Symplectic Schur Q-Functions

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Plan:

- Introduction
- Pfaffian formulas
- Tableaux description
- Positivity

Related paper:

S. Okada, Pfaffian formulas and Schur Q-function identities, arXiv:1706.01029.

Introduction

Hall–Littlewood Functions

Let λ be a partition of length $\leq n$, i.e.,

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n), \quad \lambda_i \in \mathbb{Z}, \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \lambda_n \geq 0,$$

and let $\boldsymbol{x} = (x_1, \dots, x_n)$ be indeterminates. We define the Hall–Littlewood function $P_\lambda(\boldsymbol{x}; t)$ by putting

$$P_\lambda(\boldsymbol{x}; t) = \frac{1}{v_\lambda(t)} \sum_{w \in \mathfrak{S}_n} w \left(\prod_{i=1}^{n} x_i^{\lambda_i} \prod_{1 \leq i < j \leq n} \frac{x_i - t x_j}{x_i - x_j} \right),$$

where $\mathfrak{S}_n$ is the symmetric group and

$$v_\lambda(t) = \prod_{j \geq 0} \prod_{k=1}^{m_j(\lambda)} (1 - t^k), \quad m_j(\lambda) = \#\{i : 1 \leq i \leq n,\ \lambda_i = j\}.$$

It can be shown that $P_\lambda(\boldsymbol{x}) \in \mathbb{Z}[t][x_1, \dots, x_n]^{\mathfrak{S}_n}$.

Schur Functions and Schur Q-Functions

Schur functions $s_\lambda(\boldsymbol{x})$ and Schur Q-functions $Q_\lambda(\boldsymbol{x})$ are obtained by specializing $t = 0$ and $t = -1$ in the Hall–Littlewood functions:

$$s_\lambda(\boldsymbol{x}) = P_\lambda(\boldsymbol{x}; 0), \quad Q_\lambda(\boldsymbol{x}) = 2^{l(\lambda)} P_\lambda(\boldsymbol{x}; -1),$$

where $l(\lambda)$ is the length of λ.

Schur functions $s_\lambda(\boldsymbol{x})$	Schur Q-functions $Q_\lambda(\boldsymbol{x})$
partitions	strict partitions
linear representation of $\mathfrak{S}_n$	projective representation of $\mathfrak{S}_n$
representation of $\mathfrak{gl}(n)$	representation of $\mathfrak{q}(n)$
Grassmannian	Lagrangian Grassmannian
KP hierarchy	BKP hierarchy
determinants	Pfaffians

Symplectic Hall–Littlewood Functions

The symplectic Hall–Littlewood functions (Hall–Littlewood functions associated to the root system of type C_n) are defined by

$$P^C_\lambda(\boldsymbol{x};t) = \frac{1}{W_\lambda(t)} \sum_{w \in W} w \left(\boldsymbol{x}^\lambda \prod_{\alpha \in \Delta^+} \frac{1 - t\boldsymbol{x}^{-\alpha}}{1 - \boldsymbol{x}^{-\alpha}} \right)$$

where $\lambda = \sum_{i=1}^n \lambda_i e_i$ is a dominant weight (identified with a partition of length $\leq n$), W is the Weyl group of type C_n, and

$$W_\lambda = \{w \in W : w\lambda = \lambda\}, \quad W_\lambda(t) = \sum_{w \in W_\lambda} t^{l(w)},$$

$$\Delta^+ = \left\{ e_i \pm e_j : 1 \leq i < j \leq n \right\} \cup \{2e_i : 1 \leq i \leq n\}.$$

It can be shown that

$$P^C_\lambda(\boldsymbol{x};t) \in \mathbb{Z}[t][x_1^{\pm 1}, \dots, x_n^{\pm 1}]^W.$$

Symplectic Schur and Symplectic Schur Q-functions

For a partition λ of length $\leq n$, we define the symplectic Schur function $s^C_\lambda(\boldsymbol{x})$ by

$$s^C_\lambda(\boldsymbol{x}) = P^C_\lambda(\boldsymbol{x}; 0).$$

Then $s^C_\lambda(\boldsymbol{x})$ gives the irreducible character of the symplectic group $\mathbf{Sp}_{2n}$ with highest weight λ.

For a strict partition λ of length $l \leq n$ ($\lambda_1 > \cdots > \lambda_l > 0$), we define the symplectic Schur P-functions $P^C_\lambda(\boldsymbol{x})$ and the symplectic Schur Q-functions $Q^C_\lambda(\boldsymbol{x})$ by

$$P^C_\lambda(\boldsymbol{x}) = P^C_\lambda(\boldsymbol{x}; -1), \quad Q^C_\lambda(\boldsymbol{x}) = 2^l P^C_\lambda(\boldsymbol{x}; -1),$$

respectively.

Main Results

Symplectic Schur Q-functions $Q^C_\lambda(\boldsymbol{x})$ enjoy many properties similar to those of Schur Q-functions $Q_\lambda(\boldsymbol{x})$.

- Nimmo-type formula
- Schur-type formula
- Józefiak–Pragacz-type formula for skew Q-functions
- Tableau description
- Positivity of structure constants (conjectures)

Pfaffian Formulas for Symplectic Schur Q-Functions

Nimmo-type formula

Theorem For a strict partition λ of length l, we have

$$Q_{\lambda}^{C}(\boldsymbol{x}) = \frac{1}{D^{C}(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} A^{C}(\boldsymbol{x}) & \left(f_{\lambda_j}^{C}(x_i) \right)_{\substack{1 \le i \le n \\ 1 \le j \le r}} \\ -{}^{t}\left(f_{\lambda_j}^{C}(x_i) \right)_{\substack{1 \le i \le n \\ 1 \le j \le r}} & O \end{pmatrix},$$

where $r = l$ or $l + 1$ according to whether $n + l$ is even or odd, and

$$f_{d}^{C}(x) = \begin{cases} 2(x^{d} - x^{-d})(x + x^{-1})/(x - x^{-1}) & \text{if } d \ge 1, \\ 1 & \text{if } d = 0, \end{cases}$$

$$A^{C}(\boldsymbol{x}) = \left(\frac{(x_j + x_j^{-1}) - (x_i + x_i^{-1})}{(x_j + x_j^{-1}) + (x_i + x_i^{-1})} \right)_{1 \le i, j \le n},$$

$$D^{C}(\boldsymbol{x}) = \prod_{1 \le i < j \le n} \frac{(x_j + x_j^{-1}) - (x_i - x_i^{-1})}{(x_j + x_j^{-1}) + (x_i + x_i^{-1})} \quad (= \operatorname{Pf} A^{C}(\boldsymbol{x}) \quad \text{if } n \text{ is even}).$$

Schur-type formula

Theorem For a strict partition λ, we have

$$Q^C_\lambda(\boldsymbol{x}) = \mathrm{Pf}\left(Q^C_{(\lambda_i,\lambda_j)}(\boldsymbol{x})\right)_{1\le i<j\le r},$$

where $r = l$ or $l+1$ according to whether l is even or odd, and $Q_{(r,0)}(\boldsymbol{x}) = Q_{(r)}(\boldsymbol{x})$.

Idea of Proof We apply a Pfaffian analogue of Sylvester identity (due to Knuth)

$$\mathrm{Pf}\left(\frac{\mathrm{Pf}\, X([n]\cup\{n+i,n+j\})}{\mathrm{Pf}\, X([n])}\right)_{1\le i,\, j\le r} = \frac{\mathrm{Pf}\, X}{\mathrm{Pf}\, X([n])},$$

where X is an $(n+r)\times(n+r)$ skew symmetric matrix, $[n] = \{1, 2, \cdots, n\}$ and

$$X(I) = \left(x_{i,j}\right)_{i,\, j\in I}.$$

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Proposition

$$\sum_{r=0}^{\infty} Q^C_{(r)}(\boldsymbol{x}) z^r = \prod_{i=1}^{n} \frac{(1+x_i z)(1+x_i^{-1} z)}{(1-x_i z)(1-x_i^{-1} z)}.$$

Proposition

$$Q^C_{(r,s)}(\boldsymbol{x}) = Q^C_{(r)}(\boldsymbol{x}) Q^C_{(s)}(\boldsymbol{x}) + 2\sum_{k=1}^{s}(-1)^k \left(Q^C_{(r+k)}(\boldsymbol{x}) + 2\sum_{i=1}^{k-1} Q^C_{(r+k-2i)}(\boldsymbol{x}) + Q^C_{(r-k)}(\boldsymbol{x})\right) Q^C_{(s-k)}(\boldsymbol{x}).$$

Józefiak–Pragacz-type formula

Theorem For strict partitions λ of length l and μ of length m, we put

$$Q^C_{\lambda/\mu}(\boldsymbol{x}) = \operatorname{Pf}\begin{pmatrix} \left(Q^C_{(\lambda_i,\lambda_j)}(\boldsymbol{x})\right)_{1\le i,\, j\le l} & \left(Q^C_{(\lambda_i-\mu_{r+1-j})}(\boldsymbol{x})\right)_{\substack{1\le i\le l\\ 1\le j\le r}} \\ -{}^t\!\left(Q^C_{(\lambda_i-\mu_{r+1-j})}(\boldsymbol{x})\right)_{\substack{1\le i\le l\\ 1\le j\le r}} & O \end{pmatrix},$$

where $r = m$ or $m+1$ according to whether $l+m$ is even or odd. Then we have

$$Q^C_\lambda(\boldsymbol{x},\boldsymbol{y}) = \sum_{\mu} Q^C_{\lambda/\mu}(\boldsymbol{x}) Q^C_\mu(\boldsymbol{y}),$$

where μ runs over all strict partitions.

Józefiak–Pragacz-type formula

Idea of Proof We apply a Pfaffian analogue of Ishikawa–Wakayama's minor summation formula

$$\sum_J \operatorname{Pf} B(J) \operatorname{Pf} \begin{pmatrix} A & S([m];J) \\ -{}^tS([m];J) & O \end{pmatrix} = \operatorname{Pf}\left(A - SB{}^tS\right),$$

where J runs over all even-element subsets of $[N]$,

$$B(J) = \left(b_{i,j}\right)_{i,\,j\in J}, \quad S([m];J) = \left(s_{i,j}\right)_{1\le i\le m,\, j\in J},$$

to the matrices

$$A = \left(Q^C_{(\lambda_i,\lambda_j)}(\boldsymbol{x})\right), \quad S = \left(Q^C_{(\lambda_i - j)}(\boldsymbol{x})\right), \quad B = \left(-Q^C_{(i,j)}(\boldsymbol{y})\right),$$

and use

$$Q^C_{(\lambda_i,\lambda_j)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{k,\,l\ge 0} Q^C_{(\lambda_i-k)}(\boldsymbol{x}) Q^C_{(\lambda_j-l)}(\boldsymbol{x}) Q^C_{(k,l)}(\boldsymbol{y}).$$

Tableau Description of Symplectic Schur Q-Functions

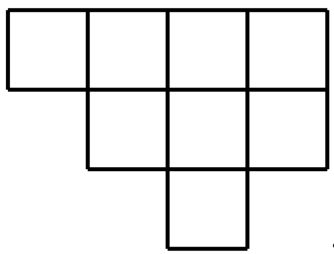
Shifted Diagram

For a strict partition λ, the shifted diagram $S(\lambda)$ is defined by

$$S(\lambda) = \{(i, j) \in \mathbb{Z}^2 : 1 \le i \le l(\lambda),\ i \le j \le i + \lambda_i - 1\}.$$

Usually we represent the shifted diagram by replacing lattice points by cells.

Example If $\lambda = (4, 3, 1)$, then the corresponding shifted diagram is depicted as



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Symplectic Primed Shifted Tableau

Definition (King–Hamel) A symplectic primed shifted tableau of shape λ is a filling of the boxes in the shifted diagram $S(\lambda)$ with entries from

$$1' < 1 < \overline{1}' < \overline{1} < 2' < 2 < \overline{2}' < \overline{2} < \cdots < n' < n < \overline{n}' < \overline{n}$$

satisfying the following conditions:

- the entries in each row and in each column are weakly increasing;
- each unprimed entry appears at most once in every column;
- each primed entry appears at most once in every row;
- at most one element from $\{k', k, \overline{k}', \overline{k}\}$ appears on the main diagonal.

Example

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \overline{2}' & 3' \\ \hline \multicolumn{1}{c|}{} & 2' & \overline{2}' & 3 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array} .$$

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To such a tableau T, we associate a monomial given by

$$\boldsymbol{x}^T = \prod_{k=1}^{n} x_k^{\#\{k', k \text{ in } T\} - \#\{\overline{k}', \overline{k} \text{ in } T\}}.$$

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Example

$T =$

1	1	$\overline{2}'$	$3'$
	$2'$	$\overline{2}'$	3
		4	

$$\boldsymbol{x}^T = x_1^2 x_2^{-1} x_3^2 x_4.$$

Tableau Description of Symplectic Schur Q-Functions

Theorem (Conjectured by King–Hamel) For a strict partition λ, we have

$$Q^C_\lambda(\boldsymbol{x}) = \sum_T \boldsymbol{x}^T,$$

where T runs over all symplectic primed shifted tableaux of shape λ.

Idea of Proof Both sides satisfy

- $Q^C_\lambda(x_1, \cdots, x_{n-1}, x_n) = \sum_\mu Q^C_\mu(x_1, \cdots, x_{n-1}) Q^C_{\lambda/\mu}(x_n),$
- $Q^C_{\lambda/\mu}(x_n) = 0$ unless $\lambda \supset \mu$ and $l(\lambda) - l(\mu) \leq 1$,
- $Q^C_{\lambda/\mu}(x_n) = \det \left(Q^C_{(\lambda_i - \mu_j)}(x_n) \right)_{1 \leq i, j \leq l(\lambda)}$ if $l(\lambda) - l(\mu) \leq 1$.

Hence the proof is reduced to the case where $\lambda = (r)$ and $\boldsymbol{x} = (x_n)$.

Positivity Conjectures

Structure Constants for Symplectic Schur P-Functions

The symplectic Schur P-functions $\{P^C_\lambda(\boldsymbol{x})\}_{\lambda:\text{strict partition of length} \le n}$ form a basis of

$$\Gamma^C_n = \left\{ f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^W : f(t, -t, x_3, \dots, x_n) \text{ is independent of } t \right\}.$$

Conjecture 1 Given two strict partitions μ and ν of length $\le n$, we can expand

$$P^C_\mu(\boldsymbol{x}) \cdot P^C_\nu(\boldsymbol{x}) = \sum_\lambda \widetilde{f}^\lambda_{\mu,\nu} P^C_\lambda(\boldsymbol{x}),$$

where λ runs over all strict partitions of length $\le n$. Then the structure constants $\widetilde{f}^\lambda_{\mu,\nu}$ are nonnegative integers.

It can be proved that Conjecture 1 is true if $l(\nu) = 1$ (Pieri-type rule).

Pieri Rule for Symplectic P-functions

Theorem Let μ and λ be strict partitions of length $\leq n$ and let r be a positive integer. Then we have

(1) $\widetilde{f}^{\lambda}_{\mu,(r)} = 0$ unless $l(\lambda) = l(\mu)$ or $l(\mu) + 1$.

(2) If $l(\lambda) = l(\mu)$ or $l(\mu) + 1$, then

$$\widetilde{f}^{\lambda}_{\mu,(r)} = \sum_{\kappa} 2^{a(\mu,\kappa)+a(\lambda,\kappa)-\chi[l(\mu)>l(\kappa)]-1},$$

where κ runs over all strict partitions satisfying

$$\mu_1 \geq \kappa_1 \geq \mu_2 \geq \kappa_2 \geq \dots, \quad \lambda_1 \geq \kappa_1 \geq \lambda_2 \geq \kappa_2 \geq \dots,$$
$$(|\mu| - |\kappa|) + (|\lambda| - |\kappa|) = r,$$

and

$$a(\mu, \kappa) = \#\{i : \mu_i > \kappa_i > \mu_{i+1}\}, \quad a(\lambda, \kappa) = \#\{i : \lambda_i > \kappa_i > \lambda_{i+1}\},$$
$$\chi[l(\mu) > l(\kappa)] = \begin{cases} 1 & \text{if } l(\mu) > l(\kappa), \\ 0 & \text{otherwise.} \end{cases}$$

Positivity Conjectures for symplectic P-functions

Conjecture 2 For a strict partition of length $\leq n$, we can expand

$$P_\lambda(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}) = \sum_\mu c_{\lambda,\mu} P^C_\mu(\boldsymbol{x}),$$

where μ runs over all strict partitions of length $\leq n$. Then the coefficients $c_{\lambda,\mu}$ are nonnegative integers.

Known Case If $l(\lambda) \leq 2$, then Conjecture 2 is true.

Positivity Conjectures for symplectic P-functions

Conjecture 2 For a strict partition of length $\leq n$, we can expand

$$P_\lambda(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}) = \sum_\mu c_{\lambda,\mu} P_\mu^C(\boldsymbol{x}),$$

where μ runs over all strict partitions of length $\leq n$. Then the coefficients $c_{\lambda,\mu}$ are nonnegative integers.

Known Case If $l(\lambda) \leq 2$, then Conjecture 2 is true.

Conjecture 3 For a strict partition of length $\leq n$, we can expand

$$P_\lambda^C(\boldsymbol{x}) = \sum_\mu g_{\lambda,\mu} s_\mu^C(\boldsymbol{x}),$$

where μ runs over all partitions of length $\leq n$. Then the coefficients $g_{\lambda,\mu}$ are nonnegative integers.

Known Case If $l(\lambda) = 1$ or n, then Conjecture 3 is true.