Enumeration of Graphs on Surfaces

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Part I

- Decomposition along connectivity
  - ▷ Recursive method
  - ▷ Singularity analysis
  - ▷ Saddle-point method

Part II

- Core-Kernel approach
  - ▷ Combinatorial Laplace's method

- Gaussian matrix integral method

Graphs on Surfaces

- Let $\mathbb{S}_g$ be the orientable surface of genus g
- Graphs on $\mathbb{S}_g$

  = Graphs that are *embeddable* on $\mathbb{S}_g$

  = Graphs that can be drawn on $\mathbb{S}_g$ without crossing edges

  Examples include

  ▷ Forests = acyclic graphs

  ▷ Planar graphs = graphs that are embeddable on the sphere $\mathbb{S}_0$

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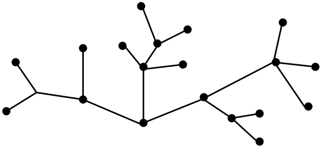
- Vertex-labelled graphs on $\mathbb{S}_g$ with vertex set $[n] := \{1, \cdots, n\}$

Recursive Method

How many *trees (= acyclic connected graphs)* are there?



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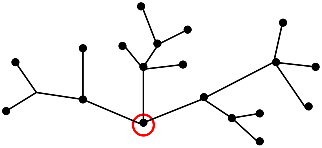

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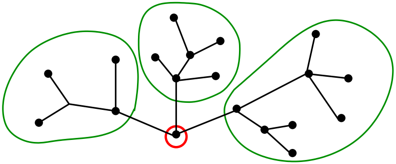

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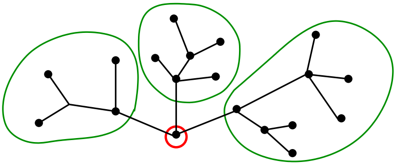

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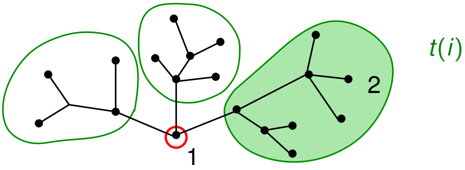

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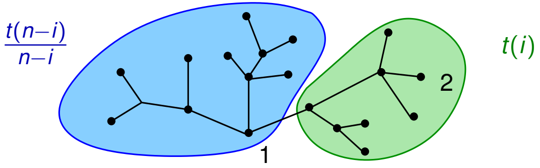

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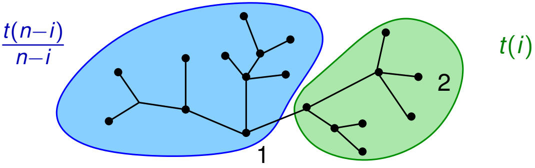

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▷ Polynomial time algorithm to compute the exact number

- planar graphs [BODIRSKY–GRÖPL–K. 07]

Recursive Method

▷ Uniform sampling algorithm [FLAJOLET–ZIMMERMAN–VAN CUTSEM 94]

Generate(n): returns a random tree on $[n]$

choose a root vertex r with probability $\frac{1}{n}$

return Generate(n, r)

Generate(n, r): returns a random tree on $[n]$ with the root vertex r

choose the order i of the subtree with prob. $\frac{n}{t(n)} \binom{n-2}{i-1} t(i) \frac{t(n-i)}{(n-i)}$

let $s = \min([n] \setminus \{r\})$

choose a random subset $\{s\} \subseteq \{w_1, \ldots, w_i\} \subseteq [n] \setminus \{r\}$ (with rel. order)

let $\{v_1, \ldots, v_{n-i}\} = [n] \setminus \{w_1, \ldots, w_i\}$ (with relative order)

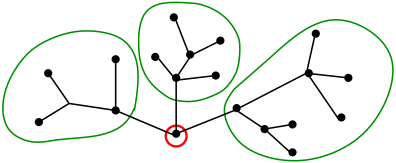
$T_1 =$ **Generate**(i); relabel vertex j in T_1 with w_j (r' = root vertex of T_1)

$T_2 =$ **Generate**$(n - i, r)$; relabel vertex $j \neq r$ in T_2 with v_j

return $T_1 \cup T_2 \cup \{(r, w_{r'})\}$ with marked r

- planar graphs [BODIRSKY–GRÖPL–K. 07]

Generating Function for Rooted Trees



Let $T(z)$ be the exponential generating function for rooted trees:

$$T(z) = \sum_{n} \frac{t(n)}{n!} z^n$$

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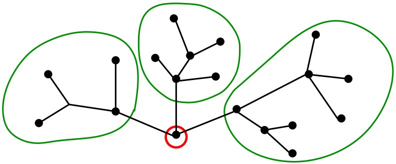

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$$T(z) \;=\; \sum_{n} \frac{t(n)}{n!}\, z^n$$

Then

$$T(z) = z\left(\qquad\qquad\qquad\qquad \frac{T(z)^3}{3!} \qquad \right)$$

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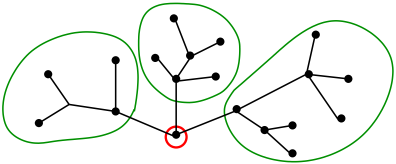

Let $T(z)$ be the exponential generating function for rooted trees:

$$T(z) \;=\; \sum_n \frac{t(n)}{n!}\, z^n$$

Then

$$T(z) = z\left(1 + T(z) + \frac{T(z)^2}{2!} + \frac{T(z)^3}{3!} + \cdots\right) \;=\; z\, e^{T(z)}$$

Singularity Analysis for Rooted Trees

[FLAJOLET–SEDGEWICK 09]

View $T : z \to T(z)$ as a complex-valued function (which is analytic at $z = 0$).

Let $z = \psi(u)$ be the functional inverse of $u = T(z)$, i.e.

$$\psi \circ T = T \circ \psi = \mathrm{Id}.$$

Then we have $\psi(u) = ue^{-u}$, since $T(z) = z\, e^{T(z)}$, and

$$\exists \quad u_0 \in (0, \infty) \quad \text{with} \quad \psi'(u_0) = 0,\ \psi''(u_0) \neq 0,$$

in fact, $u_0 = 1$, $z_0 = \psi(1) = e^{-1}$, $\psi'(1) = 0$, $\psi''(1) = -e^{-1}$.

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The Taylor expansion of the function $\psi : u \to ue^{-u}$ at $u_0 = 1$

$$\psi(u) \;=\; \psi(u_0) + \frac{1}{2}\psi''(u_0)(u - u_0)^2 + \cdots \;=\; \frac{1}{e} - \frac{1}{2e}(u-1)^2 + \cdots$$

Singularity Analysis for Rooted Trees

From local quadratic dependency between $z = \psi(u)$ and $u = T(z)$

$$(u-1)^2 \sim\ -2e\Big(\psi(u) - \frac{1}{e}\Big)$$

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$$(T(z) - 1)^2 \ = \ (u - 1)^2 \sim \ -2e\Big(\psi(u) - \frac{1}{e}\Big) \ = \ 2\Big(1 - ez\Big)$$

and the property that $T(z)$ is increasing along the real axis, we obtain

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Applying Transfer Theorem to Δ-analytic function $T(z)$, we obtain

$$\frac{t(n)}{n!} \;=\; [z^n]\,T(z) \;\sim\; -[z^n]\Big(2\,(1-ez)\Big)^{1/2} = \;\frac{1}{\sqrt{2\pi}}\,n^{-3/2}\,e^n$$

and the number $t(n)$ of rooted trees with vertex set $[n]$ satisfies

$$t(n) \;\sim\; \frac{1}{\sqrt{2\pi}}\,n^{-3/2}\,e^n\,n!$$

Generating Functions for Planar Graphs

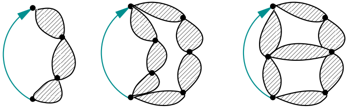
Graphs $=$ set of connected components

$$G(z) = \exp(C(z))$$

Connected graphs $\Longleftrightarrow$ block structure [HARARY–PALMER 78]

$$zC'(z) = z\exp(B'(zC'(z)))$$

2-connected graphs $\Longleftrightarrow$ networks [TRAKHTENBROT 58; TUTTE 63; WALSH 82]



$$\frac{\partial B(z,y)}{\partial y} = \frac{z^2(1+N(z,y))}{2(1+y)}$$

$$\frac{zN^2}{1+zN} - \log\frac{1+N}{1+y} + \frac{M(z,N)}{2zN^2} = 0$$

3-conn. planar graphs $\Longleftrightarrow$ c-nets [MULLIN–SCHELLENBERG 68]

$$M(z,y) = z^2y^2\left(\frac{1}{1+zy} + \frac{1}{1+y} - 1 - \frac{(1+u)^2(1+v)^2}{(1+u+v)^3}\right)$$

$$u = zy(1+v)^2, \quad v = y(1+u)^2$$

Singularity Analysis

Difficulty: analytic integration of implicitly defined function

$$B(z,y) = \frac{z^2}{2}\int_0^y \frac{1+N(z,t)}{1+t}\,dt, \qquad \frac{zN^2}{1+zN} - \log\frac{1+N}{1+y} + \frac{M(z,N)}{2z^2N} = 0$$

$$N(z,y) = \textit{analytic part} + g(y)(1-z/\rho(y))^{3/2}$$

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▷ inverse function of y [Bodirsky–Giménez–K.–Noy 07 (SP); Giménez–Noy 09 (planar)]

$$y \;=\; y(z,N) \;:=\; -1 + (1+N)\exp\left(\frac{zN^2}{1+zN} + \frac{M(z,N)}{2z^2N}\right)$$

$$B(z,y) \;=\; \mathit{analytic\ part} + h(y)\Big(1 - z/\rho(y)\Big)^{5/2}$$

▷ 'meta-theorem' with help of implicit functions [Drmota 09]

▷ dissymmetry theorem for tree-like structures [Chapuy–Fusy–K.–Shoilekova 08]

$$C(z) \;=\; C_\circ(z) \;+\; C_{\circ-\circ}(z) \;-\; C_{\circ\to\circ}(z)$$

'combinatorial ' integration instead of analytic one $C(z) \;=\; \int_0^z C'(t)\,dt$

Asymptotic Number of Graphs on Surfaces

[Giménez–Noy 09]

The number $p(n)$ of planar graphs with vertex set $[n]$ satisfies

$$p(n) \quad \sim \quad \alpha \, n^{-\frac{7}{2}} \, \gamma^n \, n!$$

where $\alpha > 0$ and $\gamma \doteq 27.23$ are analytic constants.

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[Chapuy–Fusy–Giménez–Mohar–Noy 11]

The number $s_g(n)$ of graphs on $\mathbb{S}_g$ with vertex set $[n]$ satisfies

$$s_g(n) \quad \sim \quad \alpha_g\, n^{\frac{5g}{2}-\frac{7}{2}}\, \gamma^n\, n!$$

where $\alpha_g > 0$ is an analytic constant and γ is the same as for the planar case.

Part I

Decomposition along connectivity

- Recursive method
- Singularity analysis
- Saddle-point method

Block-Stable Graphs

- A block of a graph G is a maximal 2-connected subgraph of G.

- A class $\mathcal{G}$ of graphs is block-stable if it

  (1) contains a graph consisting of one edge and its two end vertices

  (2) satisfies property that G belongs to $\mathcal{G}$ iff all its blocks belong to $\mathcal{G}$

- Examples of classes of block-stable graphs include

  classes of graphs specified by a finite list of forbidden 2-conn. minors

  - ▷ Forests = class of graphs with K_3 as a forbidden minor
  - ▷ Planar graphs = class of graphs with K_5, $K_{3,3}$ as forbidden minors

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- Classes of vertex-labelled block-stable graphs with vertex set $[n]$

Generating Functions for Block-Stable Graphs

Let $\mathcal{S}(n)$ denote a class of block-stable graphs with vertex set $[n]$ and

let $S(z) \; = \; \sum_n \frac{|\mathcal{S}(n)|}{n!} \, z^n$ be its exponential generating function.

Then $S(z)$ features a universal behaviour

$$\begin{aligned} S(z) &= G(\phi(z)) \\ G(z) &= \exp(z - zB'(z) + B(z)) \\ f(z) &= \exp(B'(z)) \\ \phi(z) &= z\, f(\phi(z)) \end{aligned}$$

where $B(z)$ = generating function for 2-connected ones

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- ▷ Forests: $B(z) = \frac{z^2}{2}$
- ▷ Planar graphs:

$$B(z) = B_0 + B_2\left(1 - \frac{z}{\rho}\right) + B_4\left(1 - \frac{z}{\rho}\right)^2 + B_5\left(1 - \frac{z}{\rho}\right)^{5/2} + \cdots$$

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Lagrange Inversion and Cauchy's Coefficient Formula

Let $f(z)$, $\phi(z)$ and $G(z)$ be power series with $f_0 \neq 0$ that satisfy

$$\phi = z\, f(\phi(z)).$$

By Lagrange Inversion Formula we have

$$[z^n]\,\phi(z) = \frac{1}{n}\,[z^{n-1}]\,f(z)^n$$

$$[z^n]\,G(\phi(z)) = \frac{1}{n}\,[z^{n-1}]\,G'(z)f(z)^n.$$

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Furthermore, by Cauchy's coefficient formula we have

$$[z^n]\,\phi(z) \;=\; \frac{1}{n}\,\frac{1}{2\pi i}\oint_{|z|=r} z^{-n}\, f(z)^n\, dz$$

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for $r > 0$ smaller than the radii of convergence of $f(z)$ and $G(z)$.

Saddle-Point Method I

From $\phi = z\, f(\phi(z))$ we have

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we let $r > 0$ be a simple saddle-point of $a(z) = -\log z + \log f(z)$, i.e.

$$a'(r) = 0, \quad a''(r) \neq 0$$

Then we have

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$$\sim \frac{1}{\sqrt{2\pi a''(r)}}\, n^{-3/2}\, \exp(n\,a(r))$$

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$$a'(r) = 0, \quad a''(r) \neq 0 \quad \Longleftrightarrow \quad f(r) = r\, f'(r), \quad f''(r) \neq 0$$

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$$\sim \frac{1}{\sqrt{2\pi a''(r)}}\, n^{-3/2}\ \exp(n\, a(r)) = \frac{1}{\sqrt{2\pi\, f''(r)/f(r)}}\, n^{-3/2}\ \left(r^{-1}\, f(r)\right)^n$$

Application to Rooted Trees

The generating function $T(z) = \sum_n \frac{t(n)}{n!} z^n$ for rooted trees satisfies

$$T(z) = z\, e^{T(z)}$$

By Lagrange Inversion Formula we have

$$\frac{t(n)}{n!} = [z^n]\, T(z) = \frac{1}{n} [z^{n-1}]\, e^{nz} = \frac{1}{n} [z^{n-1}] \sum_k \frac{(n\,z)^k}{k!} = \frac{1}{n} \frac{n^{n-1}}{(n-1)!}$$

and so we obtain Cayley's formula for rooted trees

$$t(n) = n^{n-1}$$

Applying the saddle-point method we obtain

$$\frac{t(n)}{n!} = [z^n]\, T(z) \sim \frac{1}{\sqrt{2\pi\, f''(r)/f(r)}}\, n^{-3/2} \left(r^{-1} f(r)\right)^n = \frac{1}{\sqrt{2\pi}}\, n^{-3/2}\, e^n$$

because $f(z) = e^z$ and $r = 1$, and thus

$$t(n) \sim \frac{1}{\sqrt{2\pi}}\, n^{-3/2}\, e^n\, n!$$

Saddle-Point Method II

By Lagrange inversion formula and Cauchy's coefficient formula, we have

$$
\begin{aligned}
[z^n]G(\phi(z)) &= \frac{1}{n}\,[z^{n-1}]\;G'(z)\;f(z)^n \\
&= \frac{1}{n}\,\frac{1}{2\pi i}\oint_{|z|=r} G'(z)\;z^{-n}\;f(z)^n\;dz \\
&= \frac{1}{n}\,\frac{1}{2\pi i}\oint_{|z|=r} G'(z)\;\exp\left(n\;\left(-\log z + \log f(z)\right)\right)\;dz.
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\end{aligned}$$

If $\qquad f(r) = r\,f'(r), \quad f''(r) \neq 0, \quad G'(r) = 0 \quad \text{and} \quad G''(r) \neq 0,$

then

$$\begin{aligned}
[z^n]G(\phi(z)) &= \frac{1}{n^2}\,\frac{1}{2\pi i}\oint_{|z|=r} h(z)\;z^{-n}\;f(z)^n\;dz \\
&\sim \frac{h(r)}{\sqrt{2\pi\,f''(r)/f(r)}}\;n^{-5/2}\;\left(r^{-1}f(r)\right)^n,
\end{aligned}$$

where $\qquad h(z) = \dfrac{d}{dz}\dfrac{z\,G'(z)}{1 - zf'(z)/f(z)}$

Generating Functions for Block-Stable Graphs

For the generating function $S(z)$ for a class of block-stable graphs we have

$$\begin{aligned} S(z) &= G(\phi(z)) \\ G(z) &= \exp(z - zB'(z) + B(z)) \\ f(z) &= \exp(B'(z)) \\ \phi(z) &= z\, f(\phi(z)) \end{aligned}$$

where $B(z)$ = generating ftn for 2-conn. ones (with dominant singularity ρ)

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- subcritical class
  - ▷ $\rho\, B''(\rho) > 1 \implies \exists\, r \in (0, \rho)$ s.t. $r\, B''(r) = 1,\ 1 + r^2\, B'''(r) > 0$
  - ▷ forests, series-parallel graphs, . . .
- the other class
  - ▷ $\rho\, B''(\rho) \le 1 \implies \forall\, r \in (0, \rho),\ r\, B''(r) \neq 1$
  - ▷ planar graphs

Saddle-Point Method for Subcritical Classes

[Hwang–K. 17+]

Subcritical class

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$$\Longrightarrow \ \exists\, r \in (0,\rho) \ \text{ s.t. } \ f(r) = r\,f'(r),\ f''(r) \neq 0,\ G'(r) = 0,\ G''(r) \neq 0$$

$$\Longrightarrow \ [z^n]G(\phi(z)) \sim \frac{h(r)}{\sqrt{2\pi\, f''(r)/f(r)}}\, n^{-5/2} \left(r^{-1} f(r)\right)^n,$$

where

$$h(z) = \left(1 + z\left(1 - zB''(z)\right)\right)\exp\left(z - zB'(z) + B(z)\right)$$

For any subcritical class of block-stable graphs

$$[z^n]G(\phi(z)) \sim \frac{r\,\exp(r - rB'(r) + B(r))}{\sqrt{2\pi\,(1 + r^2 B'''(r))}}\, n^{-5/2} \left(r^{-1}\ \exp(B'(r))\right)^n$$

Saddle-Point Method for Subcritical Classes

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e.g. for forests,
$$f(n) \sim \frac{e^{-1/2}}{\sqrt{2\pi}}\, n^{-5/2}\, e^n\, n!$$

Saddle-Point Method for Planar Graphs

The class of planar graphs does not belong to the subcritical class, i.e.

$$\rho\, B''(\rho) < 1 \implies \forall\, r \in (0, \rho),\;\; r\, B''(r) \neq 1$$

Saddle-Point Method for Planar Graphs

The class of planar graphs does not belong to the subcritical class, i.e.

$$\rho\, B''(\rho) \,<\, 1 \implies \forall\, r \in (0,\rho),\quad r\, B''(r) \neq 1$$

[Hwang–K. 17+]

Applying the singular expansion of $B(z)$

$$B(z) \,=\, B_0 + B_2\left(1-\frac{z}{\rho}\right) + B_4\left(1-\frac{z}{\rho}\right)^2 + B_5\left(1-\frac{z}{\rho}\right)^{5/2} + \cdots$$

we obtain that the number of planar graphs with vertex set $[n]$ satisfies

$$\begin{aligned}
&[z^n]G(\phi(z)) \\
&= \frac{1}{n^2}\,\frac{1}{2\pi i}\oint_{|z|=r} h(z)\, z^{-n}\, f(z)^n\, dz \\
&= \frac{1}{n^2}\,\frac{1}{2\pi i}\oint_{|z|=r} \left(1+z\left(1-zB''(z)\right)\right)e^{z-zB'(z)+B(z)}\, z^{-n}\, e^{B'(z)n}\, dz \\
&\sim \frac{B_5 e^{\rho+B_0+B_2}}{\Gamma(-5/2)}\left(1-\frac{2B_4}{\rho}\right)^{-5/2} n^{-7/2}\left(\rho e^{\rho^{-1}B_2}\right)^{-n} \\
&\sim \alpha\, n^{-7/2}\,\gamma^n
\end{aligned}$$

Part I

- Decomposition along connectivity
  - ▷ Recursive method
  - ▷ Singularity analysis
  - ▷ Saddle-point method

Part II

- Core-Kernel approach
  - ▷ Combinatorial Laplace's method
- Gaussian matrix integral method

Graphs on Surfaces

$$\mathbb{S}_g \quad = \quad \text{the orientable surface of genus } g \geq 0$$

$$\mathcal{S}_g(n, m) \quad = \quad \{\, \text{graphs embeddable on } \mathbb{S}_g \text{ with vertex set } [n] \text{ and } m \text{ edges} \,\}$$

where $m = d\,\frac{n}{2}$ for the average degree $d \in (0, 6)$

Dense Graphs on $\mathbb{S}_g$

Let $d \in (2, 6)$ be a constant (independent of n).

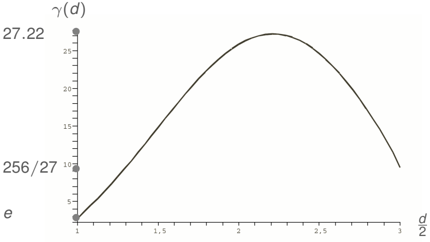
[Giménez–Noy 09] for $g = 0$

[Chapuy–Fusy–Giménez–Mohar–Noy 11] for $g \geq 1$

The number of graphs on $\mathbb{S}_g$ with vertex set $[n]$ and $m = d\,\frac{n}{2}$ edges satisfies

$$\left|\mathcal{S}_g\left(n, d\,\frac{n}{2}\right)\right| \quad \sim \quad \alpha_g(d)\, n^{\frac{5}{2}g-4}\, \gamma(d)^n\, n!$$

where $\alpha_g(d) > 0$ and $\gamma(d)$ is the same as the planar case



Sparse Graphs on $\mathbb{S}_g$

Let

$$
\begin{aligned}
\mathcal{F}(n,m) &= \{\, \text{acylic graphs with vertex set } [n] \text{ and } m \text{ edges} \,\} \\
\mathcal{S}_g(n,m) &= \{\, \text{graphs on } \mathbb{S}_g \text{ with vertex set } [n] \text{ and } m \text{ edges} \,\} \\
\mathcal{G}(n,m) &= \{\, \text{graphs with vertex set } [n] \text{ and } m \text{ edges} \,\}
\end{aligned}
$$

Note that

$$\mathcal{F}(n,m) \quad \subset \quad \mathcal{S}_g(n,m) \quad \subset \quad \mathcal{G}(n,m)$$

For $d \in (0,1)$ we have

$$\left|\mathcal{G}\left(n, d\,\frac{n}{2}\right)\right| \sim \left|\mathcal{F}\left(n, d\,\frac{n}{2}\right)\right| \sim c(d)\, n^{-3}\, \beta(d)^n\, n!$$

and therefore

$$\left|\mathcal{S}_g\left(n, d\,\frac{n}{2}\right)\right| \sim c(d)\, n^{-3}\, \beta(d)^n\, n!$$

Sparse Graphs on $\mathbb{S}_g$

[K.–Łuczak 12] for $g = 0$; [K.–Moßhammer–Sprüssel 17+] for $g \geq 0$

Let $d = d(n) \in (1-\epsilon, 2+\epsilon)$ for $\epsilon = \epsilon(n) > 0$ with $\epsilon = o(1)$.

- (1) If $(d-1)n^{1/3} \to -\infty$, then ...

  (2) If $(d-1)n^{1/3} \to c \in \mathbb{R}$, then ...

  (3) If $n^{-1/3} \ll d-1 \ll 1$, then

$$\left|\mathcal{S}_g\left(n, d\,\frac{n}{2}\right)\right| = \left(\frac{e}{2-d}\right)^{(2-d)\frac{n}{2}} n^{d\frac{n}{2}-\frac{1}{2}}\, e^{O((d-1)n^{1/3})}.$$

- (4) If d converges to a constant in $(1,2)$, then

$$\left|\mathcal{S}_g\left(n, d\,\frac{n}{2}\right)\right| = \left(\frac{e}{2-d}\right)^{(2-d)\frac{n}{2}} n^{d\frac{n}{2}}\, e^{O(n^{1/3})}.$$

- (5) If $(d-2)n^{2/5} \to -\infty$, then ...

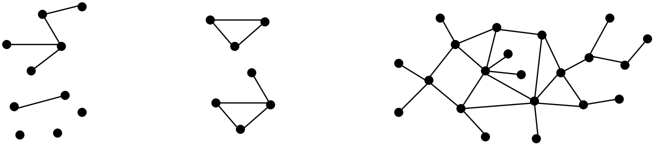
  (6) If $(d-2)n^{2/5} \to c \in \mathbb{R}$, then ...

  (7) If $n^{-2/5} \ll d-2 \ll (\log n)^{-2/3}$, then

$$\left|\mathcal{S}_g\left(n, d\,\frac{n}{2}\right)\right| = \left(d-2\right)^{-\frac{3}{4}(d-2)n} n^{n}\, e^{O((d-2)n)}.$$

Component Structure of Graphs on $\mathbb{S}_g$

Component structure of a graph from $\mathcal{S}_g(n, m)$



tree components *unicyclic compopnents* *complex components*

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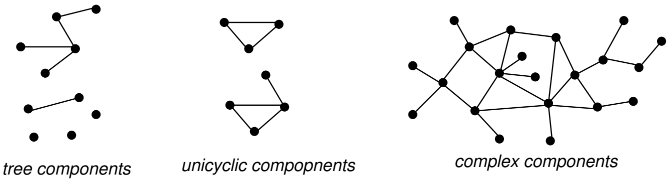


$$S_g(n, m) \;=\; \#\text{ graphs on } \mathbb{S}_g \text{ with vertex set } [n] \text{ and } m \text{ edges}$$

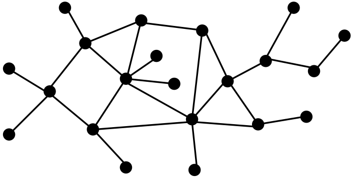
$$= \sum_{k,\ell} \binom{n}{k} C_g(k, k+\ell)\, U(n-k, m-k-\ell)$$

$C_g(k, k+\ell) \;=\; \#$ complex graphs with vertex set $[k]$ and $k+\ell$ edges

$U(n-k, m-k-\ell) \;=\; \#$ graphs consisting of tree or unicyclic components with vertex set $[n-k]$ and $m-k-\ell$ edges

Core-Kernel Approach

- Complex graph G



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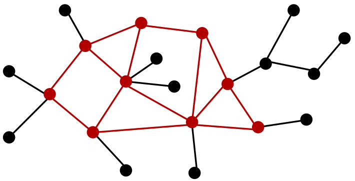

$\Longrightarrow$ 2-Core $=$ maximal subgraph of G with minimum degree at least two

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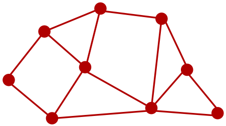

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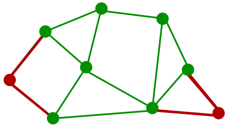

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Core-Kernel Approach

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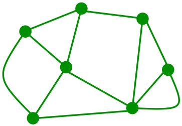

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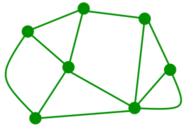

$\Longrightarrow$ 2-Core $=$ maximal subgraph of G with minimum degree at least two

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- G is embeddable on $\mathbb{S}_g$ if and only if its kernel is embeddable on $\mathbb{S}_g$

Complex Graphs on $\mathbb{S}_g$

- Construct complex graph G on $\mathbb{S}_g$

$C_g(k, k+\ell) = \#$ complex graphs on $\mathbb{S}_g$ with vertex set $[k]$ and $k+\ell$ edges

Complex Graphs on $\mathbb{S}_g$

- Construct complex graph G on $\mathbb{S}_g$ by
  - ▷ choosing the kernel of G from the set of possible candidates

$$C_g(k, k+\ell) = \#\text{ complex graphs on } \mathbb{S}_g \text{ with vertex set } [k] \text{ and } k+\ell \text{ edges}$$

$$= \sum_{i,j} K_g(2\ell - j)$$

Complex Graphs on $\mathbb{S}_g$

- Construct complex graph G on $\mathbb{S}_g$ by
  - choosing the kernel of G from the set of possible candidates
  - putting on its edges vertices of degree two to obtain the 2-core of G

$$C_g(k, k+\ell) \; = \; \# \text{ complex graphs on } \mathbb{S}_g \text{ with vertex set } [k] \text{ and } k+\ell \text{ edges}$$

$$= \sum_{i,j} K_g(2\ell - j)\, \frac{(k)_i}{(2\ell - j)!} \binom{i - a\,\ell - 1}{3\ell - j - 1}$$

Complex Graphs on $\mathbb{S}_g$

- Construct complex graph G on $\mathbb{S}_g$ by
  - choosing the kernel of G from the set of possible candidates
  - putting on its edges vertices of degree two to obtain the 2-core of G
  - adding a forest rooted at vertices of the 2-core of G

$C_g(k, k+\ell) = \#$ complex graphs on $\mathbb{S}_g$ with vertex set $[k]$ and $k+\ell$ edges

$$= \sum_{i,j} K_g(2\ell - j) \frac{(k)_i}{(2\ell - j)!} \binom{i - a\,\ell - 1}{3\ell - j - 1} i\, k^{k-i-1}$$

Combinatorial Laplace's method

In order to to analyse a sum of the form

$$S(n) \;=\; \sum_{i \in I_n} Q(i)\, R(n-i)$$

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we let $A_n(i) = \log(Q(i)\,R(n-i))$ and $r > 0$ be a simple 'saddle-point' of $A(i)$

$$A_n'(r) = 0, \quad A_n''(r) \neq 0 \qquad\qquad (\textit{in fact } A_n''(r) < 0)$$

and estimate

$$S(n) \;=\; \sum_{i\in I_n} \exp\left(A_n(i)\right) \;=\; \sum_{i\in I_n} \exp\left(A_n(r) + \frac{A_n''(r)}{2}(i-r)^2 + \cdots\right)$$

$$\sim\; \exp\left(A_n(r)\right) \sum_{i=r+O\left(\sqrt{1/|A_n''(r)|}\right)} \exp\left(-\frac{|A_n''(r)|}{2}(i-r)^2 + \cdots\right)$$

$$\sim\; \exp\left(A_n(r)\right) \sqrt{\frac{2\pi}{|A_n''(r)|}}$$

Combinatorial Laplace's method

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$$\begin{aligned} S(n) \;&=\; \sum_{i\in I_n} \exp\left(A_n(i)\right) \;=\; \sum_{i\in I_n} \exp\left(A_n(r) + \frac{A_n''(r)}{2}(i-r)^2 + \cdots\right)\right) \\ &\sim\; \exp\left(A_n(r)\right) \sum_{i=r+O\left(\sqrt{1/|A_n''(r)|}\right)} \exp\left(-\frac{|A_n''(r)|}{2}(i-r)^2 + \cdots\right) \\ &\sim\; \exp\left(A_n(r)\right)\,\sqrt{\frac{2\pi}{|A_n''(r)|}} \qquad\qquad (\textit{ this would be an ideal scenario}) \end{aligned}$$

Hunt for Optimal Main Contribution

[K.– MOSSHAMMER–SPRÜSSEL 17+]

In order to to analyse a sum of the form

$$S(n) \;=\; \sum_{i \in I_n} \exp\left(A_n(i)\right)$$

we determine an *optimal* interval $J_n \subset I_n$ in the sense that it should be

- ▷ large enough so as to provide the main contribution to $S(n)$, i.e.

$$\sum_{i \in I_n \setminus J_n} A_n(i) = o\left(S(n)\right)$$

- ▷ as small as possible so as to yield stronger concentration results

Complex Graphs on $\mathbb{S}_g$

$C_g(k, k+\ell) = \#$ complex graphs on $\mathbb{S}_g$ with vertex set $[k]$ and $k+\ell$ edges

$$= \sum_{i,j} K_g(2\ell - j) \frac{(k)_i}{(2\ell - j)!} \binom{i - a\,\ell - 1}{3\ell - j - 1} i\, k^{k-i-1}$$

in which the main contribution comes from the terms

$$\textit{core-size } i = (1 + O(\sqrt{\ell/k}) + O(1/\sqrt{\ell}))\sqrt{3k\ell} \quad \textit{and} \quad j = \Theta(\sqrt{\ell^3/k})$$

Graphs on $\mathbb{S}_g$

$$|\mathcal{S}_g(n,m)| \;=\; \#\text{ graphs on } \mathbb{S}_g \text{ with vertex set } [n] \text{ and } m = d\,\tfrac{n}{2} \text{ edges}$$

$$\text{for } n^{-1/3} \ll d-1 \ll 1$$

$$= \sum_{k,\ell} \binom{n}{k} C_g(k, k+\ell)\, U(n-k, m-k-\ell)$$

$$= \left(\frac{e}{2-d}\right)^{(2-d)\frac{n}{2}} n^{d\frac{n}{2}-\frac{1}{2}}\, e^{O((d-1)n^{1/3})}$$

in which the main contribution comes from the terms

complex-size $k = (1+o(1))\,(d-1)\,n$

excess & kernel-size $\ell = \Theta(\,(d-1)\,n^{1/3})$

core-size $i = \Theta(\,(d-1)\,n^{2/3})$

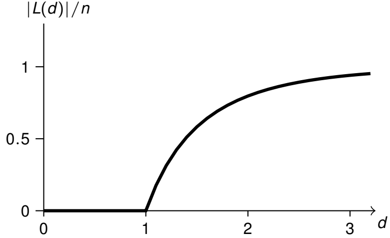
Two Critical Periods in 'Evolution' of $\boldsymbol{R}_g(n, m)$

$L(d) \;=\; \#$ vertices in largest component in $\boldsymbol{R}_g(n, m)$ with $m = d\,\frac{n}{2}$

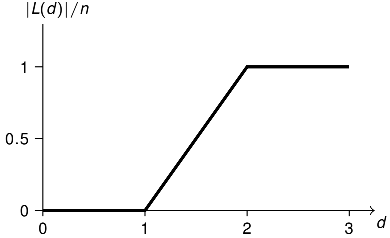
where $d \in (0, 6)$

[K.–Łuczak 12] for $g = 0$

[K.–Moßhammer–Sprüssel 17+] for $g \geq 1$



$\boldsymbol{G}(n, m)$



$\boldsymbol{R}_g(n, m)$

critical period: $d = 1 + O(n^{-\frac{1}{3}})$

first critical period: $d = 1 + O(n^{-\frac{1}{3}})$

second critical period: $d = 2 + O(n^{-\frac{2}{5}})$

Part I

- Decompose along connectivity
  - ▷ Recursive method
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  - ▷ Saddle-point method

Part II

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- Gaussian matrix integral method

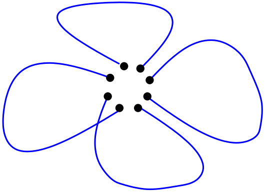
Gaussian Integral

The Gaussian integral is defined by

$$\langle f \rangle \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-\frac{x^2}{2}}\, dx.$$

For example, we have

$$\langle x^n \rangle \;=\; \begin{cases} (n-1)!! & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$



Gaussian Matrix Integral

Let $\mathcal{H}_N \; = \;$ set of $N \times N$ Hermitian matrices $M = (M_{ij})$, i.e., $M_{ij} = \overline{M}_{ji}$
and $dM = \prod_i dM_{ii} \prod_{i<j} d\,\mathrm{Re}(M_{ij}) d\,\mathrm{Im}(M_{ij})$ the Haar measure on $\mathcal{H}_N$.

The Gaussian matrix integral is defined by

$$\langle f \rangle \; = \; \frac{\int_{\mathcal{H}_N} f(M) e^{-N\,\mathrm{Tr}(\frac{M^2}{2})} dM}{\int_{\mathcal{H}_N} e^{-N\,\mathrm{Tr}(\frac{M^2}{2})} dM}.$$

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Using the source integral $\langle e^{\mathrm{Tr}(MS)} \rangle \; = \; e^{\frac{\mathrm{Tr}(S^2)}{2N}}$, we obtain

$$\langle M_{ij} M_{kl} \rangle = \frac{\partial}{\partial S_{ji}} \frac{\partial}{\partial S_{lk}} \langle e^{\mathrm{Tr}(MS)} \rangle \Big|_{S=0} = \frac{\partial}{\partial S_{ji}} \frac{\partial}{\partial S_{lk}} e^{\frac{\mathrm{Tr}(S^2)}{2N}} \Big|_{S=0} = \frac{\delta_{il}\delta_{jk}}{N}$$

where M_{ij} are the entries of the Hermitian matrix $M = (M_{ij}) \in \mathcal{H}_N$.

Gaussian Matrix Integral and Wick's Theorem

Recall that for a Hermitian matrix $M = (M_{ij}) \in \mathcal{H}_N$ we have

$$\langle M_{ij} M_{kl} \rangle = \frac{\partial}{\partial S_{ji}} \frac{\partial}{\partial S_{lk}} \langle e^{\mathrm{Tr}(MS)} \rangle \Big|_{S=0} = \frac{\partial}{\partial S_{ji}} \frac{\partial}{\partial S_{lk}} e^{\frac{\mathrm{Tr}(S^2)}{2N}} \Big|_{S=0} = \frac{\delta_{il}\delta_{jk}}{N}$$

[WICK 50]

Let $M = (M_{ij}) \in \mathcal{H}_N$ and I be a multiset of elements of $N \times N$. Then

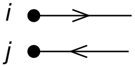
$$\begin{aligned} \langle \sum_I c_I \prod_{ij \in I} M_{ij} \rangle &= \sum_I c_I \sum_{\text{pairing } P \subset I^2} \prod_{(ij,kl) \in P} \langle M_{ij} M_{kl} \rangle \\ &= \sum_I c_I \sum_{\text{pairing } P \subset I^2} \prod_{(ij,kl) \in P} \frac{\delta_{il}\delta_{jk}}{N} \end{aligned}$$

Pictorial Interpretation

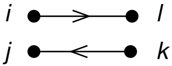
['t HOOFT 74; BRÉZIN–ITZYKSON–PARISI–ZUBER 78; DI FRANCESCO 04 . . .]

Pictorial interpretation of $\langle\, M_{ij} M_{kl} \,\rangle = \frac{\delta_{il}\delta_{jk}}{N}$

$M_{ij} \quad \Longleftrightarrow$



$\langle\, M_{ij} M_{kl} \,\rangle \;=\; \frac{1}{N} \quad \Longleftrightarrow$



and $l = i$

and $k = j$

Pictorial Interpretation for Matrix Integral of Trace

The Gaussian matrix integral of $\mathrm{Tr}(M^n) = \sum_{1\le i_1, i_2, \cdots, i_n \le N} M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_n i_1}$

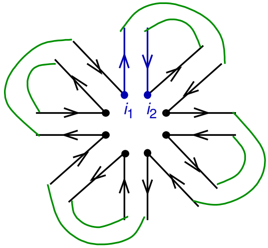
satisfies

$$\langle\, \mathrm{Tr}(M^n)\, \rangle = \langle \sum_{1\le i_1, i_2, \cdots, i_n \le N} M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_n i_1} \rangle$$

$$= \sum_{1\le i_1, i_2, \cdots, i_n \le N} \sum_{P} \prod_{(i_k i_{k+1}, i_l i_{l+1}) \in P} \frac{\delta_{i_k i_{l+1}} \delta_{i_l i_{k+1}}}{N}$$

where P is a partition of $\{ i_1 i_2, i_2 i_3, \cdots, i_n i_1 \}$ into pairs.

$$\langle\, M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_n i_1}\, \rangle \quad \Longleftrightarrow$$



Fat Graphs and Maps

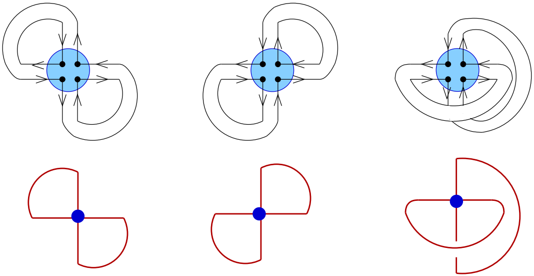
['t Hooft 74; Brézin–Itzykson–Parisi–Zuber 78; Di Francesco 04 . . .]

$$\langle \operatorname{Tr}(M^n) \rangle = \sum_{1 \le i_1, i_2, \cdots, i_n \le N} \sum_{P} \prod_{(i_k i_{k+1}, i_l i_{l+1}) \in P} \frac{\delta_{i_k i_{l+1}} \delta_{i_l i_{k+1}}}{N}.$$

A pairing P with non-zero contribution to $\langle \operatorname{Tr}(M^n) \rangle$

$\Longleftrightarrow$ a fat graph with one island and $n/2$ fat edges ordered cyclically.

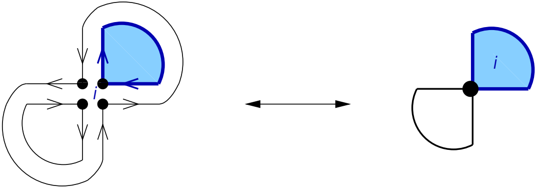
It defines uniquely an embedding on a surface: a map!



Matrix Integral for Maps

Let F be a map with one vertex, $e(F)$ edges, and $f(F)$ faces.

- The edges contribute $N^{-e(F)}$, since each edge contributes N^{-1}.
- The faces contribute $N^{f(F)}$, since each face attains independently any index from 1 to N.



Thus

$$\langle \mathrm{Tr}(M^n) \rangle = \sum_{1 \le i_1, i_2, \cdots, i_n \le N} \sum_P \prod_{(i_k i_{k+1}, i_l i_{l+1}) \in P} \frac{\delta_{i_k i_{l+1}} \delta_{i_l i_{k+1}}}{N} = \sum_F N^{-e(F) + f(F)}$$

where the sum is over all maps F with one vertex.

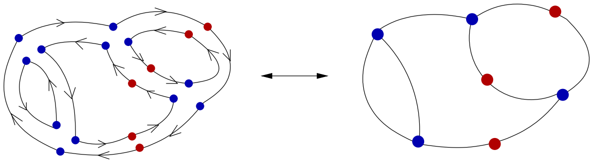
Matrix Integral for Maps and Graphs on Surfaces

For example,

$$\left\langle \left[\mathrm{Tr}(M^3) z_3\right]^4 \left[\mathrm{Tr}(M^2) z_2\right]^3 \right\rangle = \sum_F N^{f(F)-e(F)} {z_3}^4 {z_2}^3,$$

where the sum is over all maps F with

four vertices of degree 3 and three vertices of degree 2.



[K.–LOEBL 09]

The enumeration of graphs embeddable on a surface can be formulated as the Gaussian matrix integral of an ice-type partition function.

Summary

- Decomposition along connectivity
  - ▷ Recursive method
  - ▷ Singularity analysis
  - ▷ Saddle-point method
- Core-Kernel approach
  - ▷ Combinatorial Laplace's method
- Gaussian matrix integral method