Unlabelled planar graphs and symmetries of triangulations

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joint work Mihyun Kang

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Labelled planar graphs

planar graphs on $[n] = \{1, \ldots, n\}$

- arbitrary # edges [Giménez, Noy 09]
- $m = an$ edges, $a \in (1, 3)$ [Giménez, Noy 09]
- $m \leq (1 + o(1))n$ [Kang, Łuczak 12]
- 2-conn., $m = an$, $a \in (1, 3)$ [Bender, Gao, Wormald 02]
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Cubic planar graphs $\longrightarrow$ sparse random planar graphs
$\longrightarrow$ phase transitions for rand. pl. gr.

Unlabelled planar graphs

Unlabelled planar graphs

Question

Asymptotic number of unlabelled cubic planar simple graphs?

Constructive decomposition

Labelled graphs:

$$\text{planar} \quad \overset{\text{components}}{\longleftrightarrow} \quad \text{pl. connected}$$
$$\overset{\text{blocks}}{\longleftrightarrow} \quad \text{pl. 2-connected}$$
$$\overset{\text{Tutte-decomposition}}{\longleftrightarrow} \quad \text{pl. 3-connected}$$

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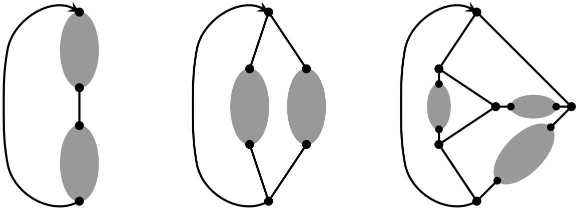

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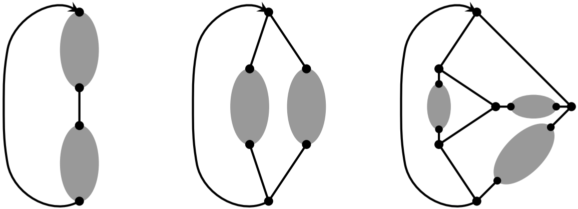

$\longrightarrow$ Equations for generating functions

Generating functions vs. cycle index sums

Example: $\mathcal{F} = \mathcal{G}$ with edges replaced by $\mathcal{H}$

Generating functions (GF):

$$F(x, y) = G\big(x, H(x, y)\big)$$

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Solution: Cycle index sums (CIS)

Information about sizes of orbits $\forall f \in \mathrm{Aut}(G)$

Replacements similar to GF

Cycle index sums

$$\left.\begin{matrix} i_n \\ j_n \\ k_n \end{matrix}\right\} \text{ orbits of length } n \text{ of } \left\{\begin{matrix} \text{vx's,} \\ \text{edges (orientation preserving),} \\ \text{edges (orientation reversing),} \end{matrix}\right.$$

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$$z(f) = a_1^{i_1} a_2^{i_2} \cdots b_1^{j_1} b_2^{j_2} \cdots c_1^{k_1} c_2^{k_2} \cdots$$
$$Z_{\mathcal{G}} = \sum_{G \in \mathcal{G}} \frac{1}{|\mathrm{Aut}(G)|} \sum_{f \in \mathrm{Aut}(G)} z(f)$$

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Typical replacement: $b_i \to Z_{\mathcal{H}}[a_j \to a_{ij}, b_j \to b_{ij}, c_j \to c_{ij}]$

Constructive decomposition for unlabelled graphs

Construction (Chapuy, Fusy, Kang, Shoilekova 08)

Functional equations for GF (labelled) or CIS (unlabelled) reducing a graph class $\mathcal{G}$ to the class of 3-connected graphs in $\mathcal{G}$.

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- Base cases (amongst others):
  - vx-rooted 3-conn. unl. cub. pl.
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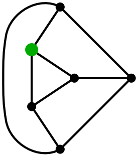

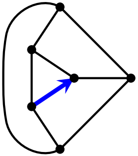

From cubic graphs to triangulations

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Every 3-conn. planar graph has a unique embedding up to orientation of the sphere.

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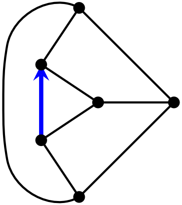
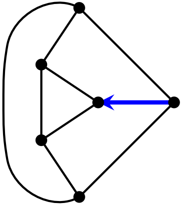
$$\begin{array}{ccc} \text{3-conn. cubic pl. graphs} & \overset{\text{Whitney}}{\longleftrightarrow} & \text{3-conn. cubic maps} \\ \text{vx-rooted/edge-rooted} & & \updownarrow \text{dual} \\ & & \text{3-conn. triangulations} \\ & & \text{face-rooted/edge-rooted} \end{array}$$

A problem

Transition 'graphs $\longleftrightarrow$ maps' is not unique.



two embeddings



one embedding

Tutte rooting

Orient an edge & fix one side as 'left' (Tutte rooting)

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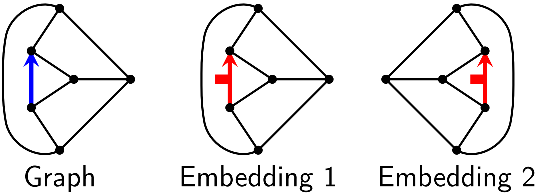

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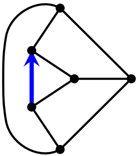
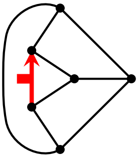
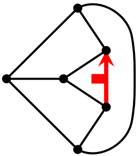


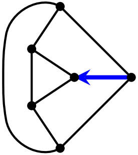
Graph



Embedding 1



Embedding 2



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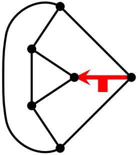

Embedding

Tutte rooting

$$\begin{aligned}\text{Edge-rooted graph} &\longleftrightarrow \text{1 or 2 Tutte-rooted maps}\\ &\longleftrightarrow \text{1 or 2 Tutte-rooted triangulations}\end{aligned}$$

Edge-rooted graph $\longleftrightarrow$ 1 or 2 Tutte-rooted maps
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Vx-rooted graph $\longleftrightarrow$ up to 6 Tutte-rooted maps/triangulations

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Problem

Describe the triangulations with a given set of symmetries.

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Problem

Describe the triangulations with a given set of symmetries.

This talk: Face-rooted triangulations.

Unlabelled triangulations

Notation

- *Cells of dim 0,1,2: vertices, edges, and faces*
- $\mathrm{Aut}(r, T)$*: automorphisms of* T *that fix the root face* r

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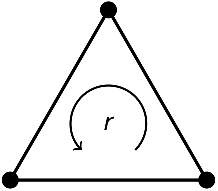
- $\mathrm{Aut}(r, T)$*: automorphisms of T that fix the root face r*

Properties of automorphisms

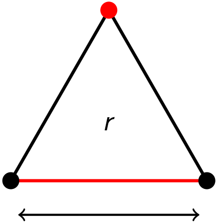
- $\varphi \in \mathrm{Aut}(r, T)$: uniquely determined by its action on the cells incident with r
- $\mathrm{Aut}(r, T)$: subgroup of the dihedral group D_3

Two types of non-trivial automorphisms:

Rotations
(no invariant cells adj. to r)

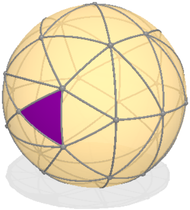


Reflections
(two invariant cells opp. at r)



Theorem (Tutte 62)

Invariant cells of a reflection: cyclic seq. $C = (c_1, \ldots, c_\ell)$ *s.t.* $\forall i$ c_{i-1} *&* c_{i+1} *lie opposite at* c_i.



Triangulations with reflective symmetries

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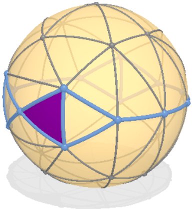

Definition

Girdle G: vx's & edges in C and on bds of faces in C

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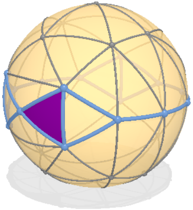

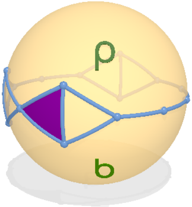

Definition

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$\Longrightarrow$ induces two near-triangulations ρ

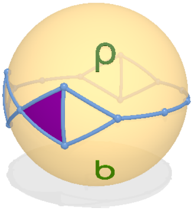
Triangulations with reflective symmetries

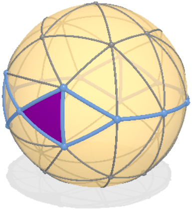
Theorem (K-S 17+)

Triangulations with reflective symmetry $\iff$ *obtained by choosing*

- *a girdle* G *and*
- *a near-triangulation* ρ *with forbidden chords*

and pasting ρ *into both sides of* G. *This is a 1-to-2 correspondence.*





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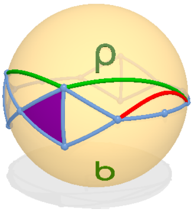
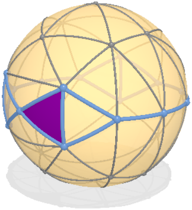
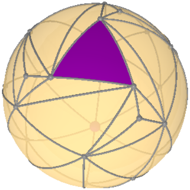




Lemma (Tutte 62)

Rotative automorphism φ: unique invariant cell $c \neq r$.



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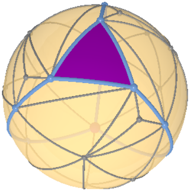

Definition

Spindle S: union of paths $P, \varphi(P), \varphi^2(P)$ from r to c

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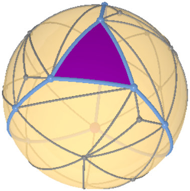

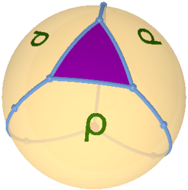

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$\Longrightarrow$ induces m isomorphic near-triangulations ρ

Theorem (K-S 17+)

Triangulations with rotative symmetry $\Longleftrightarrow$ obtained by choosing

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and pasting ρ into each segment of S.

Triangulations with rotative symmetries

Theorem (K-S 17+)

Triangulations with rotative symmetry $\iff$ *obtained by choosing*

- *a spindle* S *and*
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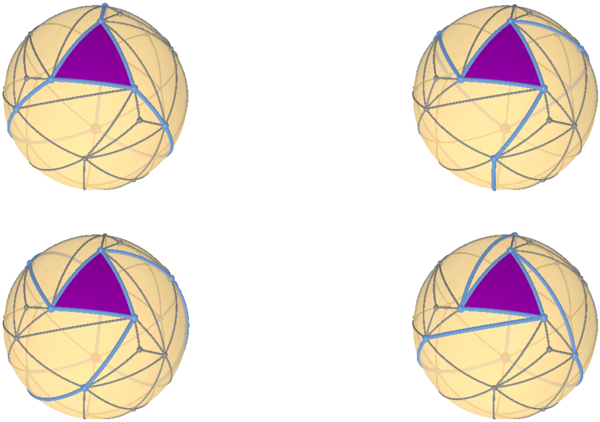
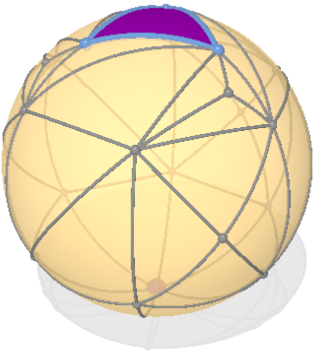
But: How many possibilities for a given triangulation?

Different spindles & near-triangulations for the same triangulation:



Triangulations with rotative symmetries

Idea:
- Construct paths simultaneously, recursively from r to c
- Always choose leftmost possible edges



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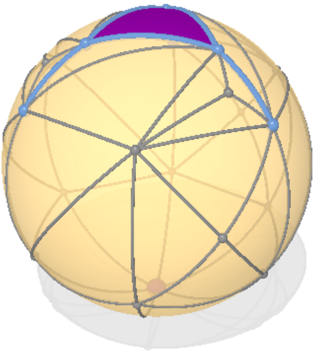

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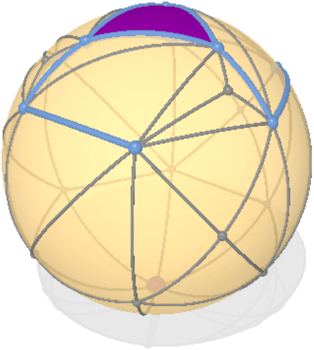

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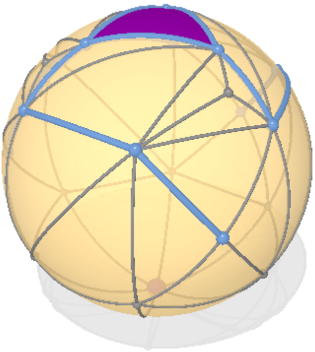

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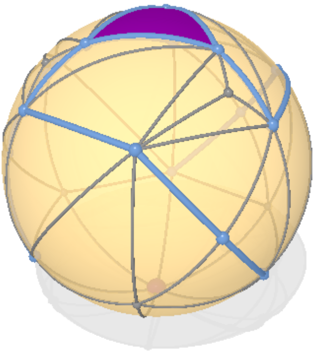

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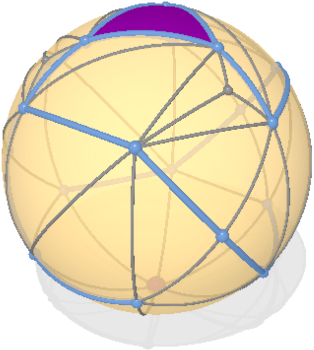

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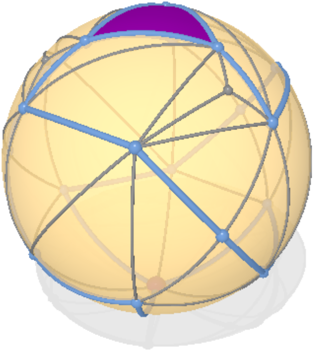

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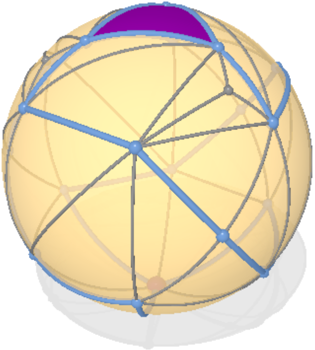
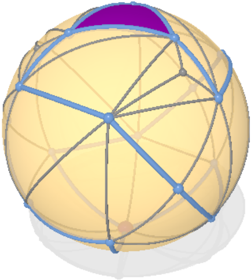
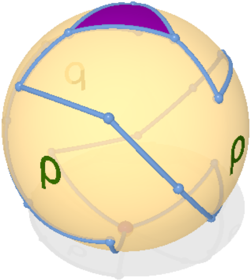


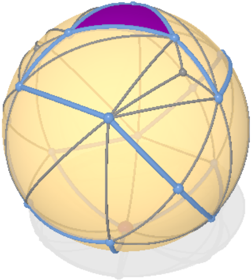
Leftmost spindle

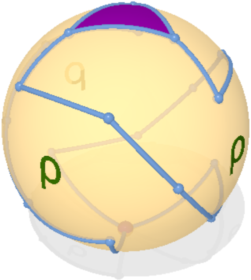
Divides triangulation into 3 isomorphic near-triangulations.





Divides triangulation into 3 isomorphic near-triangulations.





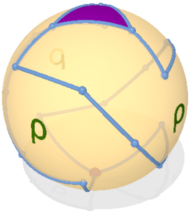
Near-triangulations have no ‘forwards chords’

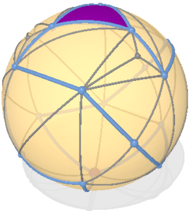
Theorem (K-S 17+)

Triangulations with rotative symmetry $\Longleftrightarrow$ obtained by choosing

- *a (leftmost) spindle S and*
- *a near-triangulation ρ without forward chords*

and pasting ρ into each face of S. This is a 1-to-1 correspondence.





Reminder

- If both reflections and rotations, then $\mathrm{Aut}(r, T) = D_3$.
- 3 reflections and 2 rotations.
- Reflection $\longrightarrow$ girdle.
- Rotation $\longrightarrow$ unique invariant cell $c \neq r$.

Reflective and rotative symmetries

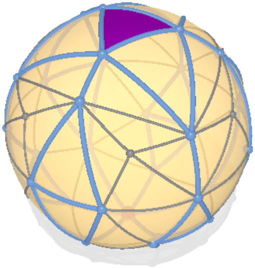
Reminder

- If both reflections and rotations, then $\mathrm{Aut}(r, T) = D_3$.
- 3 reflections and 2 rotations.
- Reflection $\longrightarrow$ girdle.
- Rotation $\longrightarrow$ unique invariant cell $c \neq r$.

- c is the same for all rotations.
- Girdles intersect only in r and c.
- All girdles are isomorphic.

Definition (K-S 17+)

Skeleton S: union of the 3 girdles



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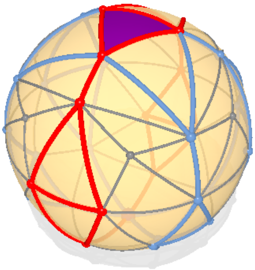

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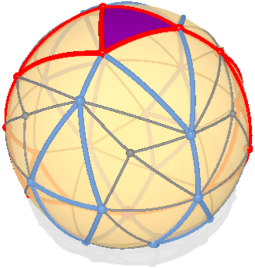

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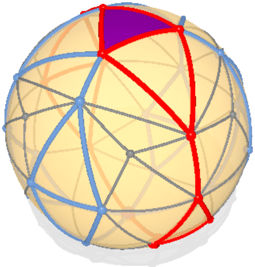
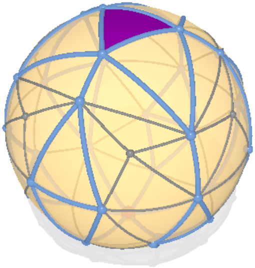


Definition (K-S 17+)

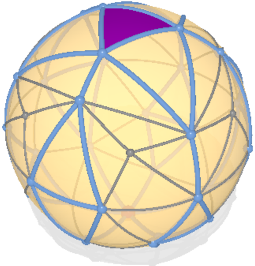
Skeleton S: union of the 3 girdles

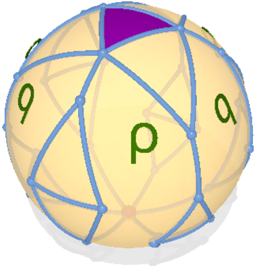


Definition (K-S 17+)

Skeleton S: union of the 3 girdles

$\Longrightarrow$ induces 6 isomorphic near-triangulations ρ



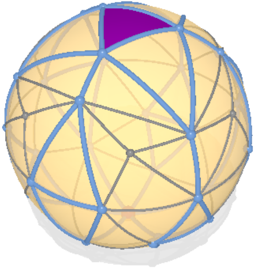


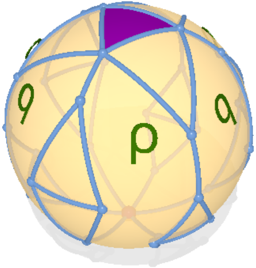
Reflective and rotative symmetries

Definition (K-S 17+)

Skeleton S: union of the 3 girdles

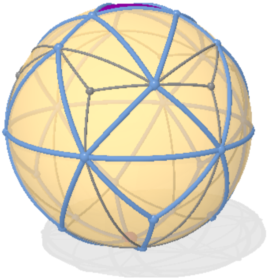
$\Longrightarrow$ induces 6 isomorphic ~~near-triangulations ρ~~

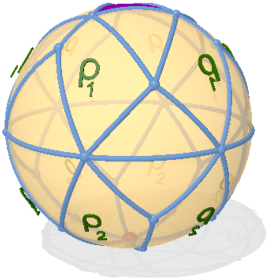




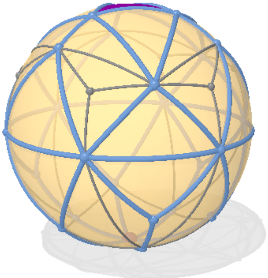
Reflective and rotative symmetries

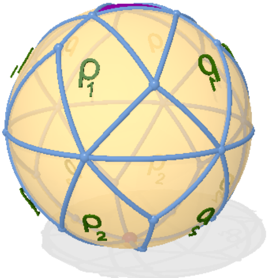
Girdles can touch:





Girdles can touch:





~~isomorphic near-triangulations ρ~~

near-triangulations $\rho_1, \ldots, \rho_\ell$, each appearing 6 times

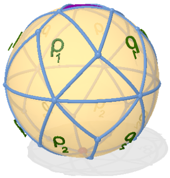
Reflective and rotative symmetries

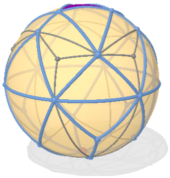
Theorem (K-S 17+)

Triangulations with both symmetries $\Longleftrightarrow$ *obtained by choosing*

- *a skeleton* S *and*
- *near-triangulations* $\rho_1, \dots, \rho_\ell$ *with forbidden chords*

and pasting $\rho_1, \dots, \rho_\ell$ *into the faces of* S. *This is a 1-to-2 correspondence.*

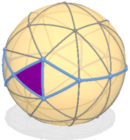


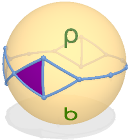


Summary

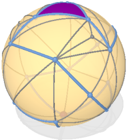
Symmetries of triangulations rooted at a face

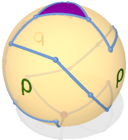
- Reflective:
  Girdle



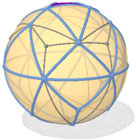


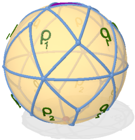
- Rotative:
  (Leftmost) spindle





- Reflective & rotative:
  Skeleton





Summary and Outlook

Roadmap:

1. Decomposition scheme for near-triangulations;
2. Cycle index sums for near-triangulations;
3. CIS for triangulations;
4. CIS for cubic 3-conn. maps;
5. CIS for cubic 3-conn. planar graphs;
6. CIS for cubic planar graphs;
7. Asymptotic numbers.