A Galoisian Approach to Counting Walks

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Workshop on "Enumerative Combinatorics"
Erwin Schrödinger International Institute for Mathematics and Physics

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- Talk 1: An Introduction to the Galois theory of difference equations
- Talk 2: Walks, Difference Equations and Elliptic Curves

- Talk 1: An Introduction to the Galois theory of difference equations

Galois theory describes the possible

algebraic and differential relations

among solutions of functional equations

- A Warmup - the Gamma function

An Introduction to the Galois theory of difference equations

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- Galois Theory of linear difference equations
- Applications to differential transcendence
- Applications to P-recursive/holonomic sequences
- General differential Galois theory of linear difference equations

The Gamma Function is Transcendental

Elementary Proof:

$\Gamma(x+1) = x\Gamma(x)$. Assume $\Gamma(x)$ is algebraic over $\mathbb{C}(x)$.

$$\Gamma(x)^n + a_{n-1}(x)\Gamma(x)^{n-1} + \ldots + a_0(x) = 0$$

with n minimal, so $a_0(x) \neq 0$.

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Minimality $\Rightarrow$ $a_0(x+1) = x^n a_0(x)$ which is impossible.

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Galoisian Proof:

If $\Gamma(x)$ is algebraic over $\mathbb{C}(x)$ then **from Galois theory** we know that for some $n \neq 0$

$$y(x+1) = x^n y(x)$$

has a nonzero solution in $\mathbb{C}(x)$, which is impossible.

Classical Galois theory of polynomial equations

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More formally:

Splitting Ring: $K = k[y_1, \ldots, y_n, (\prod_{i<j}(y_i - y_j))^{-1}]/M = k[\alpha_1, \ldots, \alpha_n]$,

M a max ideal containing $(f(y_1), \ldots, f(y_n))$

Note: K is a field and all such are isomorphic.

Galois group: $\mathrm{Gal}(K/k) = \{\sigma : K \to K \mid \sigma \text{ is a } k\text{-autom.}\}$

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Ex.

$f(y) = y^3 + py + q$ irreducible over k, K= splitting field, $\mathrm{Gal}(K/k) \subset \mathcal{S}_3$

The roots satisfy obvious relations:

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$$\mathcal{A}_3 = \{\sigma \in \mathcal{S}_3 \mid \sigma \text{ leaves } \prod_{i<j}(X_i - X_j) \text{ invariant } \}$$

$$\begin{aligned} \mathrm{Gal}(K/k) \subset \mathcal{A}_3 \quad &\Leftrightarrow \quad \prod_{i<j}(\alpha_i - \alpha_j) \in k \\ &\Leftrightarrow \quad -4p^3 - 27q^2 = \Big(\prod_{i<j}(\alpha_i - \alpha_j)\Big)^2 = a^2, a \in k \end{aligned}$$

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Def. A difference field (K, σ) is a field K together with an **automorphism** $\sigma : K \to K$. The constants are $K^\sigma = \{a \in K \mid \sigma(z) = a\}$.

σ-ring, σ-morphism, σ-subfield, σ-field extension, ... defined similarly.

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5. Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ a lattice $\ \ K = (\omega_1,\omega_2)$- periodic functions
   Let $\omega_3 \in \mathbb{C}, n\omega_3 \notin \Lambda\ \ \forall n \in \mathbb{N}$ and $\sigma(f(\omega)) = f(\omega + \omega_3)\ \ K^\sigma = \mathbb{C}$

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6. Let $(E, \oplus)$ be an elliptic curve, e.g., zeros of $y^2 = x^3 - ax - b$
   $K =$ function field of $E, \;\; \sigma(f(X)) = f(X \oplus \Omega), \; \Omega$ a nontorsion pt.,
   $K^\sigma = \mathbb{C}$. Note: 5. and 6. are the same.

Matrix Equations and Solutions

Let (K, σ) be a σ-field and g a solution of

$$\mathcal{L}(y) = \sigma^n(y) + a_{n-1}\sigma^{n-1}(y) + \cdots + a_0 y = 0,$$

with $a_0 \neq 0$, $a_i \in K$. Then, $Z := \begin{pmatrix} g \\ \sigma(g) \\ \vdots \\ \sigma^{n-1}(g) \end{pmatrix}$ satisfies $\sigma(Y) = A_{\mathcal{L}} Y$ with

$$A_{\mathcal{L}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & \cdots & -\frac{a_{n-1}}{a_n} \end{pmatrix} \in \mathrm{GL}_n(K).$$

We will consider matrix equations $\sigma(Y) = AY, \quad A \in \mathrm{GL}_n(K)$.

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Fact: If $U_1,\ U_2 \in \mathrm{GL}_n(K)$ are fund. solution matrices of $\sigma(Y) = AY$, then

$$U_1 = U_2 D$$

for some $D \in \mathrm{GL}_n(K^\sigma)$.

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Let $M =$ max σ-ideal in $K[Y, \frac{1}{\det(Y)}]$

$$R = K[Y, \frac{1}{\det(Y)}]/M = k[Z, \frac{1}{\det(Z)}] \quad = \sigma\text{-Picard-Vessiot Ring}$$

- M is radical $\Rightarrow$ R is reduced
- If $C = K^\sigma = \{c \in K \mid \sigma(c) = c\}$ is alg closed $\Rightarrow$ R is unique and $R^\sigma = C$

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Ex. $k = \mathbb{C} \ \ \sigma(y) = -y$

$$R = \mathbb{C}[y, \frac{1}{y}]/(y^2 - 1) = \mathbb{C}[y, \frac{1}{y}]/(y - 1) \oplus \mathbb{C}[y, \frac{1}{y}]/(y + 1)$$

R has zero divisors: $(y - 1)(y + 1) = 0$.

Galois group

Def. Let R be a Picard-Vessiot field extension for $\sigma(Y) = AY$ over K and let $C = K^{\sigma}$. The Galois group $G(R/K)$ of R over K is defined to be

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Let $U \in \mathrm{GL}_n(R)$ be a fund. sol. matrix and $\tau \in \mathrm{Gal}(R/K)$. Then

$$\sigma(\tau(U) = A\tau(U)$$

So $\tau(U)$ is a fund. sol. matrix and so $\exists\ [\tau]_U \in \mathrm{GL}_n(C)$ s.t. $\tau(U) = U[\tau]_U$.

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Fact: The map $\rho : \mathrm{Gal}(R/K) \to \mathrm{GL}_n(C)$ given by $\rho(\tau) = [\tau]_U$ is a group homomorphism whose image is a linear algebraic group, that is, there is a set of polynomials $P \subset C[x_{i,j}, \frac{1}{\det(x_{i,j})}]$ such that

$$\mathrm{Gal}(K_A|K) = \{g \in \mathrm{GL}_n(C) \mid p(g) = 0 \text{ for all } p \in P\}$$

Examples of linear algebraic groups

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3. $(C, +) = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in C\}$.
4. $(C^n, +) =$

$$\{\begin{pmatrix} A_1 & 0 & 0 & \ldots & 0 \\ 0 & A_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & A_n \end{pmatrix} \mid A_i = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}, a_i \in C\}$$

The linear algebraic subgroups of $(C^n, +)$ are the vector subspaces.

If G is a proper linear algebraic subgroup of $(C^n, +)$ then

$$G \subset \{(a_1, \ldots a_n) \mid c_1 a_1 + \ldots + c_n a_n = 0 \text{ for some } c_i \in C\}$$

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Ex. $K = \mathbb{C}$, $\sigma = \textit{identity}$.

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Ex.

$$\sigma(y) - y = f, f \in K \Leftrightarrow \sigma \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$
$$\phi \in \mathrm{Gal}(R/K) \Rightarrow \phi(y) = y + c_\phi, c_\phi \in C$$
$$\mathrm{Gal}_\sigma = (C, +) \text{ or } \{0\}$$

Main results of Galois theory

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There is also a *Galois correspondence* between linear algebraic subgroups of G and certain subfields of the "quotient field" of R.

Thm. (Roques 2007)

For $q \in \mathbb{C}$ with $|q| > 1$. Let $y_1(x), y_2(x)$ two linearly independent solutions of

$$y(q^2x) - \frac{2ax-2}{a^2x-1}y(qx) - \frac{x-1}{a^2x-q^2x}y(x) = 0$$

with $a \notin q^{\mathbb{Z}}$ and $a^2 \in q^{\mathbb{Z}}$. Then, $y_1(x), y_2(x), y_1(qx)$ are algebraically independent over $\mathbb{C}(x)$.

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  so $\tau(y^n) = c^n y^n = y^n$ so $y^n \in K$ and

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Proof: Use the Thm. with $n = 1$.

Let $K = \cup_{n=1}^{\infty} \mathbb{C}(x^{1/p^n}), L = \cup_{n=1}^{\infty} \mathbb{C}((x^{1/p^n}))\sigma(x) = x^p, b = -x$.

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- Apply σ and subtract:
  $\sigma(g) - g = \sigma(\sum_{i=0}^n c_i y_i) - \sum_{i=0}^n c_i y_i = \sum_{i=0}^n c_i(y_i + b_i) - \sum_{i=0}^n c_i y_i = \sum_{i=0}^n c_i b_i$

Differential transcendence

Thm. (Hölder 1887) The Gamma function is differentially transcendental over $\mathbb{C}(x)$.

Many other proofs: Bank, Bierberbach, Hilbert, Ostrowski, Rosenlicht, ...

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Idea of Galoisian proof:

- $\Gamma(x)$ is differentially algebraic over $\mathbb{C}(x)$ $\Leftrightarrow$ $\Phi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is.
- $\Phi(x+1) - \Phi(x) = \frac{1}{x} \Rightarrow \Phi'(x+1) - \Phi'(x) = \frac{-1}{x^2} \Rightarrow \ \dots$

  $\Rightarrow \Phi^{(n)}(x+1) - \Phi^{(n)}(x) = \frac{(-1)^n}{x^{n+1}}$
- Can characterize algebraic dependence among solutions of equations of the form $\sigma(z_i) - z_i = b_i$.

Def. A difference-differential field or $\sigma\delta$-field (K, σ, δ) is a difference field (K, σ) together with a derivation $\delta : K \to K$ commuting with σ.

$\delta(a + b) = \delta(a) + \delta(b), \ \delta(ab) = \delta(a)b + a\delta(b)$ and $\delta(\sigma(a)) = \sigma(\delta(a)$

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- $K = \mathbb{C}(x), \ \ \sigma(x) = x + 1, \ \ \delta = \frac{d}{dx}$
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- $K = \cup_{n=1}^{\infty}\mathbb{C}(x^{1/p^n})(\log x),\ \ \sigma(x^{1/p^n}) = x^{1/p^{n-1}}, \delta = x \log x \frac{d}{dx}$

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$$z = \sum_{n=0}^{\infty} B_n x^n \text{ satisfies } z(\frac{x}{x+1}) - xz(x) = 1$$

$B_n =$ number of partitions of $\{1, \ldots, n\}$ (Bell numbers)

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- Galois theory $\quad \Rightarrow \quad$ conclusion.

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Sequences

C - field

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$\mathcal{S} := \text{SEQ}/\sim \quad (a_0, a_1, \ldots) \sim (b_0, b_1, \ldots)$ if $a_i = b_i \ \forall i \gg 0$

$(\mathcal{S}, \sigma)$ is a difference ring

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$\sigma((a_0, a_1, a_2, \ldots)) = (a_1, a_2, \ldots)$ is a homomorphism, NOT injective

$\mathcal{S} := \text{SEQ}/\sim \quad (a_0, a_1, \ldots) \sim (b_0, b_1, \ldots)$ if $a_i = b_i \; \forall i \gg 0$

$(\mathcal{S}, \sigma)$ is a difference ring

$C \hookrightarrow \mathcal{S} \;\; c \mapsto (c, c, \ldots)$

$(C(x), x \mapsto x + 1) \hookrightarrow (\mathcal{S}, \sigma) \;\; f(x) \mapsto (f(0), f(1), \ldots)$

Sequences

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$\mathcal{S}$ is universal for $C(x)$:

For $\mathbb{C}(x) \subset \mathcal{S}$ and $A \in \mathrm{GL}_n(\mathbb{C}(x))$,

there exists a PV-ring R for $\sigma(Y) = AY$ with $R \subset \mathcal{S}$.

Def. w is the interlacing of $u = (u_0, u_1, \ldots)$ and $v = (v_0, v_1, \ldots)$ if

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Thm. Let $u, v, \in \mathcal{S}$ each satisfy a linear difference equation over $\mathbb{C}(x) \subset \mathcal{S}$.

- (Larson-Taft) If $uv = 0$ there exist $u_1, \ldots, u_t, v_0, \ldots, v_t \in \mathcal{S}$ such that u (resp. v) is the interlacing of the u_i (resp. v_i) and for all i either $u_i = 0$ or $v_i = 0$

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- (Conj. of Larson-Taft) If u is invertible in $\mathcal{S}$ and $1/u$ also satisfies a linear difference equation over $\mathbb{C}(x)$ then u is the interlacing of hypergeometric sequences u_i, i.e. $\sigma(u_i)/u_i \in \mathbb{C}(x)$.

General differential Galois theory of linear difference equations

The above Galois theory is good enough to determine differential properties of solutions of equations of the form

$$\sigma(y) - y = b \quad \text{or} \quad \sigma(y) = by.$$

For higher order equations, a Galois theory whose groups are differential algebraic groups has been developed (Hardouin - S. 2008).

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Ex. Let

$$a_n = \begin{cases} 1 \text{ if the binary rep. of } n \text{ contains no block of 0's of odd length} \\ 0 \text{ otherwise} \end{cases}$$

be the Baum-Sweet sequence. The generating function $f_{BS} = \sum_{n\geq 0} a_n x^n$ satisfies

$$f_{BS}(x^4) + xf_{BS}(x^2) - f_{BS}(x) = 0$$

Thm. (Dreyfus-Hardouin-Roques 2015) The series $f_{BS}(x^2)$ and $f_{BS}(x)$ and all their derivatives are algebraically independent over $\mathbb{C}(x)$, i.e., these series ore differentially independent.

For general in information on the Galois Theory of Difference equations:

Galois Theories of Linear Difference Equations: An Introduction

Mathematical Surveys and Monographs, Vol. 211, AMS, 2016, 171 pages

- Algebraic and Algorithmic Aspects of Linear Difference Equations - S.
- Galoisian Approach to Differential Transcendence- Hardouin
- Analytic Study of q-Difference Equations - Sauloy