Hook formulas for skew shapes

Greta Panova (University of Pennsylvania and IAS Princeton)
joint with Alejandro Morales (UCLA), Igor Pak (UCLA)

ESI Workshop I, October 2017

Standard Young Tableaux

Irreducible representations of S_n:

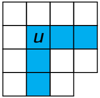
Specht modules $\mathbb{S}_\lambda$, for all $\lambda \vdash n$.

Basis for $\mathbb{S}_\lambda$: **Standard Young Tableaux** of shape λ:

$$\lambda = (2,2,1): \quad \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \\ \cline{1-1}\end{array} : \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$$

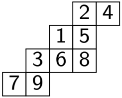
Hook-length formula [Frame-Robinson-Thrall]:

$$\dim \mathbb{S}_\lambda = \#\{\text{SYTs of shape } \lambda\} = f^\lambda = \frac{|\lambda|!}{\prod_{u \in \lambda} h_u} = \frac{5!}{4 * 3 * 2 * 1 * 1}$$

Hook length of box $u = (i,j) \in \lambda$: $h_u = \lambda_i - j + \lambda'_j - i + 1 = \#\left\{ \blacksquare \in \right.$  $\left.\right\}$

Counting skew SYTs

Outer shape λ, inner shape μ, e.g. for $\lambda = (5,4,4,2), \mu = (3,2,1)$

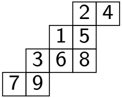


Jacobi-Trudi[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)}.$$

Counting skew SYTs

Outer shape λ, inner shape μ, e.g. for $\lambda = (5,4,4,2), \mu = (3,2,1)$



Jacobi-Trudi[Feit 1953]:

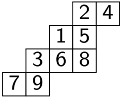
$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)}.$$

Littlewood-Richardson:

$$f^{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} f^{\nu}$$

Counting skew SYTs

Outer shape λ, inner shape μ, e.g. for $\lambda = (5,4,4,2), \mu = (3,2,1)$



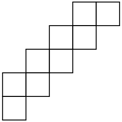
Jacobi-Trudi[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)}.$$

Littlewood-Richardson:

$$f^{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} f^{\nu}$$

No product formula, e.g. $\lambda/\mu = \delta_{n+2}/\delta_n$:



$f^{\delta_{n+2}/\delta_n} = E_{2n+1}$:

$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

Euler numbers: $2, 5, 16, 61 \ldots$.

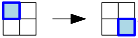
Hook-Length formula for skew shapes

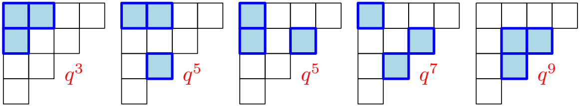
Theorem (Naruse, SLC, September 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

where $\mathcal{E}(\lambda/\mu)$ *is the set of* excited diagrams *of* λ/μ.

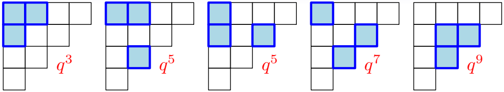
Excited diagrams:

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$



$$f^{(4321/21)} = 7! \left(\frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} \right) = 61$$

Hook-Length formula for skew shapes



$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2 \times \frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)} + \cdots$$

Theorem (Morales-Pak-P)

For skew SSYTs, we have that

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(\lambda/\mu)} q^{|T|} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in [\lambda] \setminus D} \left[\frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}} \right].$$

Theorem (Morales-Pak-P)

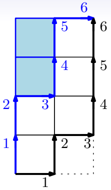
For (reverse) plane partitions of skew shape λ/μ we have that

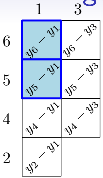
$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$

where $PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda] \setminus D, \text{ for some } D \in \mathcal{E}(\lambda/\mu)\}$ is the set of "pleasant diagrams".

Other recent proof by [M. Konvalinka]

Algebraic proof for SSYTs:



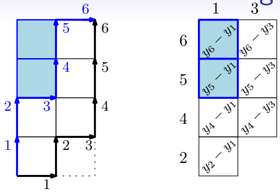


$v = 245613$, $w = 361245$

[Ikeda-Naruse, Kreiman]:
Let $w \preceq v$ be Grassmannian permutations whose unique descent is at position d with corresponding partitions $\mu \subseteq \lambda \subseteq d \times (n-d)$. Then the Schubert class X_w for w at point v is:

$$[X_w]\big|_v = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} \left(y_{v(d+j)} - y_{v(d-i+1)}\right).$$

Algebraic proof for SSYTs:



$v = 245613$, $w = 361245$

[Ikeda-Naruse, Kreiman]:
Let $w \preceq v$ be Grassmannian permutations whose unique descent is at position d with corresponding partitions $\mu \subseteq \lambda \subseteq d \times (n-d)$. Then the Schubert class X_w for w at point v is:

$$[X_w]\big|_v = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} \left(y_{v(d+j)} - y_{v(d-i+1)}\right).$$

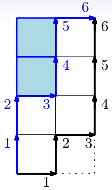
Factorial Schur functions:

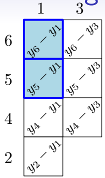
$$s_\mu^{(d)}(\mathbf{x}|\mathbf{a}) := \frac{\det\left[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\right]_{i,j=1}^{d}}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

[Knutson-Tao, Lakshmibai–Raghavan–Sankaran] Schubert class at a point:

$$[X_w]\big|_v = (-1)^{\ell(w)} s_\mu^{(d)}\left(y_{v(1)},\ldots,y_{v(d)}|y_1,\ldots,y_{n-1}\right).$$

Algebraic proof for SSYTs:





$v = 245613, \; w = 361245$

[Ikeda-Naruse, Kreiman]:
Let $w \preceq v$ be Grassmannian permutations whose unique descent is at position d with corresponding partitions $\mu \subseteq \lambda \subseteq d \times (n-d)$. Then the Schubert class X_w for w at point v is:

$$[X_w]\big|_v = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} \left(y_{v(d+j)} - y_{v(d-i+1)}\right).$$

Factorial Schur functions:

$$s^{(d)}_\mu(\mathbf{x}|\mathbf{a}) := \frac{\det\left[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\right]_{i,j=1}^{d}}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

[Knutson-Tao, Lakshmibai–Raghavan–Sankaran] Schubert class at a point:

$$[X_w]\big|_v = (-1)^{\ell(w)} s^{(d)}_\mu\left(y_{v(1)}, \ldots, y_{v(d)} | y_1, \ldots, y_{n-1}\right).$$

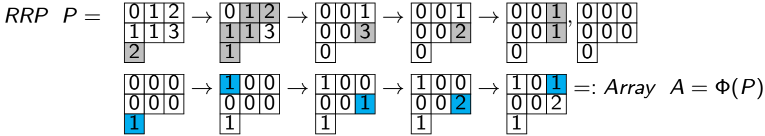
Evaluation at $y = 1, q, q^2, \ldots,$ $v(d+1-i) = \lambda_i + d + 1 - i$, $x_i \to y_{v(i)} = q^{\lambda_i+d+1-i}$
$\to$ Jacobi-Trudi

$$s^{(d)}_\mu(q^{v(1)}, \ldots | 1, q, \ldots) = \frac{\det[\prod_{r=1}^{\mu_j+d-j}(q^{\lambda_i+d+1-i} - q^r)]_{i,j=1}^{d}}{\prod_{i<j}(q^{\lambda+d+1-i} - q^{\lambda_j+d+1-j})} = \ldots.$$

$$\ldots[\textit{simplifications}]\ldots = \det[h_{\lambda_i-i-\mu_j+j}(1, q, \ldots)] = s_{\lambda/\mu}(1, q, \ldots)$$

Combinatorial proofs:

Hillman-Grassl map Φ: Reverse Plane Partitions of shape λ to Arrays of shape λ:

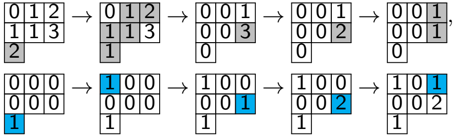


$Weight(P) = |P| = 0 + 1 + 2 + 1 + 1 + 3 + 2 = 10 =$
$= \sum_{i,j} A_{i,j} hook(i,j) = 1 * 5 + 1 * 2 + 2 * 1 + 1 * 1 =: weight(A)$

Combinatorial proofs:

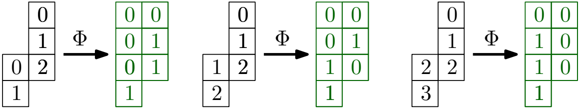
Hillman-Grassl map Φ: Reverse Plane Partitions of shape λ to Arrays of shape λ:

$RRP \ \ P =$



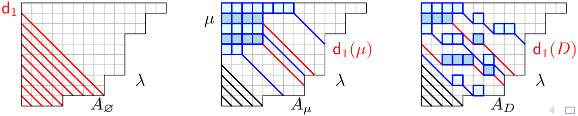
$=: Array \ \ A = \Phi(P)$

$Weight(P) = |P| = 0 + 1 + 2 + 1 + 1 + 3 + 2 = 10 =$
$= \sum_{i,j} A_{i,j} hook(i,j) = 1 * 5 + 1 * 2 + 2 * 1 + 1 * 1 =: weight(A)$

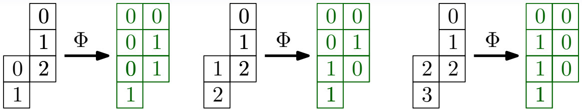


Theorem (Morales-Pak-P)

The restricted Hillman-Grassl map is a bijection from the SSYTs of shape λ/μ to the excited arrays (diagrams in $\mathcal{E}(\lambda/\mu)$ with nonzero entries on the broken diagonals) .

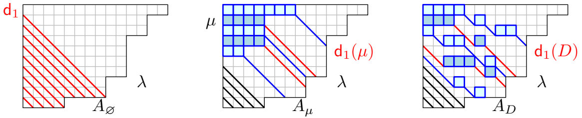


Combinatorial proofs:



Theorem (Morales-Pak-P)

The restricted Hillman-Grassl map is a bijection from the SSYTs of shape λ/μ to the excited arrays (diagrams in $\mathcal{E}(\lambda/\mu)$ with nonzero entries on the broken diagonals) .



Proof sketch:

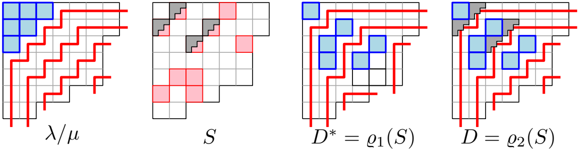
Issue: enforce 0s on μ and strict increase down columns on λ/μ.
Show $\Phi^{-1}(A)$ is column strict in λ/μ + support in λ/μ via properties of RSK
(Integer partition on kth diagonal
$(\ldots, P_{2,2+k}, P_{1,1+k}) = shape(RSK(A_k^T))$ is shape of RSK tableau on the corresponding subrectangle of A)
Thus, Φ^{-1} is injective: restricted arrays $\to$ SSYTs of shape λ/μ.
Bijective: use the algebraic identity.

Hillman-Grassl on skew RPPs

Weakly increasing rows:
Skew reverse plane partitions $\Leftrightarrow$ arrays with support *"pleasant diagrams"*:

$$PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda] \setminus D, \text{ for some } D \in \mathcal{E}(\lambda/\mu)\}$$

– subsets of complements of the excited diagrams, identified by the "high peaks".

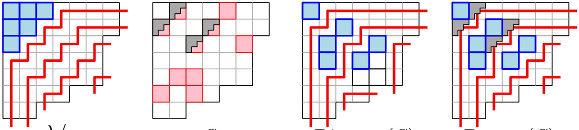


Hillman-Grassl on skew RPPs

Weakly increasing rows:
Skew reverse plane partitions $\Leftrightarrow$ arrays with support *"pleasant diagrams"*:

$$PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda] \setminus D, \text{ for some } D \in \mathcal{E}(\lambda/\mu)\}$$

– subsets of complements of the excited diagrams, identified by the "high peaks".

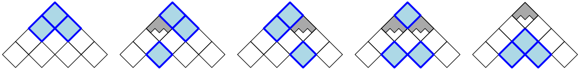


λ/μ $\quad S \quad$ $D^* = \varrho_1(S)$ $\quad D = \varrho_2(S)$

Theorem (MPP)

The HG map is a bijection between skew RPPs of shape λ/μ and arrays with certain nonzero entries (at the "high peaks"):

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$



2^7 $\quad 2^6$ $\quad 2^6$ $\quad 2^5$ $\quad 2^6$

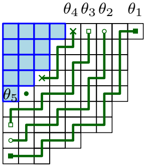
With P-partitions/limit: combinatorial proof of original Naruse Hook-Length Formula for $f^{\lambda/\mu}$..

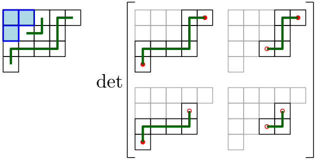
Non-intersecting lattice paths

Theorem[Lascoux-Pragacz, Hamel-Goulden] If $(\theta_1, \dots, \theta_k)$ is a Lascoux–Pragacz decomposition (i.e. maximal outer border strip decomposition) of λ/μ, then

$$s_{\lambda/\mu} = \det \left[s_{\theta_i \# \theta_j} \right]_{i,j=1}^{k}.$$

where $s_\varnothing = 1$ and $s_{\theta_i \# \theta_j} = 0$ if the $\theta_i \# \theta_j$ is undefined.
θ_1 – border strip following the inner border of λ;
θ_i – inner border of $\lambda \setminus (\theta_1 \cup \cdots \cup \theta_{i-1})$ etc until μ is hit,
then – border strips from each connected part etc.
Ordering: corners.
Strip $\theta_i \# \theta_j :=$ shape of θ_1 between the diagonals of the endpoints of θ_i and θ_j.





NHLF for border strips

Lemma (MPP)

For a border strip $\theta = \lambda/\mu$ with end points (a, b) and (c, d) we have

$$s_\theta(1, q, q^2, \ldots,) = \sum_{\substack{\gamma:(a,b)\to(c,d),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}}.$$

s  $(1, q, q^2, \ldots) = \frac{q^3}{(1-q^2)(1-q^1)(1-q^3)(1-q^1)(1-q^2)}$ $\gamma=(3,1),(3,2),(2,2),(2,3),(1,3)$:  $+ \frac{q^4}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$ 

$+ \frac{q^1}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$  $+ \frac{q^7}{(1-q)^2(1-q^3)(1-q^4)^2}$  $+ \frac{q^6}{(1-q)^2(1-q^5)(1-q^4)^2}$ 

Proofs: induction on $|\lambda/\mu|$, or [multivariate] Chevalley formula for factorial Schurs.

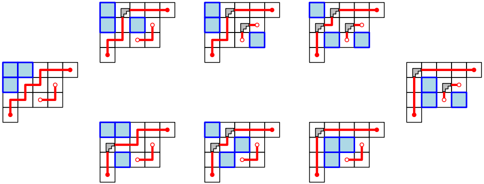
NHLF for border strips

Lemma (MPP)

For a border strip $\theta = \lambda/\mu$ with end points (a, b) and (c, d) we have

$$s_\theta(1, q, q^2, \dots,) = \sum_{\substack{\gamma:(a,b)\to(c,d),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}}.$$

Excited diagrams for $\lambda/\mu \leftrightarrow$ Non-Intersecting Lattice Paths:



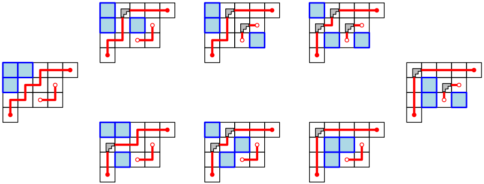
NHLF for border strips

Lemma (MPP)

For a border strip $\theta = \lambda/\mu$ with end points (a, b) and (c, d) we have

$$s_\theta(1, q, q^2, \dots,) = \sum_{\substack{\gamma:(a,b)\to(c,d),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}}.$$

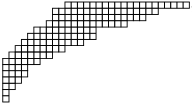
Excited diagrams for $\lambda/\mu \leftrightarrow$ Non-Intersecting Lattice Paths:

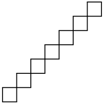
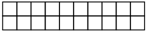


$$s_{\lambda/\mu} \underset{\text{Lascoux-Pragacz}}{=} \det\left[s_{\theta_i\#\theta_j} \right]_{i,j=1}^{k} \underset{s_{\text{Border Strip}}}{=} \det\left[\sum_{\gamma:(a_i,b_i)\to(c_j,d_j)} \prod_{u\in\gamma} \frac{q^{\cdot\cdot}}{1-q^{h_u}} \right]$$

$$\underset{Lindstrom-Gessel-Viennot}{=} \sum_{NILP:\gamma_1,\dots} \prod_{u\in\gamma_1\cup\dots} \frac{q^{\cdot\cdot}}{1-q^{h_u}} \underset{\mathcal{E}(\lambda/\mu)=NILP}{=} \sum_{D\in ED(\lambda/\mu)} \prod_{u\in D} \frac{q^{\cdot\cdot\cdot}}{1-q^{h_u}}$$

Asymptotics of the number of skew SYTs

$\lambda/\mu =$ , $|\lambda/\mu| = n \to \infty$

 $f^{\lambda/\mu} = n!$  $f^{\lambda/\emptyset} = C_{n/2} \approx 2^n \frac{2\sqrt{2}}{n^{3/2}\sqrt{\pi}}$

Question: What is the asymptotic value of $f^{\lambda/\mu}$, $|\lambda/\mu| = n$ as $n \to \infty$ and λ, μ change under various regimes:

0. If $\mu = \emptyset$, then $f^{\lambda} \sim \sqrt{n!}(1 + O(1/n))$ for $\lambda \sim$ Plancherel.

1. [Stanley, 2001]: when μ is fixed, $\lambda^n \to (a; b)$ (Frobenius limit):

$$f^{\lambda^n/\mu} \sim f^{\lambda^n} s_\mu(\rho_a^+; \rho_b^-)(1 + O(1/n)),$$

where ρ_a^+, ρ_b^- are the corresponding specializations.
Similar results in [Corteel-Goupil-Schaeffer]
[Okounkov-Olshanski]

Tool

Naruse Hook-Length formula:

$$f^{\lambda/\mu} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in D} \frac{1}{h_u}.$$

Define the "naive" hook-length formula:

$$F(\lambda/\mu) := \prod_{u \in \lambda/\mu} \frac{1}{h_u}.$$

			5	4	1
		5	3	2	
	7	4	2	1	
	4	1			
4	2				
3	1				
1					

$$F((6,5,5,3,2,2,1)/(3,2,1,1)) = \frac{1}{5{\cdot}4{\cdot}1{\cdot}5{\cdot}3{\cdot}2{\cdot}7{\cdot}4{\cdot}2{\cdot}1{\cdot}4{\cdot}1{\cdot}4{\cdot}2{\cdot}3{\cdot}1{\cdot}1}$$

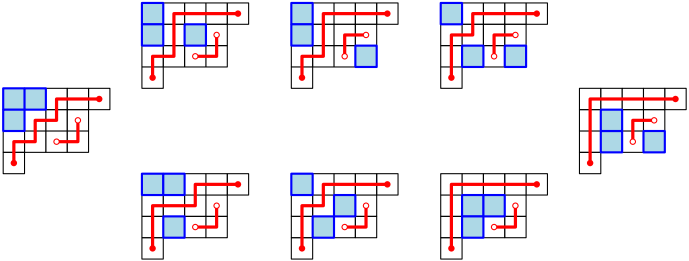
Corollary

$$F(\lambda/\mu) \le f^{\lambda/\mu} \le |\mathcal{E}(\lambda/\mu)| F(\lambda/\mu)$$

General bounds: size of $\mathcal{E}(\lambda/\mu)$

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\,\mathcal{E}(\lambda/\mu)|F(\lambda/\mu)$$

$\mathcal{E}(\lambda/\mu) = \{$ Non-intersecting Lattice Paths in λ/μ $\}$



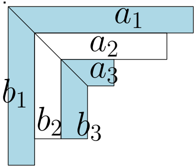
Lemma (MPP)

If $|\lambda/\mu| = n$ then $\mathcal{E}(\lambda/\mu) \leq 2^n$.

Lemma (MPP)

If d is the Durfee square size of λ, then $\mathcal{E}(\lambda/\mu) \leq n^{2d^2}$.

The "linear" regime



$a(\lambda) = (a_1, a_2, \ldots)$, $b(\lambda) = (b_1, b_2, \ldots)$ – Frobenius coordinates of λ. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta := (\beta_1, \ldots, \beta_k)$ be fixed sequences in $\mathbb{R}_+^k$.

Thoma–Vershik–Kerov (TVK) *limit* if $a_i/n \to \alpha_i$ and $b_i/n \to \beta_i$ as $n \to \infty$, for all $1 \le i \le k$.

Theorem (MPP)

Let $\{\lambda^{(n)}/\mu^{(n)}\}$ *be a sequence of skew shapes with a TVK limit, i.e. suppose* $\lambda^{(n)} \to (\alpha, \beta)$, *where* $\alpha_1, \beta_1 > 0$, *and* $\mu^{(n)} \to (\pi, \tau)$ *for some* $\alpha, \beta, \pi, \tau \in \mathbb{R}_+^k$. *Then*

$$\log f^{\lambda^{(n)}/\mu^{(n)}} = cn + o(n) \quad \textit{as} \quad n \to \infty,$$

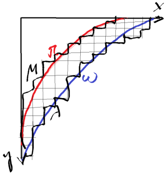
where

$$c = \gamma \log \gamma - \sum_{i=1}^{k} (\alpha_i - \pi_i) \log(\alpha_i - \pi_i) - \sum_{i=1}^{k} (\beta_i - \tau_i) \log(\beta_i - \tau_i)$$

and

$$\gamma = \sum_{i=1}^{k} (\alpha_i + \beta_i - \pi_i - \tau_i).$$

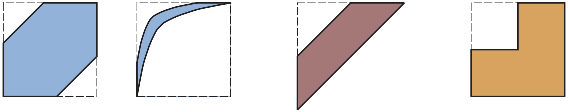
The stable shape: $\sqrt{n}$ scale



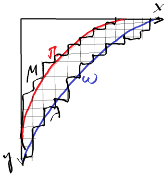
Theorem (MPP)

Let $\omega, \pi : [0, a] \to [0, b]$ be continuous non-increasing functions, and suppose that area$(\omega/\pi) = 1$. Let $\{\lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew shapes with the stable shape ω/π, i.e. $[\lambda^{(n)}]/\sqrt{n} \to \omega$, $[\mu^{(n)}]/\sqrt{n} \to \pi$. Then

$$\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2} n \log n \quad \text{as} \quad n \to \infty.$$



The stable shape: $\sqrt{n}$ scale



Theorem (MPP)

Suppose $(\sqrt{N} - L)\omega \subset [\lambda^{(n)}](\sqrt{N} + L)\omega$ *for some* $L > 0$*, and similarly for* $\mu^{(n)}$ *wrt* π*, then*

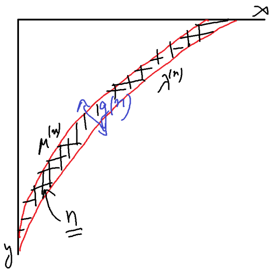
$$-\big(1+c(\omega/\pi)\big)n+o(n) \leq \log f^{\lambda^{(n)}/\mu^{(n)}} - \frac{1}{2}n\log n \leq -\big(1+c(\omega/\pi)\big)n+\log \mathcal{E}(\lambda^{(n)}/\mu^{(n)})+o(n),$$

as $n \to \infty$*, where*

$$c(\omega/\pi) = \iint_{\omega/\pi} \log h(x,y)dxdy,$$

where $h(x,y)$ *is the hook length from* (x,y) *to* ω*.*

Subpolynomial depth, "thin" shapes



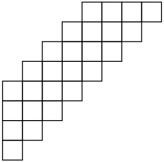
Suppose
depth$:= \max_{u \in \lambda/\mu} h_u =: g(n) = n^{o(1)}$
(subpolynomial growth).

Theorem (MPP)

Let $\{\nu_n = \lambda^{(n)}/\mu^{(n)}\}$ *be a sequence of skew partitions with a subpolynomial depth shape associated with the function* $g(n)$. *Then*

$$\log f^{\nu_n} = n \log n - \Theta\big(n \log g(n)\big) \quad \text{as} \quad n \to \infty.$$

Thick ribbons



Theorem (MPP)

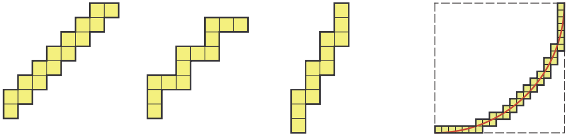
Let $\gamma_k := (\delta_{2k}/\delta_k)$, *where* $\delta_k = (k-1, k-2, \dots, 2, 1)$. *Then*

$$\frac{1}{6} - \frac{3}{2}\log 2 + \frac{1}{2}\log 3 + o(1) \leq \frac{1}{n}\left(\log f^{\gamma_k} - \frac{1}{2} n \log n\right) \leq \frac{1}{6} - \frac{7}{2}\log 2 + 2\log 3 + o(1),$$

where $n = |\gamma_k| = k(3k-1)/2$.

Question: Does there exist a c, s.t. $c = \lim_{n\to\infty} \frac{1}{n}\left(\log f^{\gamma_k} - \frac{1}{2} n \log n\right)$?
Answer: Yes (Martin Tassy's and others work in progress)
Jay Pantone's implementation (method of differential approximants) on 150+ terms of the sequence $\{\log f^{\gamma_k}\}$ to approximate $c \approx -0.1842$.

Thin ribbons



Zigzag: $\rho_k := \delta_{k+2}/\delta_k$, $E_n = |\{\sigma \in S_n : \sigma(1) < \sigma(2) > \sigma(3) < \cdots\}|$ – Euler numbers, alternating permutations.

$$f^{\rho_n} = E_{2n+1}; \qquad E_m \sim m!(2/\pi)^m 4/\pi(1 + o(1))$$

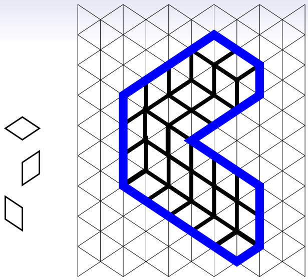
From theorem: $F(\rho_k) = n!/3^k$, $\mathcal{E}(\rho_k) = C_k$, so

$$\frac{(2k+1)!}{3^k} \leq E_{2k+1} \leq \frac{(2k+1)!C_k}{3^k}$$

Problem: If $\gamma_n := \lambda/\mu$ is a border strip (ribbon of thickness 1, n boxes) approaching a given curve γ under rescaling by n, what is $\log f^{\gamma_n} - n \log n$ in terms of γ? Is it true that $\frac{\log f^{\gamma_n} - n \log n}{n} \to c(\gamma)$ for some constant $c(\gamma)$? (Permutations with certain descent sequences)

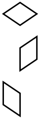
Lozenge tilings

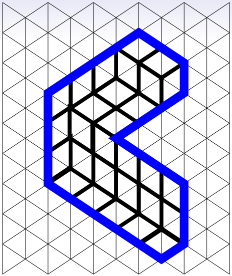
Tilings of a domain Ω (on a triangular lattice) with elementary rhombi of 3 types ("lozenges").



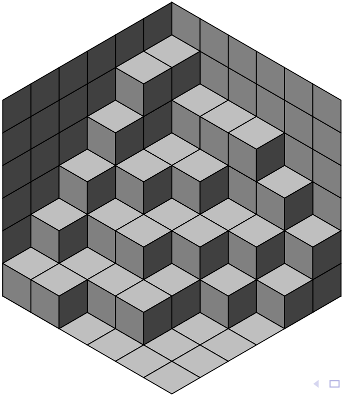
Lozenge tilings

Tilings of a domain Ω (on a triangular lattice) with elementary rhombi of 3 types ("lozenges").



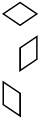


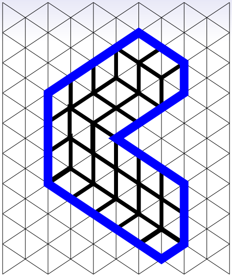
5	4	4	4	3	2
5	3	3	2	2	1
4	3	2	2	1	
3	2	2	1		
2	1	1	1		
1	1				



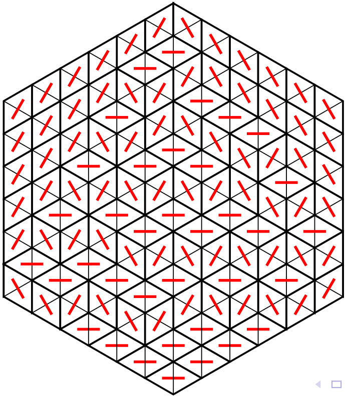
Lozenge tilings

Tilings of a domain Ω (on a triangular lattice) with elementary rhombi of 3 types ("lozenges").



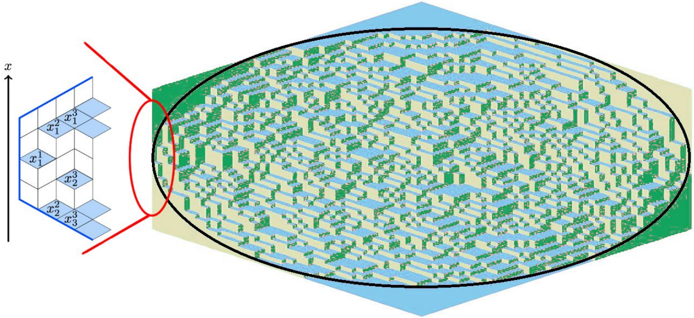


5	4	4	4	3	2
5	3	3	2	2	1
4	3	2	2	1	
3	2	2	1		
2	1	1	1		
1	1				



Classical probabilistic questions: limit behavior

Question: Fix Ω in the plane and let *grid size* $\to 0$, what are the properties of *uniformly random* tilings of Ω?



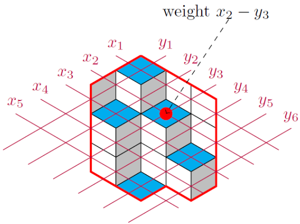
Frozen regions (polygonal domains), "limit shapes" of the surface of the height function (plane partition).

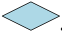
([Cohn–Larsen–Propp, 1998], [Kenyon–Okounkov, 2005], [Cohn–Kenyon–Propp, 2001; Kenyon-Okounkov-Sheffield, 2006] and newer via Schur generating functions [Borodin, Corwin, Bufetov–Gorin, Petrov, etc])

Behavior near boundary: Gaussian Unitary Ensemble eigenvalues,

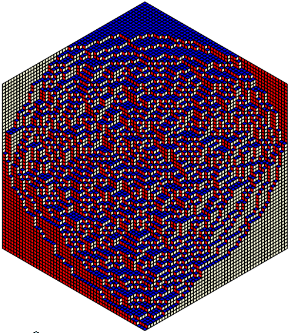
conjectured by [Okounkov-Reshetikhin, 2006], proofs – hexagon [Johansson-Nordenstam, 2006], more general shapes [Gorin-Panova, 2012]

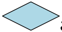
Multivariate local weights

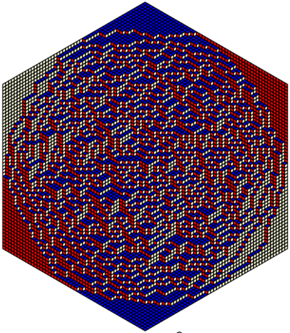


Total weight $= \prod$  at (i,j) $(x_i - y_j)$

$$(x_1 - y_1)(x_2 - y_3)(x_3 - y_5)(x_3 - y_2)(x_5 - y_5).$$



 at $(i,j) = 2N - (i+j)$



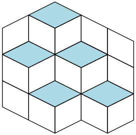
 $= 1$

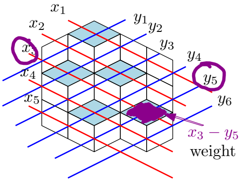
Lozenge tilings with multivariate weights

Plane partitions with base μ, height d

weights of horizontal lozenges $= x_i - y_j$

3	2	1
2	1	

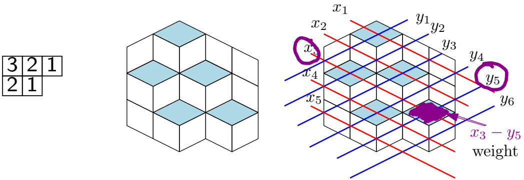




Lozenge tilings with multivariate weights

Plane partitions with base μ, height d

weights of horizontal lozenges $= x_i - y_j$



Theorem (Morales-Pak-P)

Consider tilings with base μ and height d, we have that

$$\sum_{T \in \Omega_{\mu,d}} \prod_{(i,j)\in T} (x_i - y_j) = \det[A_{i,j}(\mu,d)]_{i,j=1}^{d+\ell(\mu)},$$

where

$$A_{i,j}(\mu,d) := \begin{cases} \frac{(x_i-y_1)\cdots(x_i-y_{d+\ell(\mu)-j})}{(x_i-x_{i+1})\cdots(x_i-x_{d+\ell(\mu)})}, & \text{when } j = \ell(\mu)+1,\ldots,\ell(\mu)+d, \\ \frac{(x_i-y_1)\cdots(x_i-y_{\mu_j+d})}{(x_i-x_{i+1})\cdots(x_i-x_{d+j})}, & \text{when } j = i-d,\ldots,\ell(\mu), \\ 0, & \text{when } j < i-d. \end{cases}$$

Corollary (Krattenthaler, Stanley etc)

Consider the set $PP(\mu, d)$ of plane partitions of base μ and entries less than or equal to d. Then their volume generating function is given by the following determinantal formula

$$\sum_{P \in PP(\mu,d)} q^{|P|} = q^{\sum_r r\mu_r} \det[C_{i,j}]_{i,j=1}^{\ell+d},$$

where

$$C_{i,j} = \begin{cases} \frac{(-1)^{d+\ell-i} q^{(d-i)(d+\ell-j) - \frac{(d-i+\ell)(d-i-\ell-1)}{2}}}{(q;q)_{d+\ell-i}}, & \text{when } j = \ell+1, \ldots, \ell+d, \\ \frac{(-1)^{d+j-i} q^{(d-i)(\mu_j+d) - \frac{(d+j-i)(d-i-j-1)}{2}}}{(q;q)_{d+j-i}}, & \text{when } j = i-d, \ldots, \ell, \\ 0, & \text{when } j < i-d, \end{cases}$$

where $(q;q)_m = (1-q)\cdots(1-q^m)$ is the $q-$Pochhammer symbol.

Theorem (Morales-Pak-P)

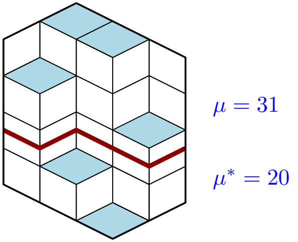
Consider tilings of the $a \times b \times c \times a \times b \times c$ (base $a \times b$, height c) hexagon with horizontal lozenges having weights $x_i - y_j$, i.e. tilings $\Omega_{a,b,c}$ with rectangular base $\mu = a \times b$ and height c. The partition function is given by

$$Z(a,b,c) := \sum_{T \in \Omega_{a,b,c}} \prod_{(i,j) \in T} (x_i - y_j) = \det \left[\begin{cases} \frac{(x_i - y_1)\cdots(x_i - y_{c+a-j})}{(x_i - x_{i+1})\cdots(x_i - x_{c+a})} & \text{if } j > a \\ \frac{(x_i - y_1)\cdots(x_i - y_{b+c})}{(x_i - x_{i+1})\ldots(x_i - x_{c+j})} & \text{if } j = i-c, \ldots, a \\ 0, & j < i-c \end{cases} \right]_{i,j=1}^{a+c}$$

Consider a path $P(d_1, \ldots)$ consisting of vertical lozenges (i.e. not the horizontal lozenges) passing through the points (i, d_i) (ith vertical line, distance of the midpoint $d_i + 1/2$ from the top axes) (necessarily $|d_i - d_{i+1}| \le 1$, $d_i \le d_{i+1}$ if $i \le b$ and $d_i \ge d_{i+1}$ if $i > b$, and $d_1 = d_{a+b}$).
The probability that such path exists is given by

$$\mathrm{Prob(path)} = \frac{\det[A_{i,j}(\mu, d)] \det[\bar{A}_{i,j}(\bar{\mu}, c-d-1)]}{Z}$$

where $d := d_1$, $\ell(\mu) = b$, $\mu_1 = a$ and μ is given by its diagonals – $(d_1 - d, d_2 - d, \ldots)$, and $\bar{\mu}$ is the complement of μ in $a \times b$. The matrix $\bar{A}$ is defined as in previous Theorem with the substitution of x_i by $x_{a+c+1-i}$ and y_j by $y_{b+c+1-j}$.

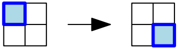


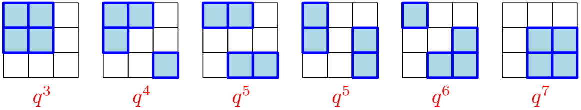
Origins: Excited diagrams and factorial Schur functions

Factorial Schur functions.

$$s_\mu^{(d)}(x|a) := \frac{\det\big[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\big]_{i,j=1}^{d}}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

where $x = (x_1, x_2, \ldots, x_d)$ and $a = (a_1, a_2, \ldots)$ is a sequence of parameters.
Excited diagrams $\mathcal{E}(\lambda/\mu)$: Start with λ/μ. Move cells of μ inside λ via:



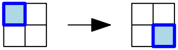


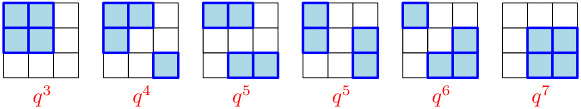
Origins: Excited diagrams and factorial Schur functions

Factorial Schur functions.

$$s_\mu^{(d)}(x|a) := \frac{\det\left[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\right]_{i,j=1}^{d}}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

where $x = (x_1, x_2, \ldots, x_d)$ and $a = (a_1, a_2, \ldots)$ is a sequence of parameters.
Excited diagrams $\mathcal{E}(\lambda/\mu)$: Start with λ/μ. Move cells of μ inside λ via:





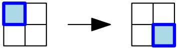
Theorem (Ikeda-Naruse Multivariate "Hook-Length Formula")

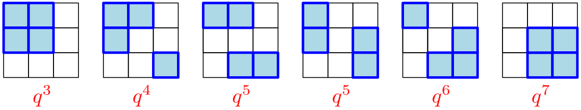
Let $\mu \subset \lambda \subset d \times (n-d)$. Let v be the Grassmannian permutation with unique descent at position d corresponding to λ, i.e. $v(d'+1-i) = \lambda_i + (d'+1-i)$ and $v(j) = d' + j - \lambda'_j$. Then

$$s_\mu^{(d)}(y_{v(1)}, \ldots, y_{v(d)} | y_1, \ldots, y_{n-1}) = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} (y_{v(d-i+1)} - y_{v(d+j)})$$

Origins: Excited diagrams and factorial Schur functions

Excited diagrams $\mathcal{E}(\lambda/\mu)$: Start with λ/μ. Move cells of μ inside λ via:

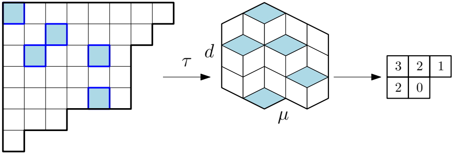


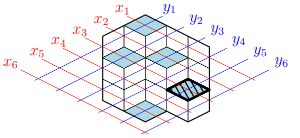


Theorem (Ikeda-Naruse Multivariate "Hook-Length Formula")

Let $\mu \subset \lambda \subset d \times (n-d)$. Let v be the Grassmannian permutation with unique descent at position d corresponding to λ, i.e. $v(d'+1-i) = \lambda_i + (d'+1-i)$ and $v(j) = d' + j - \lambda'_j$. Then

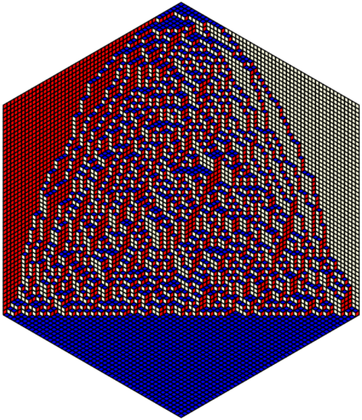
$$s_\mu^{(d)}(y_{v(1)}, \ldots, y_{v(d)} | y_1, \ldots, y_{n-1}) = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} (y_{v(d-i+1)} - y_{v(d+j)})$$

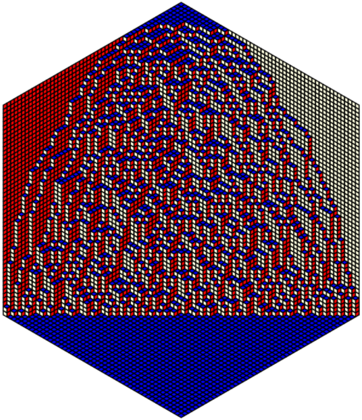




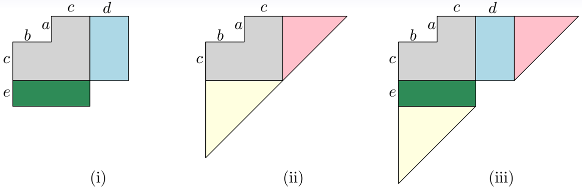
Simulation 2: base = δ_n

Weights: "hook" weights $(4n - i - j)$ versus uniform (i.e. 1).





Product formulas



$\Phi(n) := 1! \cdot 2! \cdots (n-1)!$, $\Psi(n) := 1!! \cdot 3!! \cdots (2n-3)!!$,
$\Psi(n;k) := (k+1)!! \cdot (k+3)!! \cdots (k+2n-3)!!$, $\Lambda(n) := (n-2)!(n-4)! \cdots$

Theorem (MPP)

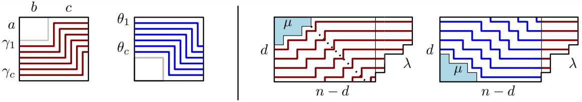
For nonnegative integers a, b, c, d, e, let n be the size of the corresponding skew shape, then for the shapes in (i), (ii), (iii) we have the following product formulas for the number of skew SYTs:

$$f^{sh(i)} = n! \frac{\Phi(a)\Phi(b)\Phi(c)\Phi(d)\Phi(e)\Phi(a+b+c)\Phi(c+d+e)\Phi(a+b+c+e+d)}{\Phi(a+b)\Phi(e+d)\Phi(a+c+d)\Phi(b+c+e)\Phi(a+b+2c+e+d)},$$

$$f^{sh(ii)} = n! \frac{\Phi(a)\Phi(b)\Phi(c)\Phi(a+b+c)}{\Phi(a+b)\Phi(b+c)\Phi(a+c)} \frac{\Psi(c)\Psi(a+b+c)}{\Psi(a+c)\Psi(b+c)\Psi(a+b+2c)},$$

$$f^{Sh(iii)} = \frac{n! \quad \Phi(a)\Phi(b)\Phi(c)\Phi(a+b+c)\,\Psi(c;d+e)\Psi(a+b+c;d+e)\,\Lambda(2a+2c)\Lambda(2b+2c)}{\Phi(a+b)\Phi(b+c)\Phi(a+c)\Psi(a+c)\Psi(b+c)\Psi(a+b+2c;d+e)\Lambda(2a+2c+d)\Lambda(2b+2c+e)}$$

Product formula reasons and consequences



Theorem (MPP)

We have the following identity for multivariate rational functions:

$$\sum_{\substack{\Gamma=(\gamma_1,\ldots,\gamma_c)\\ \gamma_p:(a+p,1)\to(p,b+c)}} \prod_{(i,j)\in\Gamma} \frac{1}{x_i - y_j} = \sum_{\substack{\Theta=(\theta_1,\ldots,\theta_c)\\ \theta_p:(p,1)\to(a+p,b+c)}} \prod_{(i,j)\in\Theta} \frac{1}{x_i - y_j}, \qquad (1)$$

where the sums are over non-intersecting lattice paths from the shapes λ/μ for $\mu_1 \le \lambda_d - d$.

Proof: symmetry of $s_\mu^{(d)}(x|y)$ in the variables x preserved under the substitution.
Corollaries: Product formulas for certain Schubert polynomial evaluations.

More problems?

- More precise asymptotics of $f^{\lambda/\mu}$ in various regimes.
- Asymptotics of lozenge tilings using the multivariate weights, new regimes?
- Asymptotics of $\frac{s_{\lambda/\mu}(x_1,\ldots,x_k,1^{n-k})}{s_{\lambda/\mu}(1^n)}$ (Schur generating functions of tilings of arbitrary domains)
- Asymptotics of Littlewood-Richardson coefficients, $c^{\lambda}_{\mu,\nu}$... (e.g. if $\lambda \vdash 2n$, $\mu, \nu \vdash n$, when is it maximal)
- Maximal $f^{\lambda/\mu}$ under constraints...

Thank you