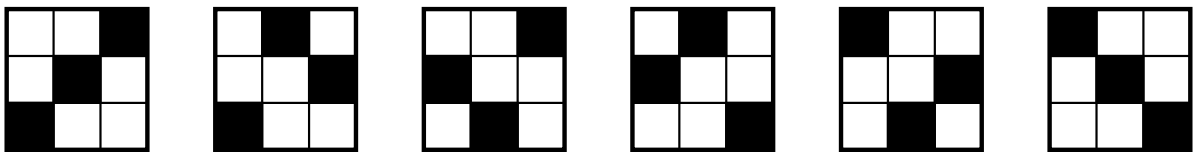
Sorting with C-machines

Jay Pantone

Dartmouth College

Hanover, NH



Erwin Schrödinger Institute
Algorithmic and Enumerative Combinatorics

October 20, 2017

Joint work with Michael Albert, Cheyne Homberger, Nathanial Shar, Vince Vatter

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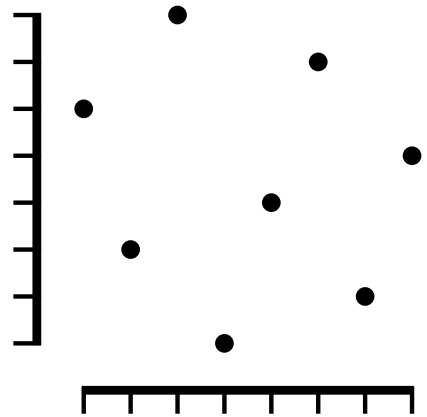

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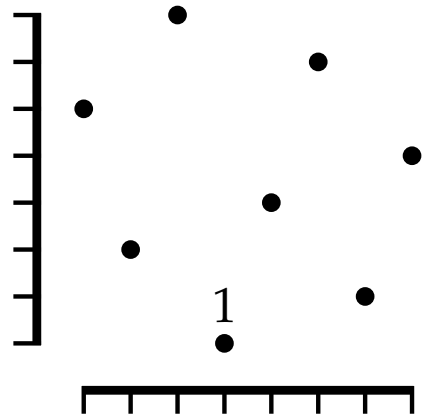

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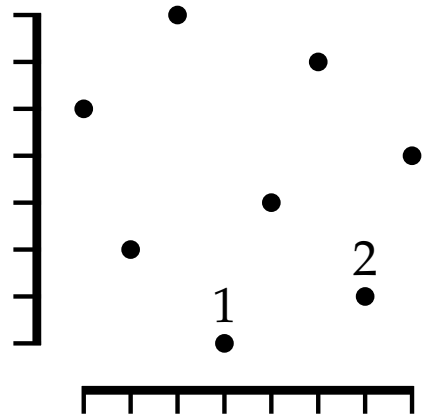

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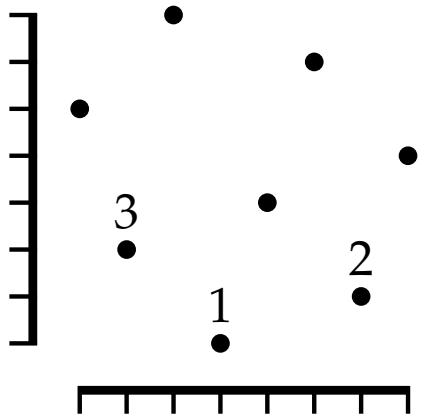

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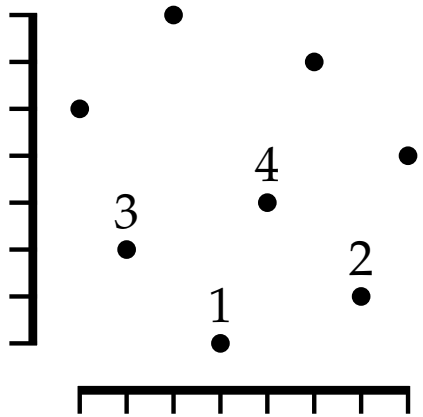

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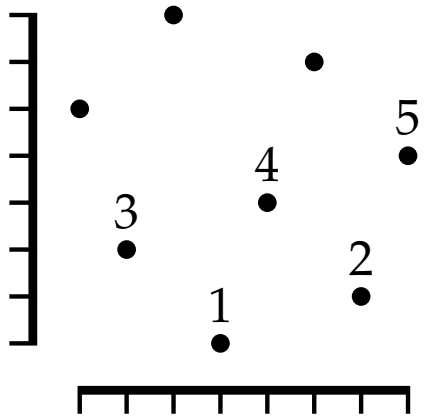

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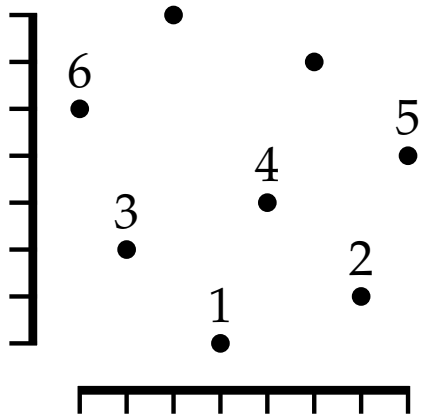

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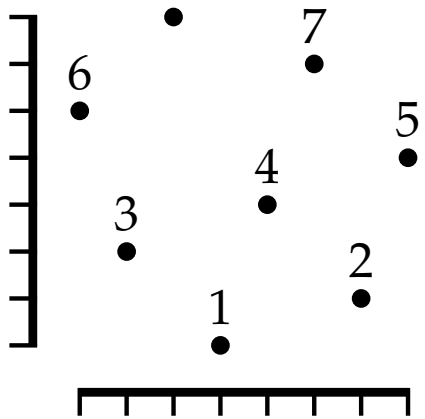

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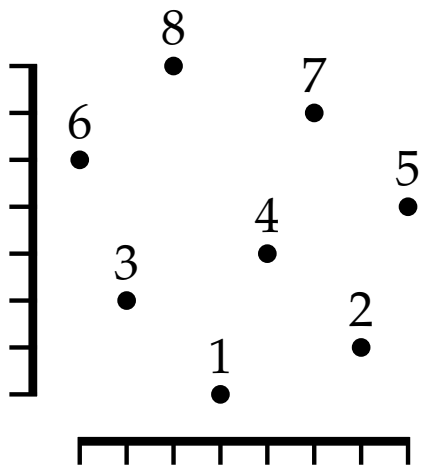

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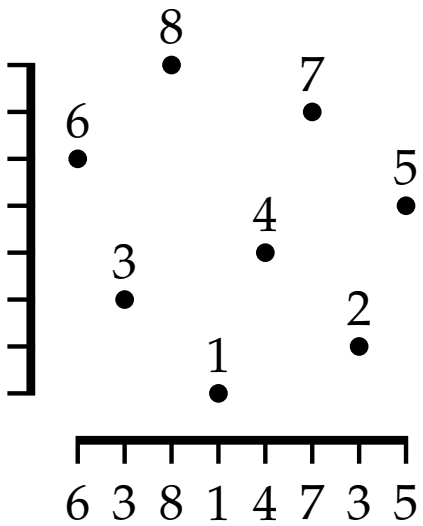

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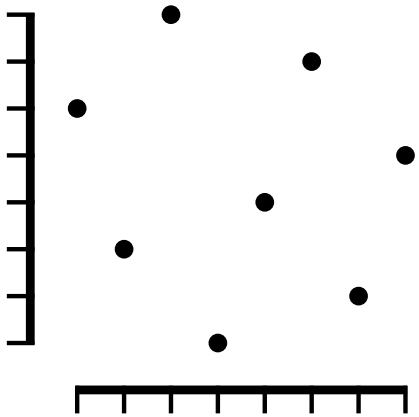

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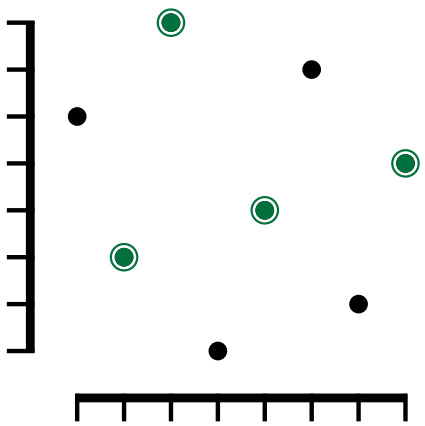

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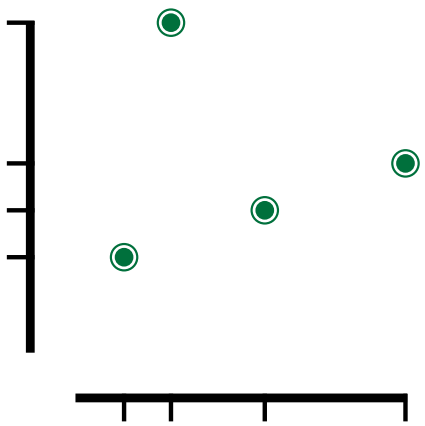

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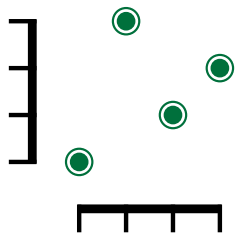
$1423 \leqslant 63814725 \iff$ 

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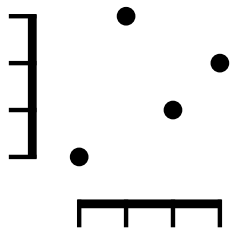
$1423 \leqslant 63814725 \iff$ 

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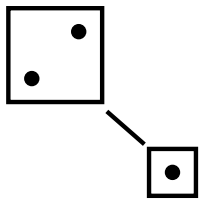

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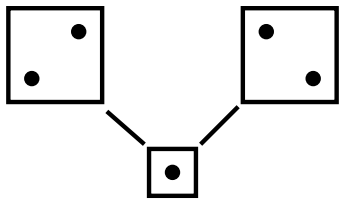

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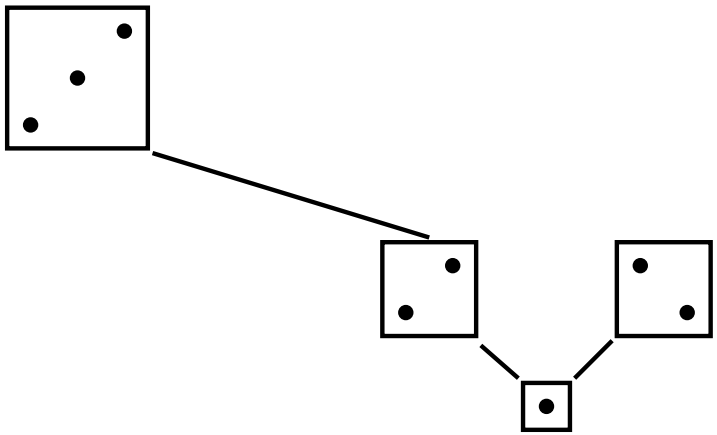

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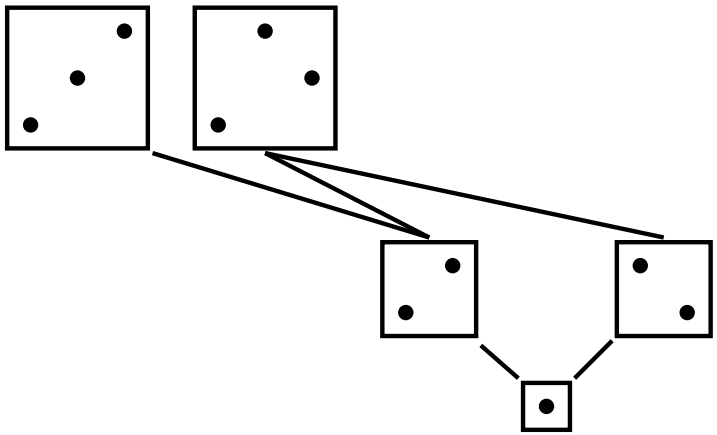

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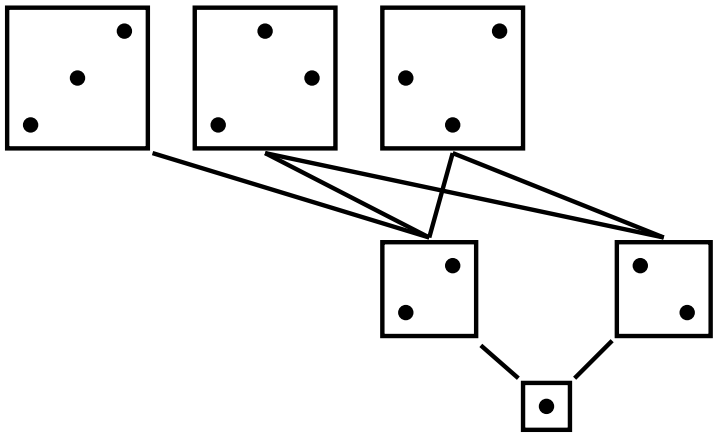

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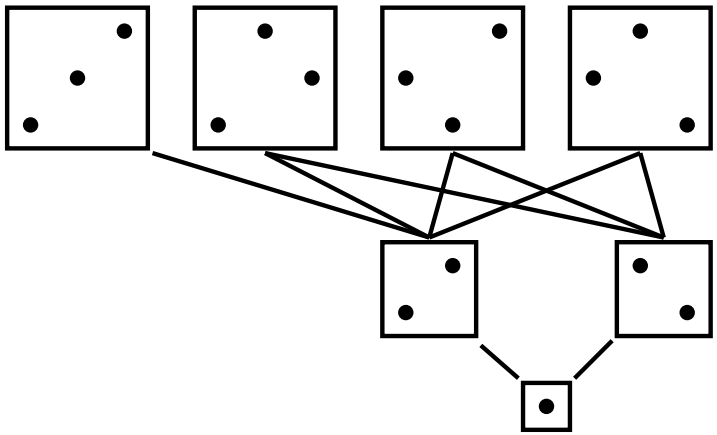

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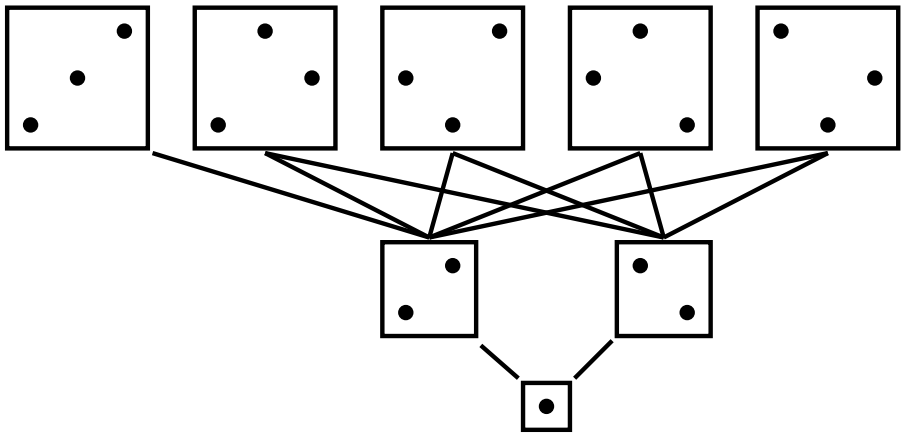

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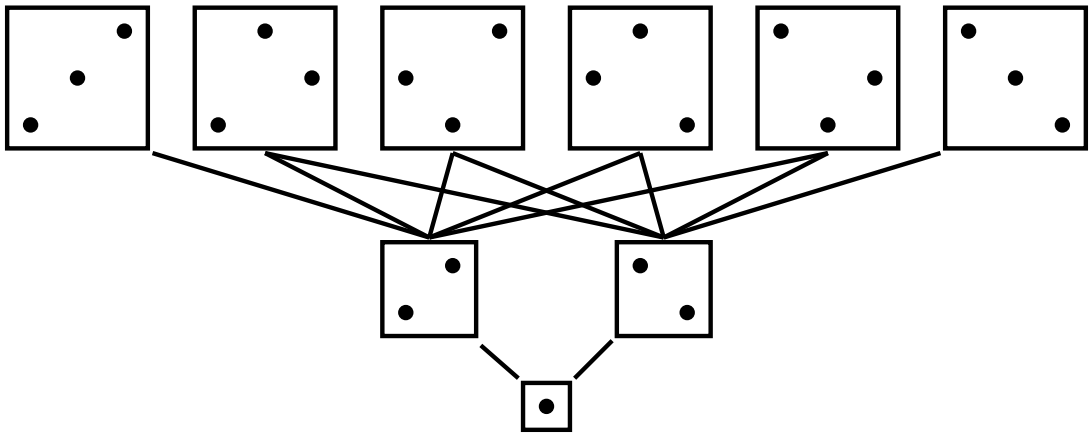

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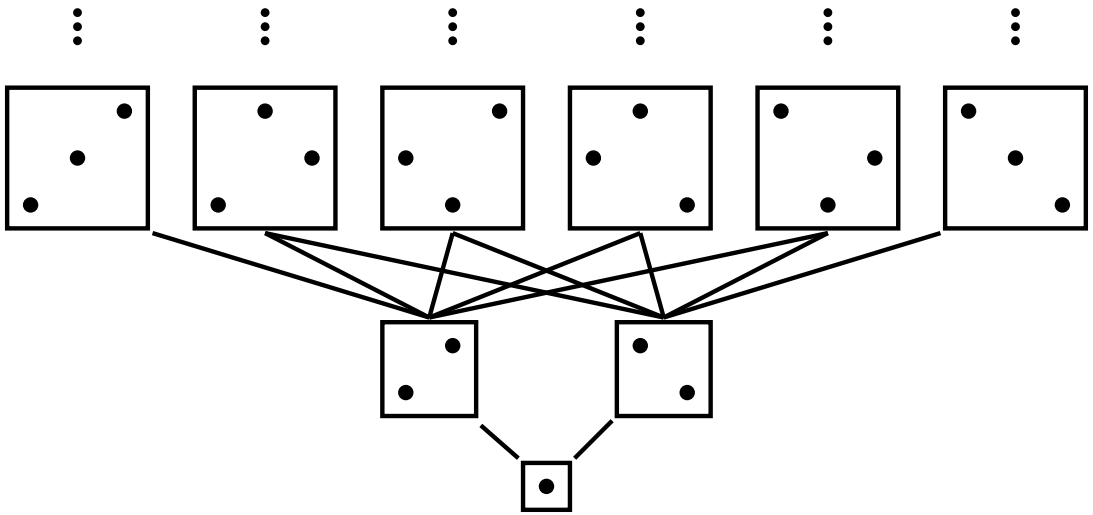

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The class with basis B is denoted $\mathrm{Av}(B)$.

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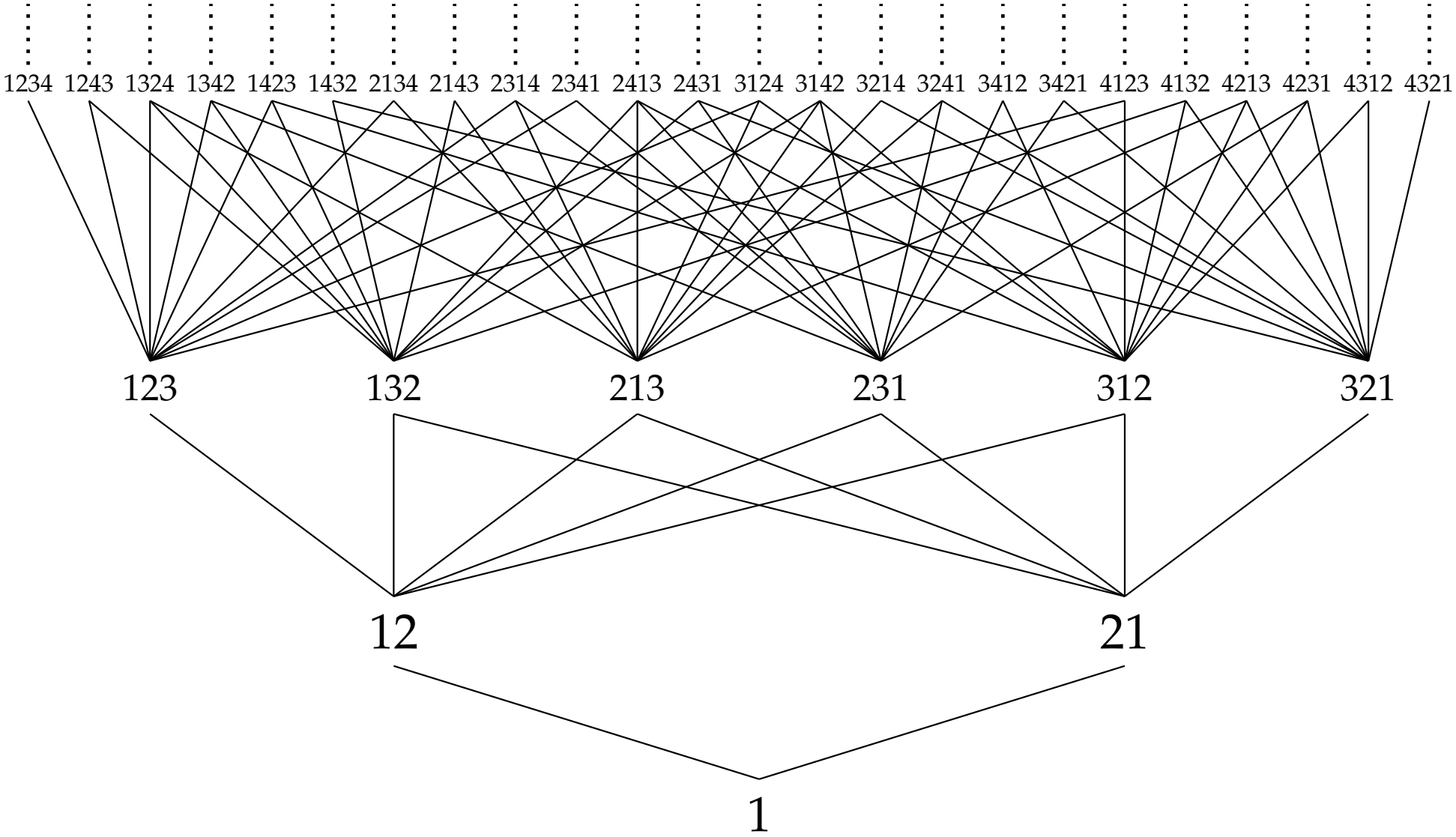

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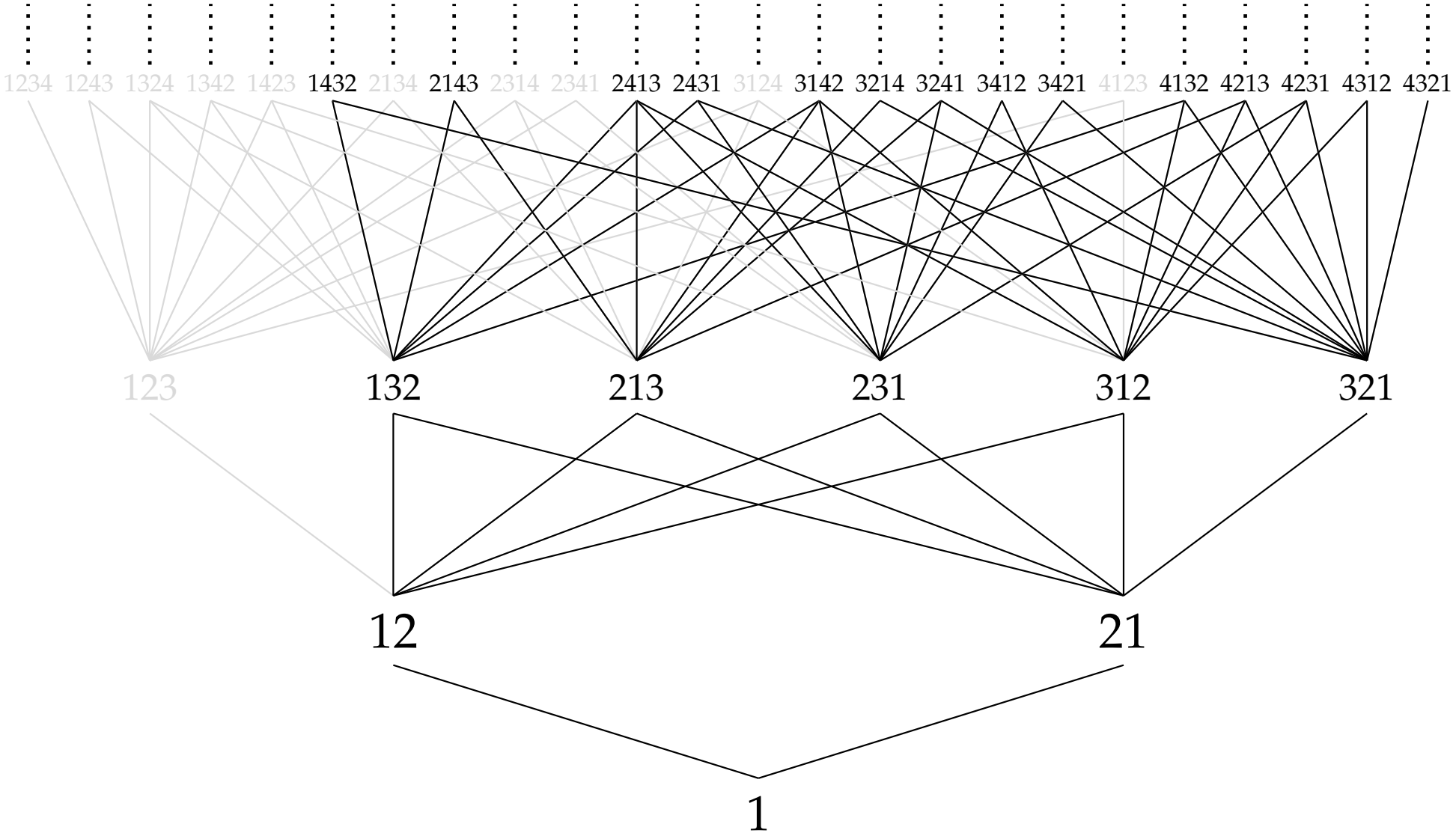

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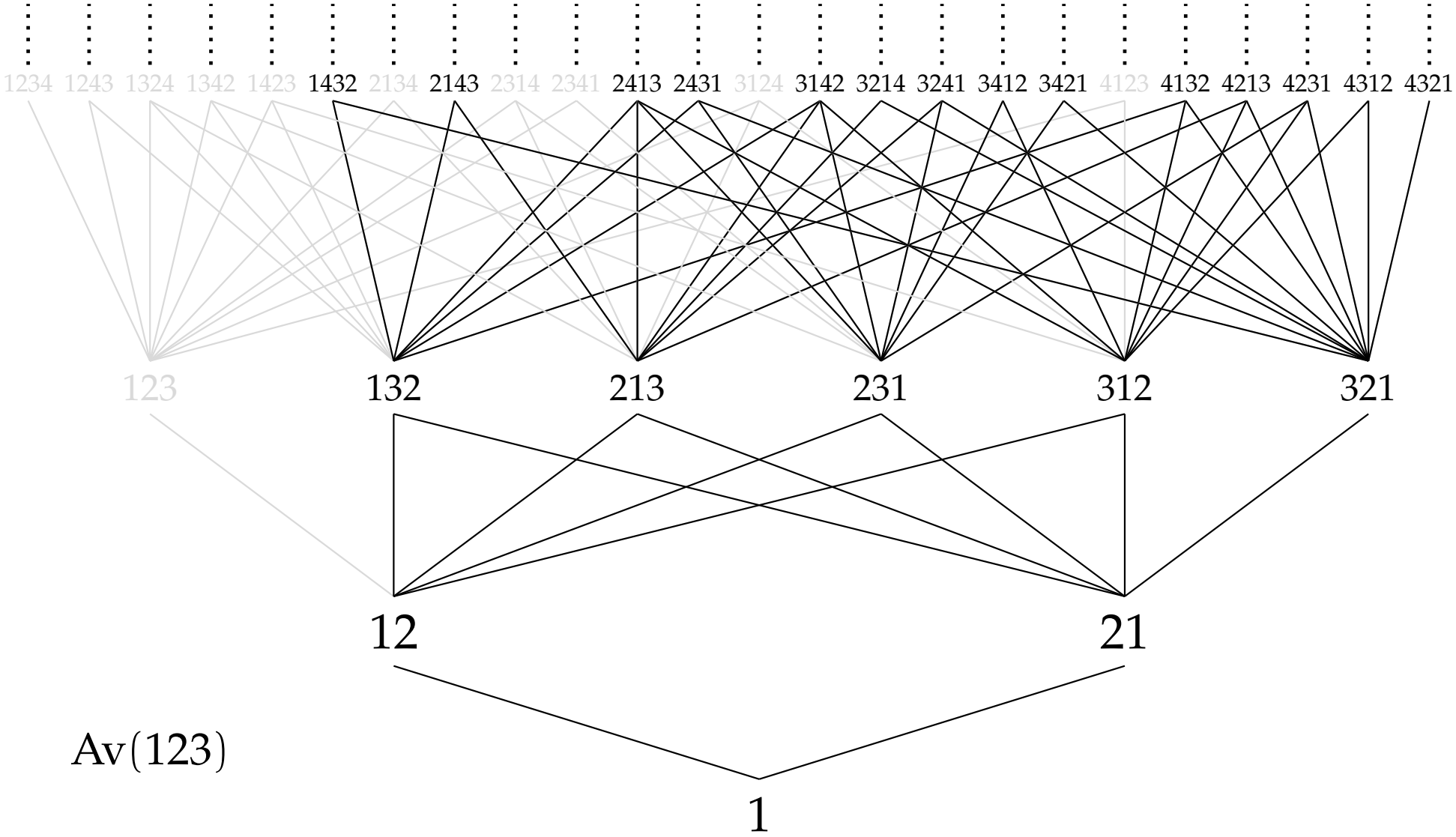

Av(123)

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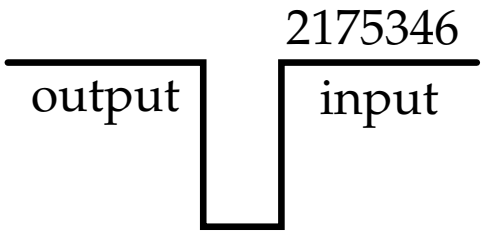

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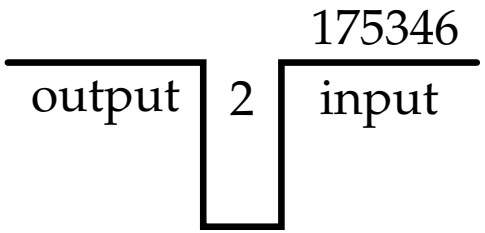

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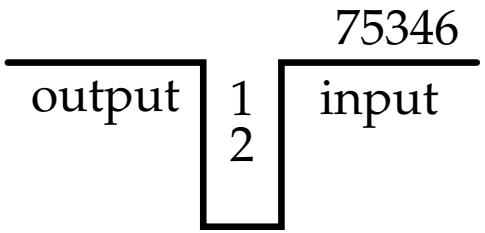

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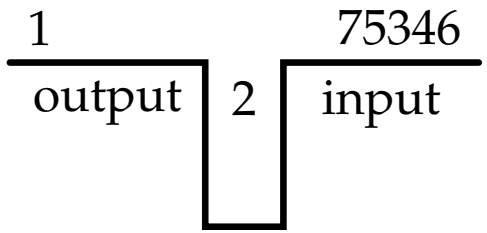

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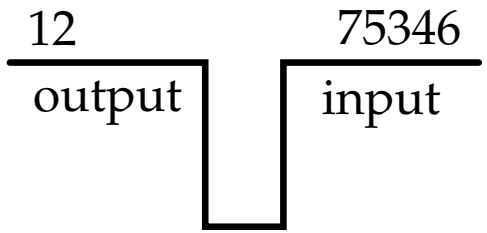

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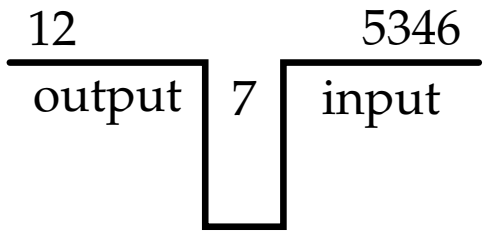

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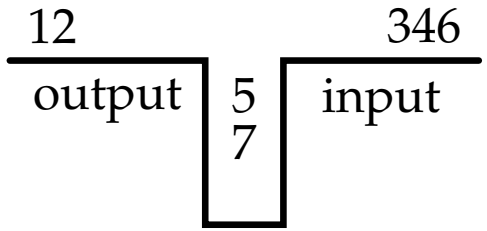

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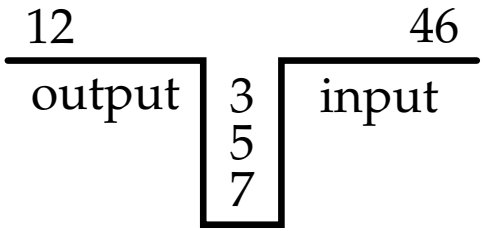

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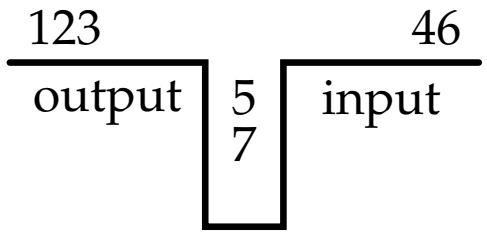

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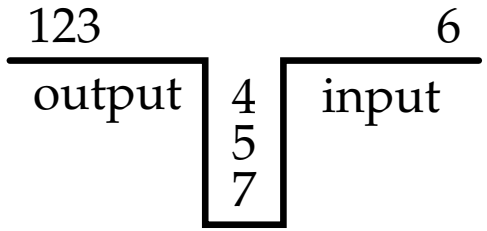

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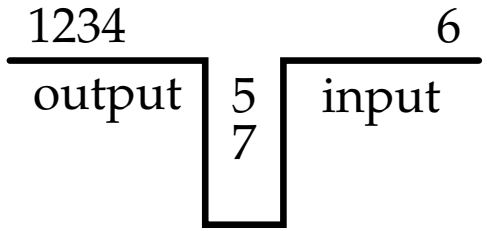

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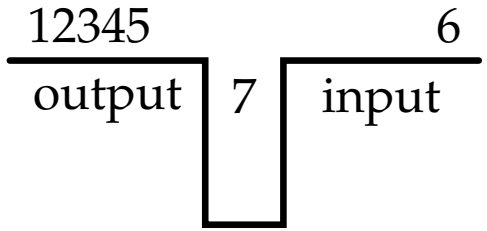

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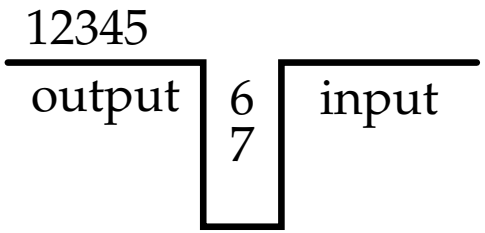

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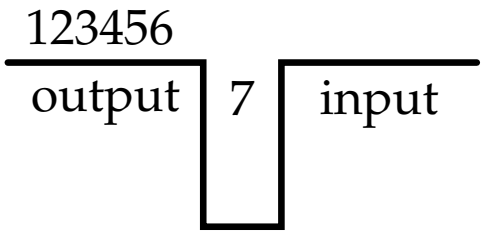

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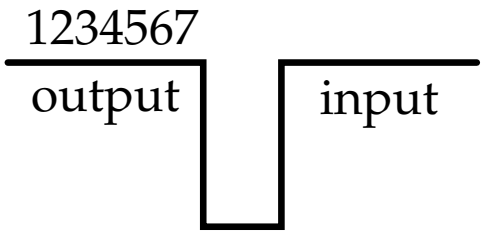

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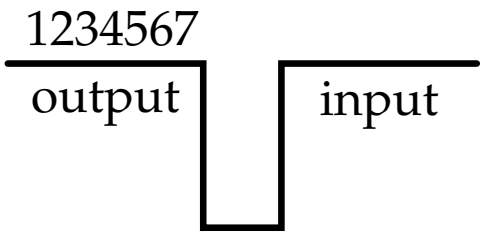

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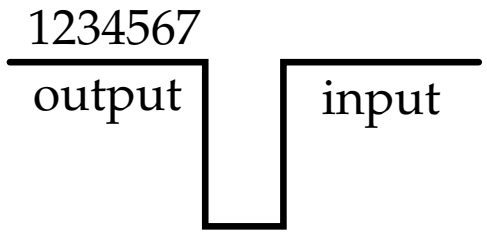

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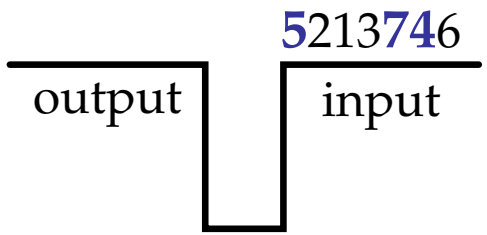

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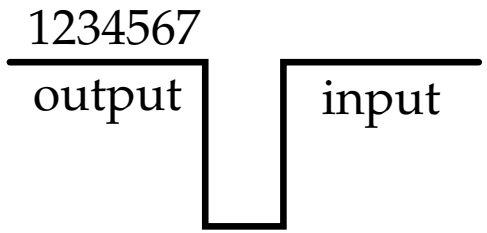

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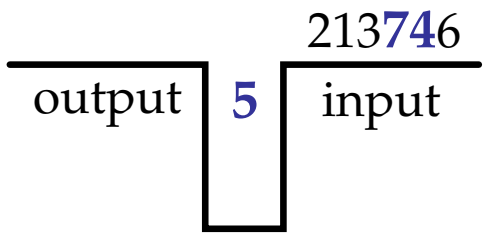

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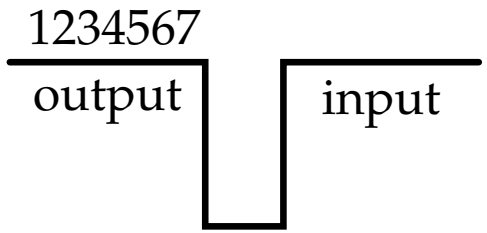

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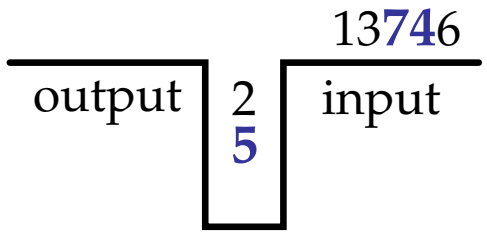

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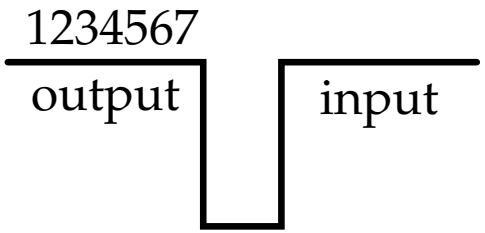

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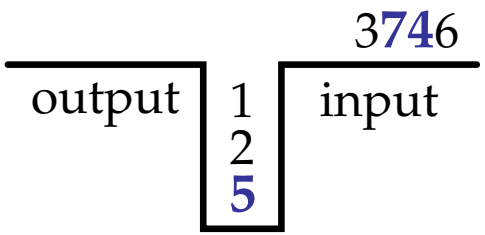

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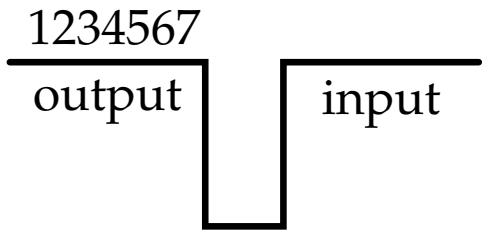

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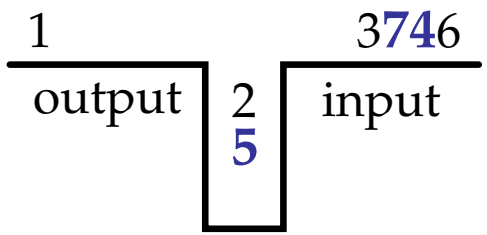

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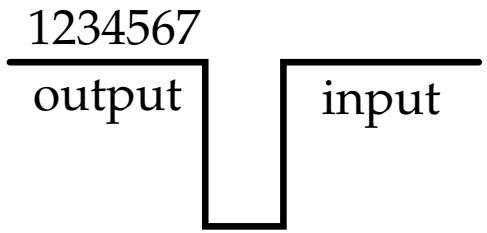

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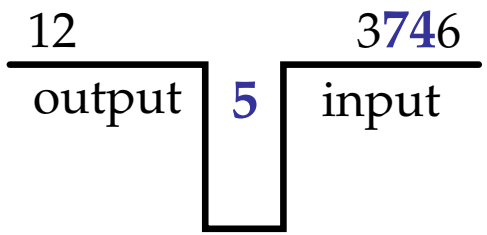

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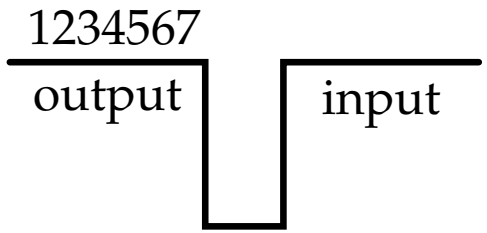

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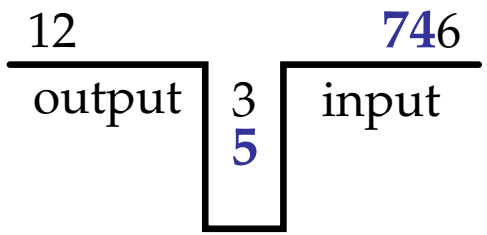

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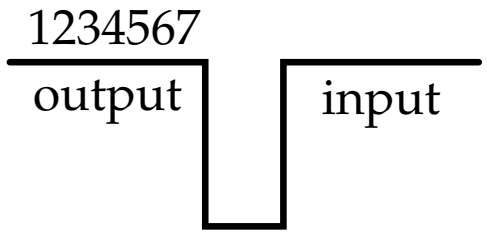

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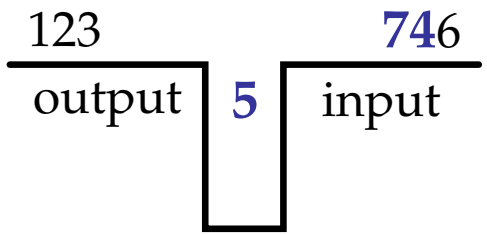

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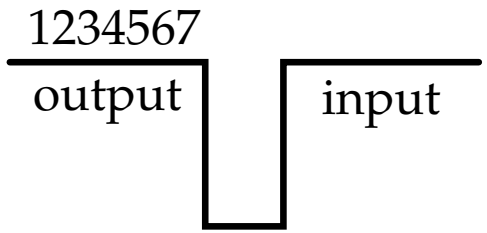

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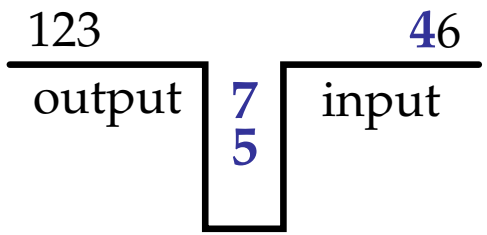

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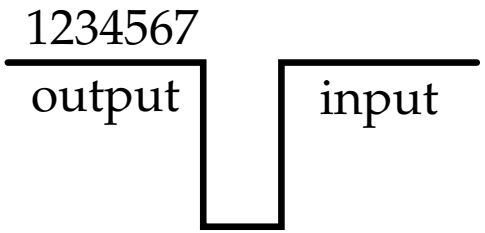

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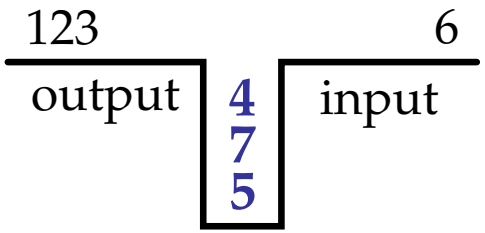

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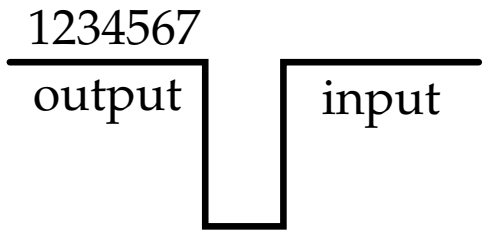

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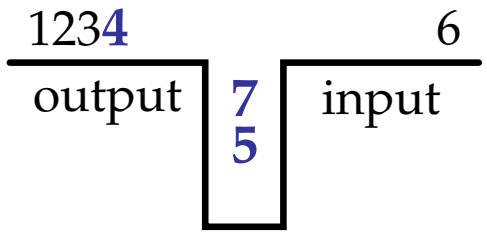

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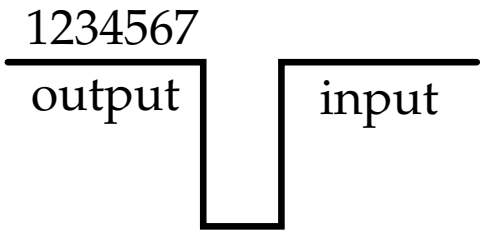

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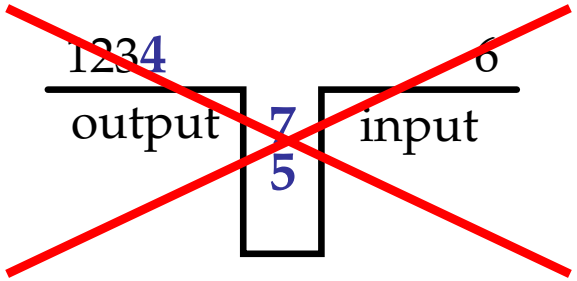

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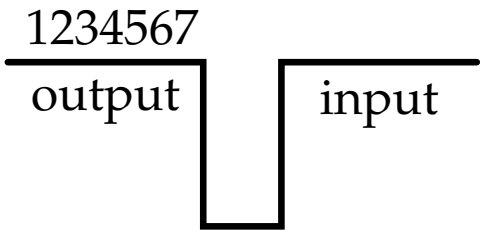

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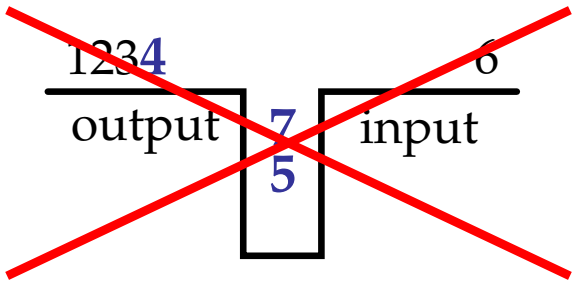
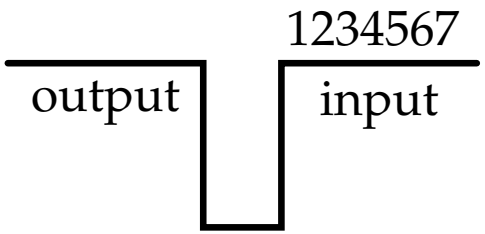


Permutations containing 231 cannot be sorted by a stack. In fact, the permutations that are stack-sortable are exactly those in $\mathrm{Av}(231)$.

Generating with a Stack

The permutation 2156473 can be generated by a stack.



(Notice the stack always holds a decreasing subpermutation when read top to bottom.)

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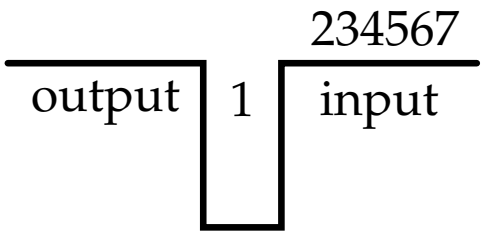

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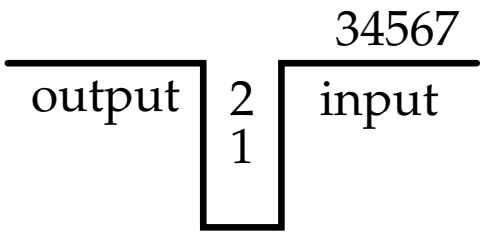

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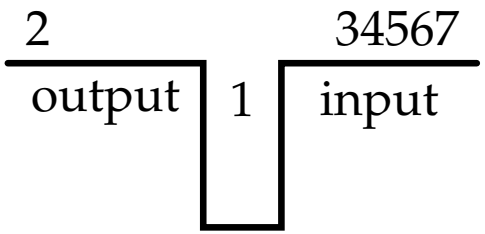

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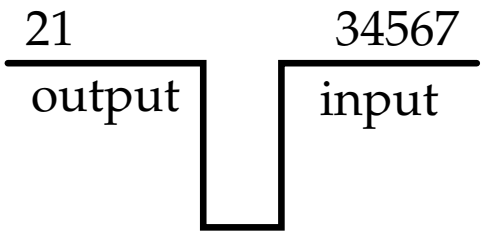

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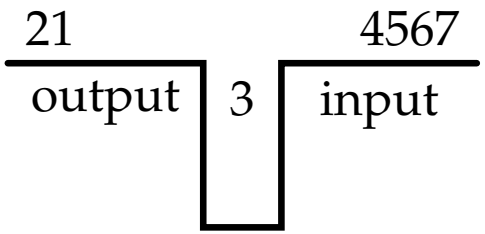

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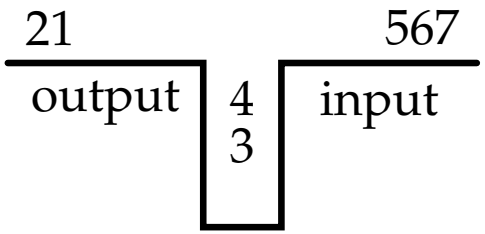

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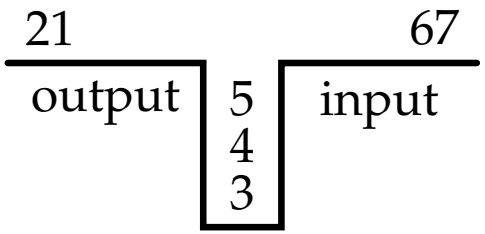

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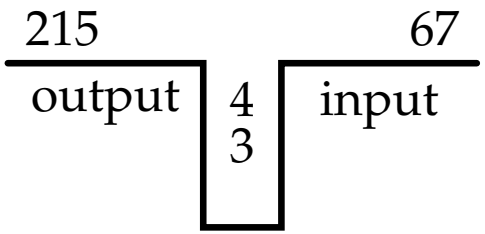

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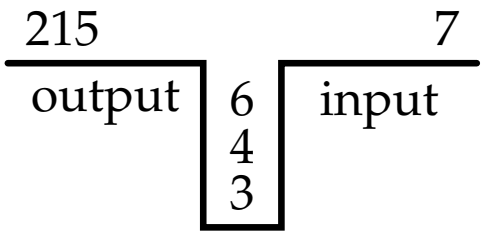

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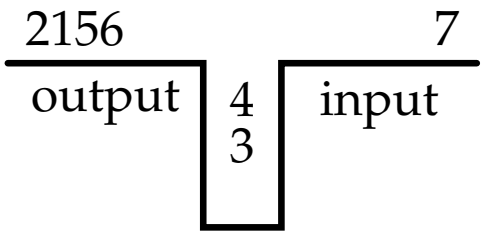

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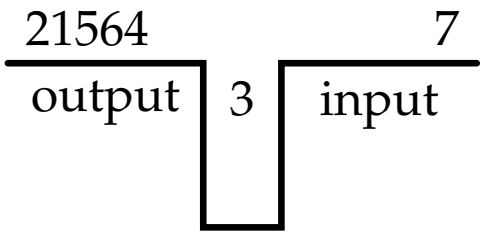

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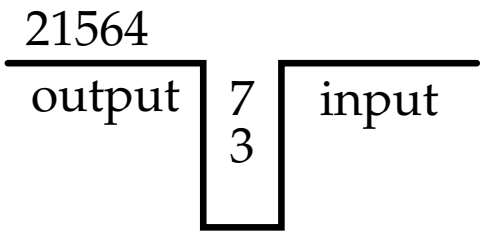

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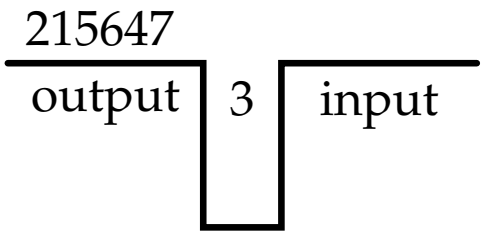

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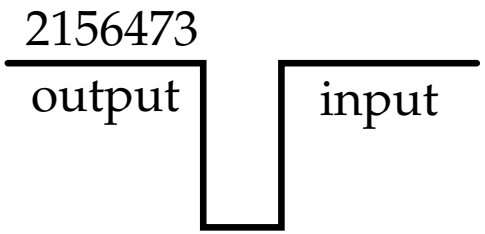

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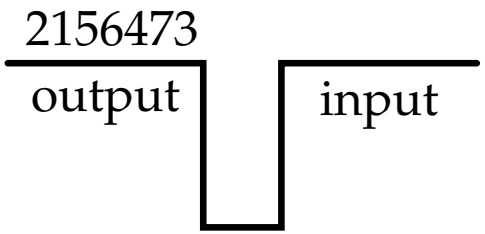

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The permutations that can be *generated* by a stack are exactly the inverses of those that can be *sorted* by a stack.

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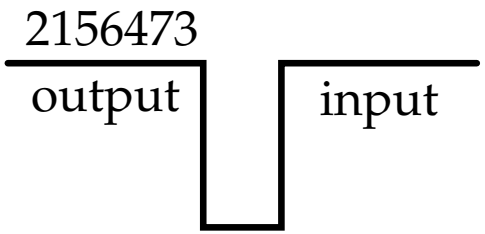

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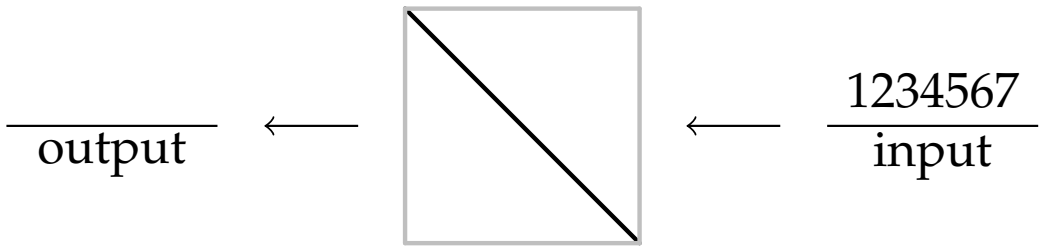
A stack can generate the permutations in $\mathrm{Av}(231^{-1}) = \mathrm{Av}(312)$.

Generalizing Stacks

The stack can be thought of as a container that always holds a decreasing permutation, where at any time we can:



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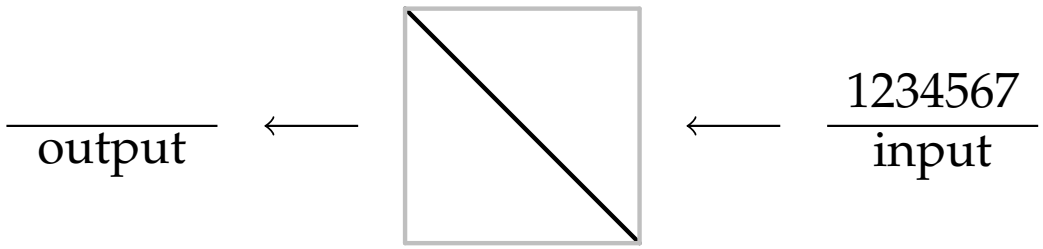
The stack can be thought of as a container that always holds a decreasing permutation, where at any time we can:

- push a new maximum entry into the container such that the container holds a decreasing permutation



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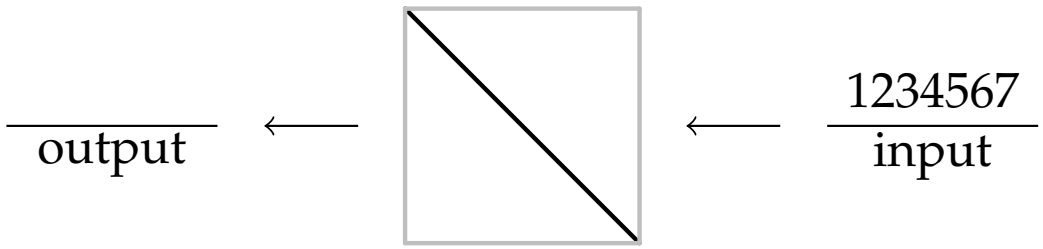

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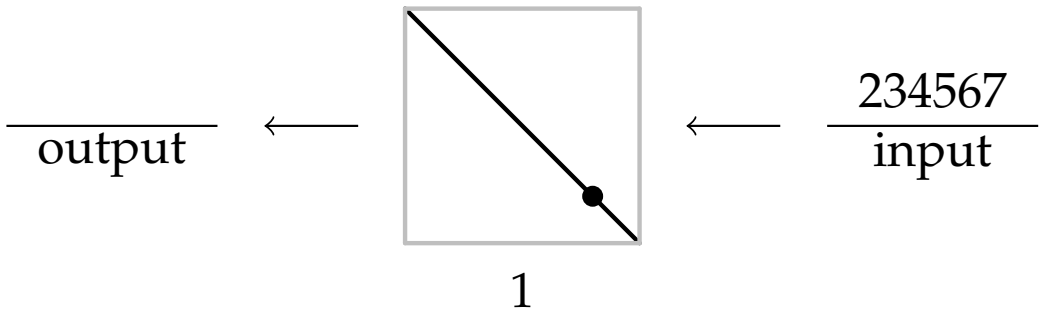

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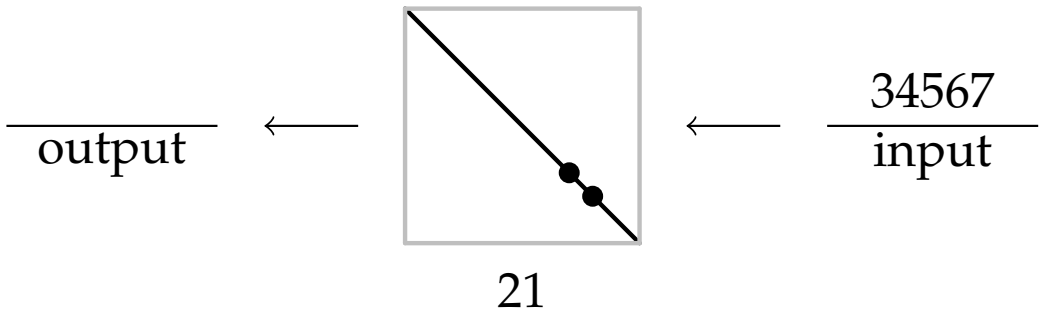

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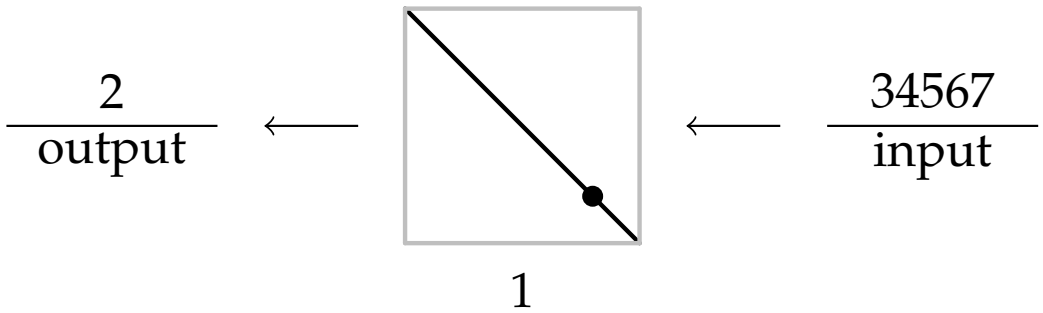

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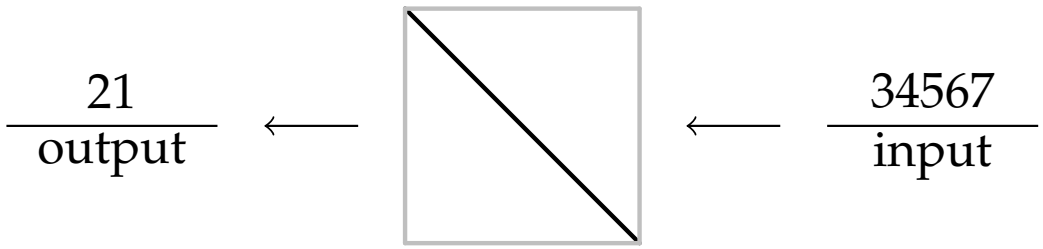

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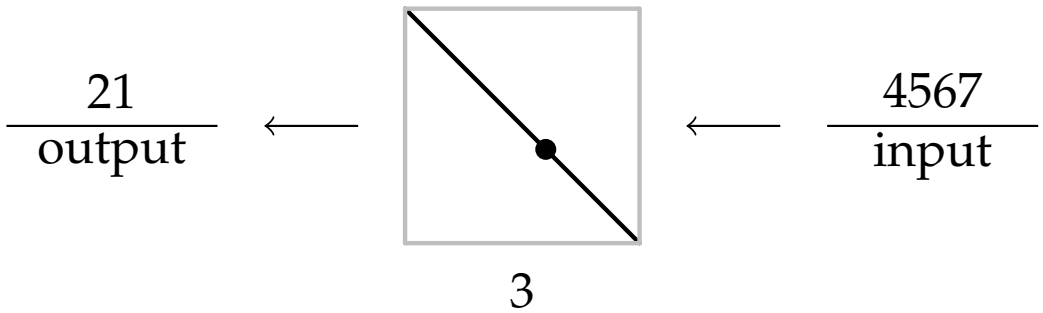

Generalizing Stacks

The stack can be thought of as a container that always holds a decreasing permutation, where at any time we can:

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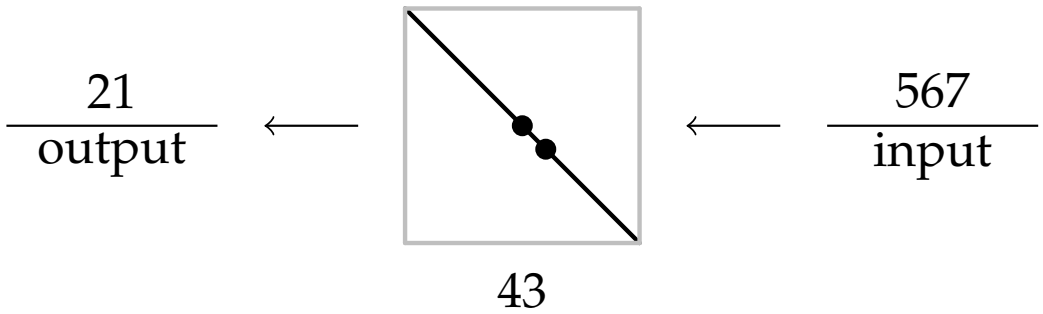

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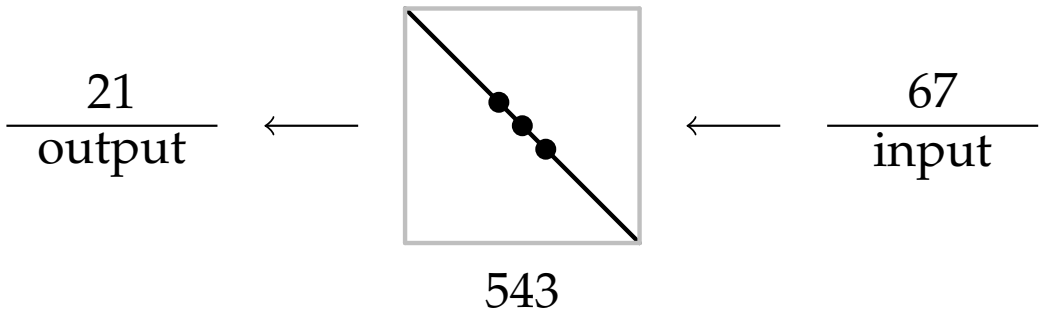

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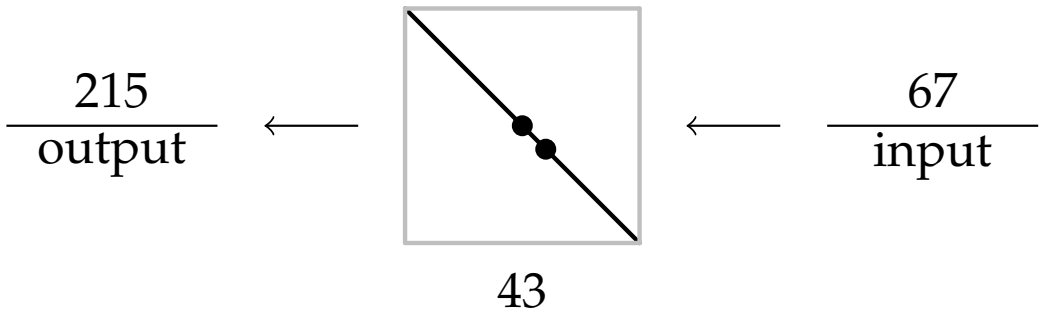

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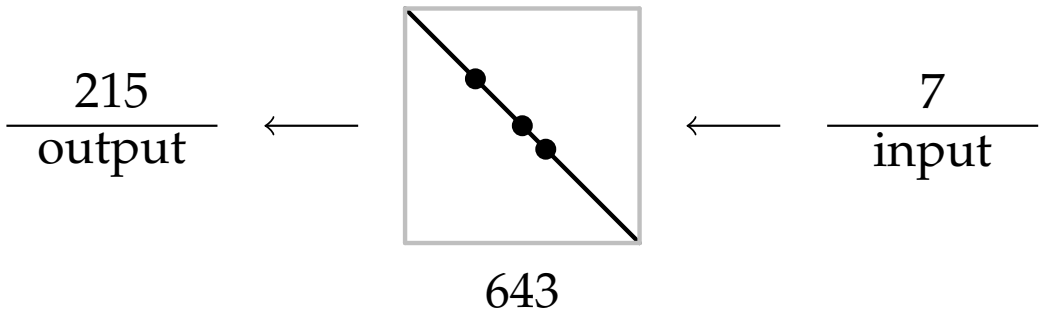

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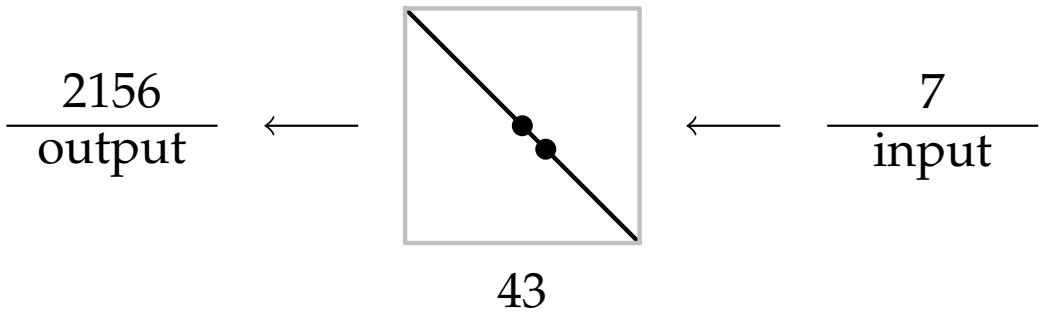

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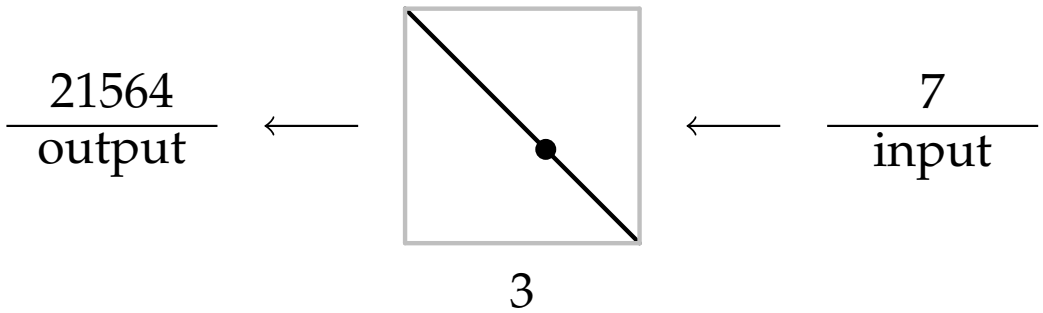

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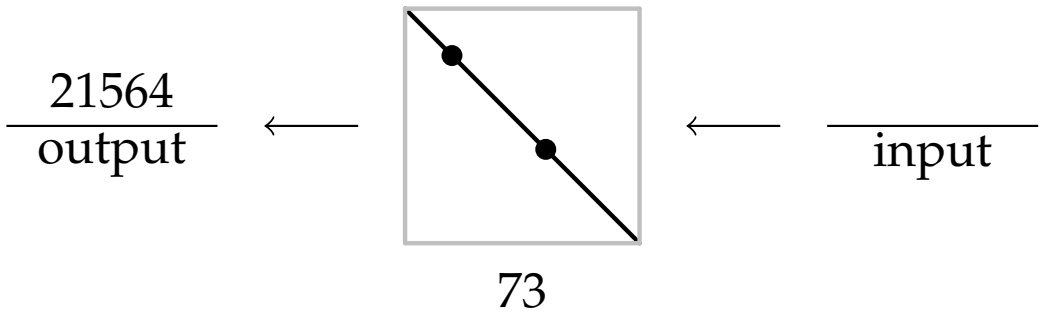

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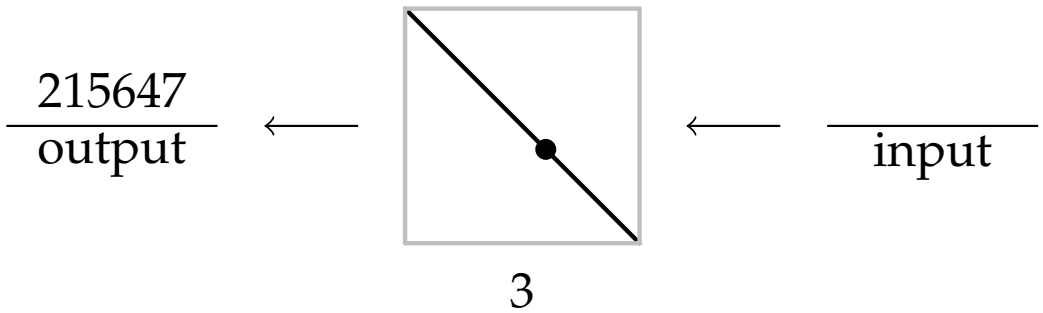

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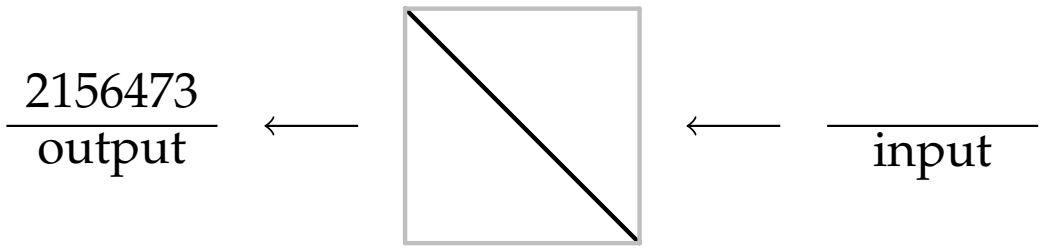

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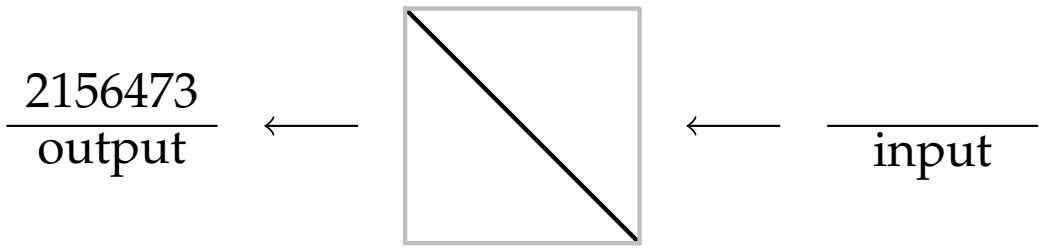

Since the container holds permutations that avoid the pattern 12, we call this the Av(12)-machine.

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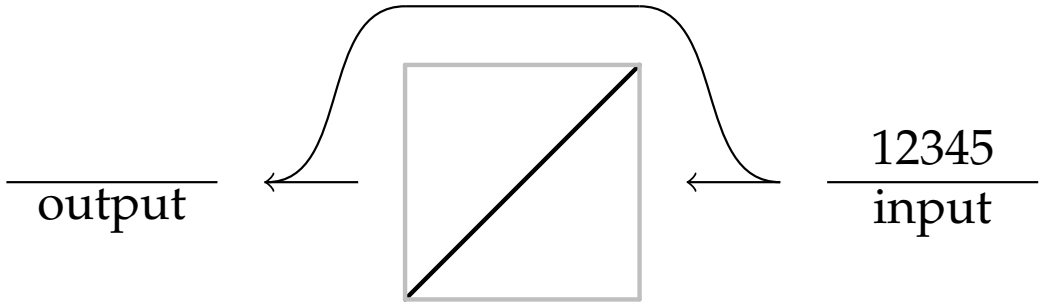

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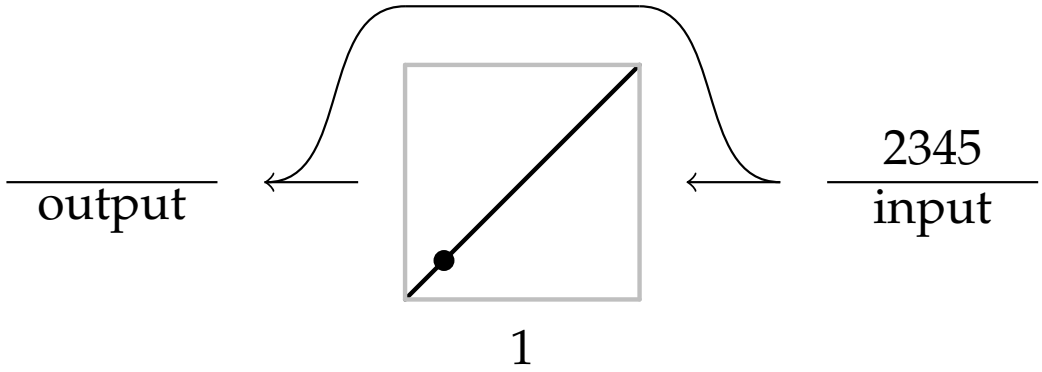

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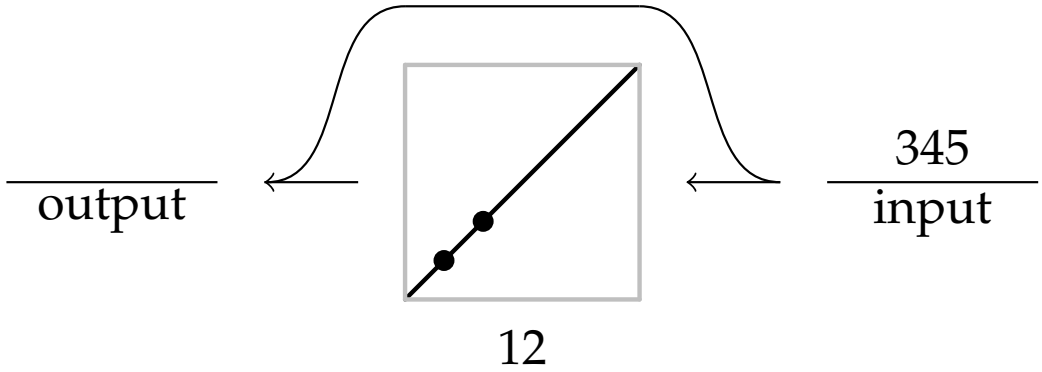

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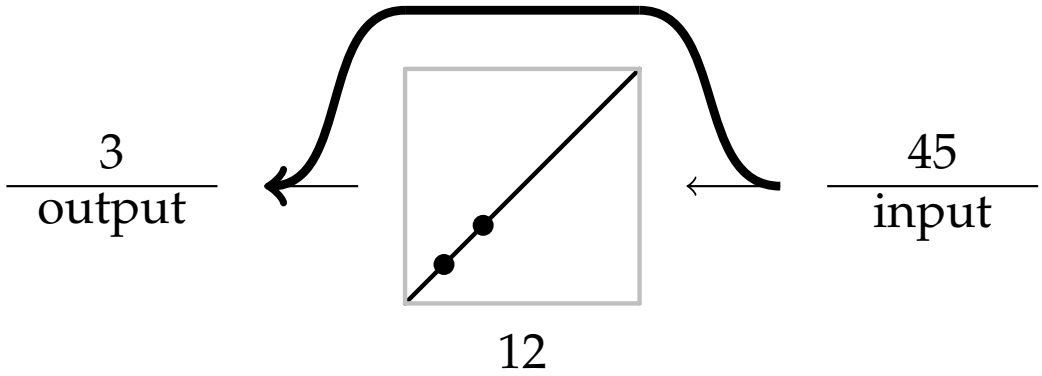

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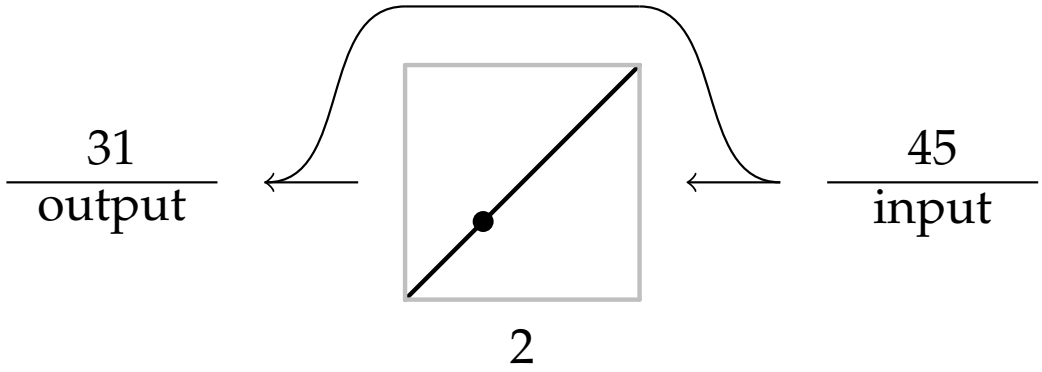

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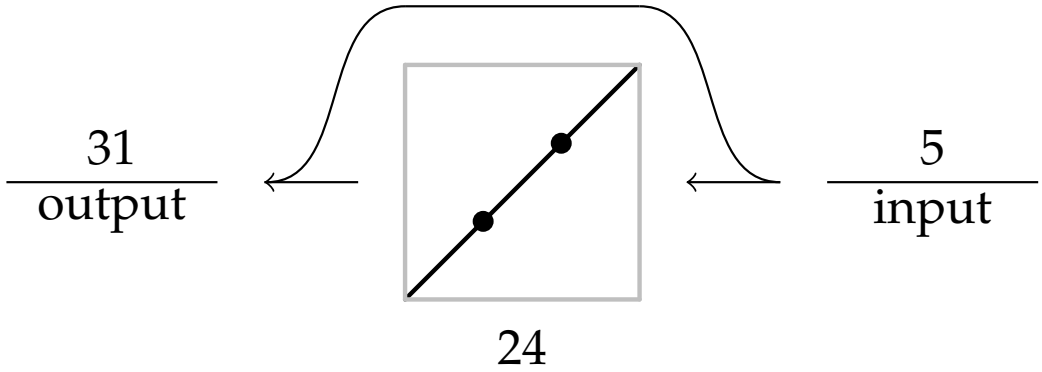

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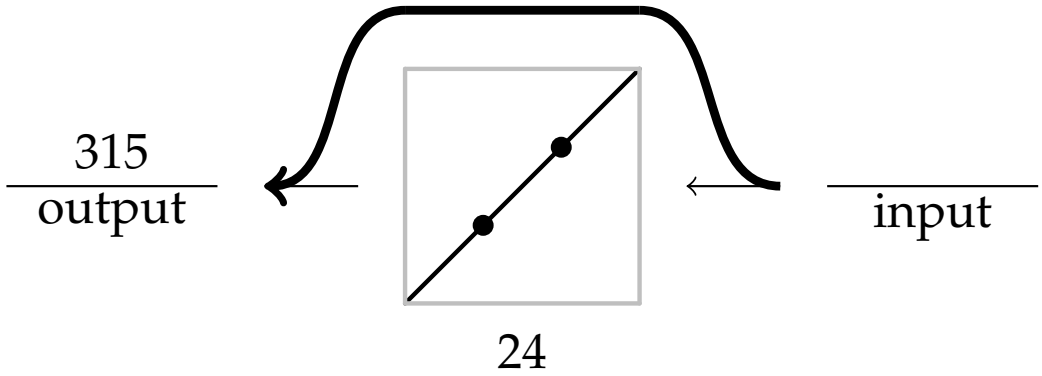

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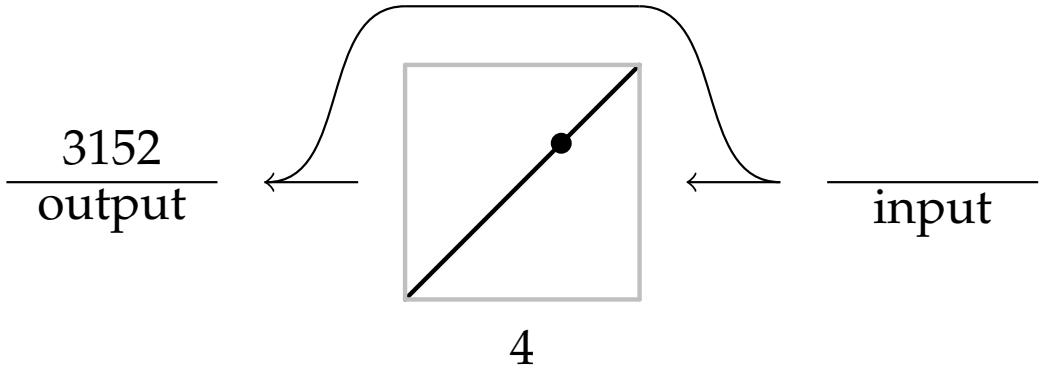

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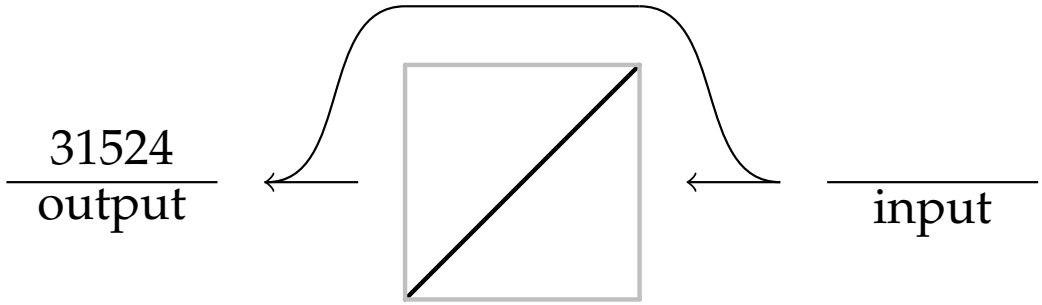

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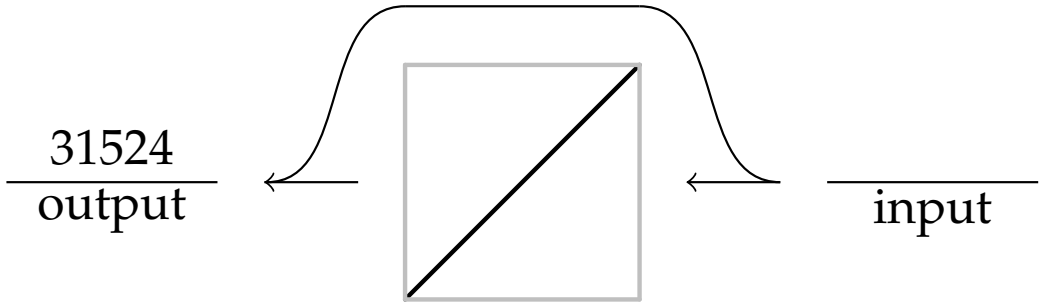
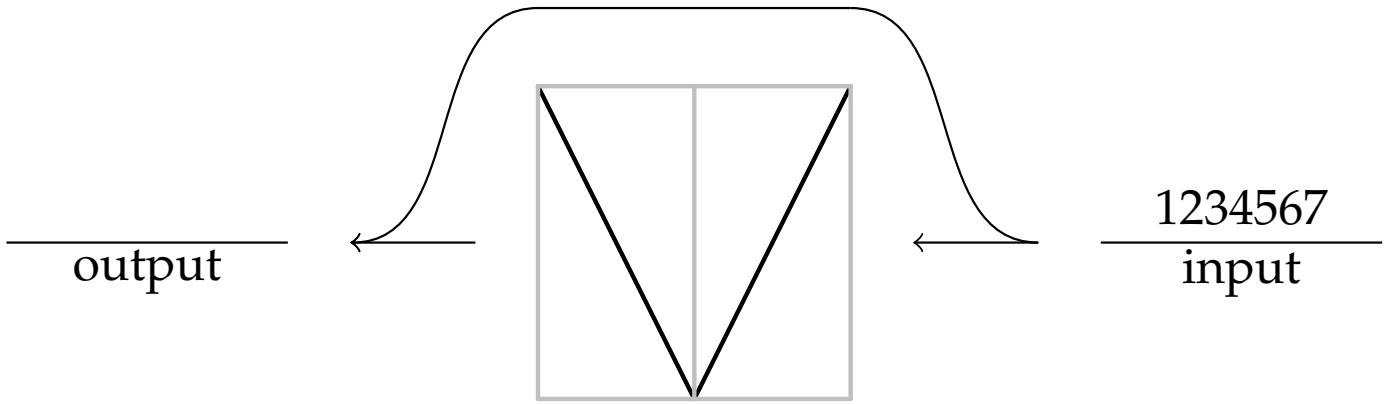


The Av(21)-machine generates the class Av(321).

Example: The Av(231, 132)-Machine



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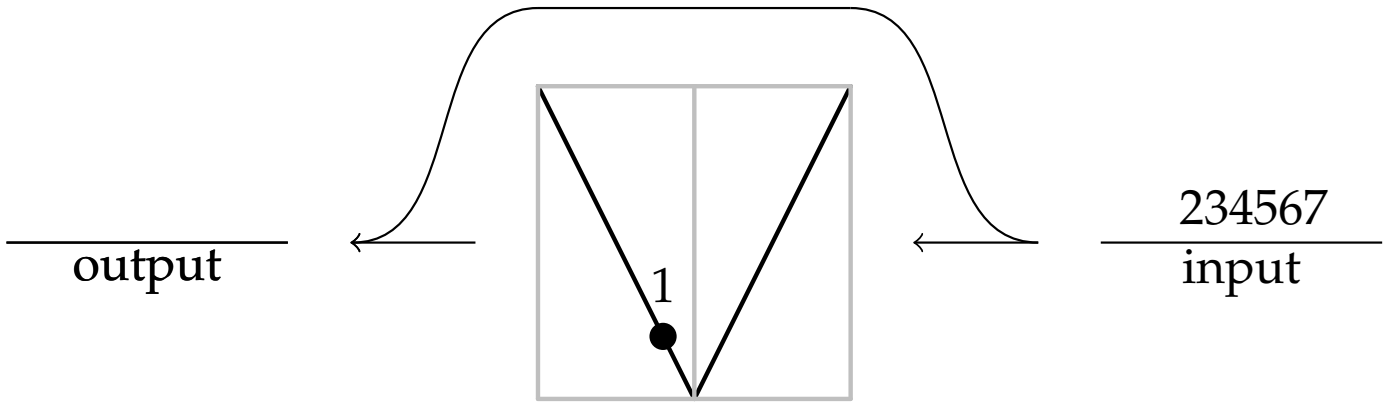

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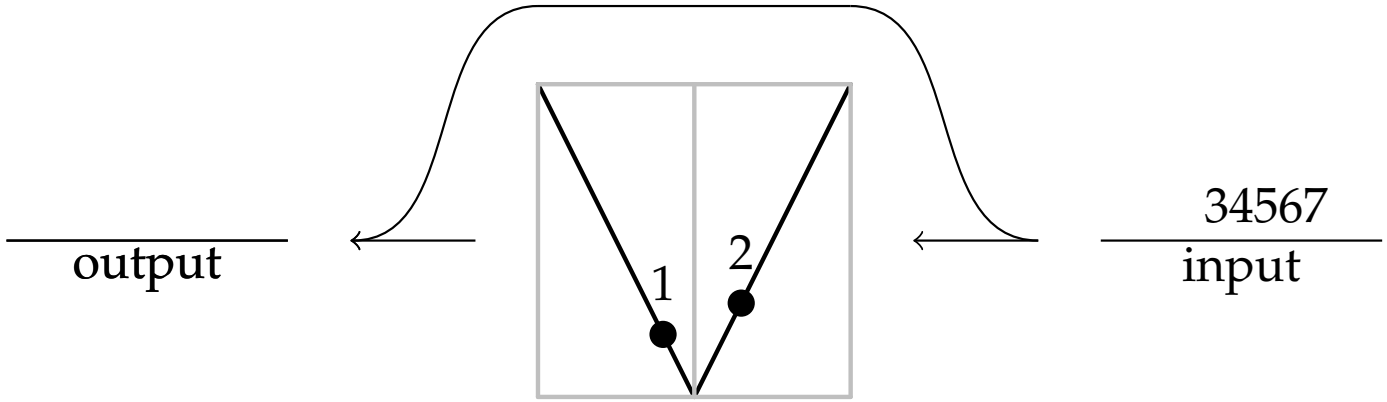

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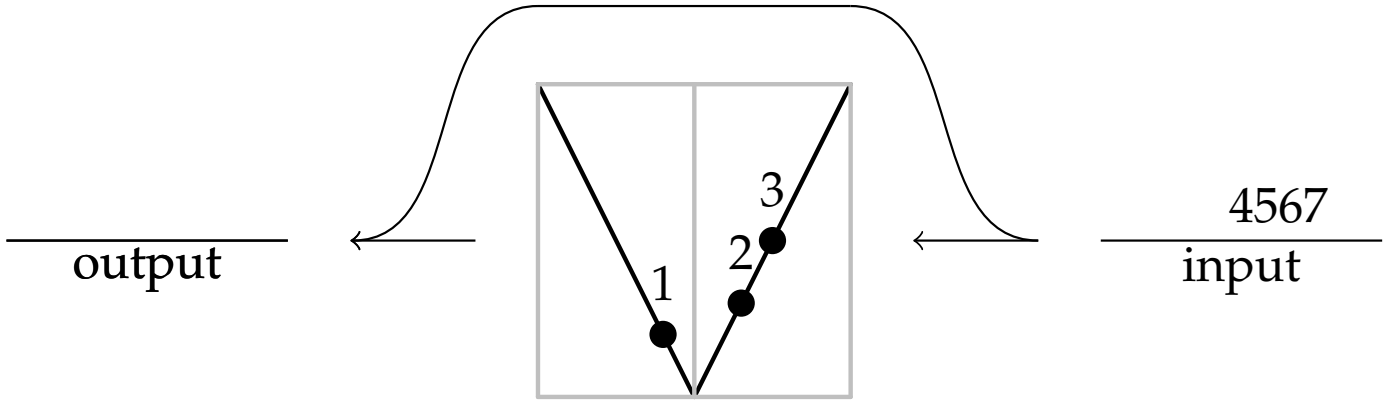

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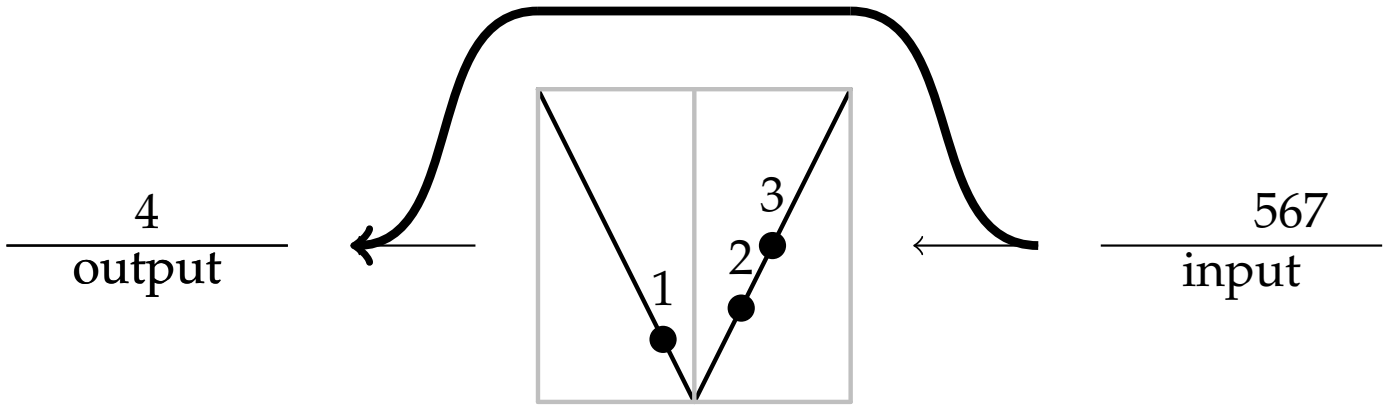

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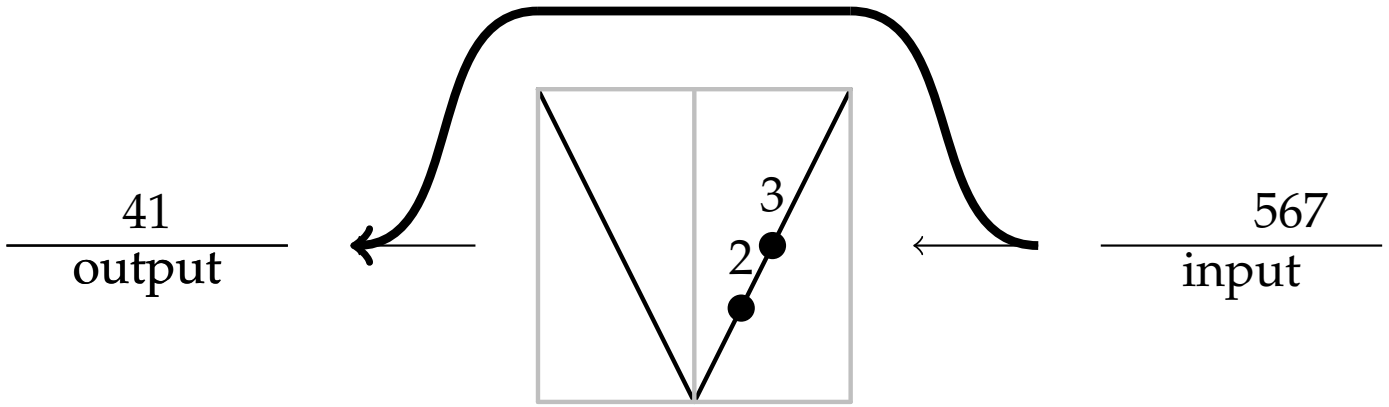

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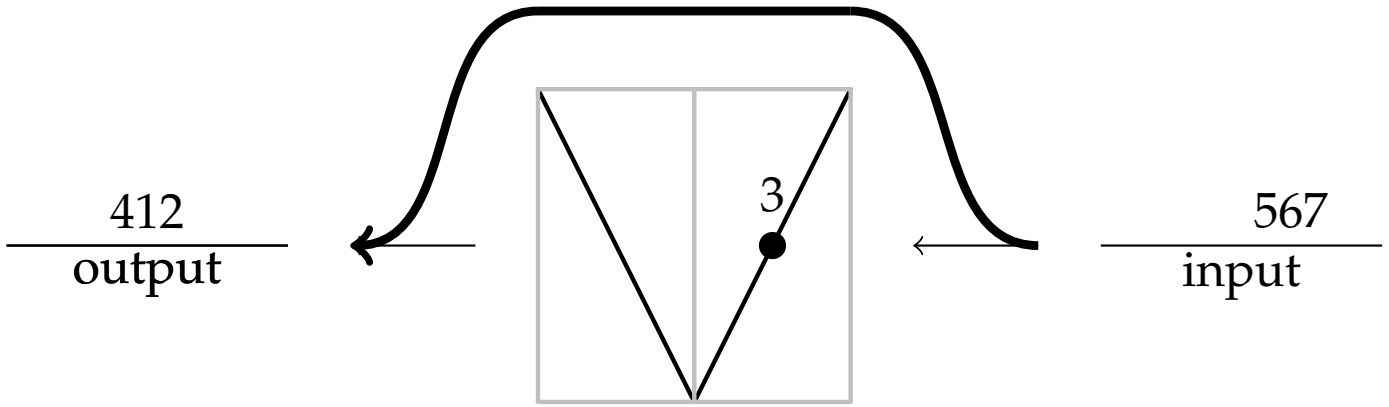

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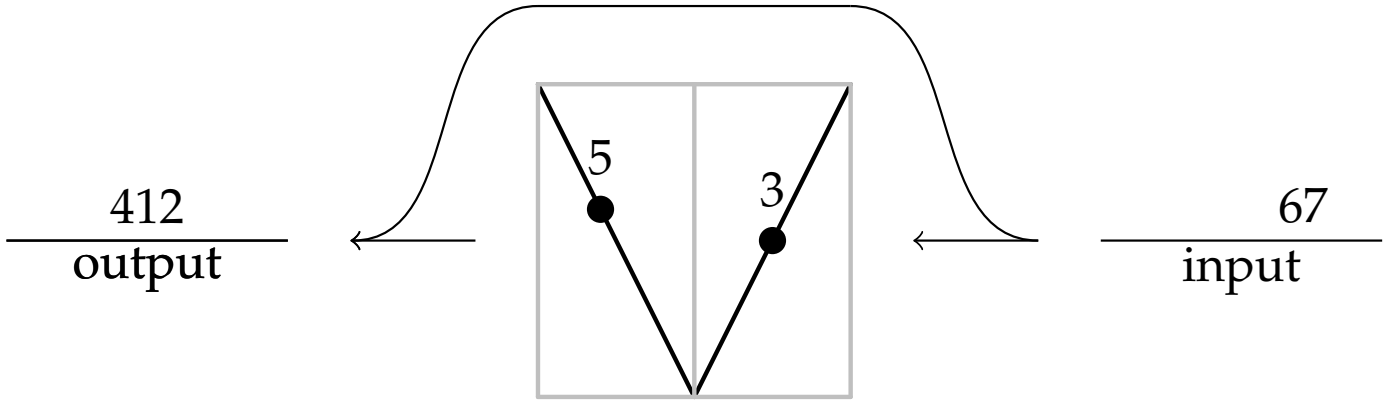

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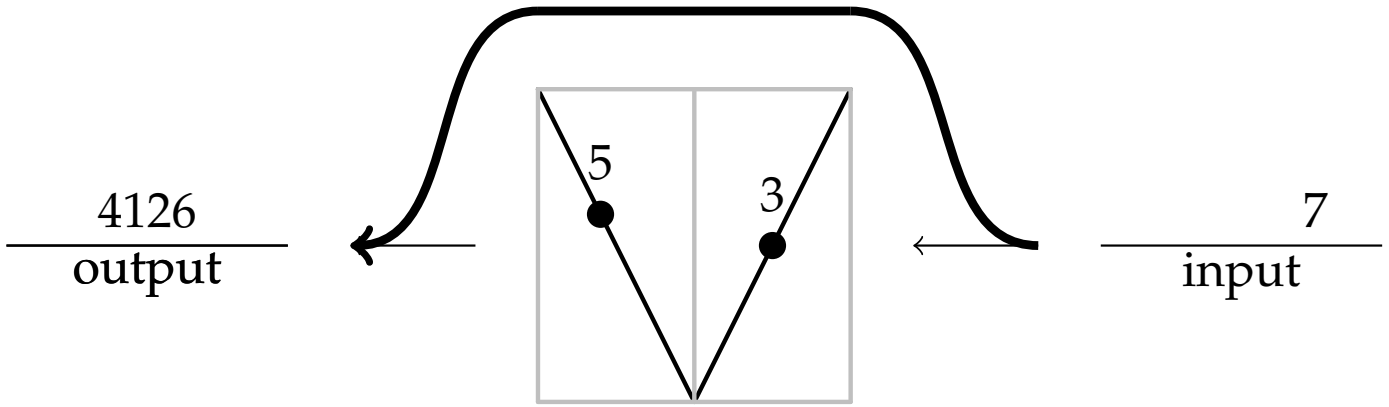

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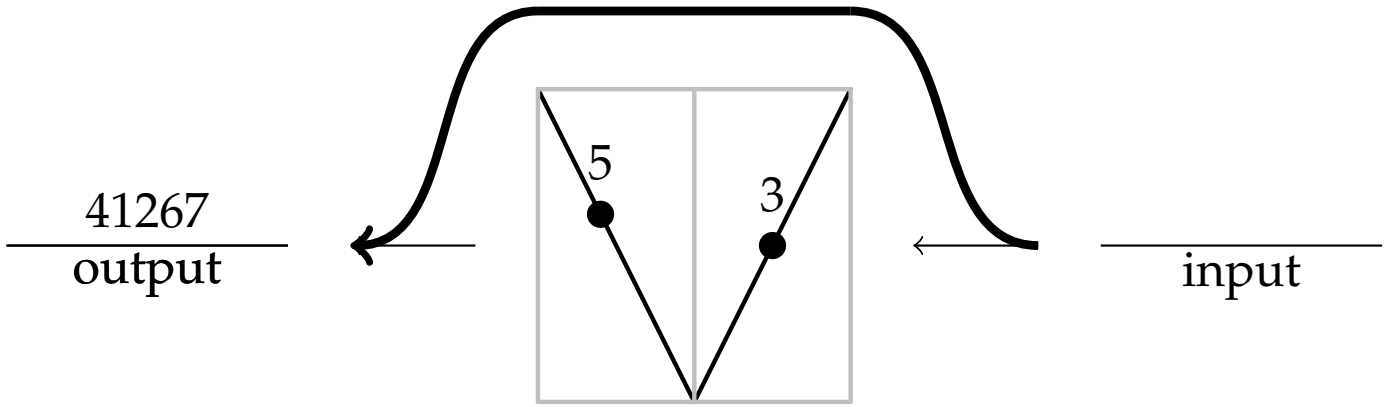

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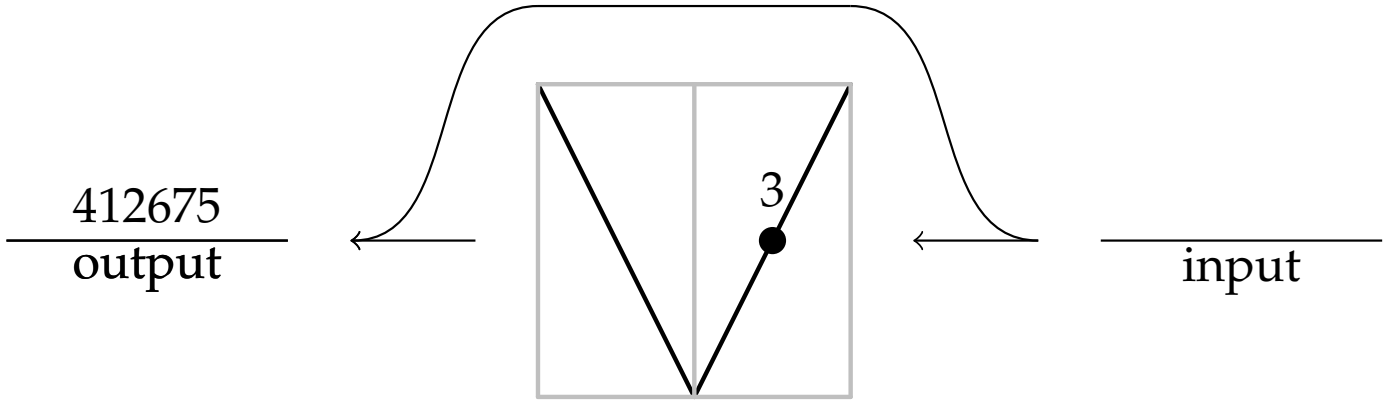

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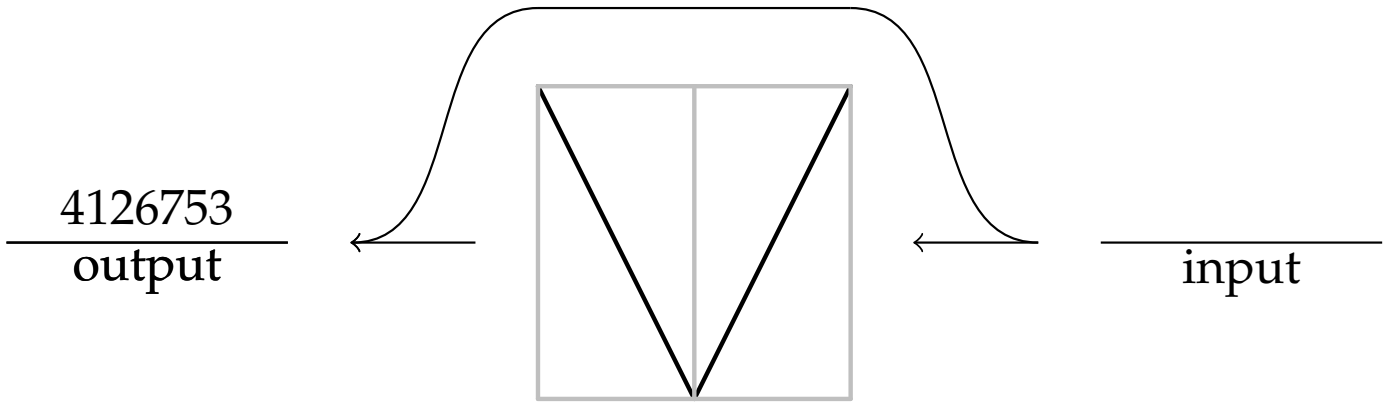

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The Av(231, 132)-machine generates the class Av(4231, 4132).

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Theorem. The $\mathrm{Av}(B)$-machine generates the class

$$\mathrm{Av}(\{{}^{+}\beta : \beta \in B\}),$$

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- **Structural Descriptions:**
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Enumerating Av(312) via the Av(12)-Machine

Av(312): $\approx 4^n$

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Instead of remembering all $\approx 4^n$ permutations of length n in $\mathrm{Av}(312)$ to build those of length $n+1$, we only need to remember the $n+1$ possible states of the machine:

[machine has no entries],
[machine has 1 entry],
...,
[machine has n entries]

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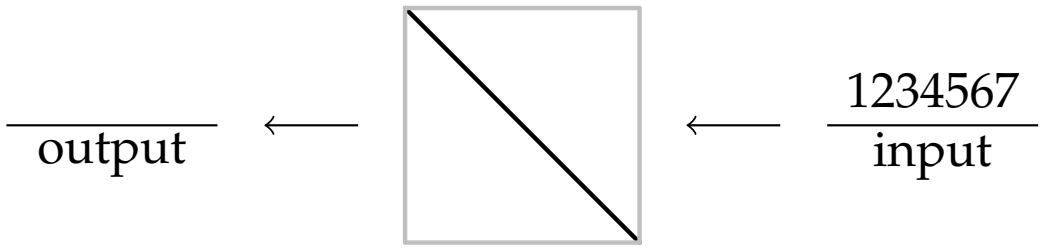

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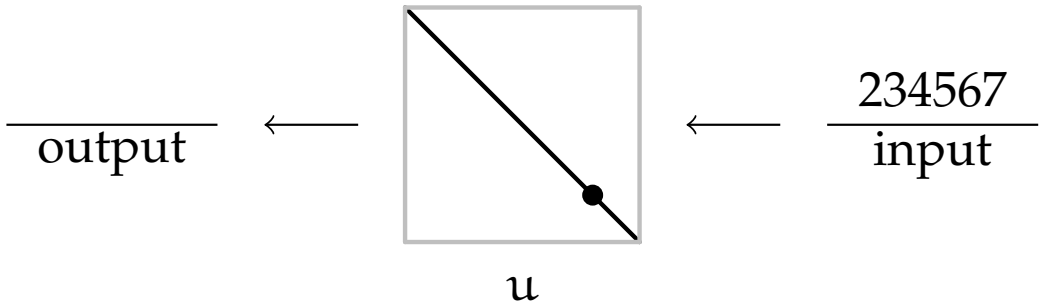

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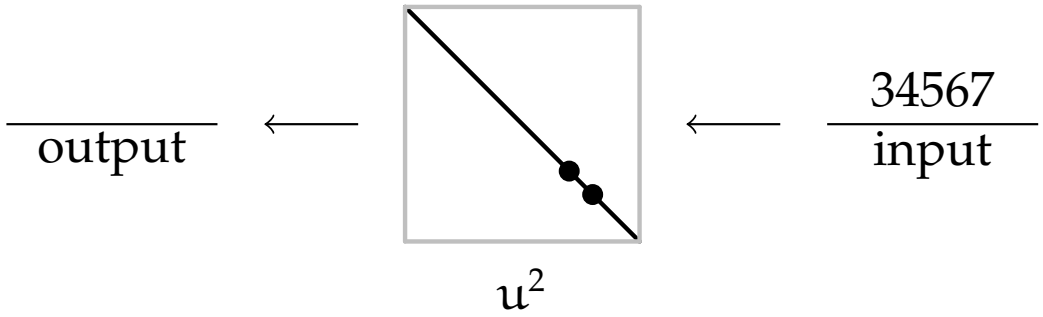

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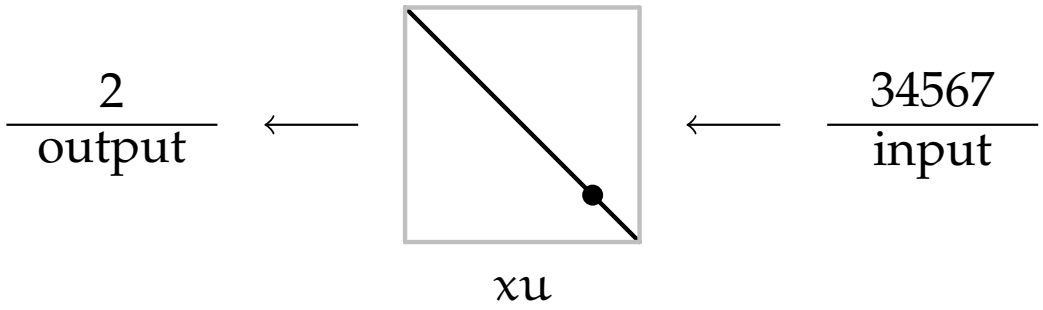

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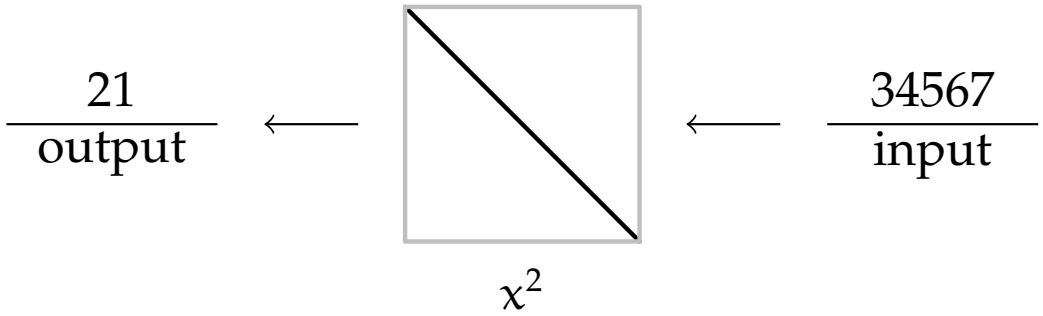

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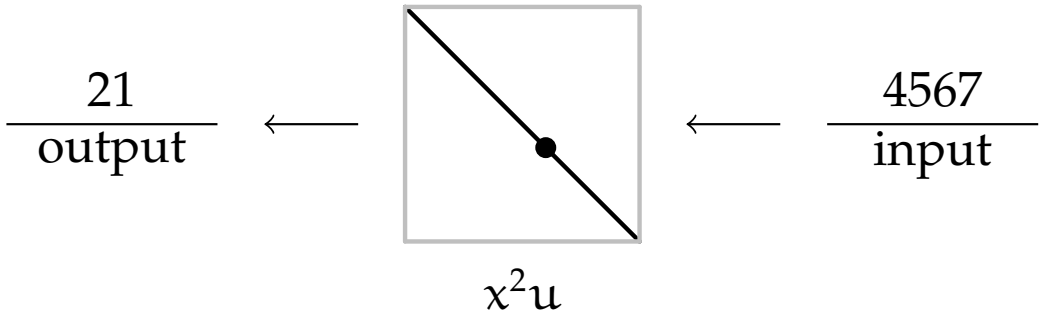

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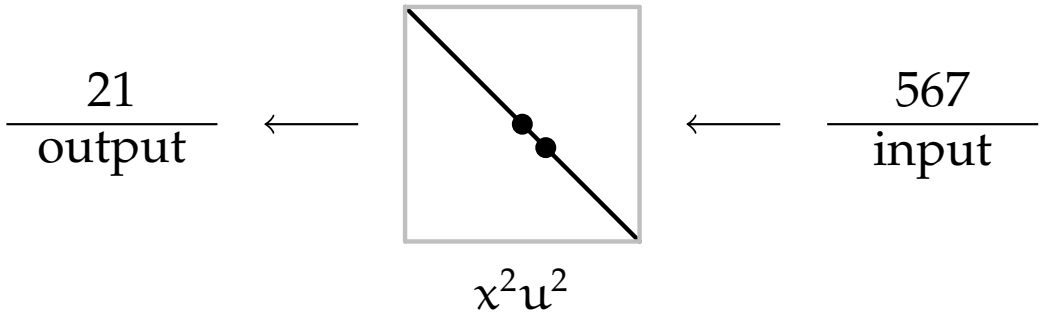

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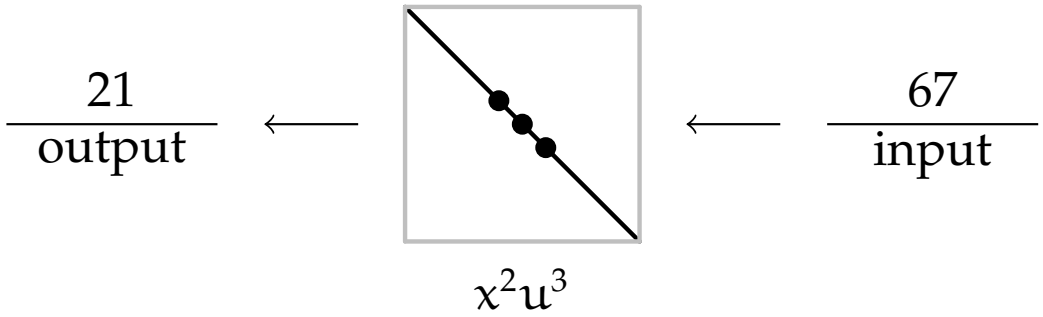

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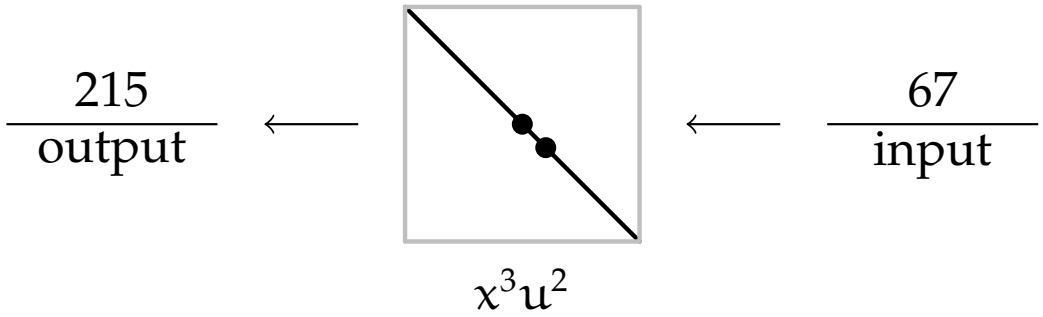

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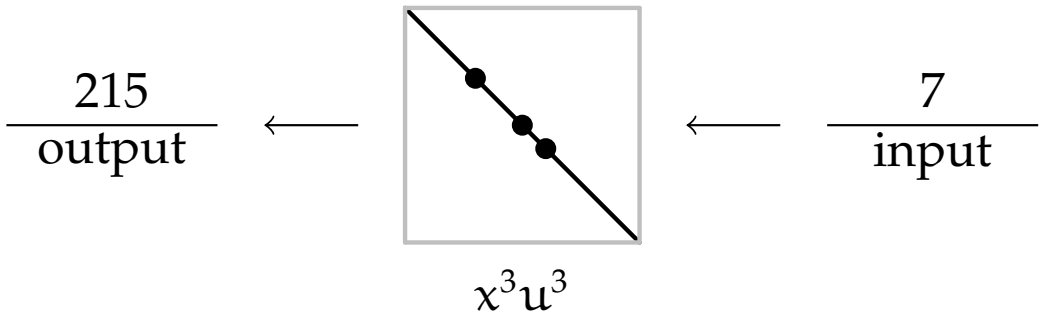

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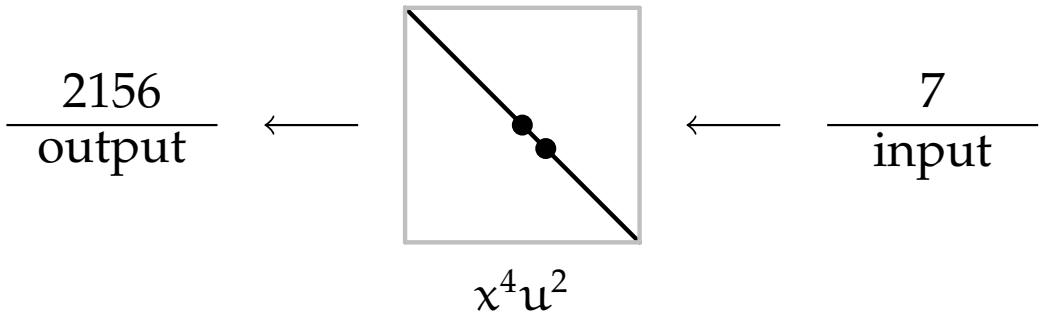

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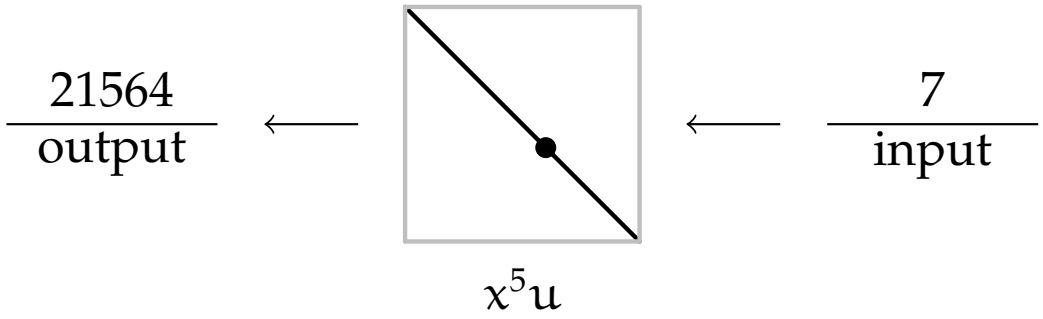

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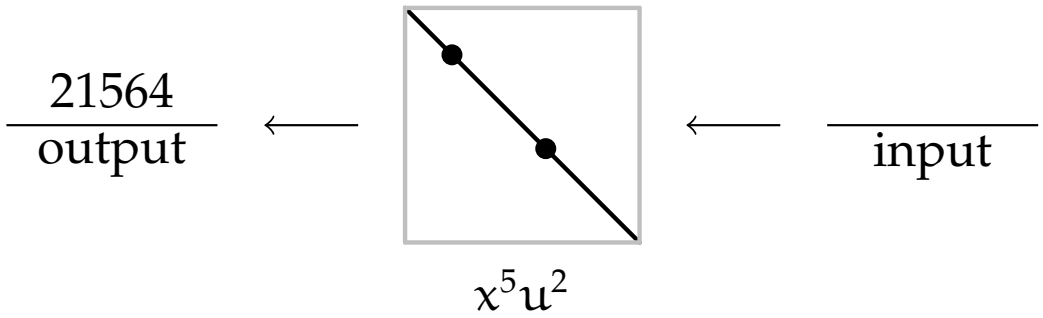

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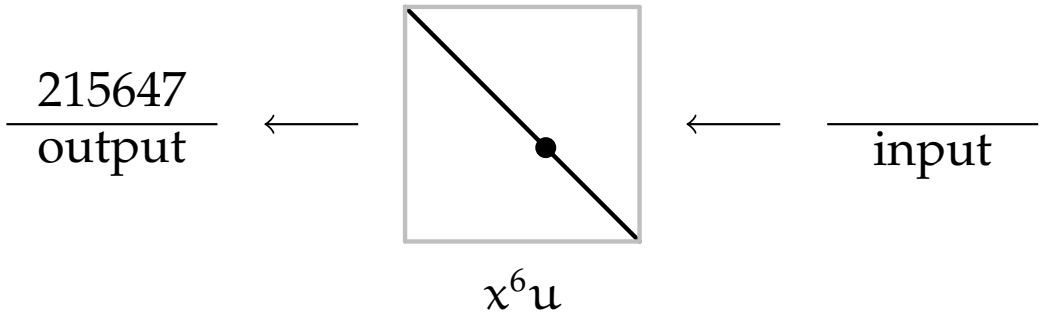

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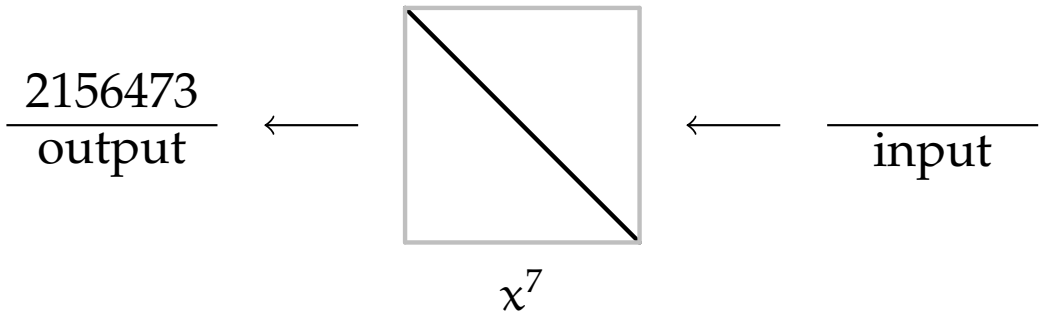

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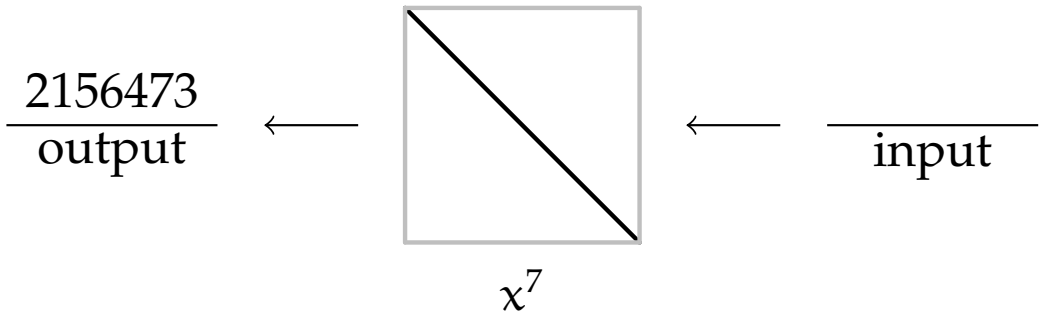

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Hence the generating function for these states is $f(x, 0)$.

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The kernel method solves: $f(x,0) = \dfrac{1-\sqrt{1-4x}}{2x}$.

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- an entry should be popped as soon as possible.

Equivalently: once you push an entry, you cannot pop any entries until you have bypassed an entry.

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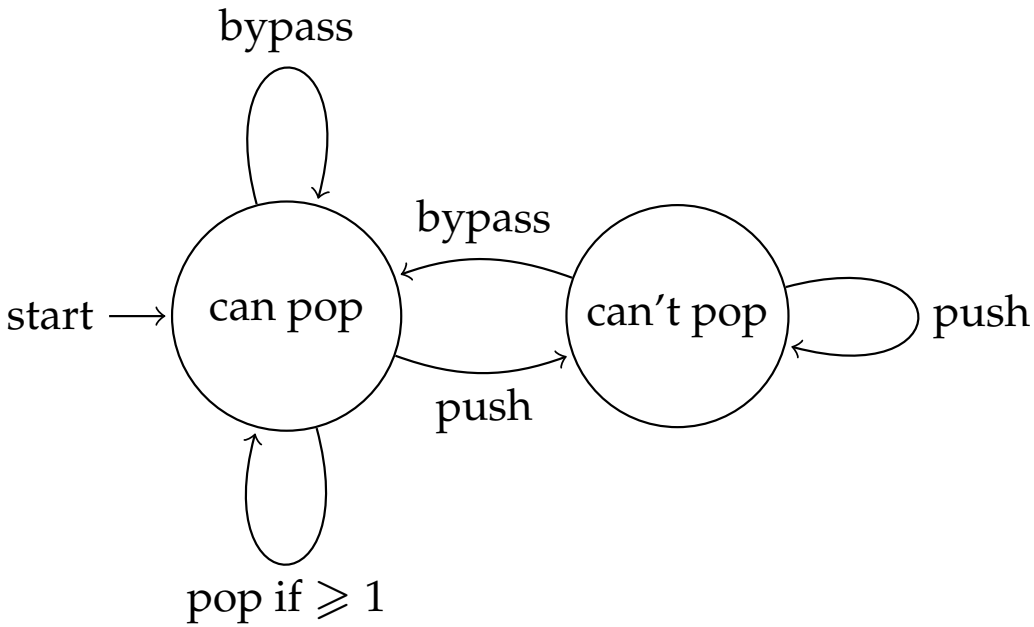

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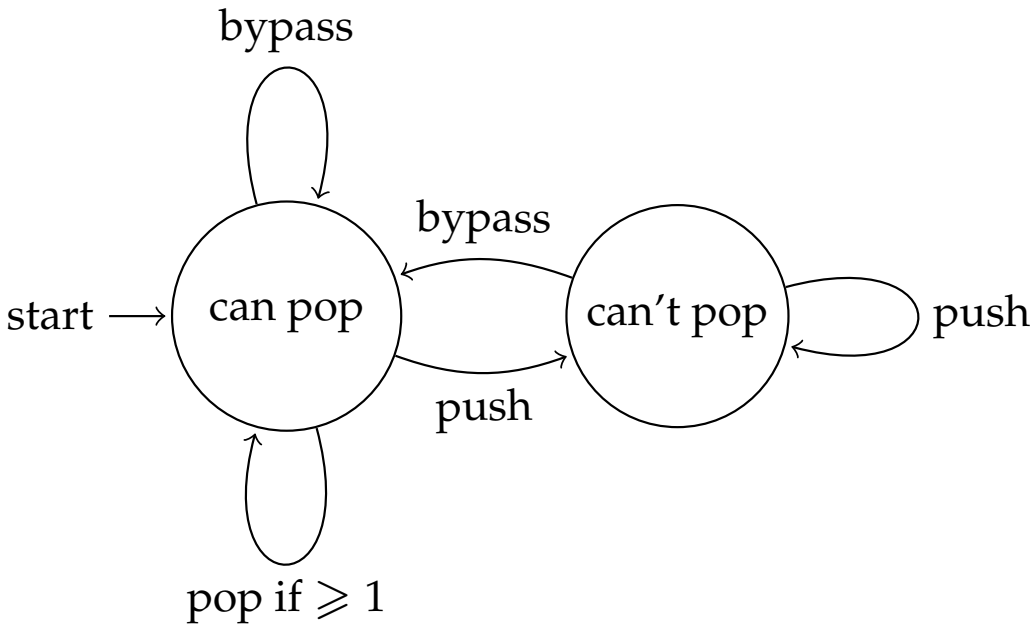

$$[\text{can pop}]: \quad f(x,u) = 1 + x(f(x,u) + g(x,u)) + \frac{x}{u}(f(x,u) - f(x,0)),$$

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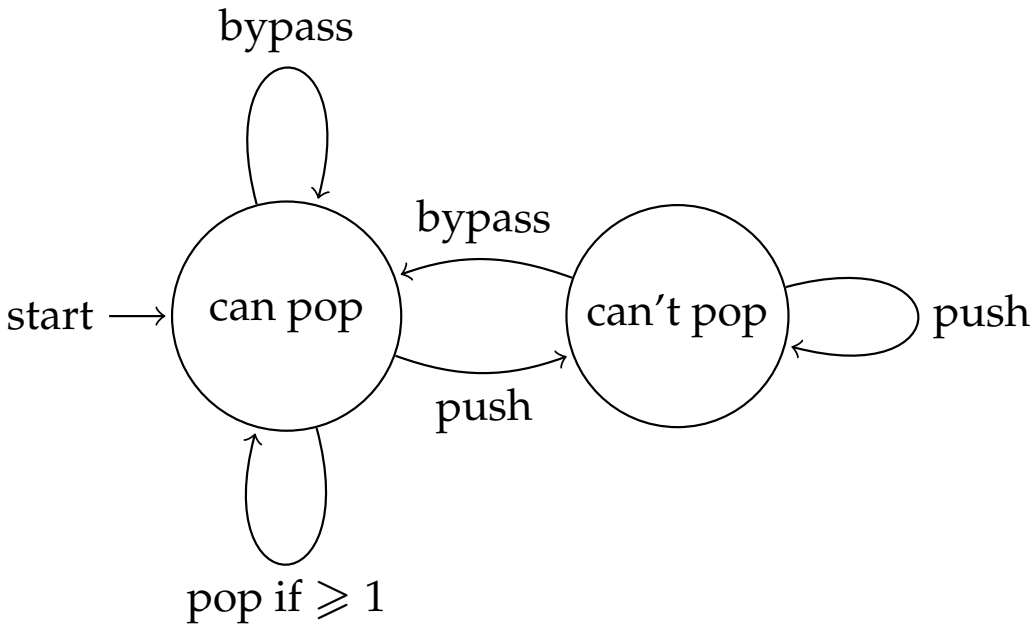

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Solve $g(x,u)$ in terms of $f(x,u) \to$ substitute into $f(x,u) \to$ kernel method $\to$ same answer!

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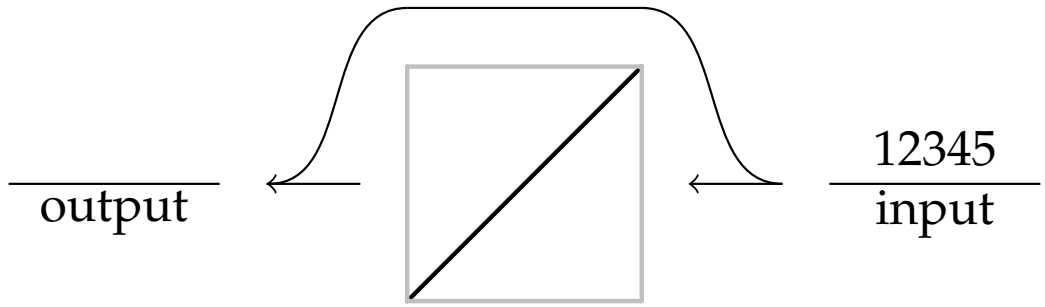
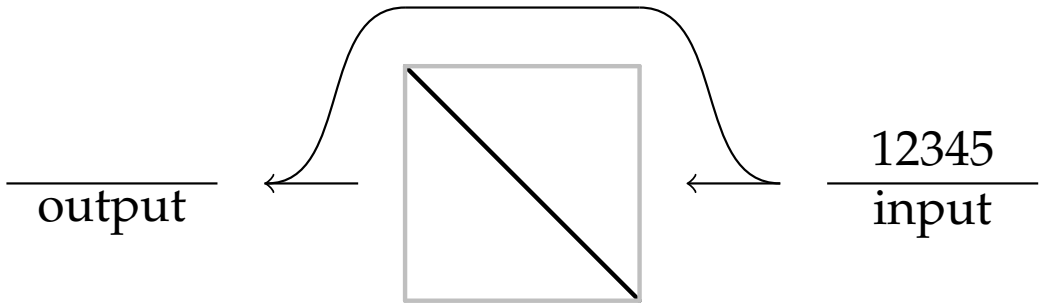
(To make this easier to describe, we tack the bypass back onto the Av(12)-Machine, even though it makes no difference.)

Bijection





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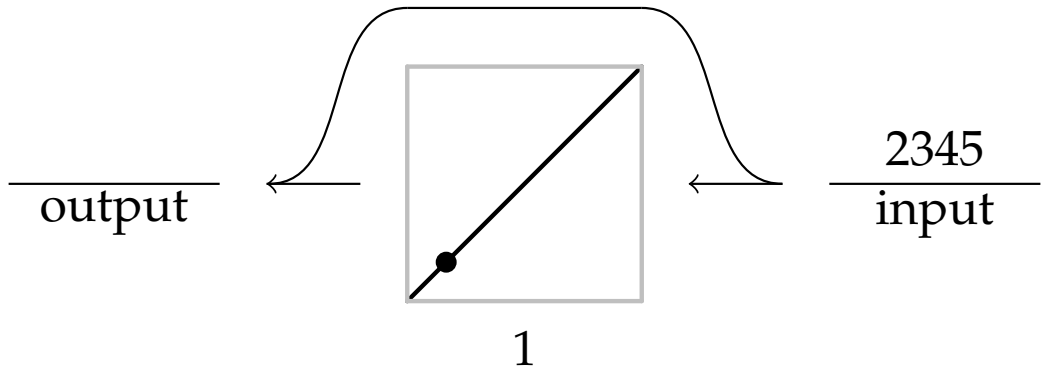

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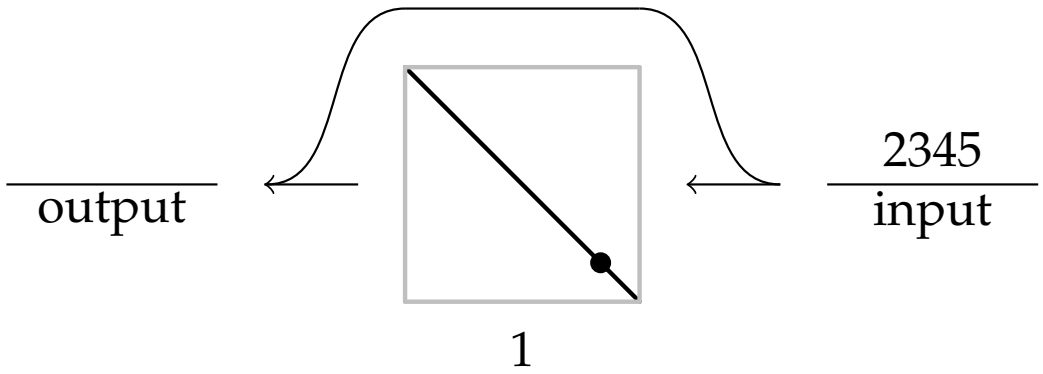

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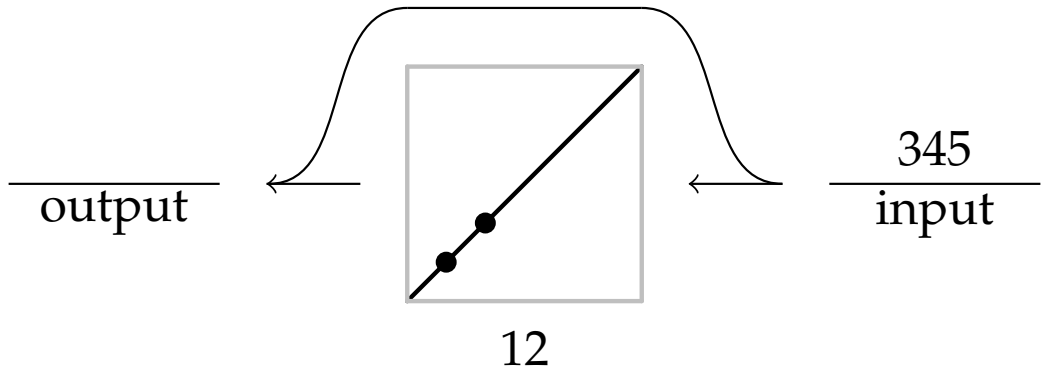

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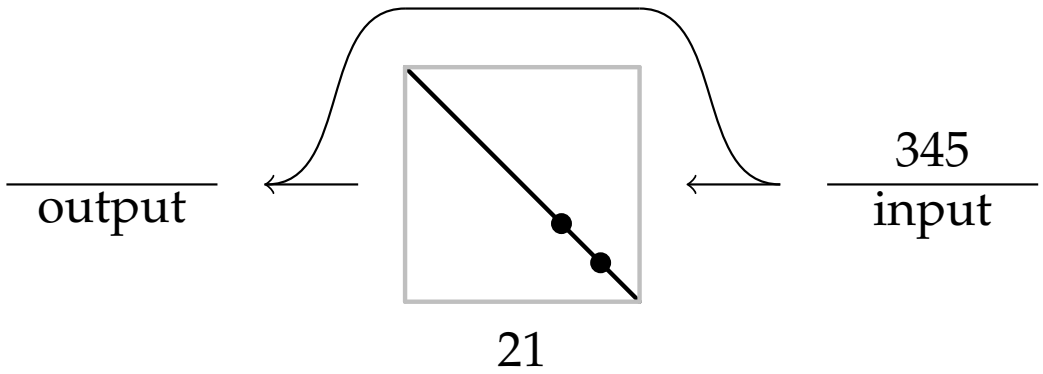

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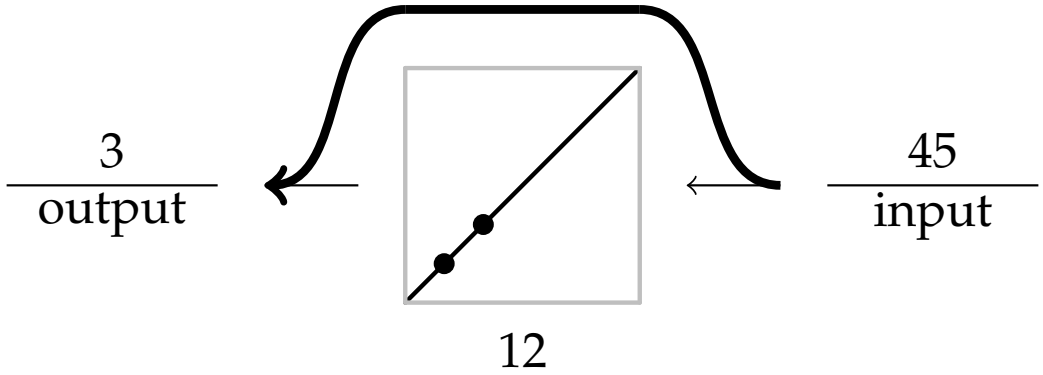
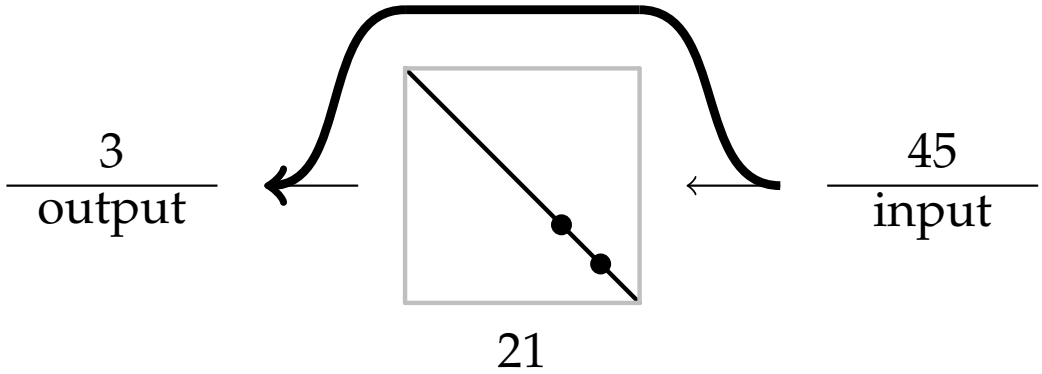


bypass



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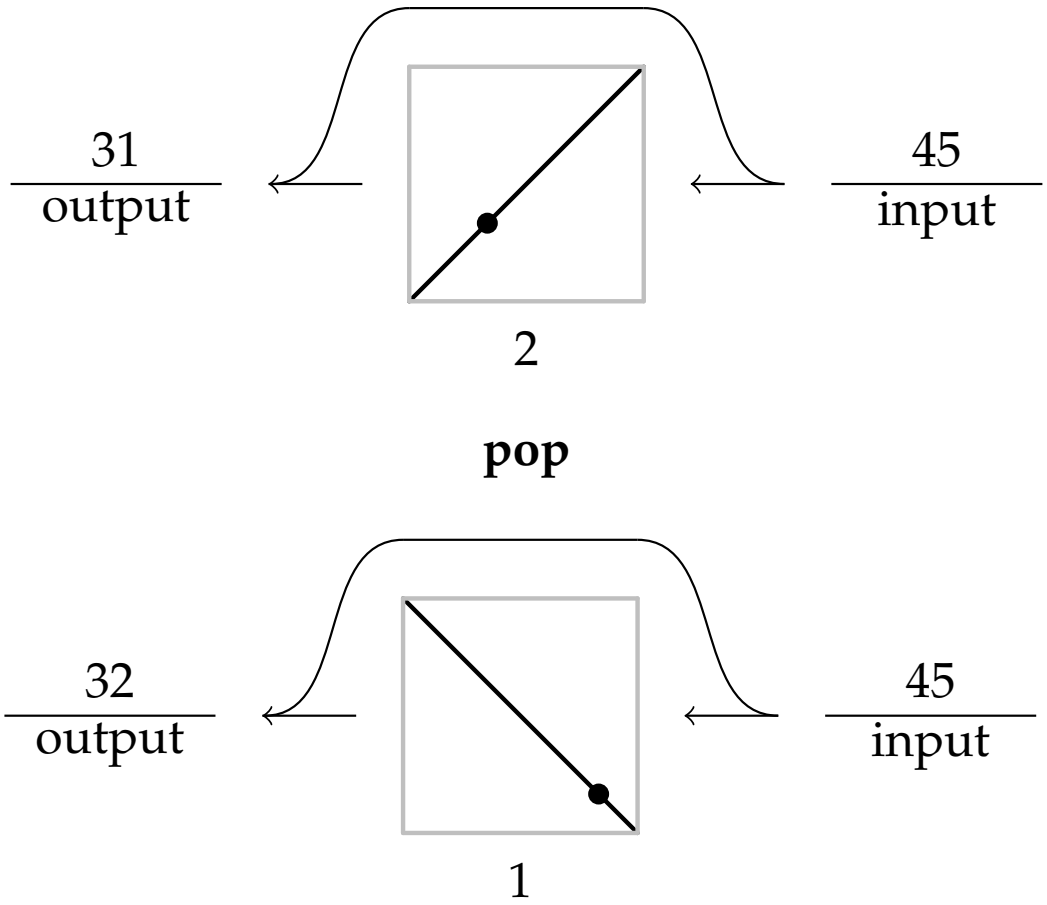

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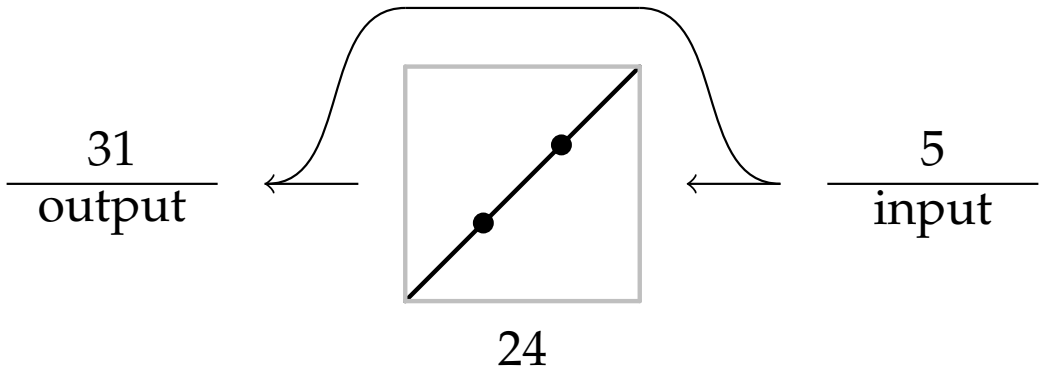

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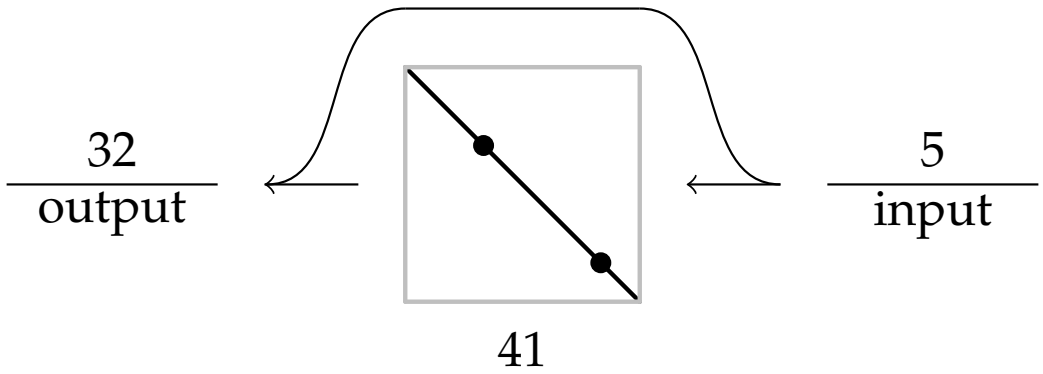

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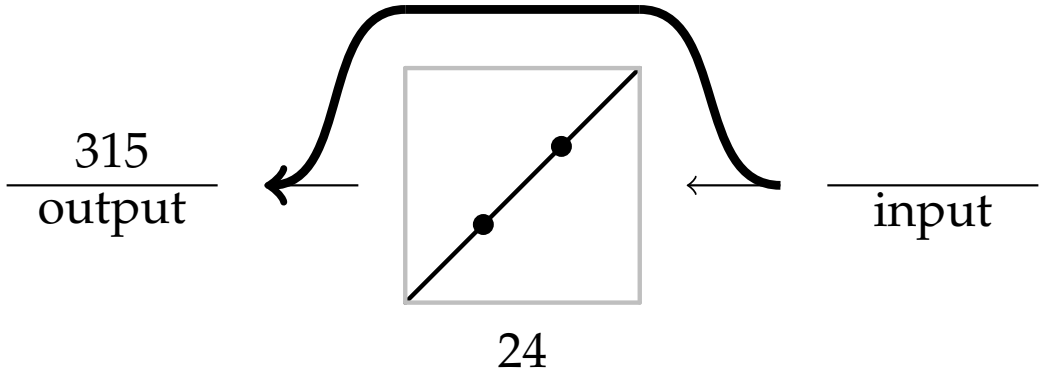

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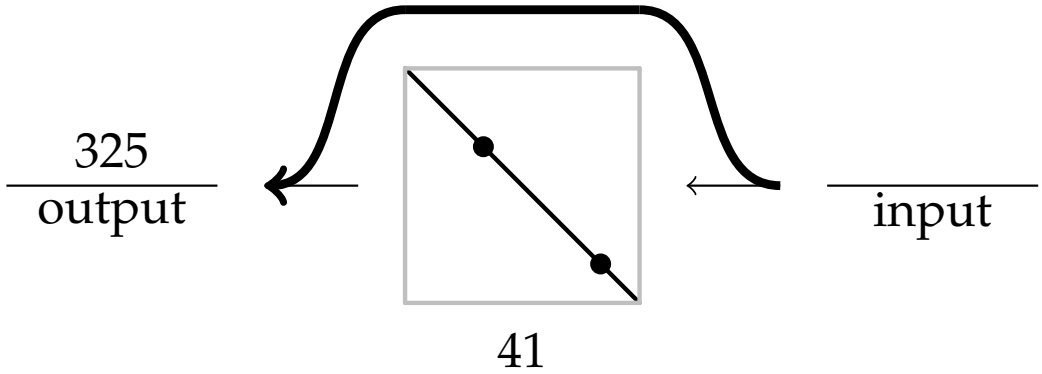

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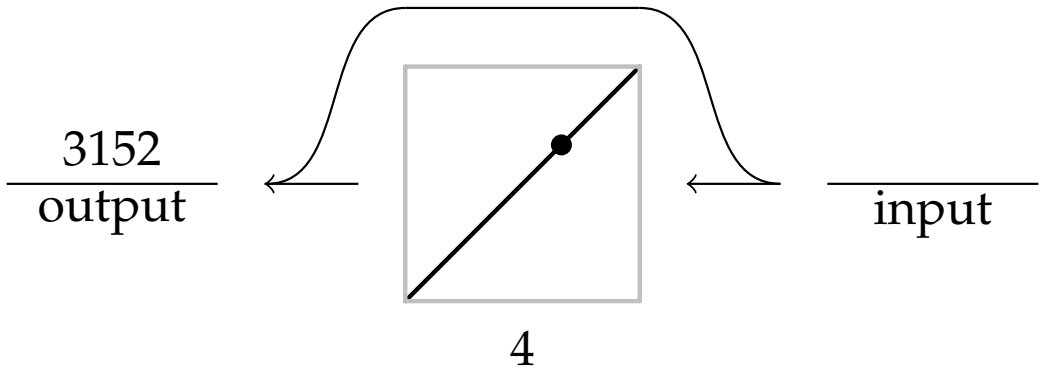

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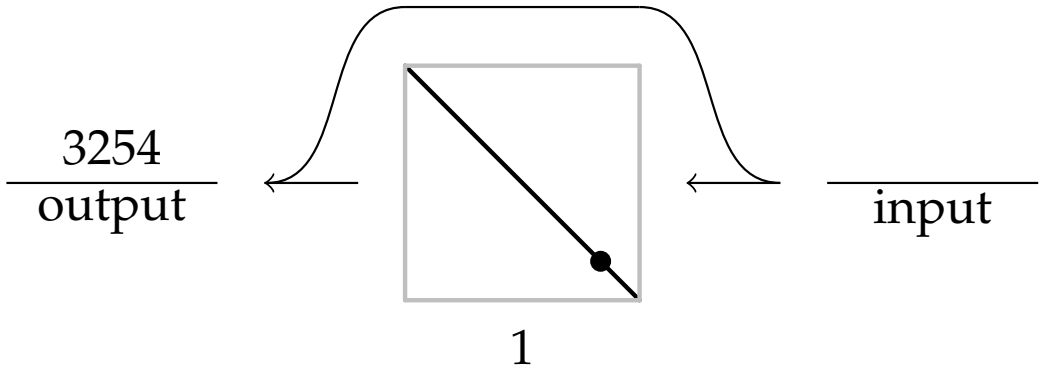

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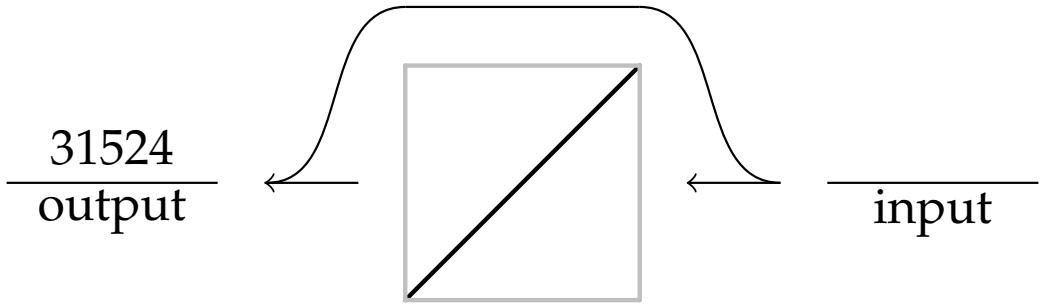

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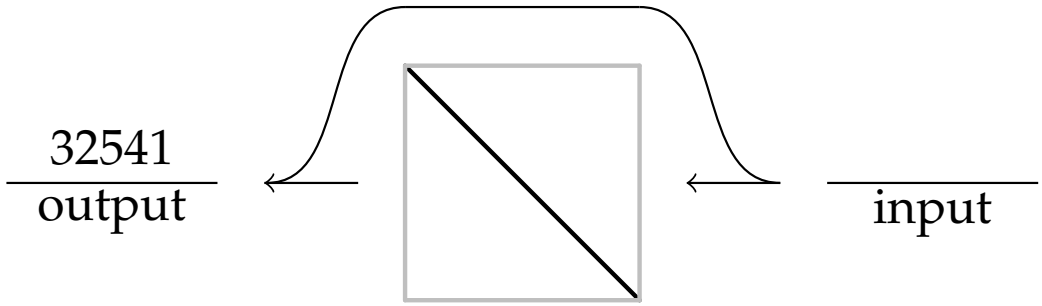

Random Sampling

Any combinatorial family that can be counted with *dynamic programming* can be sampled from uniformly at random.

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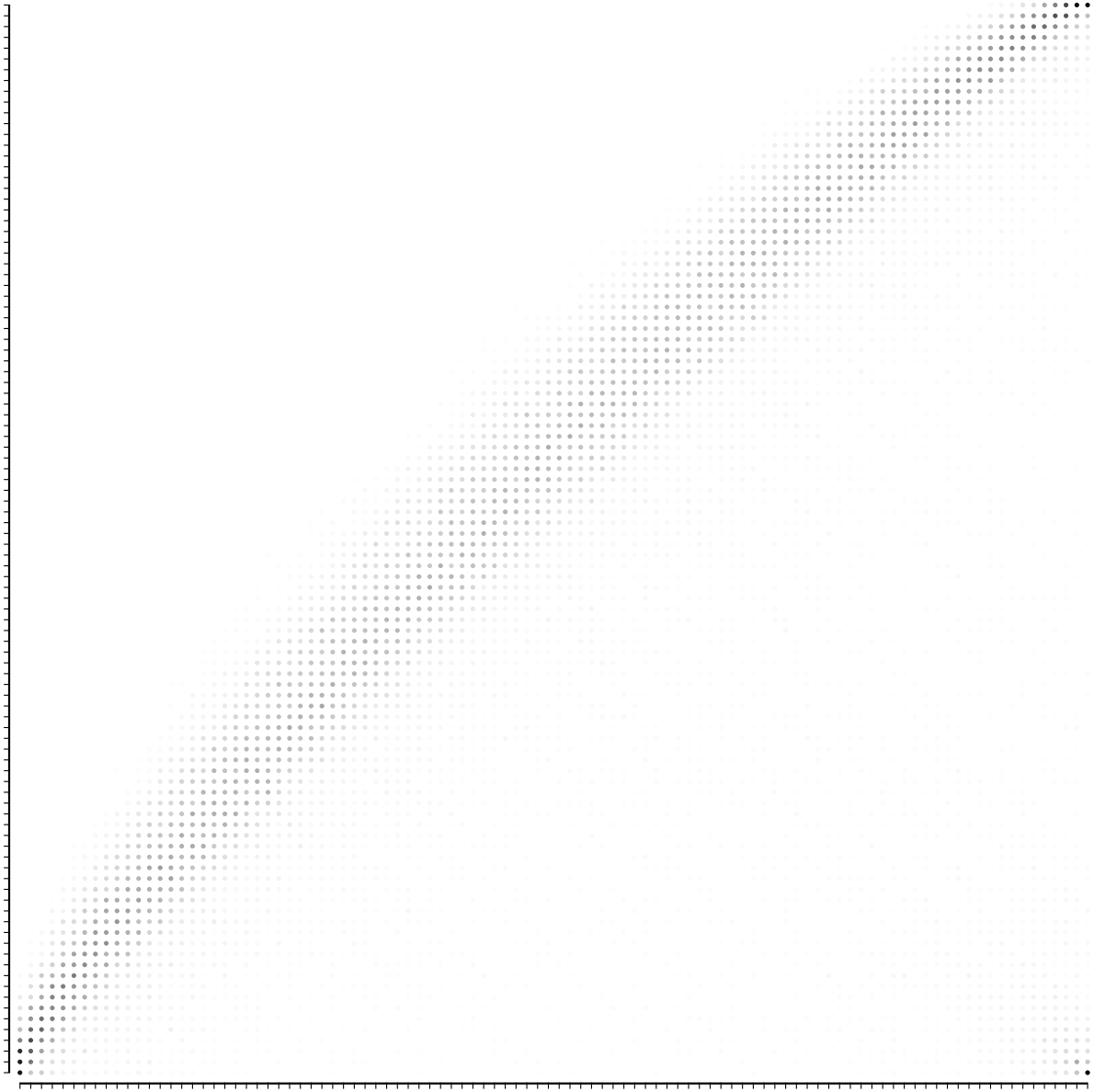

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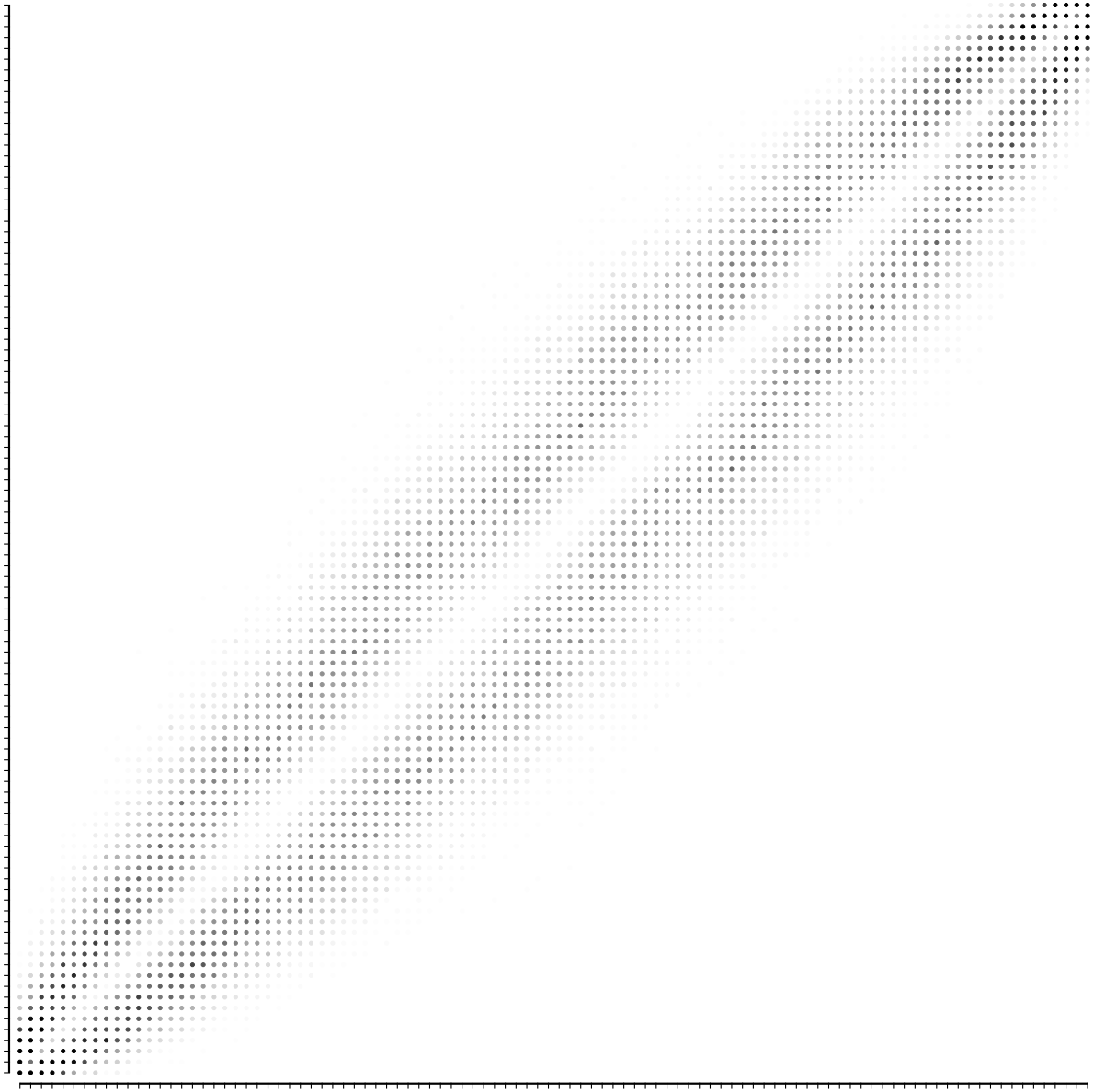

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Of these 38 classes, the exact enumerations are known for 35 of them. (More on this later!)

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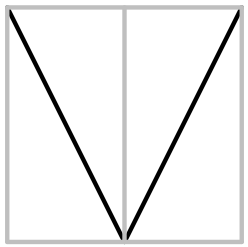

Enumerating the Schröder Classes

A small adjustment to the earlier functional equations gives:

$$f(x,u) = 1 + x(f(x,u) + g(x,u)) + \frac{x}{u}(f(x,u) - f(x,0)),$$
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An application of the kernel method gives the generating function for the Schröder numbers.

$$f(x,0) = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2}$$

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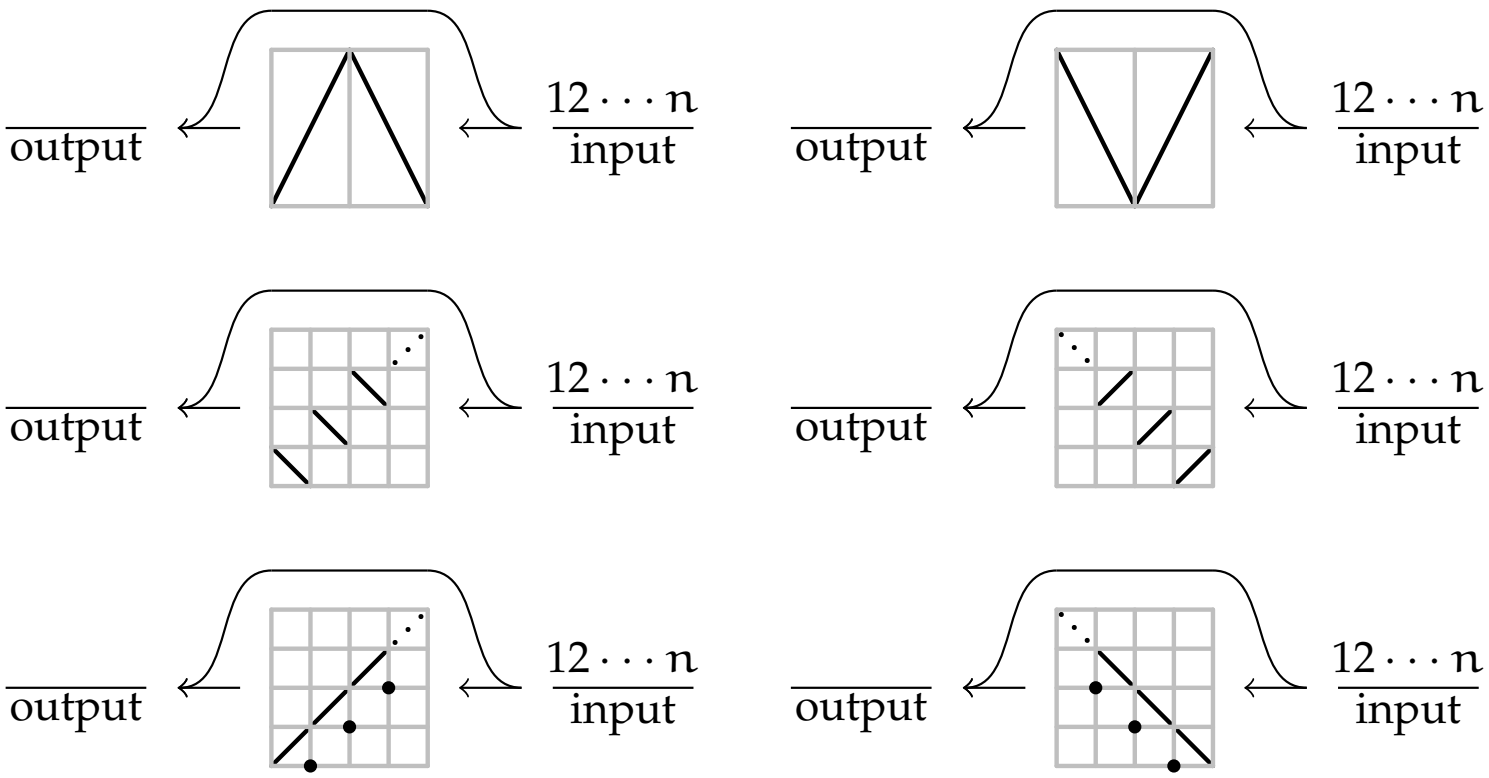

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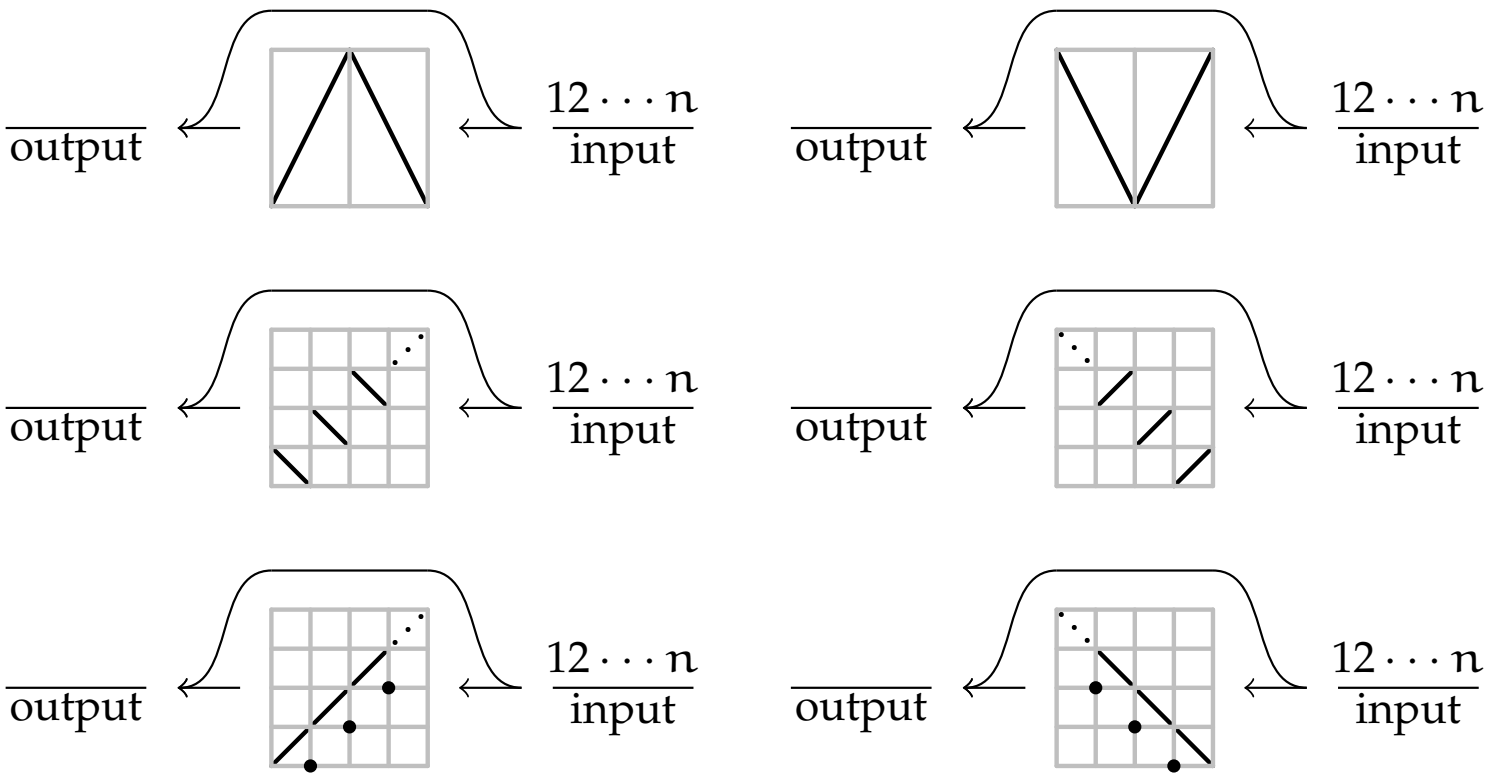

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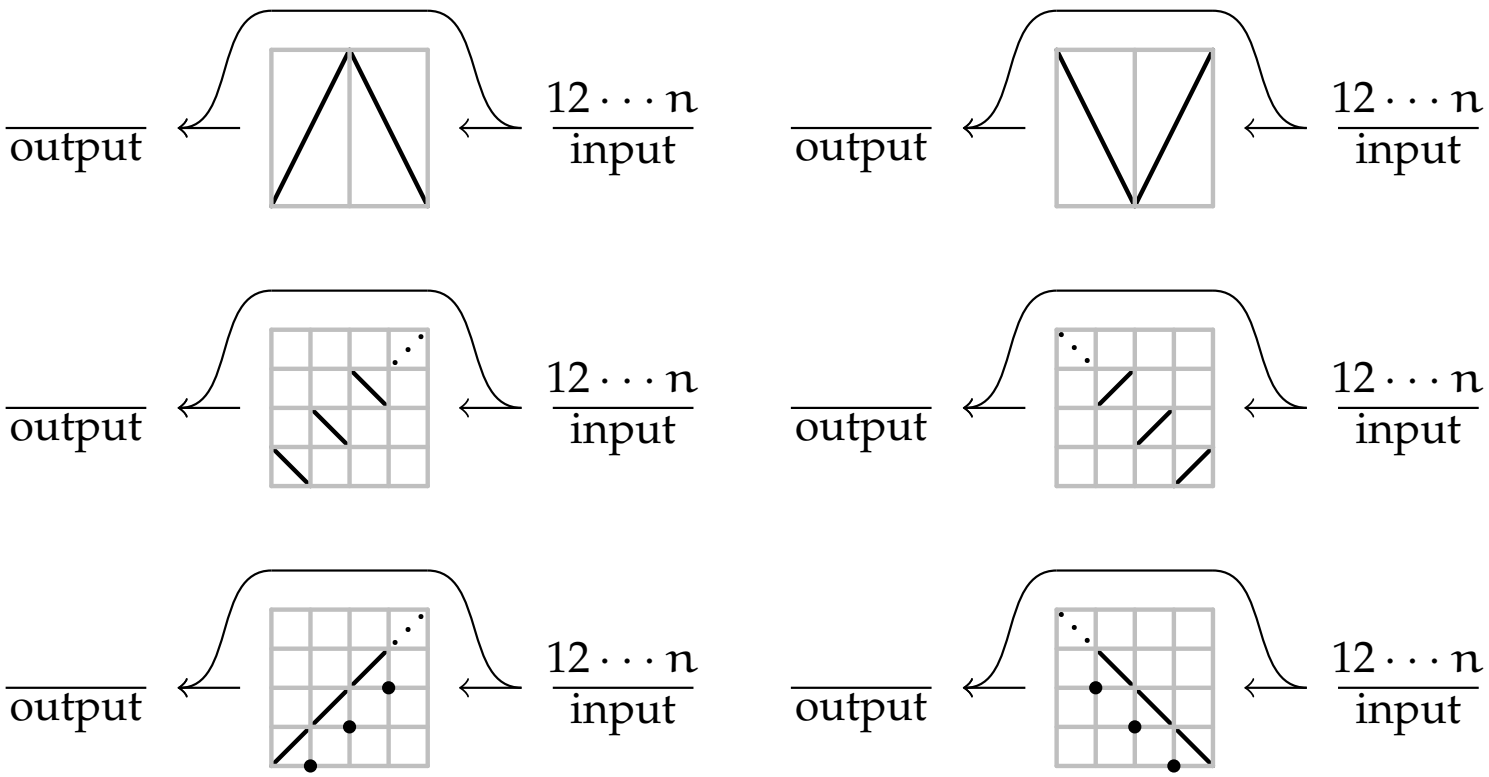

All are enumerated by the Schröder numbers.

Free bijections between every pair of classes.

2×4 Classes

Enumerations are known for all but three:

- $\mathrm{Av}(4123, 4231)$
- $\mathrm{Av}(4123, 4312)$
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These three can all be modeled with $\mathcal{C}$-machines.

$\mathrm{Av}(4123, 4231)$

$\mathrm{Av}(4123, 4231)$ is generated by the $\mathrm{Av}(123, 231)$-machine.

The class $\mathrm{Av}(123, 231)$ is a geometric grid class.

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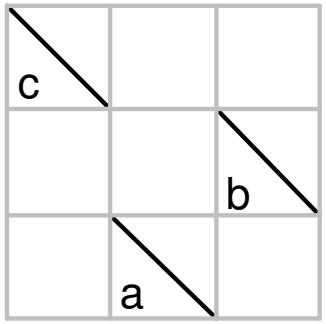

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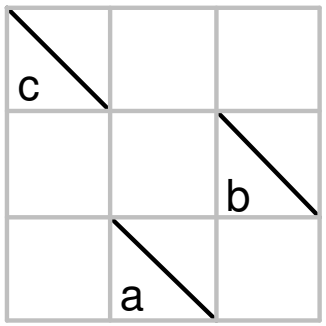
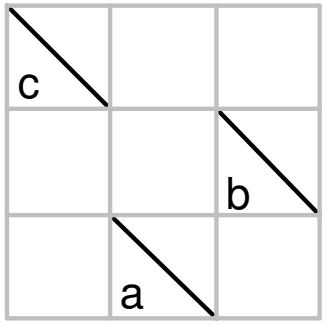


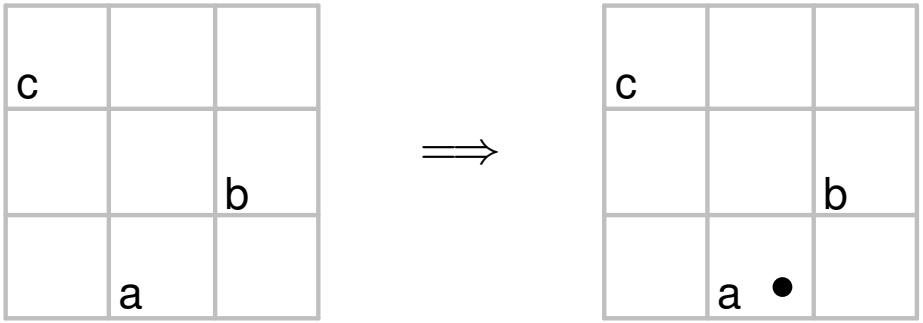
The states of the machine can be represented by 4-tuples (a, b, c, P) where a, b, and c are the number of a, b, c entries, and P is a boolean representing whether we can or can't pop.

$\mathrm{Av}(4123, 4231)$



We can describe the state transitions:

Av(4123, 4231)



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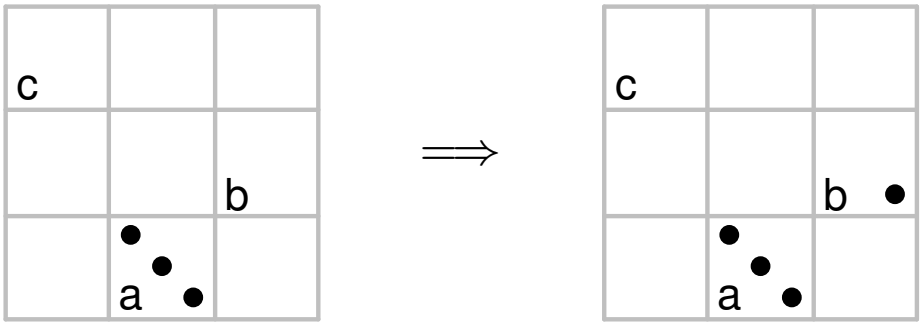

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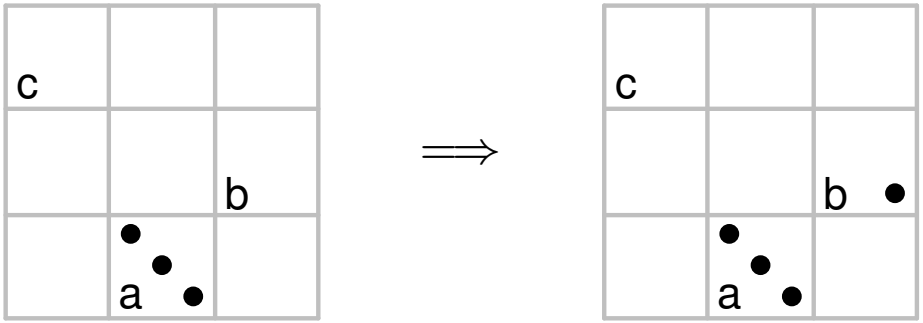

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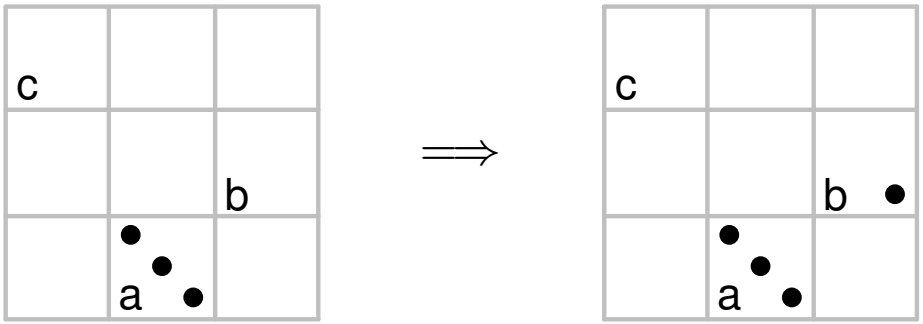

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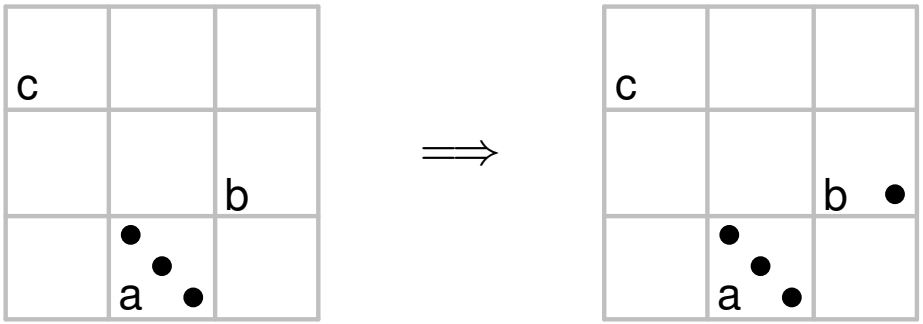

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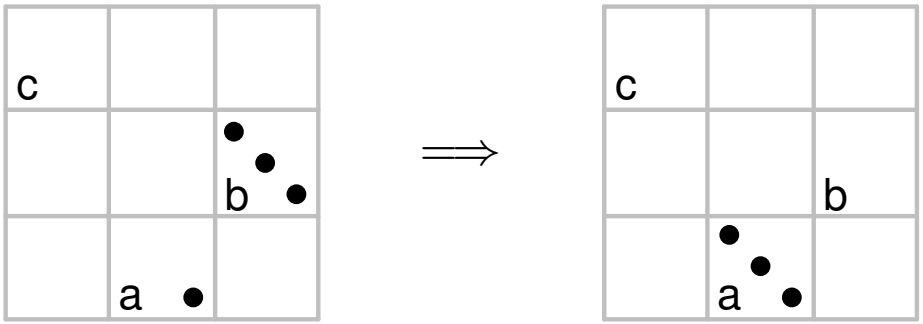

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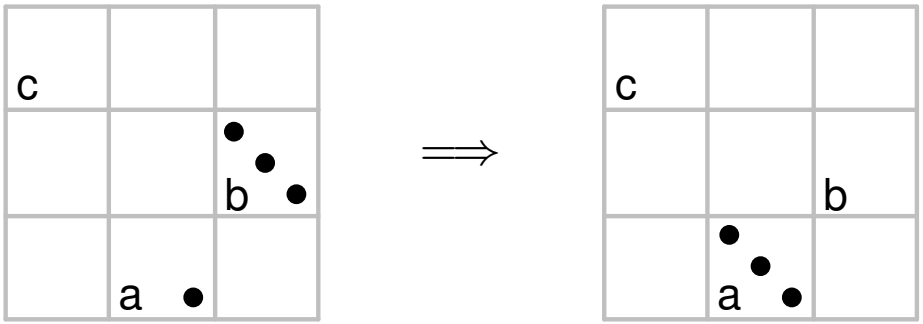

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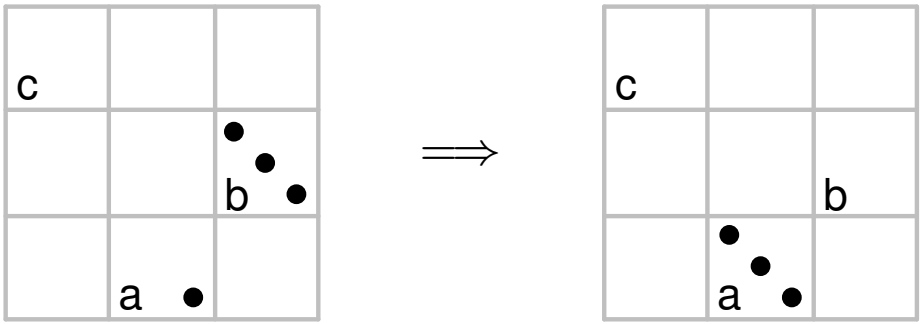

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The Av(123, 231)-Machine

With a little work, we can translate the state transitions into a set of functional equations:

$$A(a,x) = 1 + \frac{x}{a}(A(a,x) - A(0,x)) + aA(a,x) + xB(0,a,x),$$

$$B(a,b,x) = \frac{1}{a}(A(a,x) - A(0,x))\frac{bC}{1-b} + B(a,b,x)\frac{bC}{1-b} + \frac{x}{a}(1+C)(B(a,b,x) - B(0,b,x)).$$

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The generating function for the class is $A(0,x)$, but we have no idea how to solve these equations.

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Using the state transitions, dynamic programming, and Amazon cloud computing, we have computed the first 1000 terms of Av(4123, 4231).

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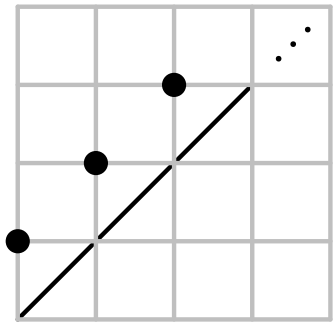
Despite having 1000 initial terms of Av(4123, 4231), we can't guess any generating function!

Three More Exotic Machines



600 terms

$\mathrm{Av}(4231, 4321)$

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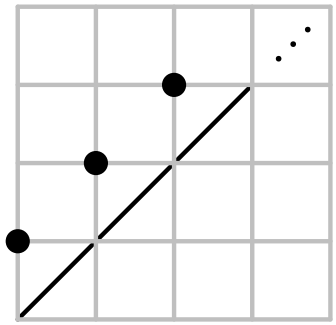

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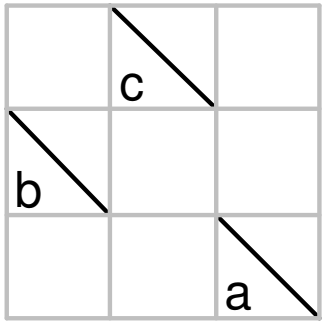

1000 terms
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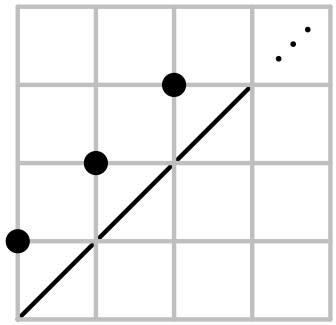
Three More Exotic Machines



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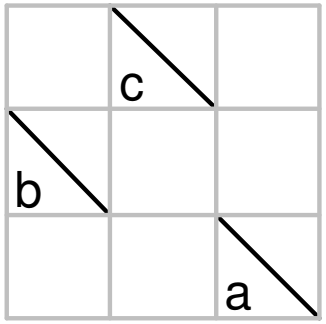

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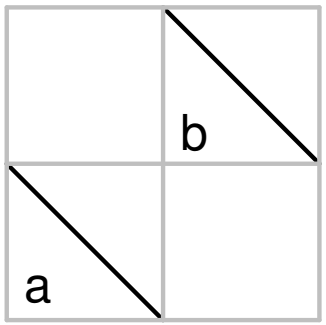

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Still...

Even though we don't have a generating function, we can still:

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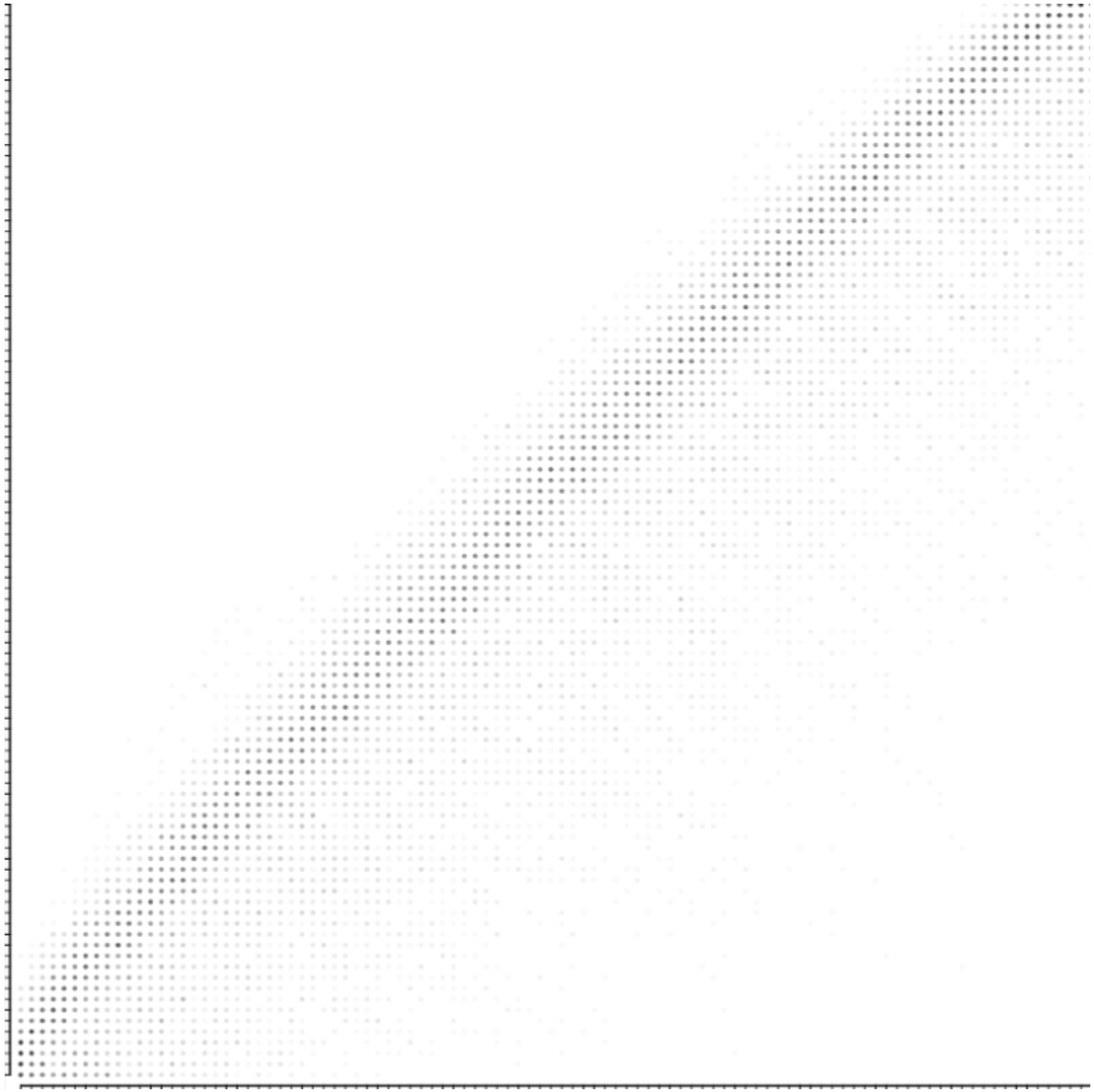
- Sample uniformly at random

Still...

Even though we don't have a generating function, we can still:

- Sample uniformly at random
- Use many initial terms to approximate the singularity structure and asymptotic behavior

$\mathrm{Av}(4123, 4231)$



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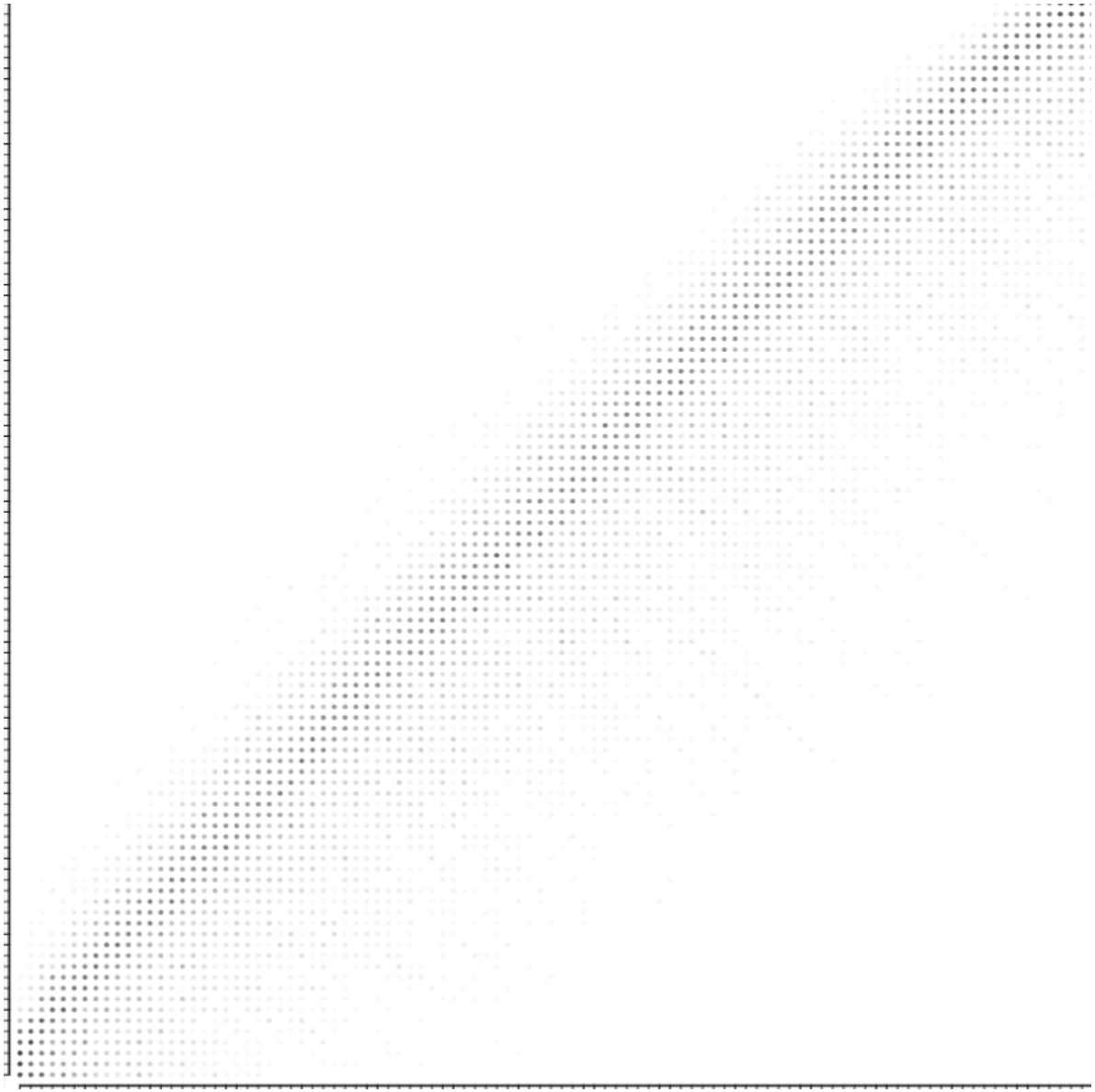

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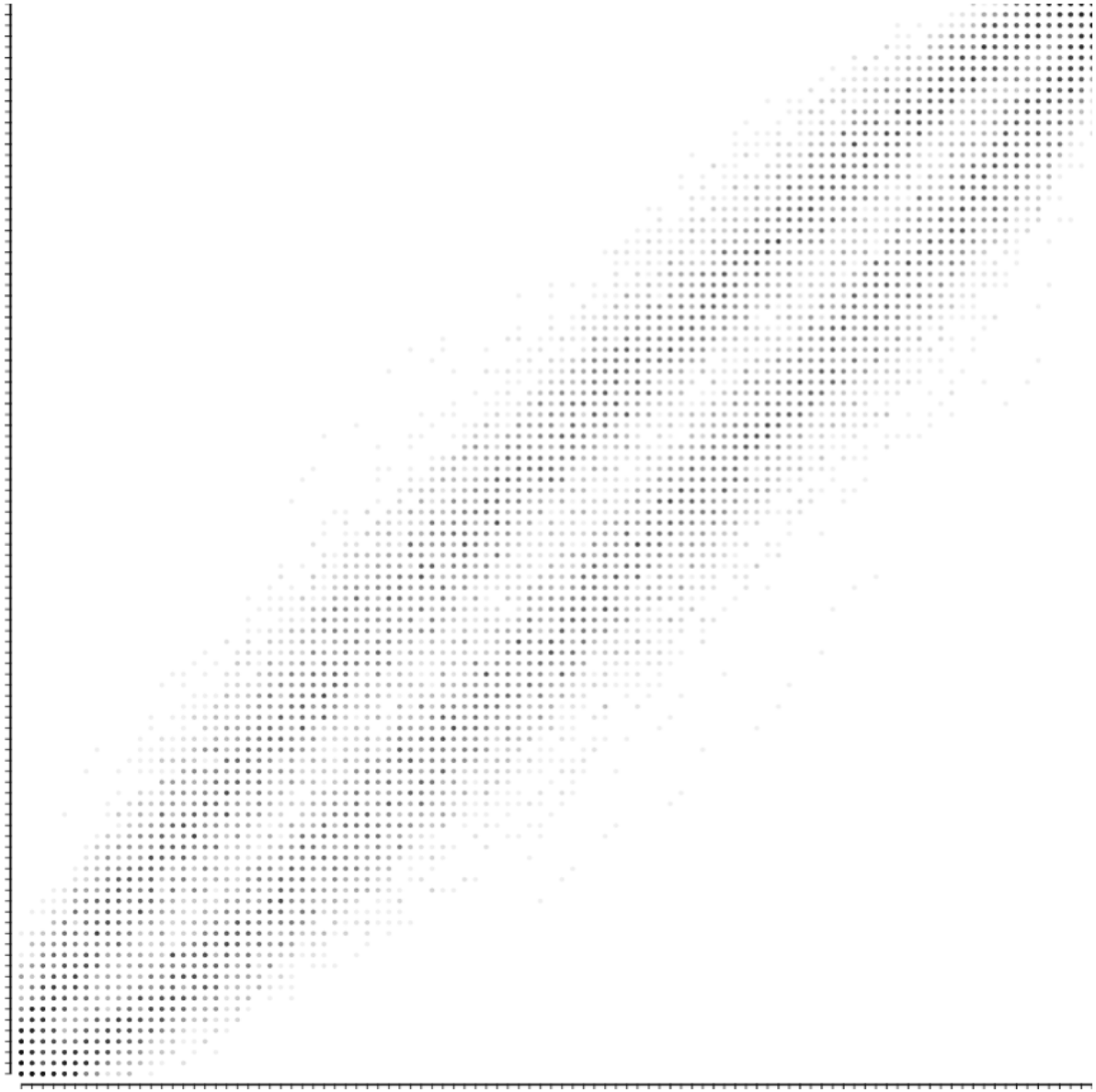

Singularity Structure – Av(4123,4231,4312)

Dominant singularity at

$$x = 0.22379330134707544472559413080653078228254418356103196904827043215739850630915030$$

with critical exponent

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Other singularities:

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- 0.24257431785781985397922441858376486736156311385549281450394241750 60...
- 0.24474485438214052759717305161330543366386588708085807333992 8...
- 0.24600369400712937477975290028099807844398...
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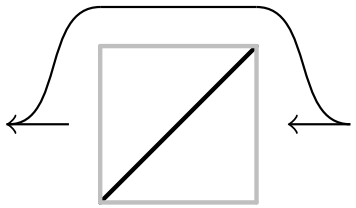
$$\Longrightarrow \sim C(4.468408...)^n$$

State Compression



4^n permutations $\longrightarrow$ n (1 integer)

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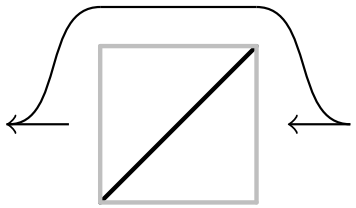

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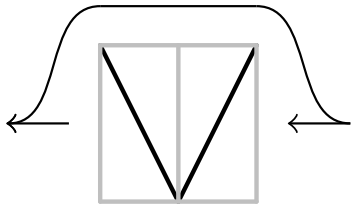
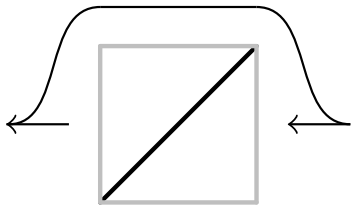


$5.828^n \longrightarrow 2^n \longrightarrow n$ (1 integer)

State Compression



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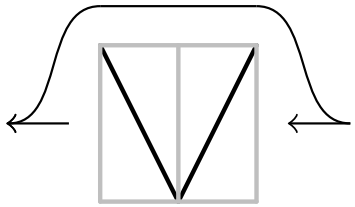
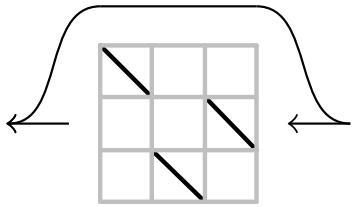
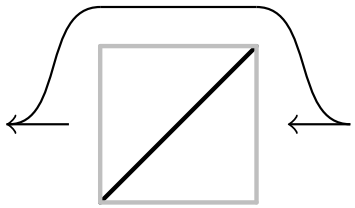


$5.828^n \longrightarrow 2^n \longrightarrow n$ (1 integer)

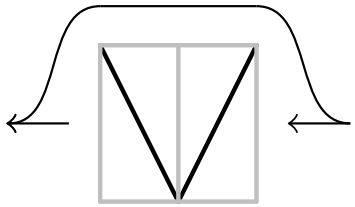


$4.977^n \longrightarrow n^3$ (3 integers)

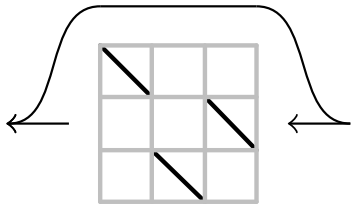
State Compression



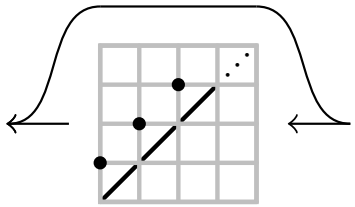
4^n permutations $\longrightarrow$ n (1 integer)



$5.828^n \longrightarrow 2^n \longrightarrow n$ (1 integer)



$4.977^n \longrightarrow n^3$ (3 integers)



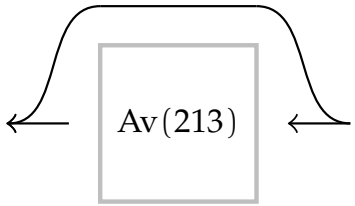
$5.892^n \longrightarrow 2^n \longrightarrow n^3$ (3 integers)

State Compression

$$11.6^n(?) \longrightarrow 4^n \longrightarrow ???$$

State Compression

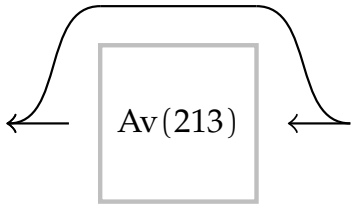
$$11.6^n(?) \longrightarrow 4^n \longrightarrow ???$$



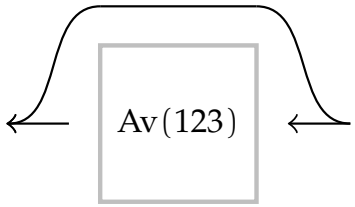
$$8^n \longrightarrow 4^n \longrightarrow ???$$

State Compression

$11.6^n(?) \longrightarrow 4^n \longrightarrow ???$



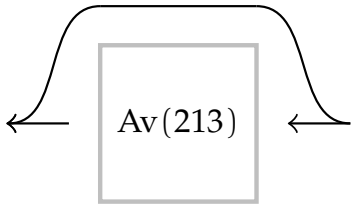
$8^n \longrightarrow 4^n \longrightarrow ???$



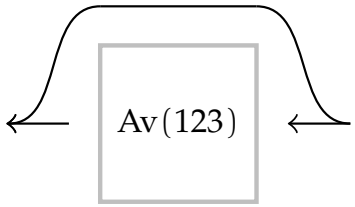
$9^n \longrightarrow 4^n \longrightarrow ???$

State Compression

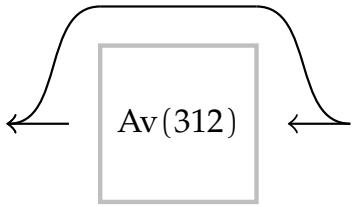
$11.6^n(?) \longrightarrow 4^n \longrightarrow ???$



$8^n \longrightarrow 4^n \longrightarrow ???$

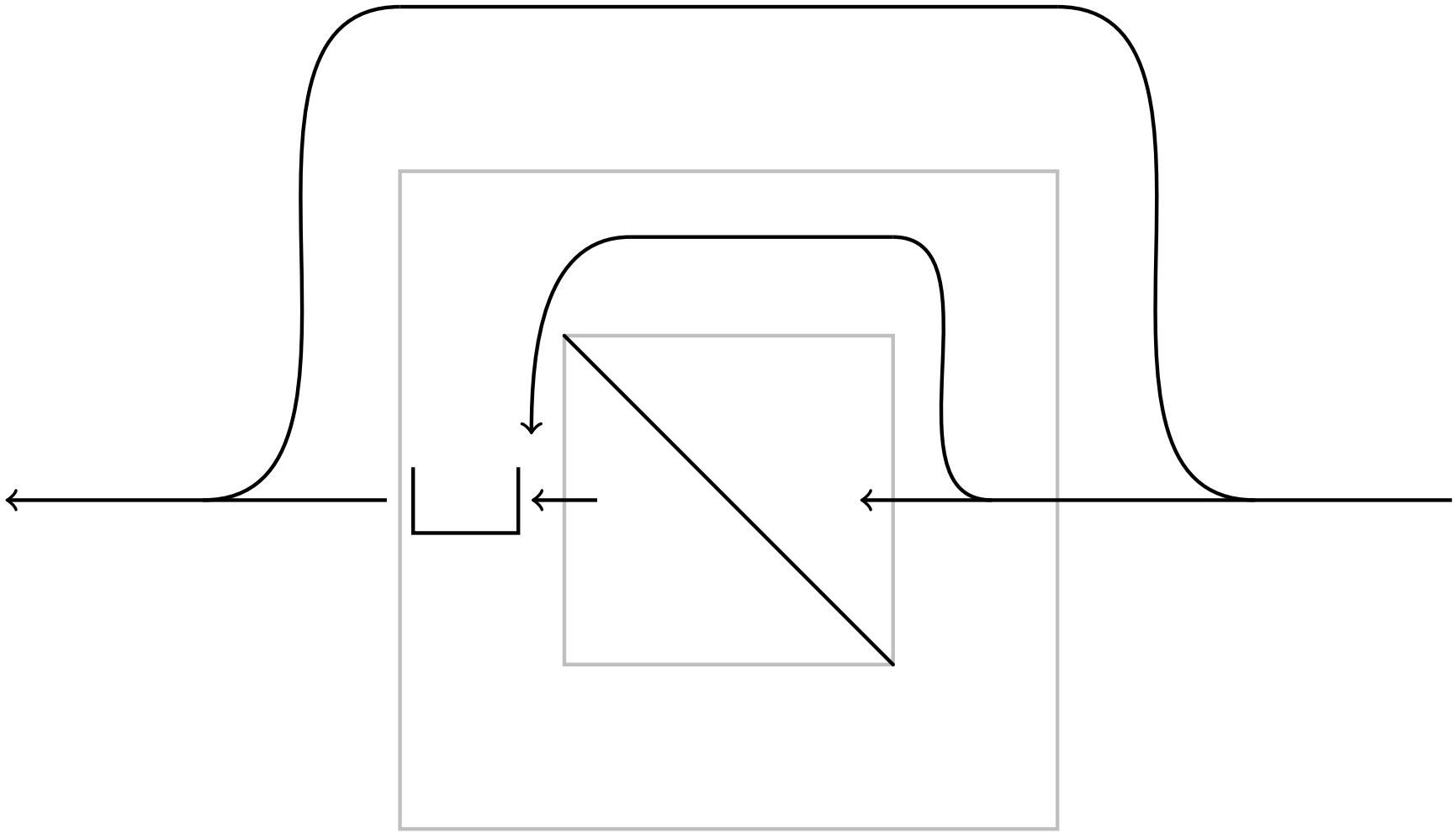


$9^n \longrightarrow 4^n \longrightarrow ???$



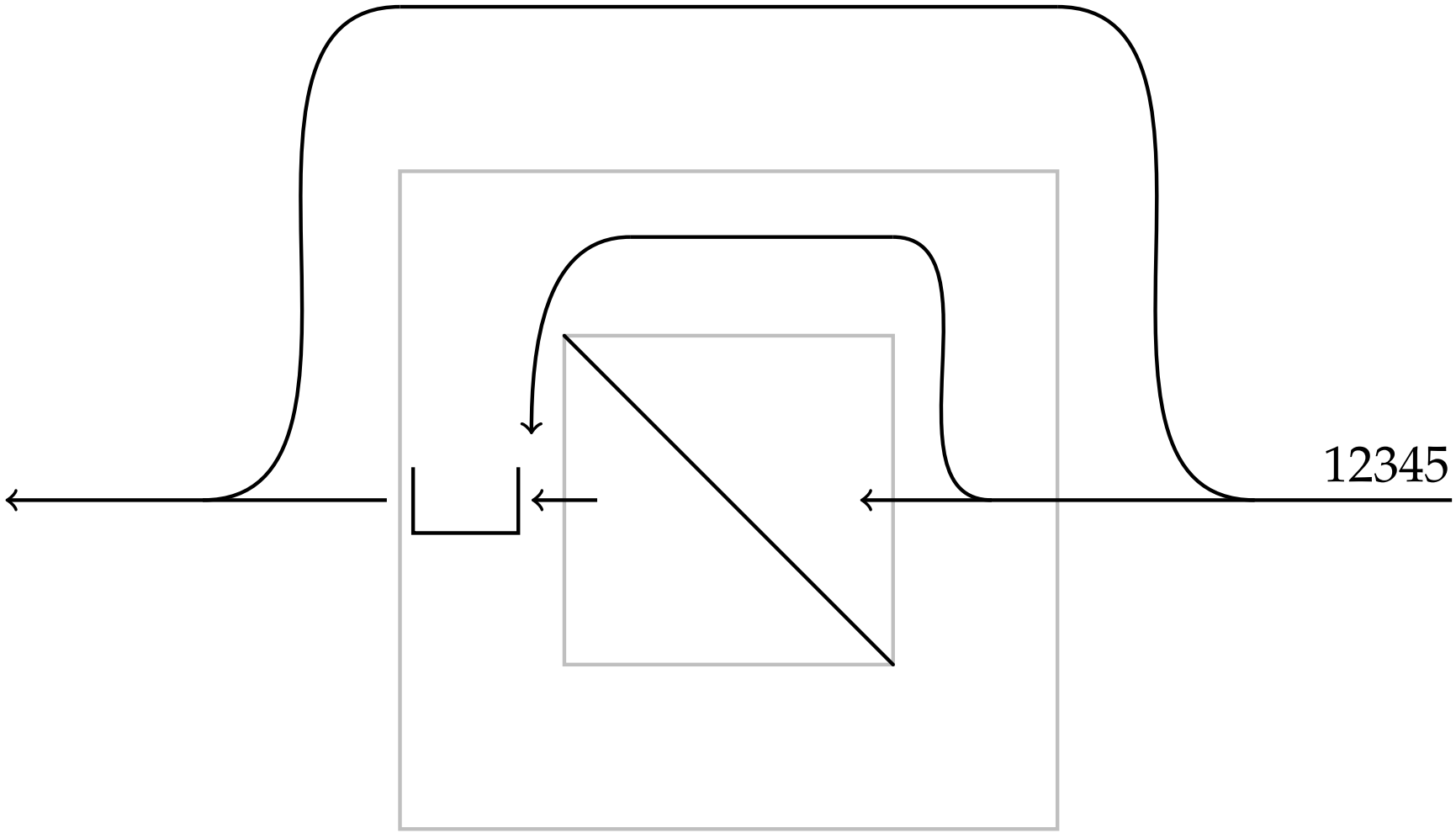
$9^n \longrightarrow 4^n \longrightarrow n^2$ (2 integers) !!!

Nested Av(12)-Machine



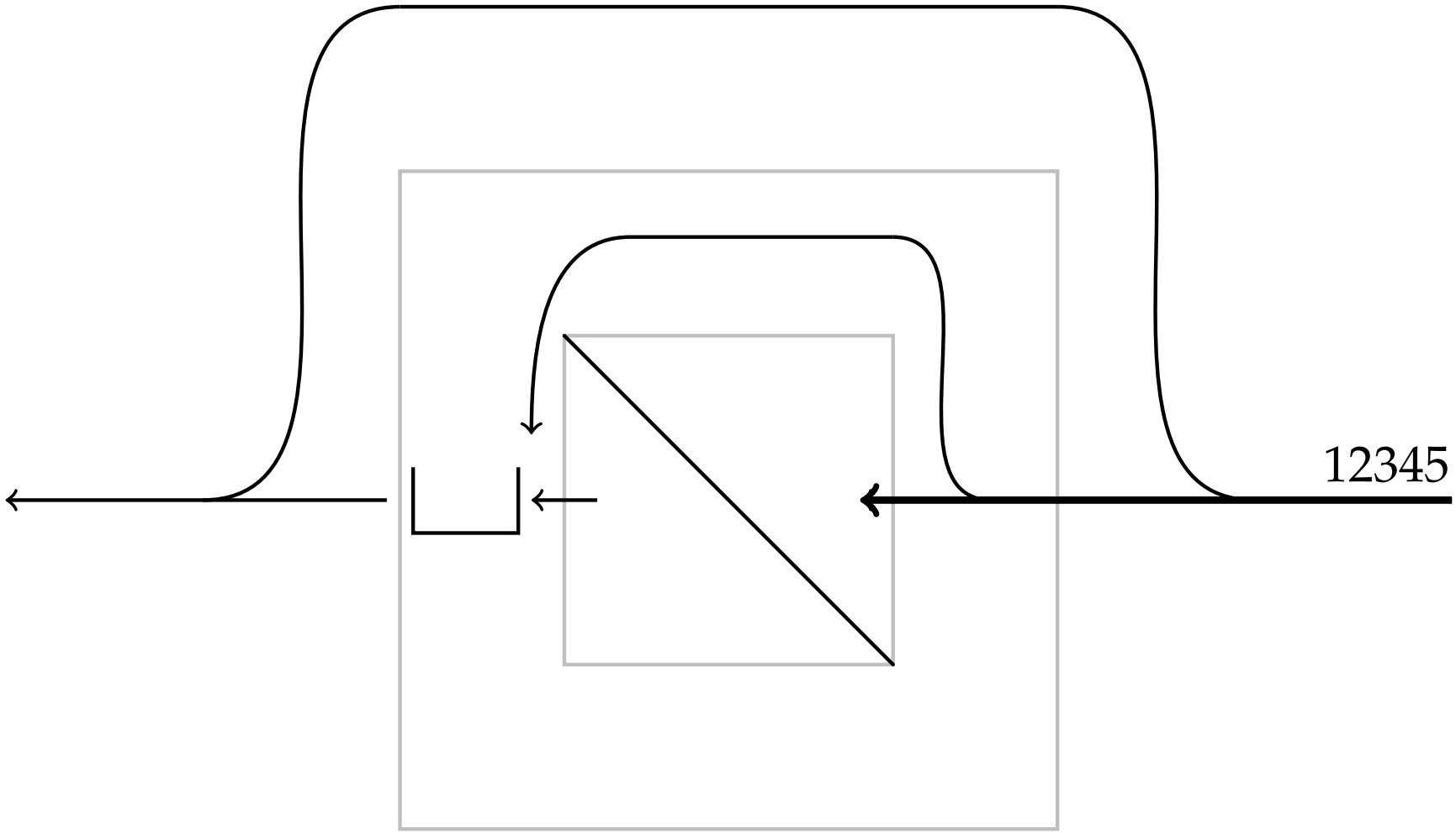
Generates Av(4312)

Nested Av(12)-Machine



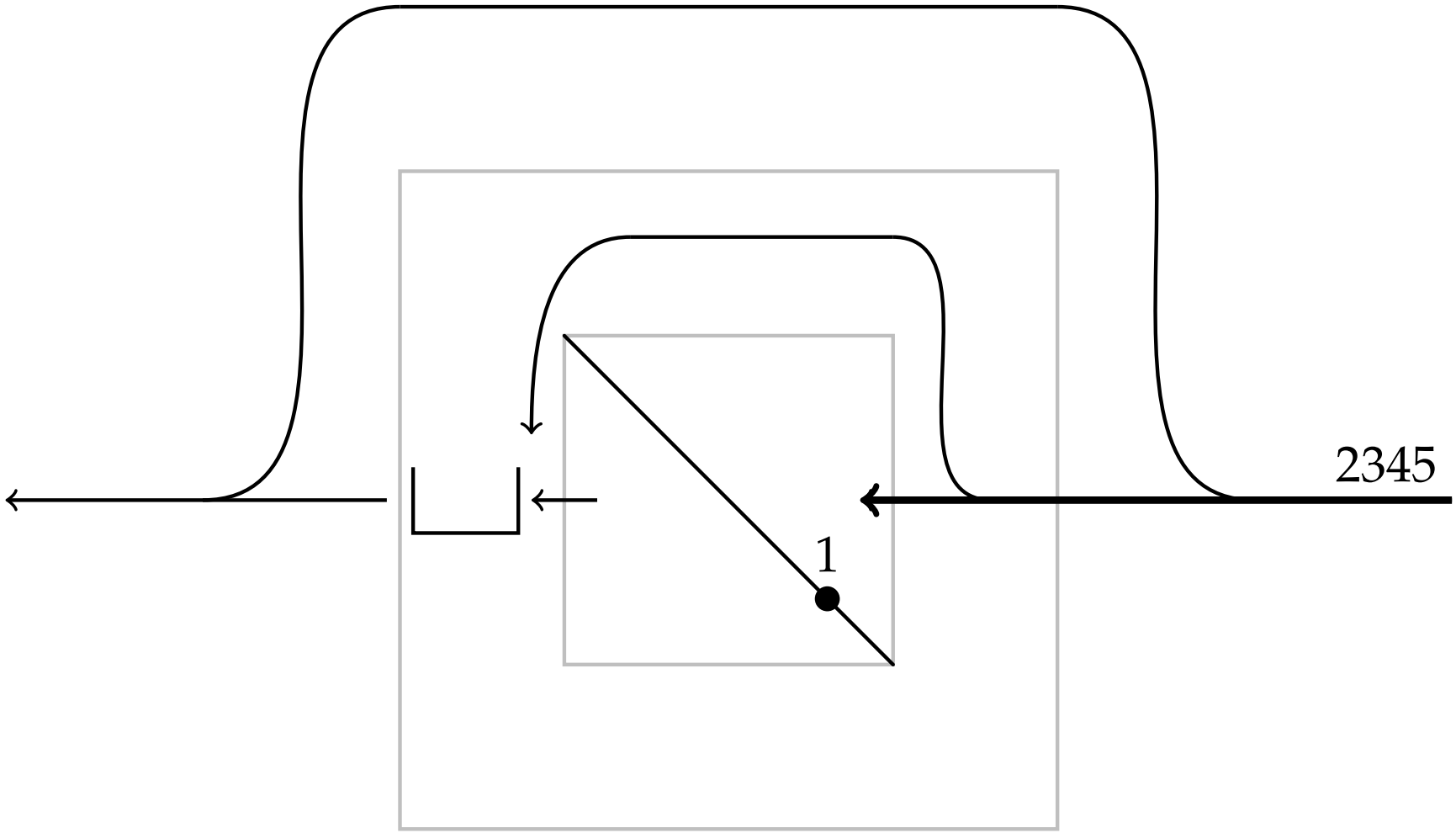
Generates Av(4312)

Nested Av(12)-Machine



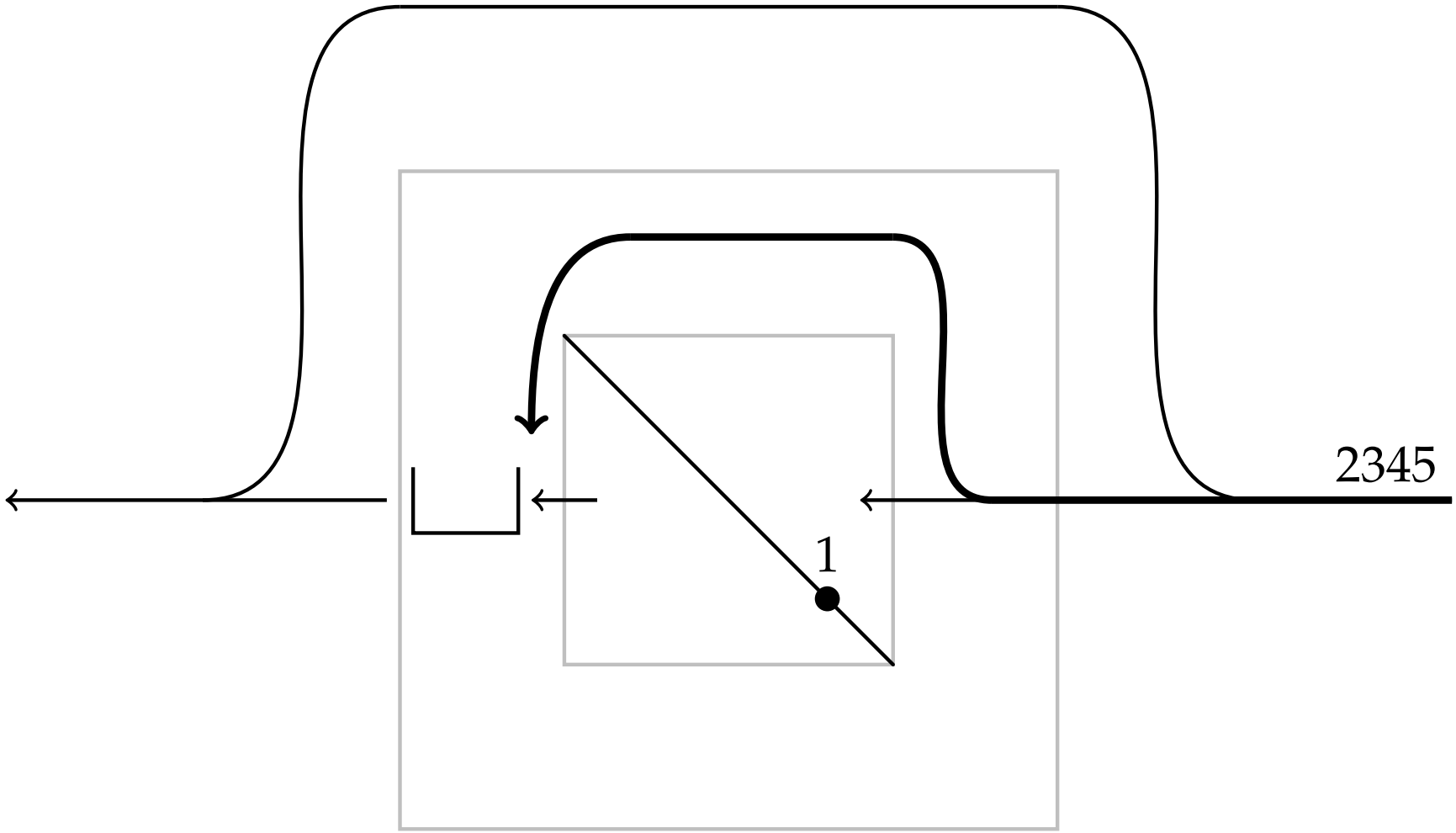
Generates Av(4312)

Nested Av(12)-Machine



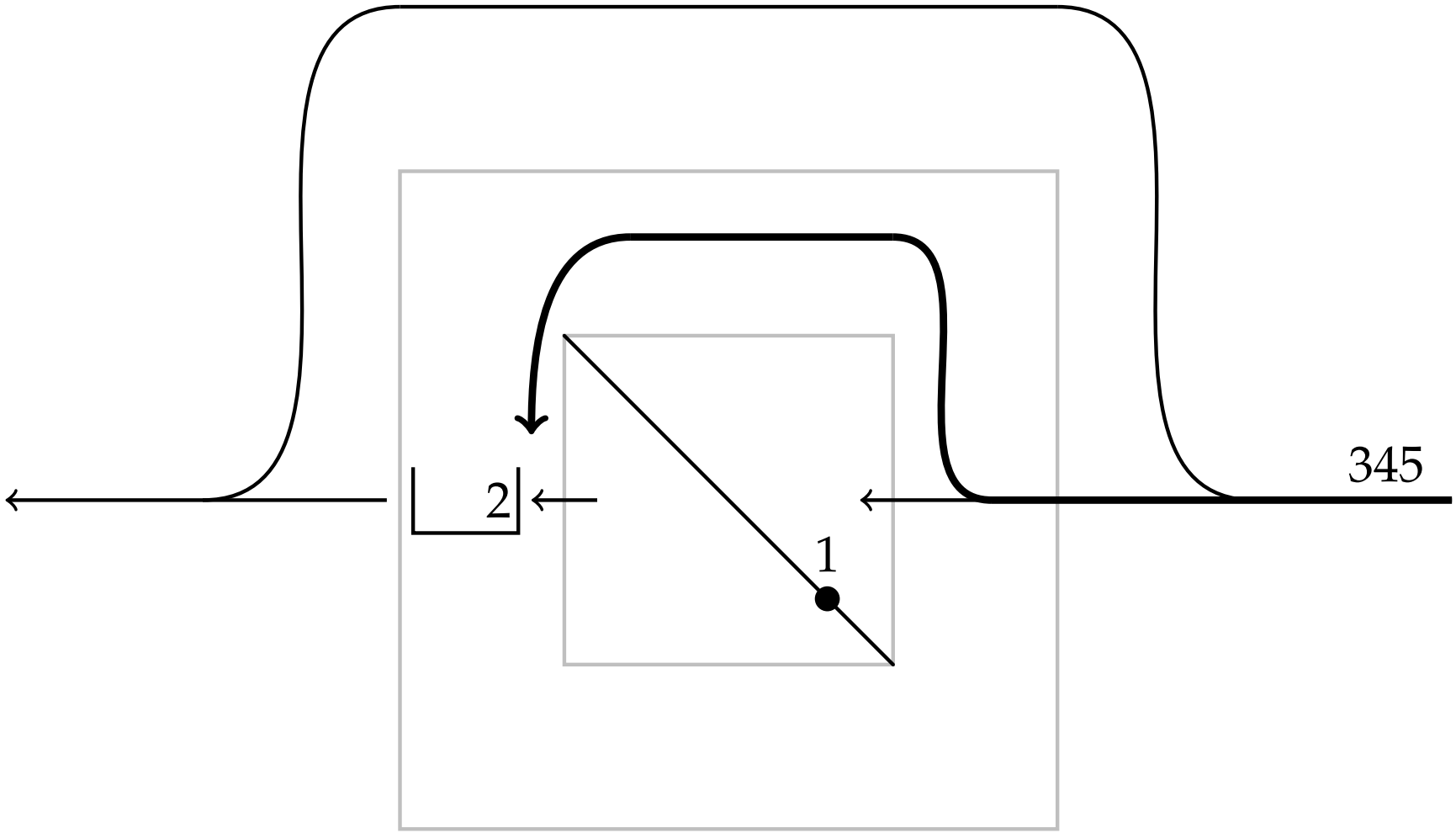
Generates Av(4312)

Nested Av(12)-Machine



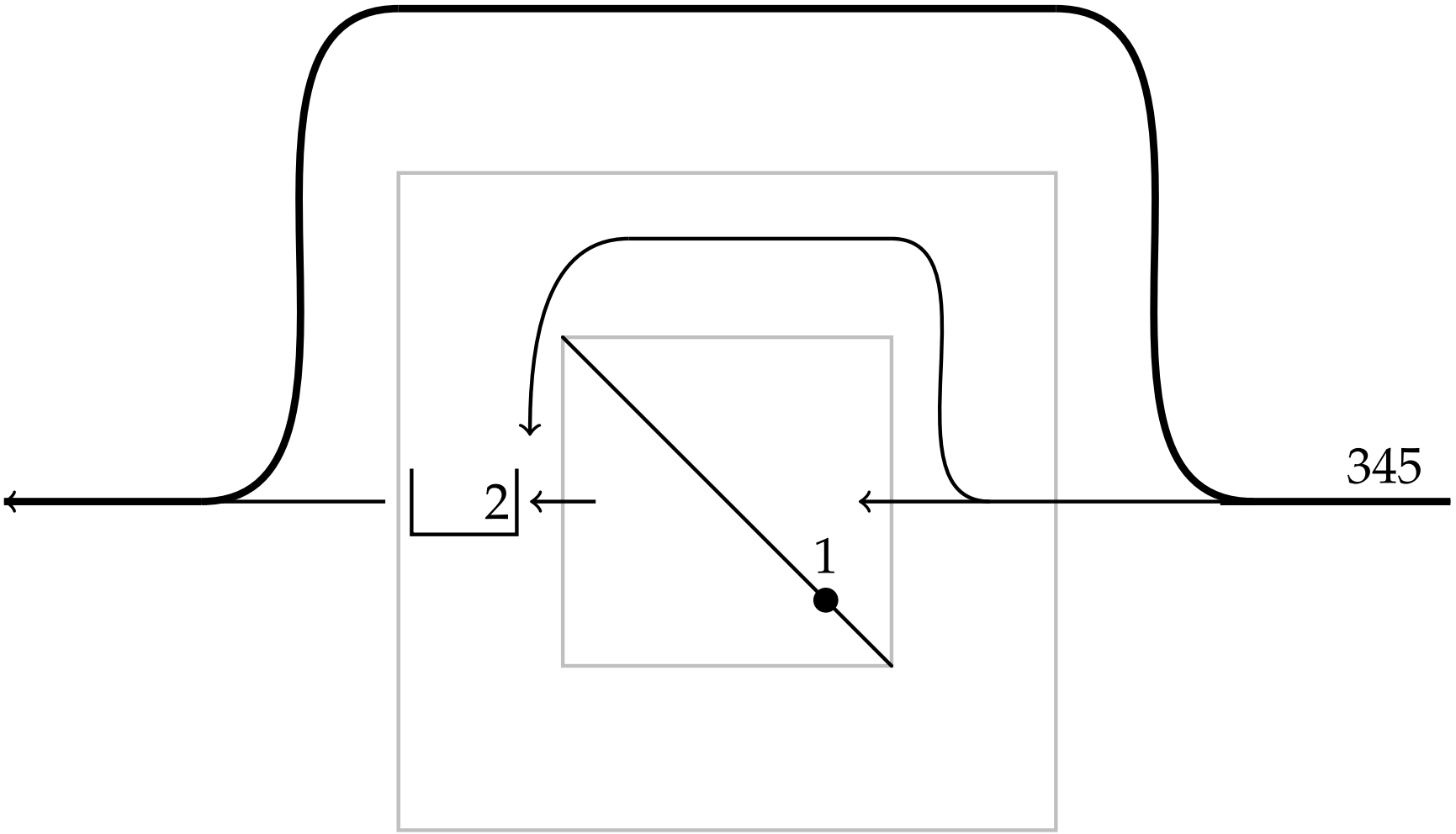
Generates Av(4312)

Nested Av(12)-Machine



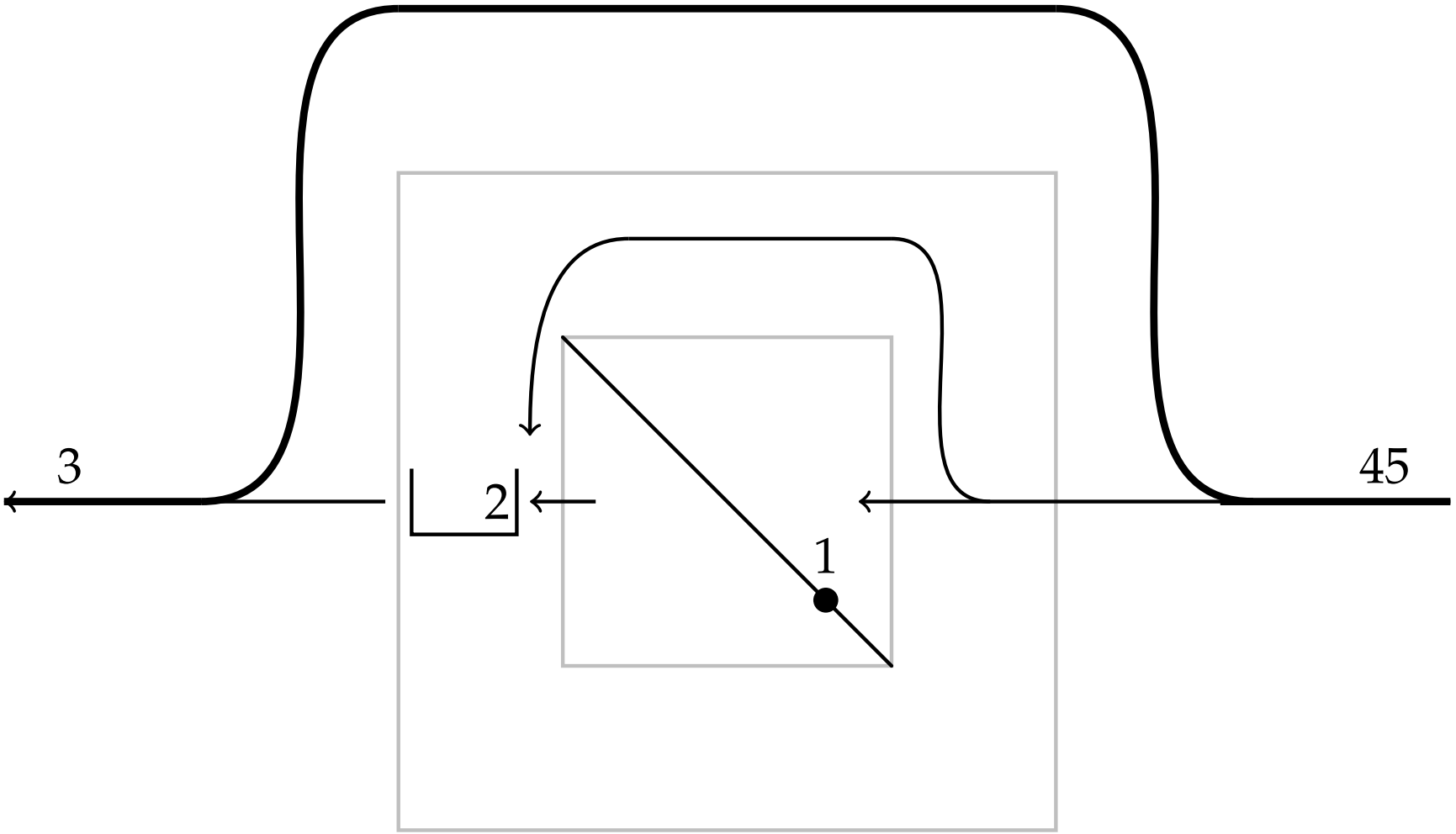
Generates Av(4312)

Nested Av(12)-Machine



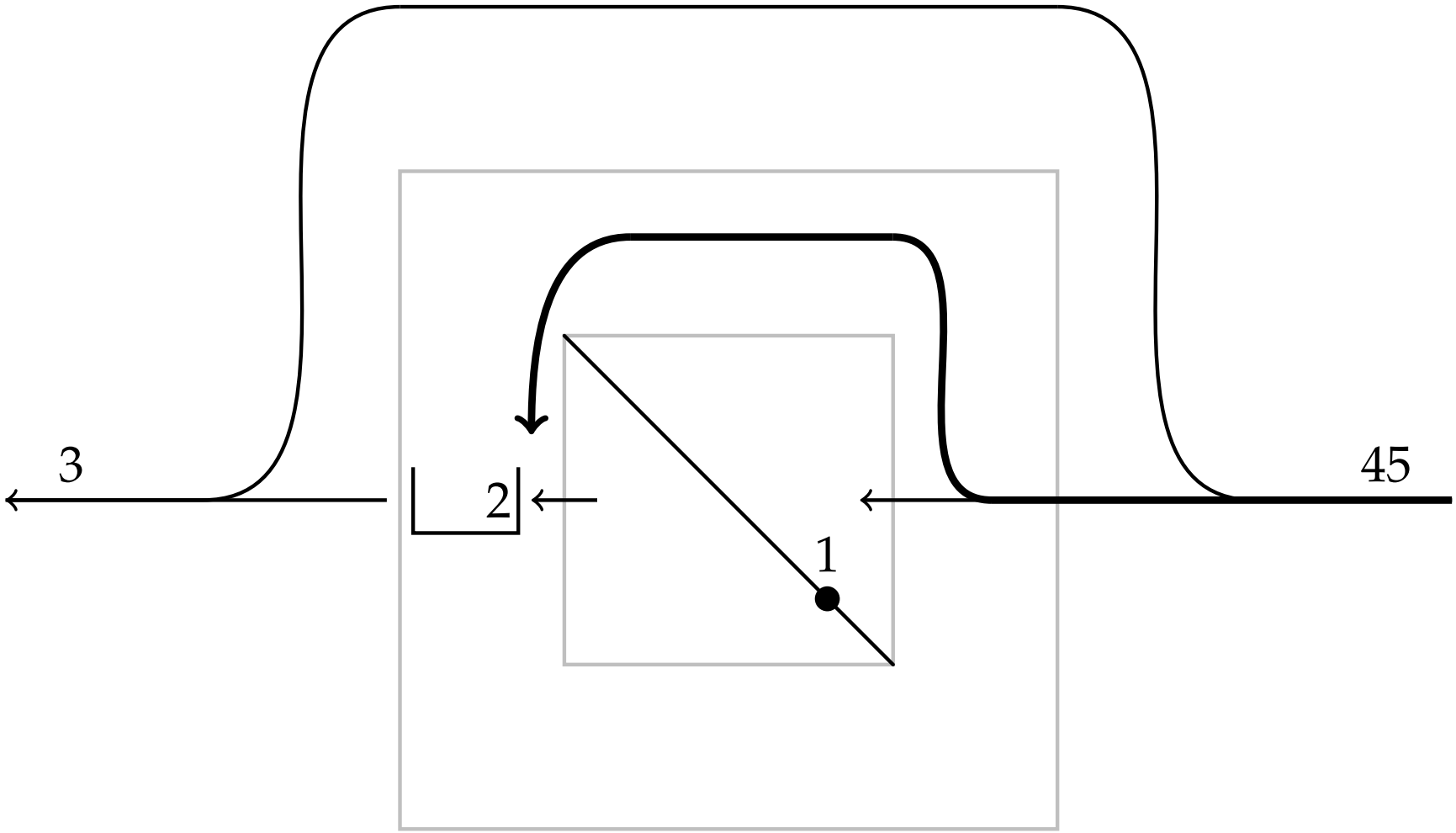
Generates Av(4312)

Nested Av(12)-Machine



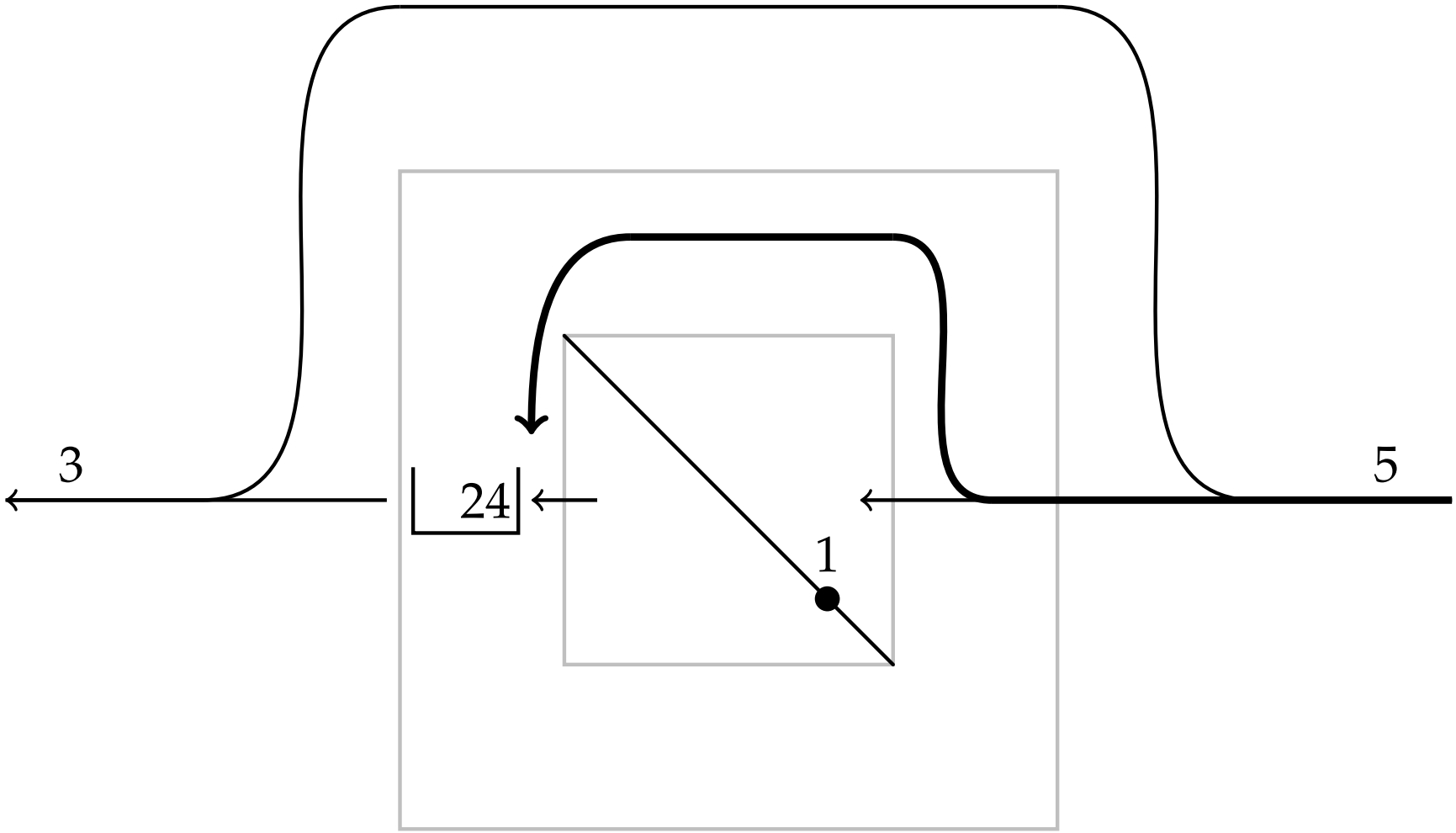
Generates Av(4312)

Nested Av(12)-Machine



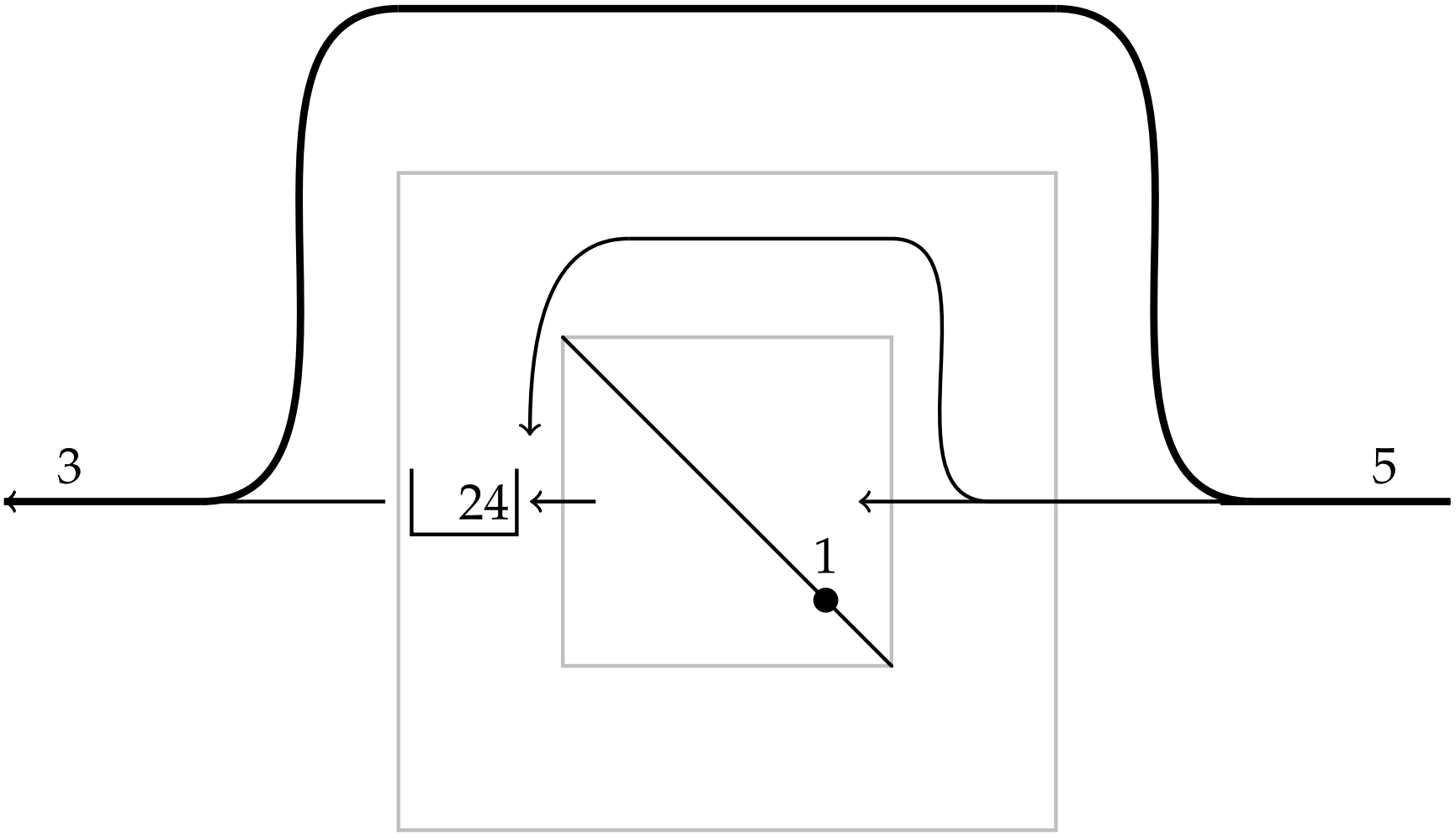
Generates Av(4312)

Nested Av(12)-Machine



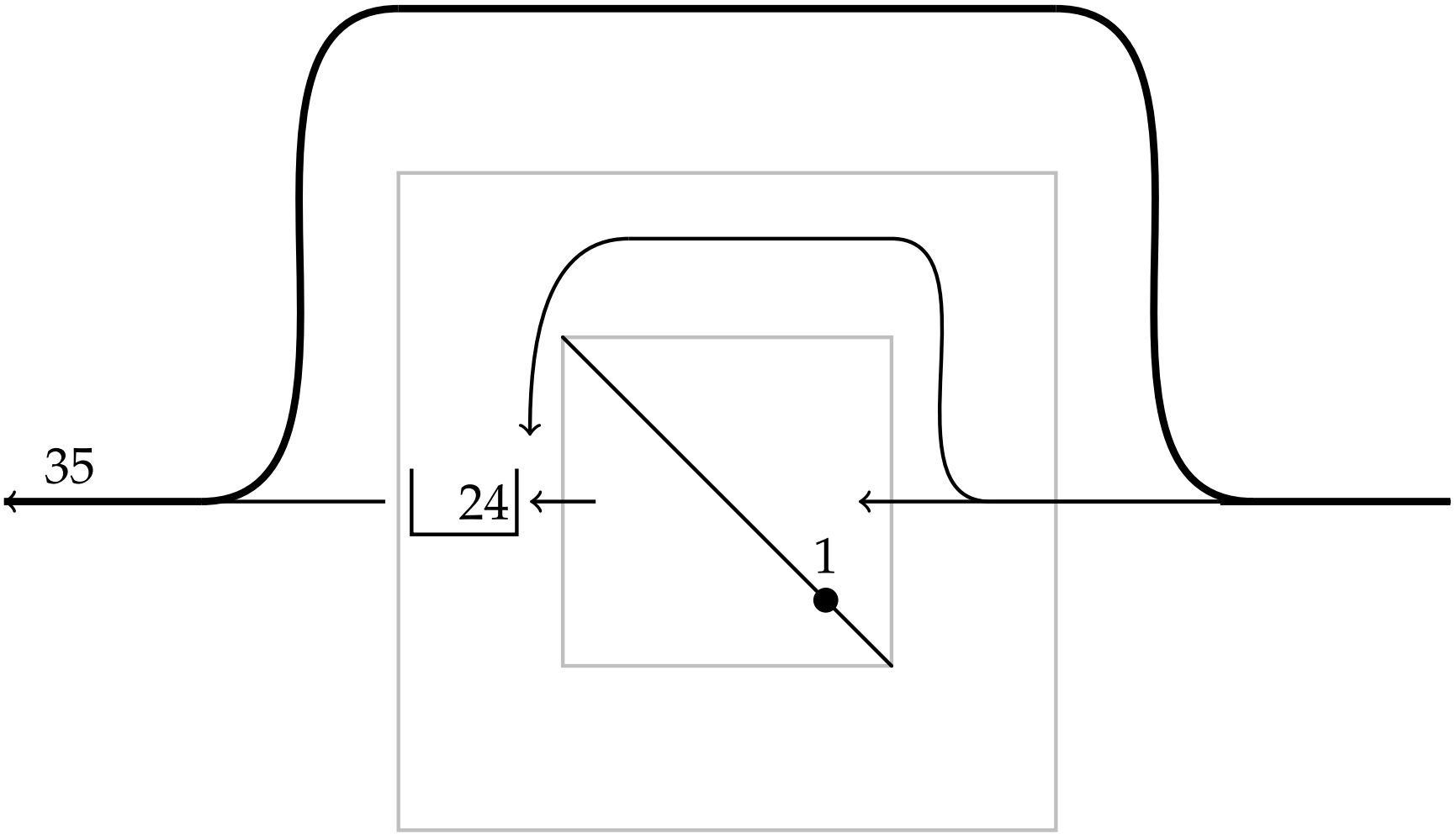
Generates Av(4312)

Nested Av(12)-Machine



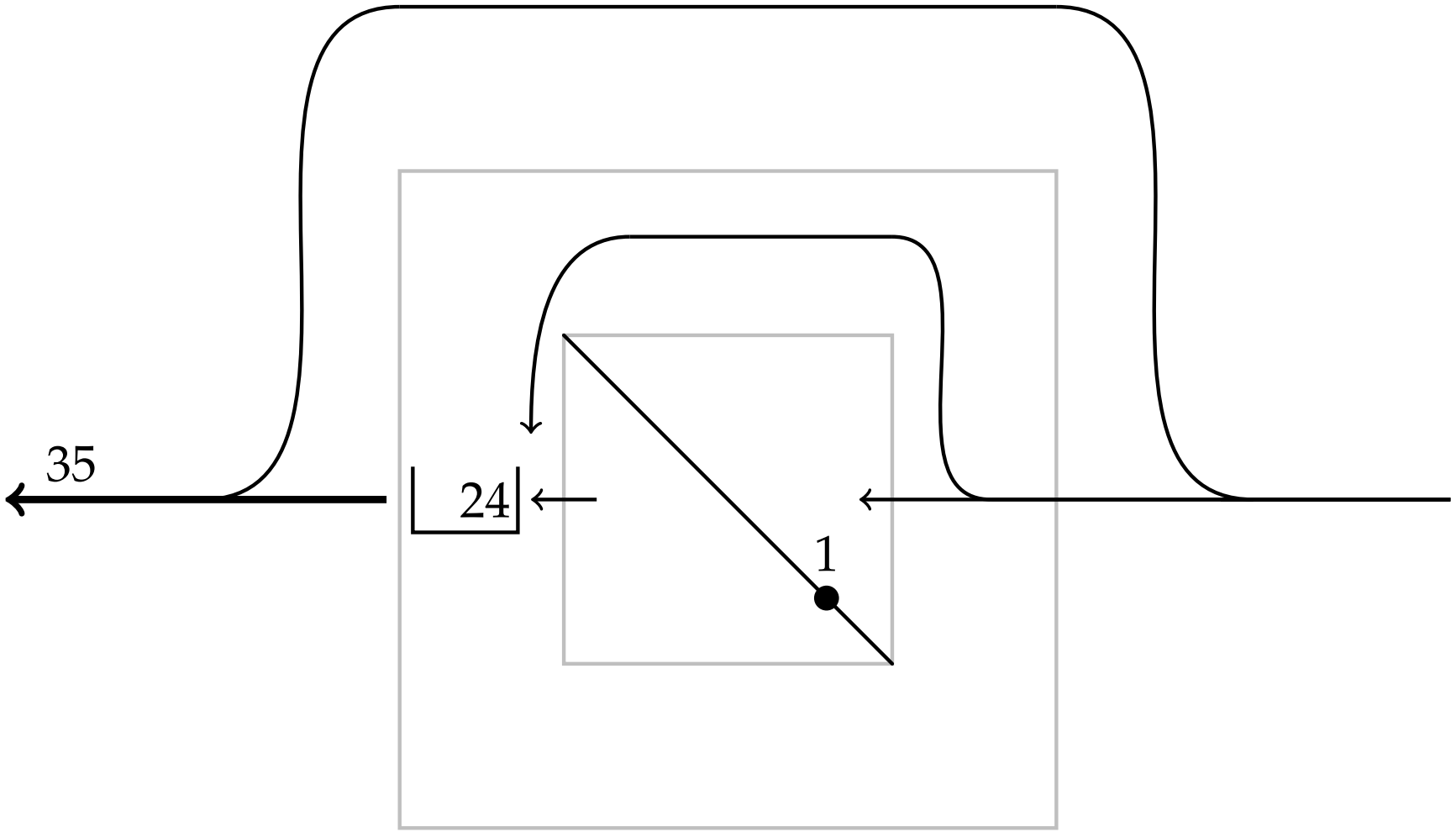
Generates Av(4312)

Nested Av(12)-Machine



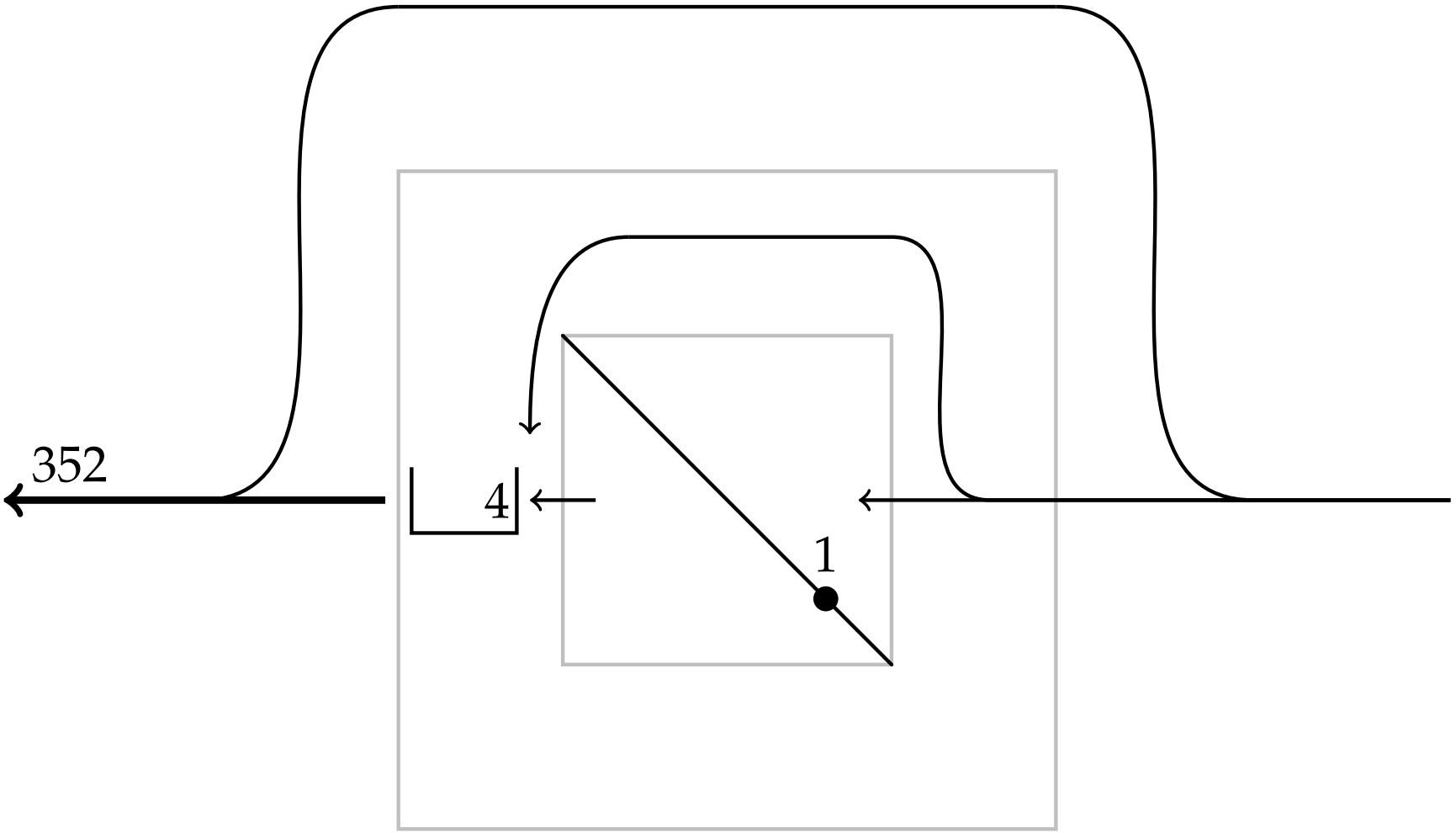
Generates Av(4312)

Nested Av(12)-Machine



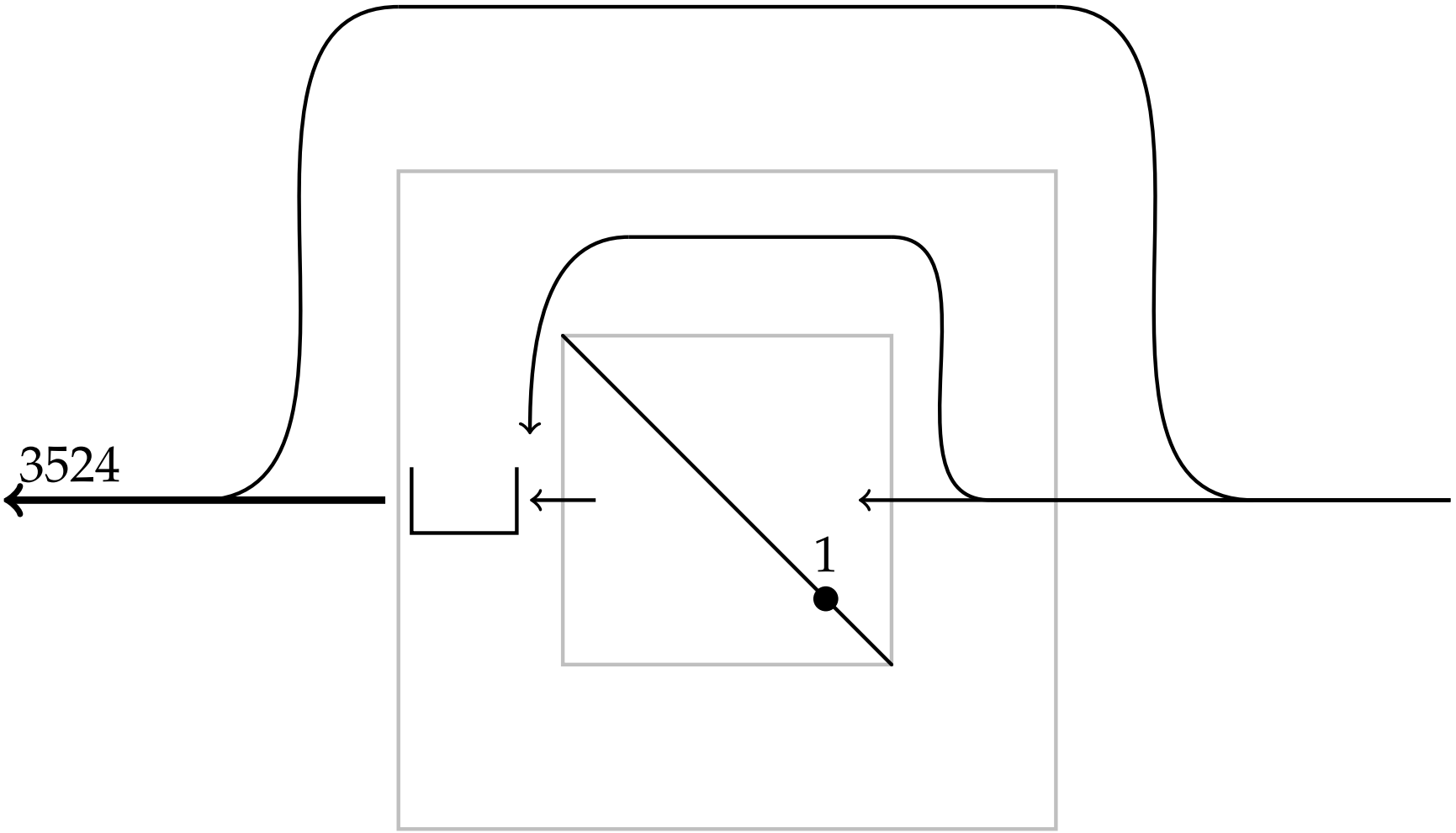
Generates Av(4312)

Nested Av(12)-Machine



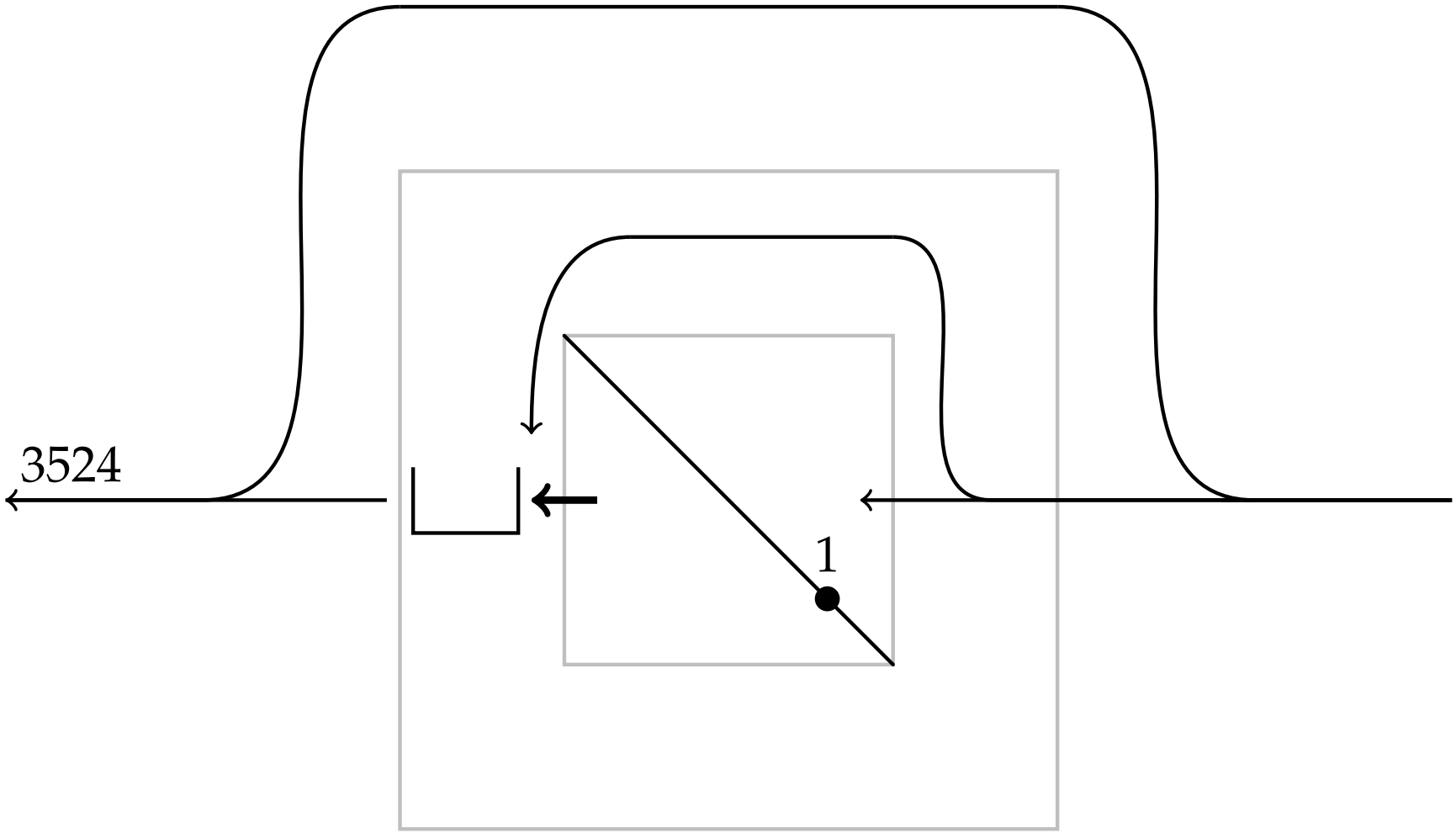
Generates Av(4312)

Nested Av(12)-Machine



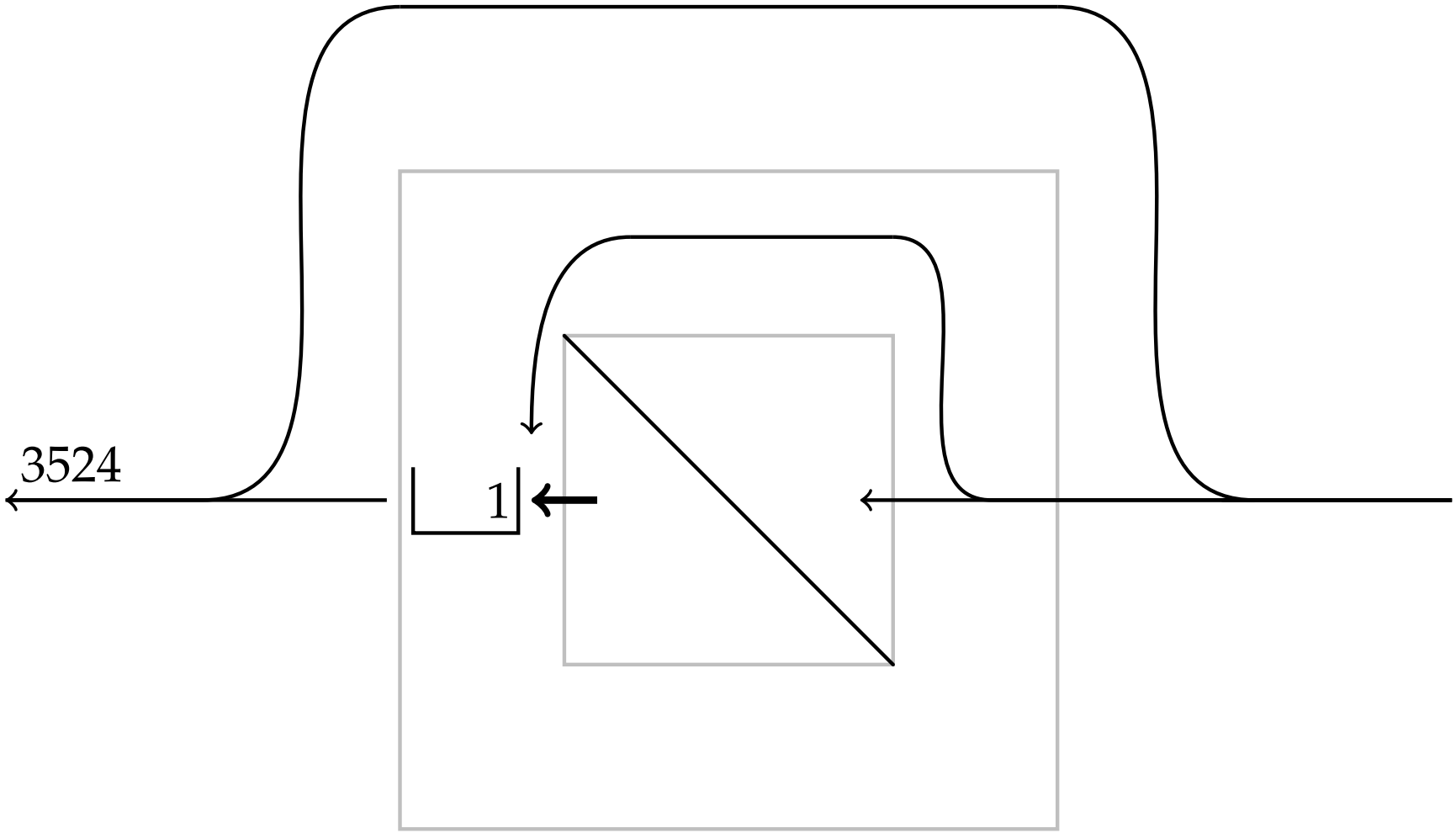
Generates Av(4312)

Nested Av(12)-Machine



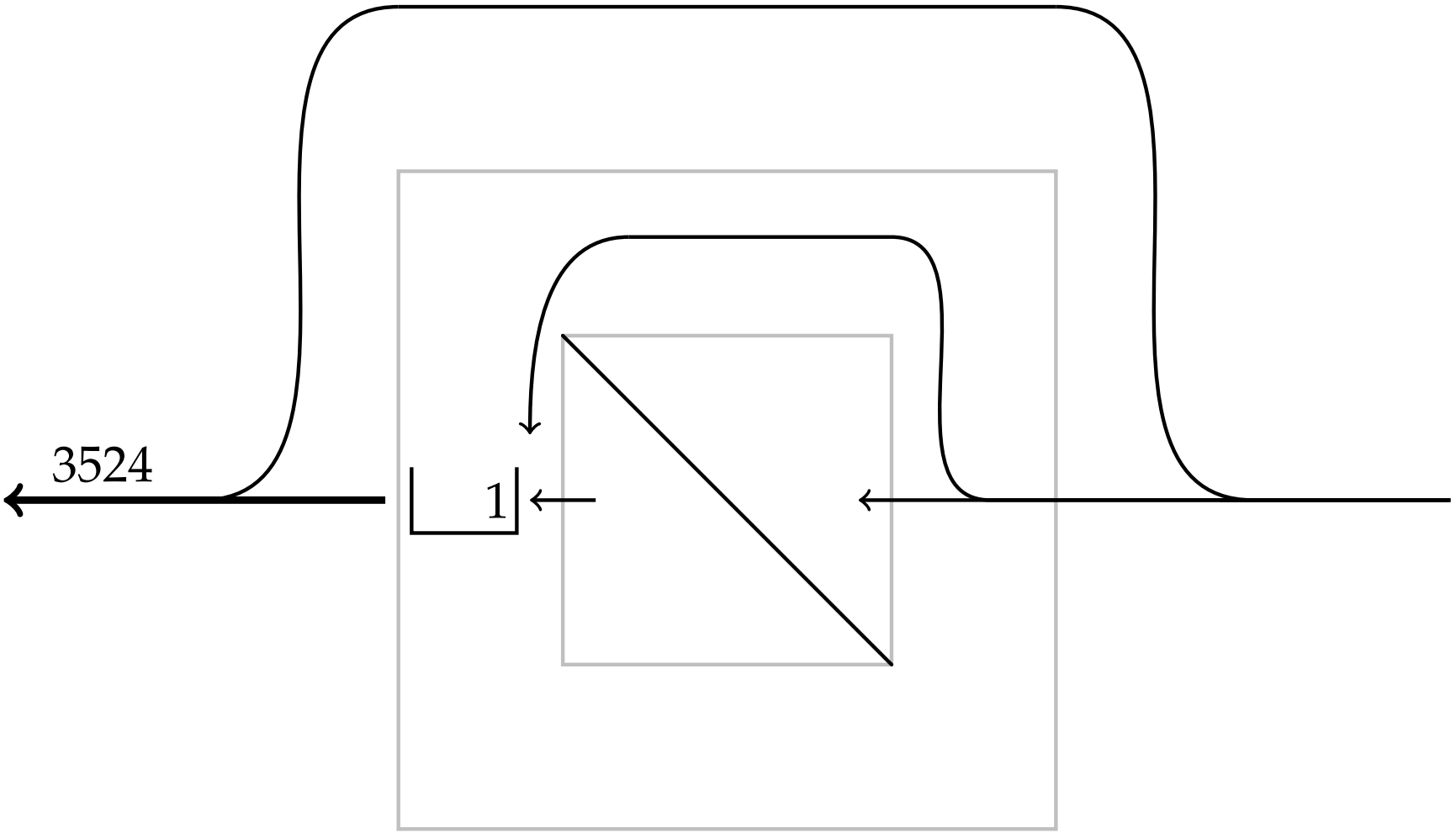
Generates Av(4312)

Nested Av(12)-Machine



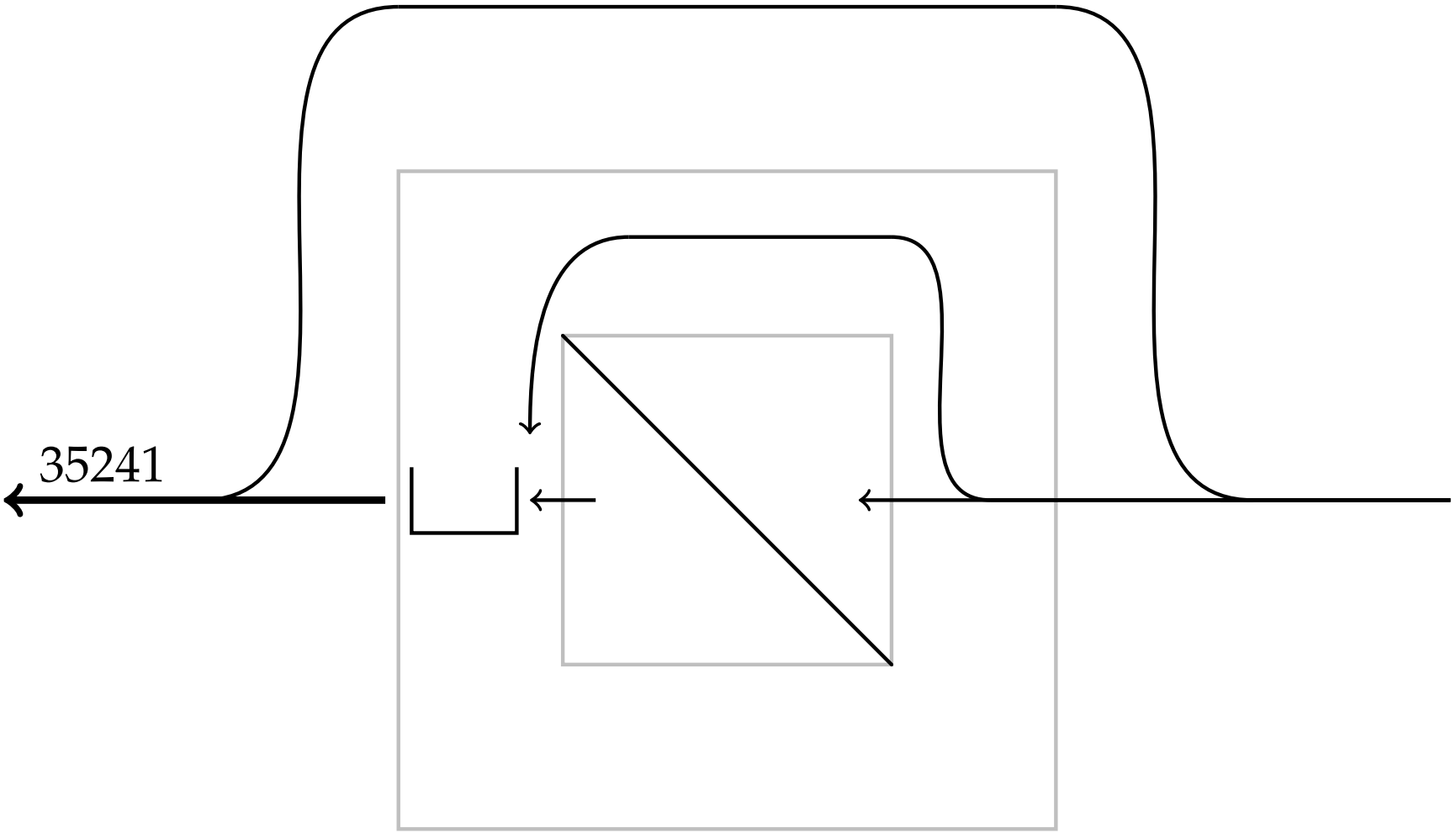
Generates Av(4312)

Nested Av(12)-Machine



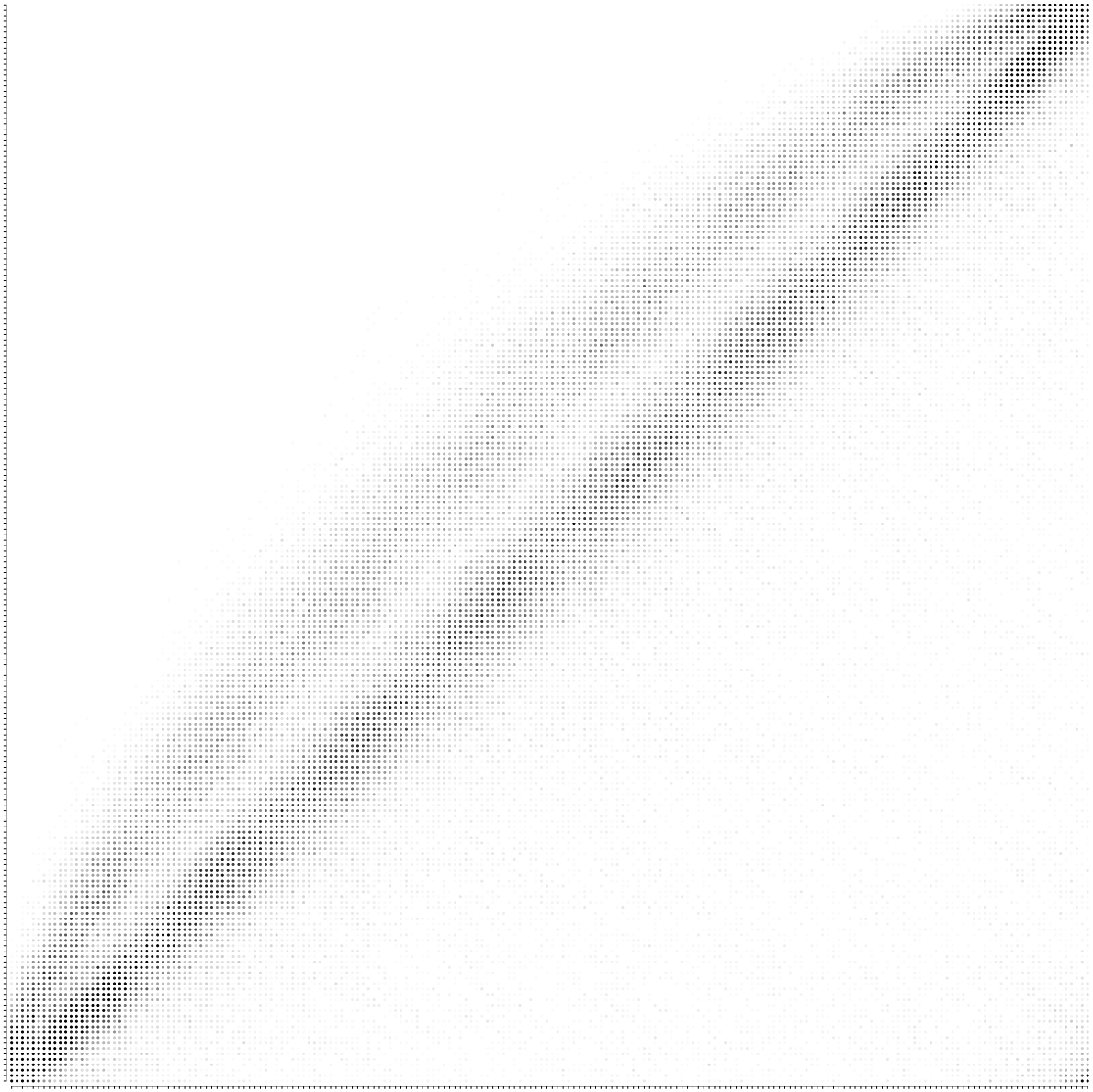
Generates Av(4312)

Nested Av(12)-Machine

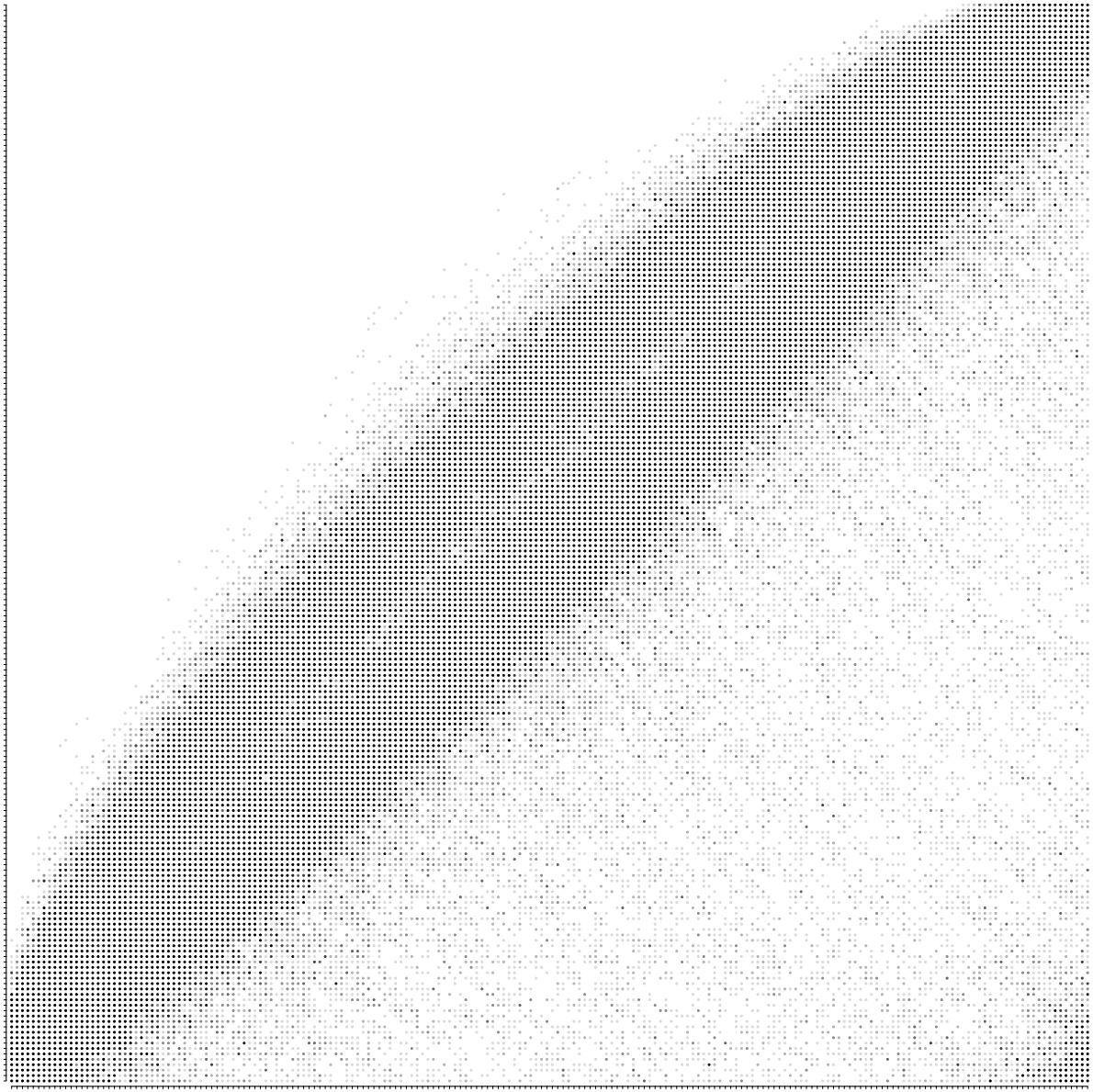


Generates Av(4312)

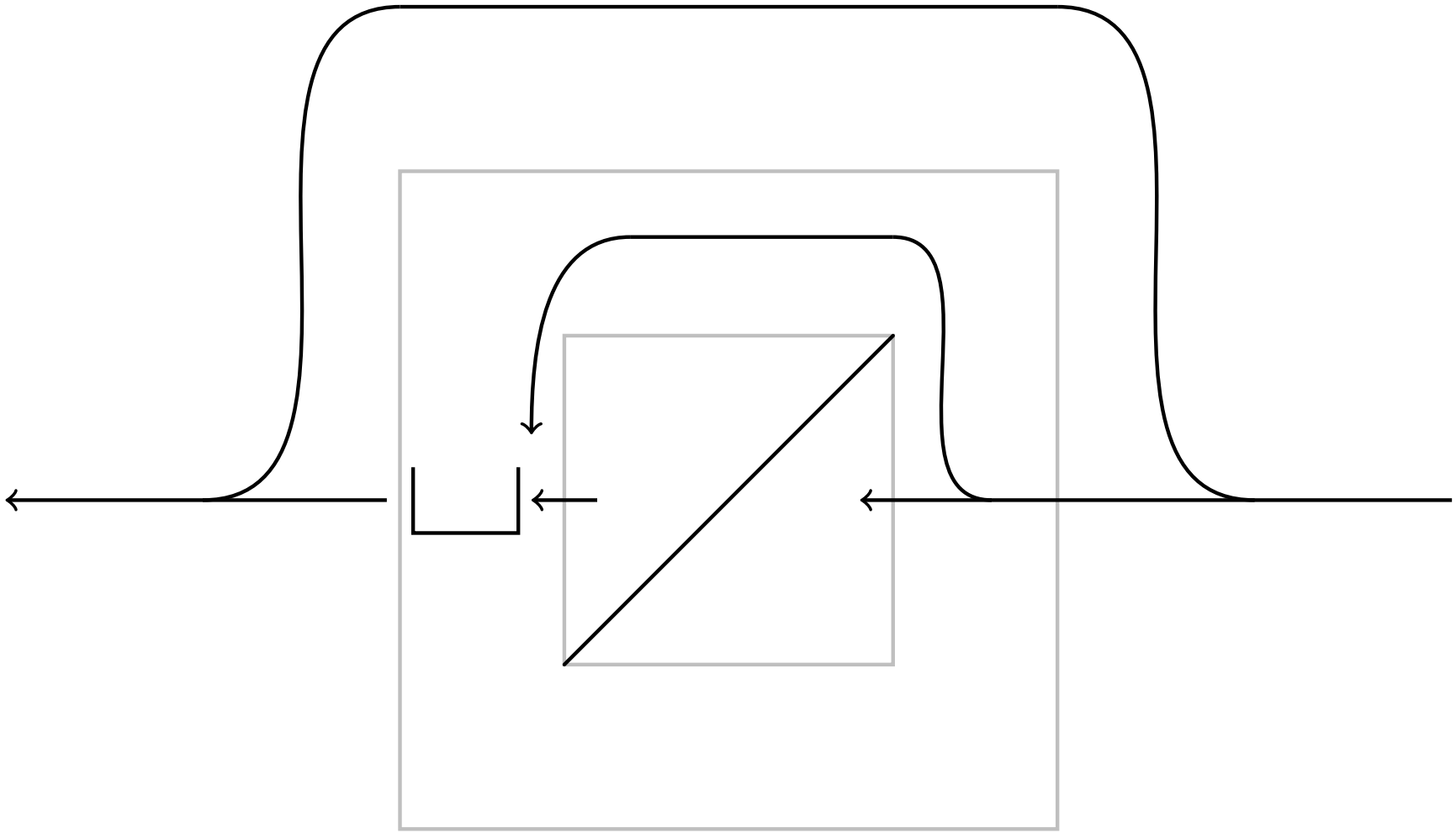
Random Sampling



Random Sampling

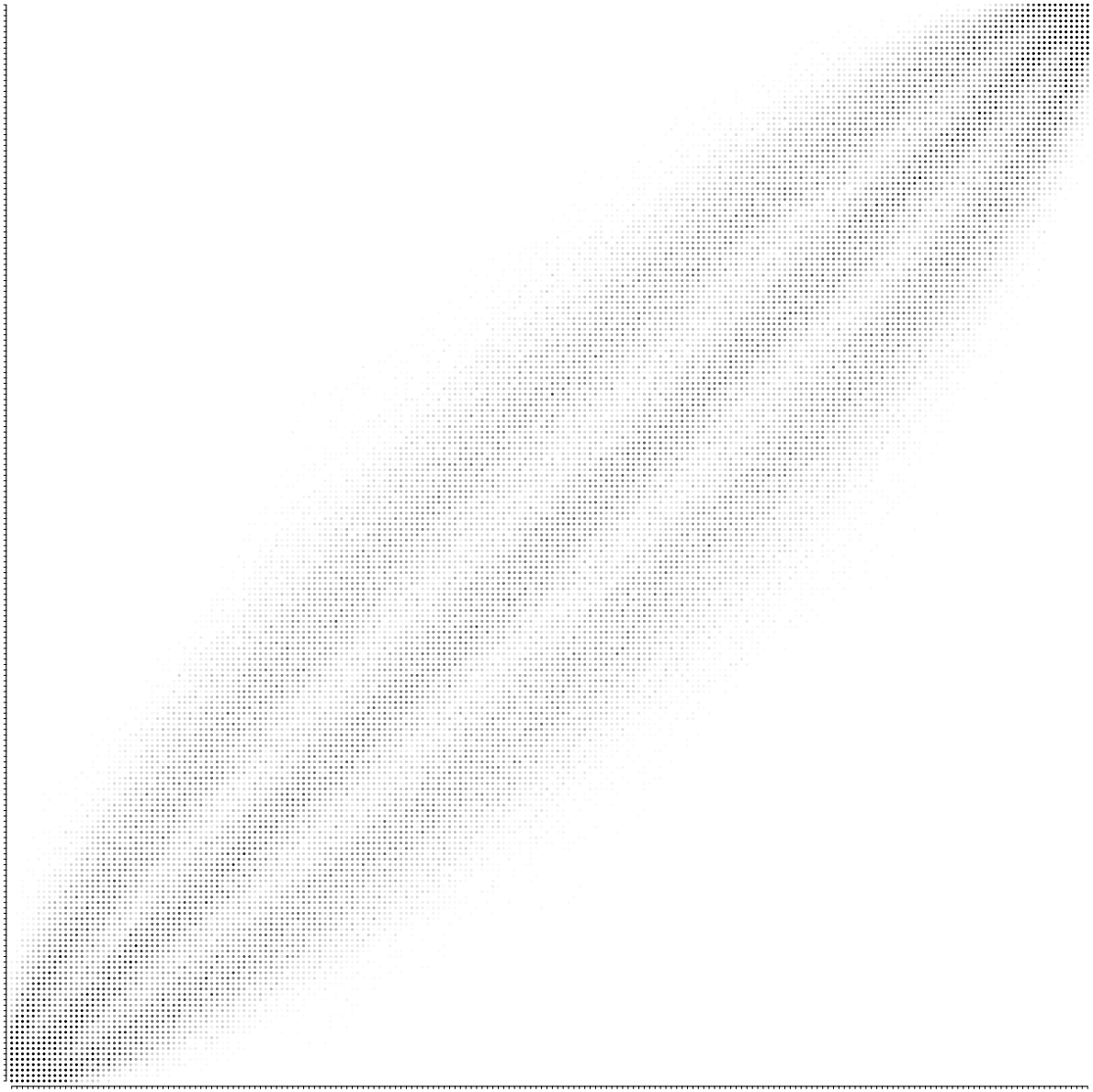


Nested Av(21)-Machine



Generates Av(4321) Free bijection to Av(4312)!

Random Sampling



What about Av(4231)?

Nesting doesn't work.

What about Av(4231)?

Nesting doesn't work.

Some heuristics suggest state compression isn't possible with a standard 𝒞-Machine.

What about Av(4231)?

Nesting doesn't work.

Some heuristics suggest state compression isn't possible with a standard 𝒞-Machine.

Perhaps a more complicated machine: second bypass, add a buffer or a stack, multiple moves at once, ...

Thanks!