Convolutions as solutions of linear recurrences

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Outline

Operations with holonomic sequences 1

Notation:

$\mathbb{N}$	...	the set of nonnegative integers
$\mathbb{K}$	...	algebraically closed field of characteristic 0
$\mathbb{K}^{\mathbb{N}}$	...	the set of all sequences over $\mathbb{K}$
$\mathcal{P}(\mathbb{K})$	...	the set of all *P-recursive* or *holonomic* sequences over $\mathbb{K}$

Operations with holonomic sequences 1

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$\mathcal{P}(\mathbb{K})$	...	the set of all *P-recursive* or *holonomic* sequences over $\mathbb{K}$

Definition

A sequence $\langle a_n \rangle_{n=0}^{\infty} \in \mathbb{K}^{\mathbb{N}}$ is *P-recursive* or *holonomic* if there are $d \in \mathbb{N}$ and $p_0, p_1, \ldots, p_d \in \mathbb{K}[n]$, $p_0 p_d \neq 0$, such that

$$p_d(n)a_{n+d} + p_{d-1}(n)a_{n+d-1} + \cdots + p_0(n)a_n \ = \ 0$$

for all $n \in \mathbb{N}$.

Operations with holonomic sequences 2

Definition

A sequence $a \in \mathbb{K}^{\mathbb{N}}$ is *hypergeometric* if:

1. $\exists p, q \in \mathbb{K}[n] \setminus \{0\}$:

$$p(n)\, a_{n+1} + q(n)\, a_n \;=\; 0$$

for all $n \geq 0$,

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for all $n \geq 0$,
2. $\exists N \in \mathbb{N}:\;\; a_n \neq 0$ for all $n \geq N$.

Operations with holonomic sequences 2

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   for all $n \geq 0$,

2. $\exists N \in \mathbb{N}$: $a_n \neq 0$ for all $n \geq N$.

Equivalently:

$$a \text{ hypergeometric} \iff \exists r \in \mathbb{K}(n)^* : \frac{a_{n+1}}{a_n} = r(n) \text{ a.e.}$$

Operations with holonomic sequences 3

Example

Some hypergeometric sequences:

- $a_n = c^n, \quad c \in \mathbb{K}^*$
- $a_n = r(n)$ a.e., $\quad r \in \mathbb{K}(n),\ r \neq 0$
- $a_n = n!$
- $a_n = \binom{2n}{n}$

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Notation: $\mathcal{H}(\mathbb{K}) \ldots$ all hypergeometric sequences in $\mathbb{K}^{\mathbb{N}}$

Research project:

Operations with holonomic sequences 4

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- hypergeometric sequences,
- operations which preserve holonomicity.

Operations with holonomic sequences 5

Theorem

$\mathcal{P}(\mathbb{K})$ is closed under the following unary operations $a \mapsto c$:

1. *scalar multiplication:* $c_n = \lambda a_n \quad (\lambda \in \mathbb{K})$
2. *shift:* $c_n = a_{n+1}$
3. *inverse shift:* $c_n = \begin{cases} a_{n-1}, & n \geq 1, \\ 0, & n = 0 \end{cases}$
4. *difference:* $c_n = a_{n+1} - a_n$
5. *partial summation:* $c_n = \sum_{k=0}^{n} a_k$
6. *multisection:* $c_n = a_{kn+r} \quad (k \in \mathbb{N} \setminus \{0\},\ 0 \leq r \leq k-1)$

Operations with holonomic sequences 6

Theorem

$\mathcal{P}(\mathbb{K})$ *closed under the following binary operations* $(a, b) \mapsto c$*:*

7. *addition:* $c_n = a_n + b_n$
8. *multiplication:* $c_n = a_n b_n$
9. *convolution:* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Theorem

$\mathcal{P}(\mathbb{K})$ *is closed under*

10 *interlacing:* $(a^{(0)}, a^{(1)}, \ldots, a^{(d-1)}) \mapsto c$, *where*

$$c_n = a_{n \operatorname{div} d}^{(n \bmod d)} \quad (d \in \mathbb{N})$$

Operations with holonomic sequences 7

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$$c_n \;=\; a_{n \operatorname{div} d}^{(n \bmod d)} \qquad (d \in \mathbb{N})$$

Example

The interlacing of $a, b \in \mathbb{K}^{\mathbb{N}}$ is the sequence

$$c \;=\; \langle a_0, b_0, a_1, b_1, a_2, b_2, \ldots \rangle$$

Explicitly representable sequences 1

Definition

$\mathcal{A}(\mathbb{K})$ is the least subring of $\mathbb{K}^{\mathbb{N}}$ containing $\mathcal{H}(\mathbb{K})$, closed under

- shift, inverse shift,
- partial summation.

The elements of $\mathcal{A}(\mathbb{K})$ are *d'Alembertian sequences*.

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Example

Some d'Alembertian sequences:

- Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$
- Derangement numbers $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
- $a_n = \frac{(n+1)!}{2^n} \sum_{k=0}^{n} \frac{2^k}{k+1}$

Explicitly representable sequences 2

Definition

$\mathcal{L}(\mathbb{K})$ is the least subring of $\mathbb{K}^{\mathbb{N}}$ containing $\mathcal{H}(\mathbb{K})$, closed under

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The elements of $\mathcal{L}(\mathbb{K})$ are *Liouvillian sequences*.

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The elements of $\mathcal{L}(\mathbb{K})$ are *Liouvillian sequences*.

Example

The sequence

$$n!! \; = \begin{cases} 2^k k!, & n = 2k, \\ \frac{(2k+1)!}{2^k k!}, & n = 2k+1 \end{cases}$$

is Liouvillian (as an interlacing of two hypergeometric sequences).

Explicitly representable sequences 3

P-recursive: $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$

Liouvillian: $n!!$

d'Alembertian: $n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

hypergeometric: $n!$

quasirational: $\frac{2^n}{n+1} + 1$

rational: $\frac{1}{n+1}$

C-recursive: $n2^n + 1$

polynomial: $n + 1$

geometric: 2^n

Explicitly representable sequences 4

Theorem

$\mathcal{L}(\mathbb{K})$ is closed under the following operations:

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7. *addition*
8. *multiplication*
9. *interlacing*

Convolutions of Liouvillian sequences 1

Question: What about *convolution*

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The convolution of $1/n!$ with itself

$$\frac{1}{n!} * \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} =$$

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The convolution of $1/n!$ with itself

$$\frac{1}{n!} * \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} = \frac{2^n}{n!}$$

is hypergeometric.

Convolutions of Liouvillian sequences 2

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$$2c_{n+1} - (n+2)c_n \;=\; 2(n+1)!, \qquad c_0 = 1$$

Convolutions of Liouvillian sequences 2

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$$c_n \ = \ \frac{(n+1)!}{2^n} \sum_{k=0}^{n} \frac{2^k}{k+1}.$$

Convolutions of Liouvillian sequences 2

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Note: c_n is d'Alembertian, *not* hypergeometric.

Convolutions of Liouvillian sequences 3

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The convolution of $n!$ with $1/n!$

$$c_n \;:=\; n! * \frac{1}{n!} \;=\; \sum_{k=0}^{n} \frac{k!}{(n-k)!}$$

Convolutions of Liouvillian sequences 3

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The convolution of $n!$ with $1/n!$

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satisfies

$$c_{n+2} - (n+2)c_{n+1} + c_n = \frac{1}{(n+2)!}$$

and

$$(n+3)c_{n+3} - (n^2+6n+10)c_{n+2} + (2n+5)c_{n+1} - c_n = 0.$$

Convolutions of Liouvillian sequences 3

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Note: This equation has *no* nonzero Liouvillian solution!

Convolutions of Liouvillian sequences 4

Definition

Sequences $a, b \in \mathbb{K}^{\mathbb{N}}$ are similar if

$$\exists N \in \mathbb{N} \ \ \forall n \geq N \colon a_n = b_n.$$

Notation: $a \sim b$.

Convolutions of Liouvillian sequences 4

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Definition

An operation f on $\mathbb{K}^{\mathbb{N}}$ is local if $\sim$ is a congruence w.r.t. f.

Convolutions of Liouvillian sequences 5

Proposition

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6. *addition*
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8. *interlacing*

Convolutions of Liouvillian sequences 6

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Convolutions of Liouvillian sequences 6

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$$\sum_{k=0}^{n} a_k = 0 \quad \not\sim \quad \sum_{k=0}^{n} b_k = 1.$$

Convolutions of Liouvillian sequences 6

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Corollary

Convolution is not local.

Convolutions of Liouvillian sequences 7

Lemma

If $a \sim a'$ *then*

$$\sum_{k=0}^{n} a_k \sim \sum_{k=0}^{n} a'_k + C$$

for some $C \in \mathbb{K}$.

Convolutions of Liouvillian sequences 7

Lemma

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for some $C \in \mathbb{K}$.

Lemma

Let $\mathcal{C} \subseteq \mathbb{K}^{\mathbb{N}}$ be closed under scalar multiplication, inverse shift, addition, and similarity. Assume $a, b, a * b \in \mathcal{C}$, $a' \sim a$ and $b' \sim b$. Then $a' * b' \in \mathcal{C}$.

Convolutions of Liouvillian sequences 8

Definition

$\mathcal{A}_{rat}(\mathbb{K})$ is the least subring of $\mathbb{K}^{\mathbb{N}}$ containing $\mathbb{K}(n)$, closed under

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The elements of $\mathcal{A}_{rat}(\mathbb{K})$ are *rationally d'Alembertian sequences*.

Convolutions of Liouvillian sequences 8

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Example

- Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ are rationally d'Alembertian.

Convolutions of Liouvillian sequences 9

Example

- Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ are rationally d'Alembertian.
- The interlacing of H_n with $H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2}$ is rationally Liouvillian.

Convolutions of Liouvillian sequences 10

Theorem

The convolution of a d'Alembertian sequence with a rationally d'Alembertian sequence is d'Alembertian.

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Inverse Zeilberger's problem 1

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Easier question: *Given* $a \in \mathcal{H}(\mathbb{K})$, how to find solutions of the form $a * b$ where $b \in \mathcal{H}(\mathbb{K})$?

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Operator notation:

$$E : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}} \qquad \text{shift operator,}$$
$$(E^k a)_n = a_{n+k} \qquad (k \in \mathbb{N})$$

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Operator notation:

$E : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$	shift operator,
$(E^k a)_n = a_{n+k}$	$(k \in \mathbb{N})$
$L = \sum_{k=0}^{d} p_k(n)\, E^k \in \mathbb{K}[n]\langle E\rangle$	linear recurrence operator

Inverse Zeilberger's problem 2

Example: Given $a_n = \frac{1}{n!}$ and

$$L = (n+3)E^3 - (n^2+6n+10)E^2 + (2n+5)E - 1,$$

find b such that $L(a * b) = 0$.

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Idea 1. Use generating functions and a hyperexponential given factor.

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Idea 1. Use generating functions and a hyperexponential given factor.

Definition

A sequence $a \in \mathbb{K}^{\mathbb{N}}$ is hyperexponential if $\mathrm{gf}_a(x) := \sum_{n \geq 0} a_n x^n$ satisfies

$$\mathrm{gf}'_a(x) = r(x)\mathrm{gf}_a(x)$$

for some $r \in \mathbb{K}(x)^*$.

Inverse Zeilberger's problem 3

Problem: *Given* $L \in \mathbb{K}[x]\langle E \rangle$ and *hyperexponential* a, find b such that $L(a * b) = 0$.

Assume $L(a * b) = 0$, $u = \mathrm{gf}_a$, $v = \mathrm{gf}_b$. Then $u' = ru$ and

$$(u \cdot v)^{(k)} = u \sum_{i=0}^{k} r_{i,k} v^{(i)} \qquad (1)$$

for all $k \in \mathbb{N}$, with $r_{i,k} \in \mathbb{K}(x)$ for all i, k.

Inverse Zeilberger's problem 3

1. From L compute $M \in \mathbb{K}[x, x^{-1}]\langle D \rangle$ s.t.

$$L(c) = 0 \implies M(\mathrm{gf}_c) = 0.$$

Then $M(u \cdot v) = M(\mathrm{gf}_a \cdot \mathrm{gf}_b) = M(\mathrm{gf}_{a*b}) = 0.$

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$$M_1(v) = 0.$$

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3. From M_1 compute $L_1 \in \mathbb{K}[n]\langle E, E^{-1} \rangle$ s.t.

$$M_1(v) = M_1(\mathrm{gf}_b) = 0 \implies L_1(b) = 0.$$

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$$M_1(v) = M_1(\mathrm{gf}_b) = 0 \implies L_1(b) = 0.$$

4. Return solutions b of $L_1(b) = 0$.

Inverse Zeilberger's problem 3

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1. $M = -x^{-2}(x^2D^2 - (x-1)(2x-1)D + (x-2)(x-1))$
2. $M_1 = -x^{-2}(x^2D^2 + (3x-1)D + 1)$
3. $L_1 = E^2((n+1)E - (n+1)^2)$
4. $\{Cn!;\ C \in \mathbb{K}\}$

Idea 2. Search for solution in terms of a polynomial series.

Inverse Zeilberger's problem 2

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Let $y = a * b$. Then

$$y_n \quad = \quad \sum_{k=0}^{n} a_{n-k} b_k \quad = \quad \sum_{k=0}^{n} \frac{b_k}{(n-k)!},$$

Inverse Zeilberger's problem 2

Idea 2. Search for solution in terms of a polynomial series.

Let $y = a * b$. Then

$$
\begin{aligned}
y_n &= \sum_{k=0}^{n} a_{n-k} b_k = \sum_{k=0}^{n} \frac{b_k}{(n-k)!}, \\
z_n &:= n! y_n = \sum_{k=0}^{n} \frac{n!}{(n-k)!} b_k = \sum_{k=0}^{n} c_k \binom{n}{k}
\end{aligned}
$$

where $c_k = k! b_k$.

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Step 3. Find solutions c of $\tilde{L}(c) = 0$, return $b_k = \frac{c_k}{k!}$.

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How to do step 2?

Inverse Zeilberger's problem 4

[S.A.-M.P.-A. Ryabenko, 2000]:

Inverse Zeilberger's problem 4

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$\mathcal{B} := \langle P_k(x) \rangle_{k=0}^{\infty}$ is a factorial basis for $\mathbb{K}[x]$ if

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$$LP_k = \sum_{i=-A}^{B} \alpha_{i,k} P_{k+i}$$

where $P_k = 0$ if $k < 0$ (*equivalently:* $[\alpha_{i-k,k}]_{i,k \in \mathbb{N}}$ is band-diagonal).

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where

$$\tilde{L} = \sum_{i=-B}^{A} \alpha_{-i,k+i} E_k^i$$

and $c_k = 0$ if $k < 0$.

Inverse Zeilberger's problem 6

1. $L(p(x)) = xp(x)$: compatible with any factorial basis;

$$xP_k(x) = u_k P_k(x) + v_k P_{k+1}(x),$$

so $A = 0$, $B = 1$, $\alpha_{0,k} = u_k$, $\alpha_{1,k} = v_k$

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3. $L(p(x)) = p(x+1)$: compatible with $P_k(x) = \binom{x}{k}$;

$$P_k(x+1) = P_{k-1}(x) + P_k(x),$$

so $A = 1$, $B = 0$, $\alpha_{-1,k} = \alpha_{0,k} = 1$

Inverse Zeilberger's problem 7

To compute $L \rightsquigarrow \tilde{L}$:

1. For differential operators with $P_n(x) = x^n$:

$$\begin{array}{rcl} D & \rightsquigarrow & (n+1)E_n, \\ x & \rightsquigarrow & E_n^{-1} \end{array}$$

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2. For recurrence operators with $P_n(x) = \binom{x}{n}$:

$$\begin{array}{rcl} E & \rightsquigarrow & E_n + 1, \\ x & \rightsquigarrow & n(E_n^{-1} + 1) \end{array}$$