

Let ζ be a pole of $A(z)$ of order $\nu \geq 1$ such that $\zeta - j$ is not a pole of A for all positive integers j. Let $\tilde{A}(z) = (z-\zeta)^{\nu} A(z)$, so that $\tilde{A}(\zeta) \neq 0$. If ζ is an apparent singularity for $[A]$, then there exists a positive integer k such that

$$\tilde{A}(\zeta) A(\zeta - 1) \cdots A(\zeta - k) = 0,$$

in particular, the matrix $A(\zeta - j)$ is singular for some nonnegative integer j.

Apparent Implies Removable

Theorem

Let ζ be a pole of $A(z)$ such that $\zeta - j$ is not a pole of A for all positive integers j. Then A is desingularizable at $p = z - \zeta$ if and only if

$$\tilde{A}(\zeta)A(\zeta - 1)\cdots A(\zeta - k) = 0.$$

Theorem

Let ζ be a pole of $A(z)$ such that $\zeta - j$ is not a pole of A for all positive integers j. If ζ is an apparent singularity, then A is desingularizable at $p = z - \zeta$.

Conclusion

We saw:

- What are linear difference systems?
- What are apparent singularities of solutions?
- What is desingularization?
- Algorithm for removing singularities.
- Apparent Singularities are removable.