Consequences of the fundamental theorem of calculus in differential rings

$$\partial \int = \mathrm{id} \qquad \mathrm{E} = \mathrm{id} - \int \partial$$

Georg Regensburger
joint work with Clemens Raab

JOHANNES KEPLER UNIVERSITÄT LINZ

Der Wissenschaftsfonds.

Computer Algebra in Combinatorics
ESI, Vienna
November 15, 2017

Differential operators

Symbolic computation with definite **integrals, sums, and identities** for special functions and combinatorial sequences

(Zeilberger ’90, Petkovšek-Wilf-Zeilberger ’96, Chyzak-Salvy ’98, Koepf ’98 ’14, Kauers-Koutschan-Zeilberger ’09, Kauers-Paule ’11)

Differential operators

Symbolic computation with definite **integrals, sums, and identities** for special functions and combinatorial sequences

(Zeilberger '90, Petkovšek-Wilf-Zeilberger '96, Chyzak-Salvy '98, Koepf '98 '14, Kauers-Koutschan-Zeilberger '09, Kauers-Paule '11)

Algebraic systems theory for linear systems of ordinary/partial differential, time-delay, and difference equations

Matrices/modules of/over **polynomial** and **Ore algebras**

(Oberst '90, Pommaret-Quadrat '03, Chyzak-Quadrat-Robertz '05, Gómez-Torrecillas '14, Robertz '15, Seiler-Zerz '15)

Differential operators

Symbolic computation with definite **integrals, sums, and identities** for special functions and combinatorial sequences

(Zeilberger ’90, Petkovšek-Wilf-Zeilberger ’96, Chyzak-Salvy ’98, Koepf ’98 ’14, Kauers-Koutschan-Zeilberger ’09, Kauers-Paule ’11)

Algebraic systems theory for linear systems of ordinary/partial differential, time-delay, and difference equations

Matrices/modules of/over **polynomial** and **Ore algebras**

(Oberst ’90, Pommaret-Quadrat ’03, Chyzak-Quadrat-Robertz ’05, Gómez-Torrecillas ’14, Robertz ’15, Seiler-Zerz ’15)

Euclidian algorithm and (non)commutative **Gröbner bases**

(Ore ’33, Buchberger ’65, Kandri-Rody-Weispfenning ’90, Kredel ’93, Chyzak-Salvy ’98, Chyzak ’98, Levandovskyy ’06, Koutschan ’10)

Differential operators

Symbolic computation with definite **integrals, sums, and identities** for special functions and combinatorial sequences

(Zeilberger ’90, Petkovšek-Wilf-Zeilberger ’96, Chyzak-Salvy ’98, Koepf ’98 ’14, Kauers-Koutschan-Zeilberger ’09, Kauers-Paule ’11)

Algebraic systems theory for linear systems of ordinary/partial differential, time-delay, and difference equations

Matrices/modules of/over **polynomial** and **Ore algebras**

(Oberst ’90, Pommaret-Quadrat ’03, Chyzak-Quadrat-Robertz ’05, Gómez-Torrecillas ’14, Robertz ’15, Seiler-Zerz ’15)

Euclidian algorithm and (non)commutative **Gröbner bases**

(Ore ’33, Buchberger ’65, Kandri-Rody-Weispfenning ’90, Kredel ’93, Chyzak-Salvy ’98, Chyzak ’98, Levandovskyy ’06, Koutschan ’10)

“Adding” integral operators

Algebraic approach and symbolic computation for manipulating and solving **boundary problems** for linear DEs

$$y'' = f,$$
$$y(0) = y(1) = 0$$

“Adding” integral operators

Algebraic approach and symbolic computation for
manipulating and solving **boundary problems** for linear DEs

$$y'' = f,$$
$$y(0) = y(1) = 0$$

- Differential operators
- Boundary conditions (evaluations)
- Integral operators (Green’s operators) $G\colon f \mapsto y$

(Rosenkranz ’03, ’05, Rosenkranz-R ’08, Rosenkranz-R-Tec-Buchberger ’12, Korporal-R ’14, Hossein Poor-Raab-R ’16 ’18)

“Adding” integral operators

Algebraic approach and symbolic computation for
manipulating and solving **boundary problems** for linear DEs

$$y'' = f,$$
$$y(0) = y(1) = 0$$

- Differential operators
- Boundary conditions (evaluations)
- Integral operators (Green’s operators) $G\colon f \mapsto y$

(Rosenkranz ’03, ’05, Rosenkranz-R ’08, Rosenkranz-R-Tec-Buchberger ’12, Korporal-R ’14, Hossein Poor-Raab-R ’16 ’18)

Goal: Develop constructive algebraic systems theory for **linear ordinary integro-differential equations** with **boundary conditions**

(Quadrat-R ’13, Quadrat-R ’17)

Integro-differential rings

FTC: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ and $f(a) = f(x) - \int_a^x f'(t)\,dt$

Integro-differential rings

FTC: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ and $f(a) = f(x) - \int_a^x f'(t)\,dt$

Definition

Let (R,∂) be a commutative differential ring with ring of constants K.

Integro-differential rings

FTC: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ and $f(a) = f(x) - \int_a^x f'(t)\,dt$

Definition

Let (R, ∂) be a commutative differential ring with ring of constants K.
Let $\partial R = R$ and $\int\colon R \to R$ be a K-linear operation on R such that

$$\partial \int f = f$$

for $f \in R$.

Integro-differential rings

FTC: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ and $f(a) = f(x) - \int_a^x f'(t)\,dt$

Definition

Let (R,∂) be a commutative differential ring with ring of constants K. Let $\partial R = R$ and $\int\colon R \to R$ be a K-linear operation on R such that

$$\partial\int f = f$$

for $f \in R$. We call $(R,\partial,\int)$ a **(generalized) integro-differential ring** and we define the **evaluation** operation on R by

$$\mathrm{E}f = f - \int\partial f.$$

Integro-differential rings

FTC: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ and $f(a) = f(x) - \int_a^x f'(t)\,dt$

Definition

Let (R, ∂) be a commutative differential ring with ring of constants K.
Let $\partial R = R$ and $\int\colon R \to R$ be a K-linear operation on R such that

$$\partial \int f = f$$

for $f \in R$. We call $(R, \partial, \int)$ a **(generalized) integro-differential ring** and we define the **evaluation** operation on R by

$$\mathrm{E}f = f - \int \partial f.$$

(Standard) integro-differential algebra, if E is multiplicative

$$\mathrm{E}fg = (\mathrm{E}f)\mathrm{E}g$$

(Rosenkranz '05, Rosenkranz-R '08)

Integro-differential rings

FTC: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ and $f(a) = f(x) - \int_a^x f'(t)\,dt$

Definition

Let (R, ∂) be a commutative differential ring with ring of constants K.
Let $\partial R = R$ and $\int : R \to R$ be a K-linear operation on R such that

$$\partial \int f = f$$

for $f \in R$. We call $(R, \partial, \int)$ a **(generalized) integro-differential ring** and we define the **evaluation** operation on R by

$$\mathrm{E} f = f - \int \partial f.$$

(Standard) integro-differential algebra, if E is multiplicative

$$\mathrm{E} fg = (\mathrm{E} f)\mathrm{E} g$$

(Rosenkranz '05, Rosenkranz-R '08)

Differential Rota-Baxter algebra, $\int$ satisfies the Rota-Baxter identity

$$(\int f)\int g = \int f \int g + \int (\int f) g$$

(Guo-Keigher '08)

Examples

Polynomials $K[x]$ with $\mathbb{Q} \subseteq K$

$$\partial = \tfrac{d}{dx} \quad \text{and} \quad \int x^k = \frac{x^{k+1}}{k+1}$$

Examples

Polynomials $K[x]$ with $\mathbb{Q} \subseteq K$

$$\partial = \tfrac{d}{dx} \quad \text{and} \quad \textstyle\int x^k = \frac{x^{k+1}}{k+1}$$

$\mathrm{E} = \mathrm{id} - \int \partial$ acts by

$$\mathrm{E}x^k = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

that is, $\mathrm{E}\colon p(x) \mapsto p(0)$, and is multiplicative

Examples

Polynomials $K[x]$ with $\mathbb{Q} \subseteq K$

$$\partial = \tfrac{d}{dx} \quad \text{and} \quad \int x^k = \frac{x^{k+1}}{k+1}$$

$\mathrm{E} = \mathrm{id} - \int \partial$ acts by

$$\mathrm{E}x^k = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

that is, $\mathrm{E}\colon p(x) \mapsto p(0)$, and is multiplicative

Differential algebras **closed under integration** and with **multiplicative evaluation**:

- Formal power series
- Smooth and analytic functions
- Exponential polynomials

Laurent polynomials

$R = K[x, \frac{1}{x}, \ln(x)]$ with $\partial = \frac{d}{dx}$ and $\int$ defined recursively by

$$\int x^k \ln(x)^n = \begin{cases} \frac{x^{k+1}}{k+1} & k \neq -1 \wedge n = 0 \\ & \\ & \end{cases}$$

Laurent polynomials

$R = K[x, \frac{1}{x}, \ln(x)]$ with $\partial = \frac{d}{dx}$ and $\int$ defined recursively by

$$\int x^k \ln(x)^n = \begin{cases} \frac{x^{k+1}}{k+1} & k \neq -1 \wedge n = 0 \\ \frac{x^{k+1}}{k+1} \ln(x)^n - \frac{n}{k+1} \int x^k \ln(x)^{n-1} & k \neq -1 \wedge n > 0 \\ & \end{cases}$$

Laurent polynomials

$R = K[x, \frac{1}{x}, \ln(x)]$ with $\partial = \frac{d}{dx}$ and $\int$ defined recursively by

$$\int x^k \ln(x)^n = \begin{cases} \frac{x^{k+1}}{k+1} & k \neq -1 \wedge n = 0 \\ \frac{x^{k+1}}{k+1} \ln(x)^n - \frac{n}{k+1} \int x^k \ln(x)^{n-1} & k \neq -1 \wedge n > 0 \\ \frac{\ln(x)^{n+1}}{n+1} & k = -1 \end{cases}$$

Laurent polynomials

$R = K[x, \frac{1}{x}, \ln(x)]$ with $\partial = \frac{d}{dx}$ and $\int$ defined recursively by

$$\int x^k \ln(x)^n = \begin{cases} \frac{x^{k+1}}{k+1} & k \neq -1 \wedge n = 0 \\ \frac{x^{k+1}}{k+1} \ln(x)^n - \frac{n}{k+1} \int x^k \ln(x)^{n-1} & k \neq -1 \wedge n > 0 \\ \frac{\ln(x)^{n+1}}{n+1} & k = -1 \end{cases}$$

$\mathrm{E} = \mathrm{id} - \int \partial$ acts by

$$\mathrm{E} x^k \ln(x)^n = \begin{cases} 1 & k = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and is not multiplicative:

Laurent polynomials

$R = K[x, \frac{1}{x}, \ln(x)]$ with $\partial = \frac{d}{dx}$ and $\int$ defined recursively by

$$\int x^k \ln(x)^n = \begin{cases} \frac{x^{k+1}}{k+1} & k \neq -1 \wedge n = 0 \\ \frac{x^{k+1}}{k+1} \ln(x)^n - \frac{n}{k+1} \int x^k \ln(x)^{n-1} & k \neq -1 \wedge n > 0 \\ \frac{\ln(x)^{n+1}}{n+1} & k = -1 \end{cases}$$

$\mathrm{E} = \mathrm{id} - \int \partial$ acts by

$$\mathrm{E} x^k \ln(x)^n = \begin{cases} 1 & k = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and is not multiplicative: for $f = x$ and $g = \frac{1}{x}$

$$\mathrm{E} fg = 1 \quad \text{and} \quad \mathrm{E} f = \mathrm{E} g = 0$$

Laurent polynomials

$R = K[x, \frac{1}{x}, \ln(x)]$ with $\partial = \frac{d}{dx}$ and $\int$ defined recursively by

$$\int x^k \ln(x)^n = \begin{cases} \frac{x^{k+1}}{k+1} & k \neq -1 \wedge n = 0 \\ \frac{x^{k+1}}{k+1} \ln(x)^n - \frac{n}{k+1} \int x^k \ln(x)^{n-1} & k \neq -1 \wedge n > 0 \\ \frac{\ln(x)^{n+1}}{n+1} & k = -1 \end{cases}$$

$\mathrm{E} = \mathrm{id} - \int \partial$ acts by

$$\mathrm{E} x^k \ln(x)^n = \begin{cases} 1 & k = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and is not multiplicative: for $f = x$ and $g = \frac{1}{x}$

$$\mathrm{E} fg = 1 \quad \text{and} \quad \mathrm{E} f = \mathrm{E} g = 0$$

Laurent series: $K((x))[\ln(x)]$

Integro-differential operators

Linear operators

- multiplication operators induced by $f \in R$ acting as $g \mapsto fg$
- differential operator ∂
- integral $\int$
- evaluation $\mathrm{E} = \mathrm{id} - \int \partial$

Integro-differential operators

Linear operators

- multiplication operators induced by $f \in R$ acting as $g \mapsto fg$
- differential operator ∂
- integral $\int$
- evaluation $\mathrm{E} = \mathrm{id} - \int\partial$

Understand all relations between these operators?

Integro-differential operators

Linear operators

- multiplication operators induced by $f \in R$ acting as $g \mapsto fg$
- differential operator ∂
- integral $\int$
- evaluation $\mathrm{E} = \mathrm{id} - \int\partial$

Understand all relations between these operators?

Algorithmic approach via **tensor reduction systems**
for tensor algebras and rings

(Bergman '78, Hossein Poor-Raab-R '16 '18)

Integro-differential operators

Linear operators

- multiplication operators induced by $f \in R$ acting as $g \mapsto fg$
- differential operator ∂
- integral $\int$
- evaluation $\mathrm{E} = \mathrm{id} - \int\partial$

Understand all relations between these operators?

Algorithmic approach via **tensor reduction systems**
for tensor algebras and rings

(Bergman '78, Hossein Poor-Raab-R '16 '18)

- operator product (composition) is represented by tensor product

Integro-differential operators

Linear operators

- multiplication operators induced by $f \in R$ acting as $g \mapsto fg$
- differential operator ∂
- integral $\int$
- evaluation $\mathrm{E} = \mathrm{id} - \int\partial$

Understand all relations between these operators?

Algorithmic approach via **tensor reduction systems**
for tensor algebras and rings

(Bergman ’78, Hossein Poor-Raab-R ’16 ’18)

- operator product (composition) is represented by tensor product
- allows for a basis-free treatment of multiplication operators and

Integro-differential operators

Linear operators

- multiplication operators induced by $f \in R$ acting as $g \mapsto fg$
- differential operator ∂
- integral $\int$
- evaluation $\mathrm{E} = \mathrm{id} - \int\partial$

Understand all relations between these operators?

Algorithmic approach via **tensor reduction systems**
for tensor algebras and rings

(Bergman '78, Hossein Poor-Raab-R '16 '18)

- operator product (composition) is represented by tensor product
- allows for a basis-free treatment of multiplication operators and
- computations with general elements in integro-differential algebras

Tensor reduction systems

Reduction rule (homomorphism) for multiplication operators

$$f \otimes g \mapsto fg$$

Tensor reduction systems

Reduction rule (homomorphism) for multiplication operators

$$f \otimes g \mapsto fg$$

Ambiguity

$$\begin{array}{ccc} & f \otimes g \otimes h & \\ \swarrow & & \searrow \\ fg \otimes h & & f \otimes gh \end{array}$$

Tensor reduction systems

Reduction rule (homomorphism) for multiplication operators

$$f \otimes g \mapsto fg$$

Ambiguity

S-polynomial

$$\begin{array}{ccc} & f \otimes g \otimes h & \\ \swarrow & & \searrow \\ fg \otimes h & - & f \otimes gh \end{array}$$

Tensor reduction systems

Reduction rule (homomorphism) for multiplication operators

$$f \otimes g \mapsto fg$$

Ambiguity

S-polynomial

$$\begin{tikzcd}[column sep=small] & f \otimes g \otimes h \arrow[dl] \arrow[dr] & \\ fg \otimes h & - & f \otimes gh \end{tikzcd}$$

$$\to (fg)h - f(gh) = 0 \quad \text{for all } f, g, h \in R$$

Tensor reduction systems

Reduction rule (homomorphism) for multiplication operators

$$f \otimes g \mapsto fg$$

Ambiguity

$$f \otimes g \otimes h$$

S-polynomial

$$fg \otimes h \quad - \quad f \otimes gh$$

$$\to (fg)h - f(gh) = 0 \quad \text{for all } f, g, h \in R$$

Definition

An ambiguity is **resolvable** if all S-polynomials can be reduced to zero.

Tensor reduction systems

Reduction rule (homomorphism) for multiplication operators

$$f \otimes g \mapsto fg$$

Ambiguity

$$f \otimes g \otimes h$$

S-polynomial

$$fg \otimes h \quad - \quad f \otimes gh$$

$$\to (fg)h - f(gh) = 0 \quad \text{for all } f, g, h \in R$$

Definition

An ambiguity is **resolvable** if all S-polynomials can be reduced to zero.

Theorem (Diamond Lemma for tensors)

Given a tensor reduction system, every tensor has a ***unique normal form*** *iff* ***all ambiguities are resolvable****.*
In that case, the tensor algebra factored by the reduction ideal is ***isomorphic to the algebra of irreducible tensors****.* *(Bergman '78)*

Differential operators

Reduction rules

$$f \otimes g \mapsto fg \qquad \text{and} \qquad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Differential operators

Reduction rules

$$f \otimes g \mapsto fg \qquad \text{and} \qquad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Ambiguity $\quad \partial \otimes f \otimes g$

S-polynomial $\quad (f \otimes \partial + \partial f) \otimes g \; - \; \partial \otimes fg$

Differential operators

Reduction rules

$$f \otimes g \mapsto fg \qquad \text{and} \qquad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Ambiguity $\qquad \partial \otimes f \otimes g$

S-polynomial $\qquad (f \otimes \partial + \partial f) \otimes g \; - \; \partial \otimes fg$

$$\to f \otimes g \otimes \partial + f \otimes \partial g + (\partial f)g - fg \otimes \partial - \partial(fg)$$

Differential operators

Reduction rules

$$f \otimes g \mapsto fg \qquad \text{and} \qquad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Ambiguity $\qquad \partial \otimes f \otimes g$

S-polynomial $\qquad (f \otimes \partial + \partial f) \otimes g \; - \; \partial \otimes fg$

$$\to f \otimes g \otimes \partial + f \otimes \partial g + (\partial f)g - fg \otimes \partial - \partial(fg)$$

$$\to fg \otimes \partial + f\partial g + (\partial f)g - fg \otimes \partial - \partial(fg)$$

Differential operators

Reduction rules

$$f \otimes g \mapsto fg \qquad \text{and} \qquad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Ambiguity $\qquad \partial \otimes f \otimes g$

S-polynomial $\qquad (f \otimes \partial + \partial f) \otimes g \; - \; \partial \otimes fg$

$$\begin{aligned} &\to f \otimes g \otimes \partial + f \otimes \partial g + (\partial f)g - fg \otimes \partial - \partial(fg) \\ &\qquad \to fg \otimes \partial + f\partial g + (\partial f)g - fg \otimes \partial - \partial(fg) \\ &\qquad\qquad = f\partial g + (\partial f)g - \partial(fg) = 0 \qquad \text{for all } f, g \in R \end{aligned}$$

Leibniz rule in R

Differential operators

Reduction rules

$$f \otimes g \mapsto fg \qquad \text{and} \qquad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Ambiguity $\qquad \partial \otimes f \otimes g$

S-polynomial $\qquad (f \otimes \partial + \partial f) \otimes g \; - \; \partial \otimes fg$

$$\begin{aligned} &\to f \otimes g \otimes \partial + f \otimes \partial g + (\partial f)g - fg \otimes \partial - \partial(fg) \\ &\to fg \otimes \partial + f\partial g + (\partial f)g - fg \otimes \partial - \partial(fg) \\ &= f\partial g + (\partial f)g - \partial(fg) = 0 \qquad \text{for all } f, g \in R \end{aligned}$$

Leibniz rule in R

Multiplication with **identity element** and **empty tensor**

$$1 \mapsto \varepsilon$$

Differential operators

Reduction rules

$$f \otimes g \mapsto fg \qquad \text{and} \qquad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Ambiguity $\quad \partial \otimes f \otimes g$

S-polynomial $\quad (f \otimes \partial + \partial f) \otimes g \; - \; \partial \otimes fg$

$$\begin{aligned}
&\to f \otimes g \otimes \partial + f \otimes \partial g + (\partial f)g - fg \otimes \partial - \partial(fg) \\
&\qquad \to fg \otimes \partial + f\partial g + (\partial f)g - fg \otimes \partial - \partial(fg) \\
&\qquad\qquad = f\partial g + (\partial f)g - \partial(fg) = 0 \qquad \text{for all } f, g \in R
\end{aligned}$$

Leibniz rule in R

Multiplication with **identity element** and **empty tensor**

$$1 \mapsto \varepsilon$$

Irreducible tensors (**normal form**)

$$f \otimes \partial^{\otimes j}$$

Rules from the FTC and their completion

Differential operators

$$1 \mapsto \varepsilon, \quad f \otimes g \mapsto fg, \quad \text{and} \quad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Rules from FTC

$$\partial \otimes \int \mapsto \varepsilon \qquad \text{and} \qquad \int \otimes \partial \mapsto \varepsilon - \mathrm{E}$$

Rules from the FTC and their completion

Differential operators

$$1 \mapsto \varepsilon, \quad f \otimes g \mapsto fg, \quad \text{and} \quad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Rules from FTC

$$\partial \otimes \int \mapsto \varepsilon \qquad \text{and} \qquad \int \otimes \partial \mapsto \varepsilon - \mathrm{E}$$

Completion: Add new rules for S-polynomials not reducing to zero

Rules from the FTC and their completion

Differential operators

$$1 \mapsto \varepsilon, \quad f \otimes g \mapsto fg, \quad \text{and} \quad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Rules from FTC

$$\partial \otimes \int \mapsto \varepsilon \qquad \text{and} \qquad \int \otimes \partial \mapsto \varepsilon - \mathrm{E}$$

Completion: Add new rules for S-polynomials not reducing to zero

Ambiguity $\quad \partial \otimes \int \otimes \partial$

S-polynomial $\quad \varepsilon \otimes \partial - \partial \otimes (\varepsilon - \mathrm{E}) = \partial \otimes \mathrm{E}$

Rules from the FTC and their completion

Differential operators

$$1 \mapsto \varepsilon, \quad f \otimes g \mapsto fg, \quad \text{and} \quad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Rules from FTC

$$\partial \otimes \int \mapsto \varepsilon \qquad \text{and} \qquad \int \otimes \partial \mapsto \varepsilon - \mathrm{E}$$

Completion: Add new rules for S-polynomials not reducing to zero

Ambiguity	$\partial \otimes \int \otimes \partial$
S-polynomial	$\varepsilon \otimes \partial - \partial \otimes (\varepsilon - \mathrm{E}) = \partial \otimes \mathrm{E}$
New reduction rule	$\partial \otimes \mathrm{E} \mapsto 0$

Rules from the FTC and their completion

Differential operators

$$1 \mapsto \varepsilon, \quad f \otimes g \mapsto fg, \quad \text{and} \quad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Rules from FTC

$$\partial \otimes \int \mapsto \varepsilon \qquad \text{and} \qquad \int \otimes \partial \mapsto \varepsilon - \mathrm{E}$$

Completion: Add new rules for S-polynomials not reducing to zero

Ambiguity	$\partial \otimes \int \otimes \partial$
S-polynomial	$\varepsilon \otimes \partial - \partial \otimes (\varepsilon - \mathrm{E}) = \partial \otimes \mathrm{E}$
New reduction rule	$\partial \otimes \mathrm{E} \mapsto 0$
Ambiguity	$\int \otimes \partial \otimes \int$
S-polynomial	$(\varepsilon - \mathrm{E}) \otimes \int - \int \otimes \varepsilon = -\mathrm{E} \otimes \int$

Rules from the FTC and their completion

Differential operators

$$1 \mapsto \varepsilon, \quad f \otimes g \mapsto fg, \quad \text{and} \quad \partial \otimes f \mapsto f \otimes \partial + \partial f$$

Rules from FTC

$$\partial \otimes \int \mapsto \varepsilon \qquad \text{and} \qquad \int \otimes \partial \mapsto \varepsilon - \mathrm{E}$$

Completion: Add new rules for S-polynomials not reducing to zero

Ambiguity	$\partial \otimes \int \otimes \partial$
S-polynomial	$\varepsilon \otimes \partial - \partial \otimes (\varepsilon - \mathrm{E}) = \partial \otimes \mathrm{E}$
New reduction rule	$\partial \otimes \mathrm{E} \mapsto 0$
Ambiguity	$\int \otimes \partial \otimes \int$
S-polynomial	$(\varepsilon - \mathrm{E}) \otimes \int - \int \otimes \varepsilon = -\mathrm{E} \otimes \int$
New reduction rule	$\mathrm{E} \otimes \int \mapsto 0$

Completion (cont.)

Ambiguity $\int \otimes \partial \otimes f$

$$(\varepsilon - \mathrm{E}) \otimes f - \int \otimes (f \otimes \partial + \partial f)$$

Completion (cont.)

Ambiguity $\int \otimes \partial \otimes f$

$$(\varepsilon - \mathrm{E}) \otimes f - \int \otimes (f \otimes \partial + \partial f) = f - \mathrm{E} \otimes f - \int \otimes f \otimes \partial - \int \otimes \partial f$$

Completion (cont.)

Ambiguity $$\int \otimes \partial \otimes f$$

$$(\varepsilon - \mathrm{E}) \otimes f - \int \otimes (f \otimes \partial + \partial f) = f - \mathrm{E} \otimes f - \int \otimes f \otimes \partial - \int \otimes \partial f$$

New rule $$\int \otimes f \otimes \partial \mapsto f - \mathrm{E} \otimes f - \int \otimes \partial f$$

Completion (cont.)

Ambiguity $$\int \otimes \partial \otimes f$$

$$(\varepsilon - \mathrm{E}) \otimes f - \int \otimes (f \otimes \partial + \partial f) = f - \mathrm{E} \otimes f - \int \otimes f \otimes \partial - \int \otimes \partial f$$

New rule $$\int \otimes f \otimes \partial \mapsto f - \mathrm{E} \otimes f - \int \otimes \partial f$$

Integration by parts in R

$$\int f \partial g = fg - \mathrm{E} fg - \int (\partial f) g$$

Completion (cont.)

Ambiguity $$\int \otimes \partial \otimes f$$

$$(\varepsilon - \mathrm{E}) \otimes f - \int \otimes (f \otimes \partial + \partial f) = f - \mathrm{E} \otimes f - \int \otimes f \otimes \partial - \int \otimes \partial f$$

New rule $$\int \otimes f \otimes \partial \mapsto f - \mathrm{E} \otimes f - \int \otimes \partial f$$

Integration by parts in R

$$\int f \partial g = fg - \mathrm{E} fg - \int (\partial f) g$$

Ambiguity $$\int \otimes f \otimes \partial \otimes \int$$

$$(f - \mathrm{E} \otimes f - \int \otimes \partial f) \otimes \int - \int \otimes f \otimes \varepsilon$$

Completion (cont.)

Ambiguity $$\int \otimes \partial \otimes f$$

$$(\varepsilon - \mathrm{E}) \otimes f - \int \otimes (f \otimes \partial + \partial f) = f - \mathrm{E} \otimes f - \int \otimes f \otimes \partial - \int \otimes \partial f$$

New rule $$\int \otimes f \otimes \partial \mapsto f - \mathrm{E} \otimes f - \int \otimes \partial f$$

Integration by parts in R

$$\int f \partial g = fg - \mathrm{E} fg - \int (\partial f) g$$

Ambiguity $$\int \otimes f \otimes \partial \otimes \int$$

$$(f - \mathrm{E} \otimes f - \int \otimes \partial f) \otimes \int - \int \otimes f \otimes \varepsilon = f \otimes \int - \mathrm{E} \otimes f \otimes \int - \int \otimes \partial f \otimes \int - \int \otimes f$$

Completion (cont.)

Ambiguity $$\int \otimes \partial \otimes f$$

$$(\varepsilon - \mathrm{E}) \otimes f - \int \otimes (f \otimes \partial + \partial f) = f - \mathrm{E} \otimes f - \int \otimes f \otimes \partial - \int \otimes \partial f$$

New rule $$\int \otimes f \otimes \partial \mapsto f - \mathrm{E} \otimes f - \int \otimes \partial f$$

Integration by parts in R

$$\int f \partial g = fg - \mathrm{E} fg - \int (\partial f) g$$

Ambiguity $$\int \otimes f \otimes \partial \otimes \int$$

$$(f - \mathrm{E} \otimes f - \int \otimes \partial f) \otimes \int - \int \otimes f \otimes \varepsilon = f \otimes \int - \mathrm{E} \otimes f \otimes \int - \int \otimes \partial f \otimes \int - \int \otimes f$$

New rule $$\int \otimes f \otimes \int \mapsto \int f \otimes \int - \int \otimes \int f - \mathrm{E} \otimes \int f \otimes \int$$

Completion (cont.)

Ambiguity $$\int \otimes \partial \otimes f$$

$$(\varepsilon - \mathrm{E}) \otimes f - \int \otimes (f \otimes \partial + \partial f) = f - \mathrm{E} \otimes f - \int \otimes f \otimes \partial - \int \otimes \partial f$$

New rule $$\int \otimes f \otimes \partial \mapsto f - \mathrm{E} \otimes f - \int \otimes \partial f$$

Integration by parts in R

$$\int f \partial g = fg - \mathrm{E} fg - \int (\partial f) g$$

Ambiguity $$\int \otimes f \otimes \partial \otimes \int$$

$$(f - \mathrm{E} \otimes f - \int \otimes \partial f) \otimes \int - \int \otimes f \otimes \varepsilon = f \otimes \int - \mathrm{E} \otimes f \otimes \int - \int \otimes \partial f \otimes \int - \int \otimes f$$

New rule $$\int \otimes f \otimes \int \mapsto \int f \otimes \int - \int \otimes \int f - \mathrm{E} \otimes \int f \otimes \int$$

Modified Rota-Baxter identity in R

$$\left(\int f\right) \int g = \int f \int g + \int \left(\int f\right) g + \color{red}{\mathrm{E} \left(\int f\right) \int g}$$

Reduction system for integro-differential operators

"Gröbner basis" of all relations

$$
\begin{array}{cr@{\;\mapsto\;}l}
K & 1 & \varepsilon \\
FF & f \otimes g & fg \\
DF & \partial \otimes f & f \otimes \partial + \partial f \\
DE & \partial \otimes \mathrm{E} & 0 \\
EE & \mathrm{E} \otimes \mathrm{E} & \mathrm{E} \\
EI & \mathrm{E} \otimes \int & 0 \\
DI & \partial \otimes \int & \varepsilon \\
IE & \int \otimes \mathrm{E} & \int 1 \otimes \mathrm{E} \\
ID & \int \otimes \partial & \varepsilon - \mathrm{E} \\
II & \int \otimes \int & \int 1 \otimes \int - \int \otimes \int 1 - \mathrm{E} \otimes \int 1 \otimes \int \\
IFE & \int \otimes f \otimes \mathrm{E} & \int f \otimes \mathrm{E} \\
EFE & \mathrm{E} \otimes f \otimes \mathrm{E} & (\mathrm{E} f)\mathrm{E} \\
IFD & \int \otimes f \otimes \partial & f - \int \otimes \partial f - \mathrm{E} \otimes f \\
IFI & \int \otimes f \otimes \int & \int f \otimes \int - \int \otimes \int f - \mathrm{E} \otimes \int f \otimes \int
\end{array}
$$

Normal forms

Theorem

*Every integro-differential operator over a differential ring $(R, \partial, \int)$ with a commutative ring of constants K has a **unique normal form**,*

Normal forms

Theorem

Every integro-differential operator over a differential ring $(R, \partial, \int)$ with a commutative ring of constants K has a ***unique normal form****,*
which can be written as a K-linear combination of tensors of the form

$$f \otimes \partial^{\otimes j}, \quad f \otimes \int \otimes g, \quad f \otimes \mathrm{E} \otimes g \otimes \partial^{\otimes j}, \quad \text{or} \quad f \otimes \mathrm{E} \otimes h \otimes \int \otimes g$$

where $j \in \mathbb{N}_0$, $f, g, h \in \int R$, and each f and g may be absent.

Normal forms

Theorem

*Every integro-differential operator over a differential ring $(R, \partial, \int)$ with a commutative ring of constants K has a **unique normal form**, which can be written as a K-linear combination of tensors of the form*

$$f \otimes \partial^{\otimes j}, \quad f \otimes \int \otimes g, \quad f \otimes \mathrm{E} \otimes g \otimes \partial^{\otimes j}, \quad \text{or} \quad f \otimes \mathrm{E} \otimes h \otimes \int \otimes g$$

where $j \in \mathbb{N}_0$, $f, g, h \in \int R$, and each f and g may be absent.
*These **irreducible tensors** form an algebra with multiplication*

$$s \cdot t = (s \otimes t){\downarrow},$$

where $t{\downarrow}$ denotes the normal form of a tensor t.

(Raab-R '17)

Normal forms

Theorem

Every integro-differential operator over a differential ring $(R, \partial, \int)$ with a commutative ring of constants K has a ***unique normal form****,*
which can be written as a K-linear combination of tensors of the form

$$f \otimes \partial^{\otimes j}, \quad f \otimes \int \otimes g, \quad f \otimes \mathrm{E} \otimes g \otimes \partial^{\otimes j}, \quad \text{or} \quad f \otimes \mathrm{E} \otimes h \otimes \int \otimes g$$

where $j \in \mathbb{N}_0$, $f, g, h \in \int R$, and each f and g may be absent.
These ***irreducible tensors*** *form an algebra with multiplication*

$$s \cdot t = (s \otimes t)\downarrow,$$

where $t\downarrow$ denotes the normal form of a tensor t.

(Raab-R '17)

An integro-differential operator (IDO) is the sum of a

differential, **integral**, and a **boundary** operator.

Variation of constants

First-order differential operator $\qquad L = \partial + a \qquad$ with $a \in R$

with invertible solution $z \in R$: $\qquad \partial z + az = 0$

Variation of constants

First-order differential operator $\qquad L = \partial + a \qquad$ with $a \in R$

with invertible solution $z \in R$: $\qquad \partial z + az = 0$

Variation of constants in operator form

$$V = z \otimes \int \otimes \frac{1}{z}$$

Variation of constants

First-order differential operator $\quad L = \partial + a \quad$ with $a \in R$

with invertible solution $z \in R$: $\quad \partial z + az = 0$

Variation of constants in operator form

$$V = z \otimes \int \otimes \tfrac{1}{z}$$

Verify V is right inverse of L

Using the Leibniz rule

$$(\partial + a) \otimes z \to z \otimes \partial + \partial z + az = z \otimes \partial$$

Variation of constants

First-order differential operator $L = \partial + a$ with $a \in R$

with invertible solution $z \in R$: $\partial z + az = 0$

Variation of constants in operator form

$$V = z \otimes \int \otimes \frac{1}{z}$$

Verify V is right inverse of L

Using the Leibniz rule

$$(\partial + a) \otimes z \to z \otimes \partial + \partial z + az = z \otimes \partial$$

and using FTC

$$L \otimes V = (\partial + a) \otimes z \otimes \int \otimes \frac{1}{z} \to z \otimes \partial \otimes \int \otimes \frac{1}{z} \to z \otimes \frac{1}{z} \to 1 \to \varepsilon$$

Variation of constants

First-order differential operator $\quad L = \partial + a \quad$ with $a \in R$

with invertible solution $z \in R$: $\quad \partial z + az = 0$

Variation of constants in operator form

$$V = z \otimes \int \otimes \tfrac{1}{z}$$

Verify V is right inverse of L

Using the Leibniz rule

$$(\partial + a) \otimes z \to z \otimes \partial + \partial z + az = z \otimes \partial$$

and using FTC

$$L \otimes V = (\partial + a) \otimes z \otimes \int \otimes \tfrac{1}{z} \to z \otimes \partial \otimes \int \otimes \tfrac{1}{z} \to z \otimes \tfrac{1}{z} \to 1 \to \varepsilon$$

Same computation for first-order systems: same rules, “just” replace **scalars** by **matrices** and **tensor algebra** by **tensor ring**

Variation of constants

First-order differential differential system $\quad L = \partial + A \quad$ with $A \in R$

with invertible solution $z \in R$: $\quad \partial z + az = 0$

Variation of constants in operator form

$$V = z \otimes \int \otimes \frac{1}{z}$$

Verify V is right inverse of L

Using the Leibniz rule

$$(\partial + a) \otimes z \to z \otimes \partial + \partial z + az = z \otimes \partial$$

and using FTC

$$L \otimes V = (\partial + a) \otimes z \otimes \int \otimes \frac{1}{z} \to z \otimes \partial \otimes \int \otimes \frac{1}{z} \to z \otimes \frac{1}{z} \to 1 \to \varepsilon$$

Same computation for first-order systems: same rules, “just” replace **scalars** by **matrices** and **tensor algebra** by **tensor ring**

Variation of constants

First-order differential differential system $\quad L = \partial + A \quad$ with $A \in R$

with invertible fundamental matrix $Z \in R$: $\quad \partial Z + AZ = 0$

Variation of constants in operator form

$$V = z \otimes \int \otimes \frac{1}{z}$$

Verify V is right inverse of L

Using the Leibniz rule

$$(\partial + a) \otimes z \to z \otimes \partial + \partial z + az = z \otimes \partial$$

and using FTC

$$L \otimes V = (\partial + a) \otimes z \otimes \int \otimes \frac{1}{z} \to z \otimes \partial \otimes \int \otimes \frac{1}{z} \to z \otimes \frac{1}{z} \to 1 \to \varepsilon$$

Same computation for first-order systems: same rules, “just” replace **scalars** by **matrices** and **tensor algebra** by **tensor ring**

Variation of constants

First-order differential differential system $L = \partial + A$ with $A \in R$

with invertible fundamental matrix $Z \in R$: $\partial Z + AZ = 0$

Variation of constants in operator form

$$V = Z \otimes \int \otimes Z^{-1}$$

Verify V is right inverse of L

Using the Leibniz rule

$$(\partial + a) \otimes z \to z \otimes \partial + \partial z + az = z \otimes \partial$$

and using FTC

$$L \otimes V = (\partial + a) \otimes z \otimes \int \otimes \tfrac{1}{z} \to z \otimes \partial \otimes \int \otimes \tfrac{1}{z} \to z \otimes \tfrac{1}{z} \to 1 \to \varepsilon$$

Same computation for first-order systems: same rules, “just” replace **scalars** by **matrices** and **tensor algebra** by **tensor ring**

Variation of constants

First-order differential differential system $L = \partial + A$ with $A \in R$

with invertible fundamental matrix $Z \in R$: $\partial Z + AZ = 0$

Variation of constants in operator form

$$V = Z \otimes \int \otimes Z^{-1}$$

Verify V is right inverse of L

Using the Leibniz rule

$$(\partial + A) \otimes Z \to Z \otimes \partial + \partial Z + AZ = Z \otimes \partial$$

and using FTC

$$L \otimes V = (\partial + A) \otimes Z \otimes \int \otimes Z^{-1} \to Z \otimes \partial \otimes \int \otimes Z^{-1} \to Z \otimes Z^{-1} \to 1 \to \varepsilon$$

Same computation for first-order systems: same rules, “just” replace **scalars** by **matrices** and **tensor algebra** by **tensor ring**

Taylor's theorem (first version)

Iterate

$$\varepsilon \equiv \mathrm{E} + \int \otimes \partial$$

$$\varepsilon \quad \equiv \quad \mathrm{E} + \int \otimes \partial$$

Taylor's theorem (first version)

Iterate

$$\varepsilon \equiv \mathrm{E} + \int \otimes \partial$$

$$\varepsilon \equiv \mathrm{E} + \int \otimes \varepsilon \otimes \partial$$

Taylor's theorem (first version)

Iterate

$$\varepsilon \equiv \mathrm{E} + \int \otimes \partial$$

$$\begin{aligned}
\varepsilon &\equiv \mathrm{E} + \int \otimes \varepsilon \otimes \partial \\
&\equiv \mathrm{E} + \int \otimes \mathrm{E} \otimes \partial + \int \otimes \int \otimes \partial \otimes \partial
\end{aligned}$$

Taylor's theorem (first version)

Iterate

$$\varepsilon \equiv \mathrm{E} + \int \otimes \partial$$

$$\begin{aligned}
\varepsilon &\equiv \mathrm{E} + \int \otimes \varepsilon \otimes \partial \\
&\equiv \mathrm{E} + \int \otimes \mathrm{E} \otimes \partial + \int \otimes \int \otimes \varepsilon \otimes \partial \otimes \partial
\end{aligned}$$

Taylor's theorem (first version)

Iterate

$$\varepsilon \equiv \mathrm{E} + \int \otimes \partial$$

$$\begin{aligned}
\varepsilon &\equiv \mathrm{E} + \int \otimes \varepsilon \otimes \partial \\
&\equiv \mathrm{E} + \int \otimes \mathrm{E} \otimes \partial + \int \otimes \int \otimes \varepsilon \otimes \partial \otimes \partial \\
&\;\vdots \\
&\equiv \mathrm{E} + \int \otimes \mathrm{E} \otimes \partial + \cdots + \int^{\otimes n} \otimes \mathrm{E} \otimes \partial^{\otimes n} + \int^{\otimes (n+1)} \otimes \partial^{\otimes (n+1)}
\end{aligned}$$

Analytic translation

$$\begin{aligned}
f(x) &= f(a) + \int_a^x f'(a) dt + \int_a^x \cdots \int_a^{t_{n-1}} f^{(n)}(a) dt_n \ldots dt_1 + R_n \\
R_n &= \int_a^x \cdots \int_a^{t_n} f^{(n+1)}(t_{n+1}) dt_{n+1} \ldots dt_1
\end{aligned}$$

Reducing multiple integrals (for multiplicative evaluation)

Assume that the evaluation $\mathrm{E} = \mathsf{id} - \int \partial$ is **multiplicative**

Reducing multiple integrals (for multiplicative evaluation)

Assume that the evaluation $\mathrm{E} = \mathrm{id} - \int\partial$ is **multiplicative**

Simplified rule for reducing double integrals

$$\int\otimes\int \to \int 1\otimes\int - \int\otimes\int 1 - \mathrm{E}\otimes\int 1\otimes\int$$

Reducing multiple integrals (for multiplicative evaluation)

Assume that the evaluation $\mathrm{E} = \mathsf{id} - \int \partial$ is **multiplicative**

Simplified rule for reducing double integrals

$$\int \otimes \int \to \int 1 \otimes \int - \int \otimes \int 1 - \mathrm{E} \otimes \int 1 \otimes \int \;\to\; \int 1 \otimes \int - \int \otimes \int 1$$

Reducing multiple integrals (for multiplicative evaluation)

Assume that the evaluation $\mathrm{E} = \mathrm{id} - \int\partial$ is **multiplicative**

Simplified rule for reducing double integrals

$$\int \otimes \int \to \int 1 \otimes \int - \int \otimes \int 1 - \mathrm{E} \otimes \int 1 \otimes \int \;\to\; \int 1 \otimes \int - \int \otimes \int 1$$

With

$$x = \int 1 \in R$$

we obtain $$n! \int^n 1 = x^n$$

Reducing multiple integrals (for multiplicative evaluation)

Assume that the evaluation $\mathrm{E} = \mathrm{id} - \int \partial$ is **multiplicative**

Simplified rule for reducing double integrals

$$\int \otimes \int \to \int 1 \otimes \int - \int \otimes \int 1 - \mathrm{E} \otimes \int 1 \otimes \int \ \to \ \int 1 \otimes \int - \int \otimes \int 1$$

With

$$x = \int 1 \in R$$

we obtain

$$n! \int^n 1 = x^n$$

If $\mathbb{Q} \subseteq R$:

Reducing multiple integrals

$$\int \otimes \int \quad \to \quad x \otimes \int - \int \otimes x$$

$$\int^{\otimes(n+1)} \quad \to \quad \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} x^k \otimes \int \otimes x^{n-k}$$

Taylor's theorem (second version)

First version of Taylor's theorem

$$\varepsilon \equiv \sum_{k=0}^{n} \int^{\otimes k} \otimes \mathrm{E} \otimes \partial^{\otimes k} + \int^{\otimes (n+1)} \otimes \partial^{\otimes (n+1)}$$

Theorem (for multiplicative evaluation)

$$\varepsilon \equiv \sum_{k=0}^{n} \frac{x^k}{k!} \otimes \mathrm{E} \otimes \partial^{\otimes k} + \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} x^k \otimes \int \otimes x^{n-k} \otimes \partial^{\otimes (n+1)}$$

Taylor's theorem (second version)

First version of Taylor's theorem

$$\varepsilon \equiv \sum_{k=0}^{n} \int^{\otimes k} \otimes \mathrm{E} \otimes \partial^{\otimes k} + \int^{\otimes (n+1)} \otimes \partial^{\otimes (n+1)}$$

Theorem (for multiplicative evaluation)

$$\varepsilon \equiv \sum_{k=0}^{n} \frac{x^k}{k!} \otimes \mathrm{E} \otimes \partial^{\otimes k} + \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} x^k \otimes \int \otimes x^{n-k} \otimes \partial^{\otimes (n+1)}$$

Analytic translation

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n \\ R_n &= \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} x^k \int_a^x t^{n-k} f^{(n+1)}(t)\,dt \end{aligned}$$

Taylor's theorem (second version)

First version of Taylor's theorem

$$\varepsilon \equiv \sum_{k=0}^{n} \int^{\otimes k} \otimes \mathrm{E} \otimes \partial^{\otimes k} + \int^{\otimes (n+1)} \otimes \partial^{\otimes (n+1)}$$

Theorem (for multiplicative evaluation)

$$\varepsilon \equiv \sum_{k=0}^{n} \frac{x^k}{k!} \otimes \mathrm{E} \otimes \partial^{\otimes k} + \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} x^k \otimes \int \otimes x^{n-k} \otimes \partial^{\otimes (n+1)}$$

Analytic translation

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n \\ R_n &= \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \end{aligned}$$

Summary and outlook

- FTC in differential algebra
- Integro-differential operators via tensor reduction systems
- Completion and normal forms
- Variations of constants and Taylor formula
- Matrix coefficients

Summary and outlook

- FTC in differential algebra
- Integro-differential operators via tensor reduction systems
- Completion and normal forms
- Variations of constants and Taylor formula
- Matrix coefficients

- Algebraic theory of boundary problems (with singularities)
- Include also shift operators (delay equations) or linear substitutions
- Applications to algebraic systems theory
- Computable integro-differential algebras (nested integrals)
- Other operator algebras (discrete analogs)