Properties of Laurent coefficients of multivariate rational functions

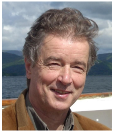
Workshop on Computer Algebra in Combinatorics
Erwin Schroedinger Institute

Armin Straub

November 14, 2017

University of South Alabama

includes joint work with



Frits Beukers
(Utrecht University)

Marc Houben
(Utrecht University)

and

Wadim Zudilin
(University of Newcastle/
Radboud Universiteit)

Goal of this talk

We introduce and advertise two questions about rational functions like

$$\frac{1}{1-(x_1+x_2+x_3)+4x_1x_2x_3} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3_{\geqslant 0}} A(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}.$$

Goal of this talk

We introduce and advertise two questions about rational functions like

$$\frac{1}{1-(x_1+x_2+x_3)+4x_1x_2x_3} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3_{\geqslant 0}} A(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}.$$

Q When has a rational function the **Gauss property**?

That is, when do the following congruences hold?

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^r}$$

Goal of this talk

We introduce and advertise two questions about rational functions like

$$\frac{1}{1-(x_1+x_2+x_3)+4x_1x_2x_3} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3_{\geqslant 0}} A(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}.$$

Q When has a rational function the **Gauss property**?

That is, when do the following congruences hold?

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^r}$$

Q When is a rational function **positive**?

That is, when is $A(\boldsymbol{n}) > 0$ for all $\boldsymbol{n}$?

Goal of this talk

We introduce and advertise two questions about rational functions like

$$\frac{1}{1-(x_1+x_2+x_3)+4x_1x_2x_3} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3_{\geqslant 0}} A(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}.$$

Q When has a rational function the **Gauss property**?

That is, when do the following congruences hold?

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^r}$$

Q When is a rational function **positive**?

That is, when is $A(\boldsymbol{n}) > 0$ for all $\boldsymbol{n}$?

In both cases, we will wonder about an explicit characterization.

These are not conjectures because our evidence is limited. Computer algebra!

Goal of this talk

We introduce and advertise two questions about rational functions like

$$\frac{1}{1-(x_1+x_2+x_3)+4x_1x_2x_3} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3_{\geqslant 0}} A(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}.$$

EG Here, the **diagonal coefficients** are the Franel numbers

$$A(n,n,n) = \sum_{k=0}^{n} \binom{n}{k}^3.$$

- As seen in previous talks, simple multivariate generating functions can be enormously useful, for instance, in computing asymptotics.
- Time permitting, more on Apéry-like numbers later...

I

Gauss congruences

The classical Gauss congruence

THM
Fermat

$$a^p \equiv a \pmod{p}$$

if p is prime.

The classical Gauss congruence

THM **Fermat**

$$a^p \equiv a \pmod{p}$$

if p is prime.

THM **Euler**

$$a^{\phi(m)} \equiv 1 \pmod{m}$$

if a is coprime to m.

The classical Gauss congruence

THM Fermat

$$a^p \equiv a \pmod{p}$$

if p is prime.

THM Euler

$$a^{\phi(m)} \equiv 1 \pmod{m}$$

if a is coprime to m.

THM Gauss

$$\sum_{d|m} \mu(\tfrac{m}{d}) a^d \equiv 0 \pmod{m}$$

Möbius function: $\mu(n) = (-1)^{\#\text{ of } p|n}$ if n is square-free, $\mu(n) = 0$ else

The classical Gauss congruence

THM Fermat

$$a^p \equiv a \pmod{p}$$

if p is prime.

THM Euler

$$a^{\phi(m)} \equiv 1 \pmod{m}$$

if a is coprime to m.

THM Gauss

$$\sum_{d|m} \mu(\tfrac{m}{d}) a^d \equiv 0 \pmod{m}$$

Möbius function: $\mu(n) = (-1)^{\#\text{ of } p|n}$ if n is square-free, $\mu(n) = 0$ else

EG If $m = p^r$ then only $d = p^r$, $d = p^{r-1}$ contribute, and we get

$$a^{p^r} \equiv a^{p^{r-1}} \pmod{p^r}.$$

DEF $a(n)$ satisfies the **Gauss congruences** if, for all primes p,

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r}.$$

Equivalently, $\displaystyle\sum_{d|m} \mu(\tfrac{m}{d}) a(d) \equiv 0 \pmod{m}$.

DEF $a(n)$ satisfies the **Gauss congruences** if, for all primes p,

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r}.$$

Equivalently, $\displaystyle\sum_{d|m} \mu(\tfrac{m}{d}) a(d) \equiv 0 \pmod{m}$.

EG

- $a(n) = a^n$

Gauss congruences

DEF $a(n)$ satisfies the **Gauss congruences** if, for all primes p,

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r}.$$

Equivalently, $$\sum_{d|m} \mu(\tfrac{m}{d}) a(d) \equiv 0 \pmod{m}.$$

EG

- $a(n) = a^n$
- $a(n) = L_n$ **Lucas numbers**: $\begin{gathered} L_{n+1} = L_n + L_{n-1} \\ L_0 = 2, L_1 = 1 \end{gathered}$

Gauss congruences

DEF $a(n)$ satisfies the **Gauss congruences** if, for all primes p,

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r}.$$

Equivalently, $\sum_{d|m} \mu(\tfrac{m}{d}) a(d) \equiv 0 \pmod{m}$.

EG

- $a(n) = a^n$
- $a(n) = L_n$ **Lucas numbers**: $\begin{array}{c} L_{n+1} = L_n + L_{n-1} \\ L_0 = 2, L_1 = 1 \end{array}$
- $a(n) = D_n$ **Delannoy numbers**: $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$

DEF $a(n)$ satisfies the **Gauss congruences** if, for all primes p,

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r}.$$

Equivalently, $$\sum_{d|m} \mu(\tfrac{m}{d}) a(d) \equiv 0 \pmod{m}.$$

EG

- $a(n) = a^n$
- $a(n) = L_n$ **Lucas numbers**: $\begin{array}{c} L_{n+1} = L_n + L_{n-1} \\ L_0 = 2, L_1 = 1 \end{array}$
- $a(n) = D_n$ **Delannoy numbers**: $D_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}$

- Later, we allow $a(n) \in \mathbb{Q}$. If the Gauss congruences hold for all but finitely many p, we say that the sequence (or its GF) has the **Gauss property**.
- Similarly, for multivariate sequences $a(\boldsymbol{n})$, we require

$$a(\boldsymbol{m}p^r) \equiv a(\boldsymbol{m}p^{r-1}) \pmod{p^r}.$$

More sequences satisfying Gauss congruences

$$a(mp^r) \equiv a(mp^{r-1}) \qquad (\operatorname{mod} p^r) \tag{G}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{"points of period } n\text{"}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

In fact, up to a positivity condition, (G) characterizes realizability.

More sequences satisfying Gauss congruences

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \tag{G}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

In fact, up to a positivity condition, (G) characterizes realizability.

- $a(n) = \operatorname{trace}(M^n)$ Jänichen '21, Schur '37; also: Arnold, Zarelua

where M is an integer matrix

More sequences satisfying Gauss congruences

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(G)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

  $$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

  Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

  In fact, up to a positivity condition, (G) characterizes realizability.

- $a(n) = \operatorname{trace}(M^n)$ Jänichen '21, Schur '37; also: Arnold, Zarelua

  where M is an integer matrix

- (G) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

Minton's theorem

THM Minton, 2014 $f \in \mathbb{Q}(x)$ has the Gauss property if and only if f is a $\mathbb{Q}$-linear combination of functions $xu'(x)/u(x)$, with $u \in \mathbb{Z}[x]$.

Minton's theorem

THM Minton, 2014 — $f \in \mathbb{Q}(x)$ has the Gauss property if and only if f is a $\mathbb{Q}$-linear combination of functions $xu'(x)/u(x)$, with $u \in \mathbb{Z}[x]$.

- If $u(x) = \prod_{i=1}^s (1 - \alpha_i x)$ then

$$x \frac{u'(x)}{u(x)} = -\sum_{i=1}^s \frac{\alpha_i x}{1 - \alpha_i x} = s - \sum_{i=1}^s \frac{1}{1 - \alpha_i x}.$$

Minton's theorem

THM Minton, 2014: $f \in \mathbb{Q}(x)$ has the Gauss property if and only if f is a $\mathbb{Q}$-linear combination of functions $xu'(x)/u(x)$, with $u \in \mathbb{Z}[x]$.

- If $u(x) = \prod_{i=1}^s (1 - \alpha_i x)$ then

$$x \frac{u'(x)}{u(x)} = -\sum_{i=1}^s \frac{\alpha_i x}{1 - \alpha_i x} = s - \sum_{i=1}^s \frac{1}{1 - \alpha_i x}.$$

- Assuming the α_i are distinct,

$$\sum_{i=1}^s \frac{1}{1 - \alpha_i x} = \sum_{n \geqslant 0} \left(\sum_{i=1}^s \alpha_i^n \right) x^n = \sum_{n \geqslant 0} \operatorname{trace}(M^n) x^n,$$

where M is the companion matrix of $\prod_{i=1}^s (x - \alpha_i) = x^s u(1/x)$.

THM Minton, 2014 $f \in \mathbb{Q}(x)$ has the Gauss property if and only if f is a $\mathbb{Q}$-linear combination of functions $xu'(x)/u(x)$, with $u \in \mathbb{Z}[x]$.

- If $u(x) = \prod_{i=1}^{s}(1-\alpha_i x)$ then

$$x\frac{u'(x)}{u(x)} = -\sum_{i=1}^{s}\frac{\alpha_i x}{1-\alpha_i x} = s - \sum_{i=1}^{s}\frac{1}{1-\alpha_i x}.$$

- Assuming the α_i are distinct,

$$\sum_{i=1}^{s}\frac{1}{1-\alpha_i x} = \sum_{n\geqslant 0}\left(\sum_{i=1}^{s}\alpha_i^n\right)x^n = \sum_{n\geqslant 0}\operatorname{trace}(M^n)x^n,$$

where M is the companion matrix of $\prod_{i=1}^{s}(x-\alpha_i) = x^s u(1/x)$.

- Minton: No new C-finite sequences with the Gauss property!
- Can we generalize from C-finite towards D-finite?

The multivariate case

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \tag{D}$$

has the Gauss property.

Interesting detail: true for any of the different Laurent expansions of multivariate rational functions

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \tag{D}$$

has the Gauss property.

Interesting detail: true for any of the different Laurent expansions of multivariate rational functions

EG Consider $Q = 1 - x - y - z + 4xyz$:

$$f_1 = Q \quad \Longrightarrow \quad (\mathrm{D}) = \frac{-x + 4xyz}{Q}$$

The multivariate case

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \qquad \text{(D)}$$

has the Gauss property.

Interesting detail: true for any of the different Laurent expansions of multivariate rational functions

EG Consider $Q = 1 - x - y - z + 4xyz$:

$$f_1 = Q \quad \Longrightarrow \quad \text{(D)} = \frac{-x + 4xyz}{Q}$$

$$f_1 = Q, \quad f_2 = 1 - 4yz \quad \Longrightarrow \quad \text{(D)} = \frac{4xyz}{Q}$$

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \tag{D}$$

has the Gauss property.

Interesting detail: true for any of the different Laurent expansions of multivariate rational functions

EG Consider $Q = 1 - x - y - z + 4xyz$:

$$f_1 = Q \quad \Longrightarrow \quad \text{(D)} = \frac{-x + 4xyz}{Q}$$

$$f_1 = Q, \quad f_2 = 1 - 4yz \quad \Longrightarrow \quad \text{(D)} = \frac{4xyz}{Q}$$

In particular, $\dfrac{1}{1 - x - y - z + 4xyz}$ has the Gauss property.

There is nothing special about 4.

The multivariate case

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \tag{D}$$

has the Gauss property.

THM BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q is linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

- Here, $N(Q)$ is the Newton polytope of Q.
- In this case, $N(Q) = \operatorname{supp}(Q) \subseteq \{0,1\}^n$.

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \tag{D}$$

has the Gauss property.

THM BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q is linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

- Here, $N(Q)$ is the Newton polytope of Q.
- In this case, $N(Q) = \operatorname{supp}(Q) \subseteq \{0,1\}^n$.

PROP BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}^{\pm 1}]$.

If P/Q has the Gauss property, then $N(P) \subseteq N(Q)$.

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \qquad \text{(D)}$$

has the Gauss property.

Q BHS

Suppose $f \in \mathbb{Q}(\boldsymbol{x})$ has the Gauss property. Can it be written as a $\mathbb{Q}$-linear combination of functions of the form (D)?

- Yes, for $n = 1$, by Minton's theorem.

The multivariate case

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \tag{D}$$

has the Gauss property.

Q BHS

Suppose $f \in \mathbb{Q}(\boldsymbol{x})$ has the Gauss property. Can it be written as a $\mathbb{Q}$-linear combination of functions of the form (D)?

- Yes, for $n = 1$, by Minton's theorem.
- Yes, for $f = P/Q$ with Q linear in all, or all but one, variables.

The multivariate case

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \tag{D}$$

has the Gauss property.

Q BHS

Suppose $f \in \mathbb{Q}(\boldsymbol{x})$ has the Gauss property. Can it be written as a $\mathbb{Q}$-linear combination of functions of the form (D)?

- Yes, for $n = 1$, by Minton's theorem.
- Yes, for $f = P/Q$ with Q linear in all, or all but one, variables.
- Yes, for $f = P/Q$ with Q in two variables and total degree 2.

The multivariate case

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \tag{D}$$

has the Gauss property.

Q BHS

Suppose $f \in \mathbb{Q}(\boldsymbol{x})$ has the Gauss property. Can it be written as a $\mathbb{Q}$-linear combination of functions of the form (D)?

- Yes, for $n = 1$, by Minton's theorem.
- Yes, for $f = P/Q$ with Q linear in all, or all but one, variables.
- Yes, for $f = P/Q$ with Q in two variables and total degree 2.

EG

Can $\dfrac{x(x + y + y^2 + 2xy^2)}{1 + 3x + 3y + 2x^2 + 2y^2 + xy - 2x^2y^2}$ be written in that form?

Application: Delannoy numbers

THM BHS Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q is linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

Application: Delannoy numbers

THM BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q is linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

EG Beukers, Houben, S 2017

The **Delannoy numbers** D_{n_1,n_2} are characterized by

$$\frac{1}{1-x-y-xy} = \sum_{n_1,n_2=0}^{\infty} D_{n_1,n_2} x^{n_1} y^{n_2}.$$

Application: Delannoy numbers

THM BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q is linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

EG Beukers, Houben, S 2017

The **Delannoy numbers** D_{n_1,n_2} are characterized by

$$\frac{1}{1-x-y-xy} = \sum_{n_1,n_2=0}^{\infty} D_{n_1,n_2} x^{n_1} y^{n_2}.$$

By the theorem, the following have the Gauss property:

$$\frac{N}{1-x-y-xy} \quad \text{with } N \in \{1, x, y, xy\}$$

Application: Delannoy numbers

THM BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q is linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

EG Beukers, Houben, S 2017

The **Delannoy numbers** D_{n_1,n_2} are characterized by

$$\frac{1}{1-x-y-xy} = \sum_{n_1,n_2=0}^{\infty} D_{n_1,n_2} x^{n_1} y^{n_2}.$$

By the theorem, the following have the Gauss property:

$$\frac{N}{1-x-y-xy} \quad \text{with } N \in \{1, x, y, xy\}$$

In other words, for $\boldsymbol{\delta} \in \{0,1\}^2$,

$$D_{\boldsymbol{m}p^r - \boldsymbol{\delta}} \equiv D_{\boldsymbol{m}p^{r-1} - \boldsymbol{\delta}} \pmod{p^r}.$$

II

Positivity

Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1, \ldots, n_d} > 0$ for all indices.

Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1, \ldots, n_d} > 0$ for all indices.

EG $\frac{1}{1-x}$ and $\frac{1}{(1-x)(1-y)}$ are positive.

Positivity of rational functions

- A rational function
$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$
is **positive** if $a_{n_1, \ldots, n_d} > 0$ for all indices.

EG $\frac{1}{1-x}$ and $\frac{1}{(1-x)(1-y)}$ are positive.

EG Szegő 1933

$$\frac{1}{(1-x)(1-y)+(1-y)(1-z)+(1-z)(1-x)}$$ is positive.

- Szegő's proof builds on an integral of a product of Bessel functions.
"the used tools, however, are disproportionate to the simplicity of the statement"
- Elementary proof by Kaluza ('33)
- Askey–Gasper ('72) use integral of product of Legendre functions.
- Ismail–Tamhankar ('79) systematize Kaluza's approach by using MacMahon's Master Theorem.
- S ('08): simple proof using a positivity-preserving operator

Historical motivation

$$\frac{1}{(1-x)(1-y)+(1-y)(1-z)+(1-z)(1-x)} = \sum_{k,m,n} A(k,m,n) x^k y^m z^n$$

- Friedrichs and Lewy conjectured positivity of $A(k,m,n)$.
- Wanted to show convergence of finite difference approximations to

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}+\frac{\partial}{\partial x}\frac{\partial}{\partial z}+\frac{\partial}{\partial y}\frac{\partial}{\partial z}\right)u(x,y,z)=0,$$

which transforms to the 2D wave equation.

Historical motivation

$$\frac{1}{(1-x)(1-y)+(1-y)(1-z)+(1-z)(1-x)} = \sum_{k,m,n} A(k,m,n) x^k y^m z^n$$

- Friedrichs and Lewy conjectured positivity of $A(k,m,n)$.
- Wanted to show convergence of finite difference approximations to

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\frac{\partial}{\partial z}\right) u(x,y,z) = 0,$$

  which transforms to the 2D wave equation.
- With $\partial/\partial x$ replaced by Δ_k, $\Delta a(k) = a(k) - a(k-1)$

$$(\Delta_k \Delta_m + \Delta_k \Delta_n + \Delta_m \Delta_n) A(k,m,n) = 0.$$

Generalizations

- Szegő also showed positivity of (and extension to any # of variables)

$$\frac{1}{\sum_{i=1}^{4}\prod_{j\neq i}(1-x_j)} = \frac{1}{(1-x_2)(1-x_3)(1-x_4)+\cdots+(1-x_1)(1-x_2)(1-x_3)}$$

Generalizations

- Szegő also showed positivity of (and extension to any # of variables)

$$\frac{1}{\sum_{i=1}^4 \prod_{j\neq i}(1-x_j)} = \frac{1}{(1-x_2)(1-x_3)(1-x_4)+\cdots+(1-x_1)(1-x_2)(1-x_3)}$$

- The Lewy–Askey problem asks for positivity of

$$\frac{1}{\sum\limits_{1\leqslant i<j\leqslant 4}(1-x_i)(1-x_j)} = \frac{1}{(1-x_1)(1-x_2)+\cdots+(1-x_3)(1-x_4)}.$$

Generalizations

- Szegő also showed positivity of (and extension to any # of variables)

$$\frac{1}{\sum_{i=1}^4 \prod_{j\neq i}(1-x_j)} = \frac{1}{(1-x_2)(1-x_3)(1-x_4)+\cdots+(1-x_1)(1-x_2)(1-x_3)}$$

- The Lewy–Askey problem asks for positivity of

$$\frac{1}{\sum\limits_{1\leqslant i<j\leqslant 4}(1-x_i)(1-x_j)} = \frac{1}{(1-x_1)(1-x_2)+\cdots+(1-x_3)(1-x_4)}.$$

- Non-negativity proved by a very general result of Scott–Sokal ('13):
  - $\frac{1}{\det\left(\sum(1-x_i)A_i\right)}$ is non-negative if $A_i \geqslant 0$ are hermitian matrices.
  - For the Lewy–Askey problem:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & e^{-i\pi/3} \\ e^{i\pi/3} & 1 \end{bmatrix}.$$

Generalizations

- Szegő also showed positivity of (and extension to any # of variables)

$$\frac{1}{\sum_{i=1}^{4}\prod_{j\neq i}(1-x_j)} = \frac{1}{(1-x_2)(1-x_3)(1-x_4)+\cdots+(1-x_1)(1-x_2)(1-x_3)}$$

- The Lewy–Askey problem asks for positivity of

$$\frac{1}{\sum\limits_{1\leqslant i<j\leqslant 4}(1-x_i)(1-x_j)} = \frac{1}{(1-x_1)(1-x_2)+\cdots+(1-x_3)(1-x_4)}.$$

- Non-negativity proved by a very general result of Scott–Sokal ('13):
  - $\dfrac{1}{\det\left(\sum(1-x_i)A_i\right)}$ is non-negative if $A_i \geqslant 0$ are hermitian matrices.
  - For the Lewy–Askey problem:

$$A_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad A_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \quad A_3 = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \quad A_4 = \begin{bmatrix}1 & e^{-i\pi/3}\\ e^{i\pi/3} & 1\end{bmatrix}.$$

Q $_{2\leqslant r\leqslant n}$ $e_r(1-\boldsymbol{x})^{-\beta}$ in n variables positive iff $\beta \geqslant (n-r)/2$ (or $\beta = 0$)?

With complete monotonicity of $e_r(\boldsymbol{x})^{-\beta}$, this is a conjecture of Scott-Sokal ('13).

Multivariate asymptotics?

Preserving positivity

- Positivity of the Askey–Gasper rational function

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

$$\frac{1}{1-(x+y+z)+4xyz}$$

Preserving positivity

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

- Positivity of the Askey–Gasper rational function

$$\frac{1}{1-(x+y+z)+4xyz}$$

implies positivity, for any $\varepsilon > 0$, of

$$\frac{1}{1-(x+y+z)+(4-\varepsilon)xyz}$$

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

- Positivity of the Askey–Gasper rational function

$$\frac{1}{(1-(x+y+z)+4xyz)^{\beta}}$$

implies positivity, for any $\varepsilon > 0$, of

for $\beta \geqslant (\sqrt{17}-3)/2 \approx 0.56$

$$\frac{1}{1-(x+y+z)+(4-\varepsilon)xyz}$$

Preserving positivity

- Positivity of the Askey–Gasper rational function

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

$$\frac{1}{(1-(x+y+z)+4xyz)^{\beta}}$$

implies positivity, for any $\varepsilon > 0$, of for $\beta \geqslant (\sqrt{17}-3)/2 \approx 0.56$

$$\frac{1}{1-(x+y+z)+(4-\varepsilon)xyz}$$

- If $F(x_1, \ldots, x_n)$ is positive, so is, for $0 \leqslant p \leqslant 1$,

Gillis–Reznick–Zeilberger '83
S '08
Kauers–Zeilberger '08

$$T_p(F) = \frac{F\left(\frac{px_1}{1-(1-p)x_1}, \ldots, \frac{px_n}{1-(1-p)x_n}\right)}{(1-(1-p)x_1)\cdots(1-(1-p)x_n)}.$$

Preserving positivity

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

- Positivity of the Askey–Gasper rational function

$$\frac{1}{(1-(x+y+z)+4xyz)^{\beta}}$$

implies positivity, for any $\varepsilon > 0$, of for $\beta \geqslant (\sqrt{17}-3)/2 \approx 0.56$

$$\frac{1}{1-(x+y+z)+(4-\varepsilon)xyz}$$

Gillis–Reznick–Zeilberger '83
S '08
Kauers–Zeilberger '08

- If $F(x_1, \ldots, x_n)$ is positive, so is, for $0 \leqslant p \leqslant 1$,

$$T_p(F) = \frac{F\left(\frac{px_1}{1-(1-p)x_1}, \ldots, \frac{px_n}{1-(1-p)x_n}\right)}{(1-(1-p)x_1)\cdots(1-(1-p)x_n)}.$$

EG
S '08

$$T_{1/2}\,\frac{1}{1-(x+y+z)+4xyz} = \frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}$$

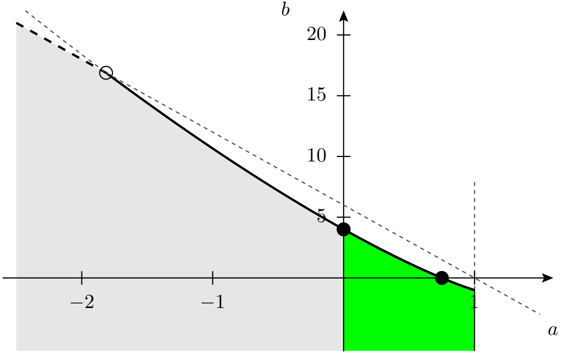
Hence, we can conclude positivity of Szegő's function $e_2(1-x, 1-y, 1-z)^{-1}$.

The case of three variables

$$h_{a,b}(x,y,z) = \frac{1}{1-(x+y+z)+a(xy+yz+zx)+bxyz}$$

CONJ S '08

$h_{a,b}$ is positive $\iff \begin{cases} b < 6(1-a) \\ b \leqslant 2-3a+2(1-a)^{3/2} \\ a \leqslant 1 \end{cases}$



- $h_{a,b}$ is positive in the green region S '08
- The conditions in the conjecture are necessary for positivity S–Zudilin '15

A conjecture of Gillis, Reznick and Zeilberger

CONJ
G-R-Z '83

For any $d \geqslant 4$, the following function is non-negative:

$$\frac{1}{1-(x_1+x_2+\ldots+x_d)+d!x_1x_2\cdots x_d}$$

A conjecture of Gillis, Reznick and Zeilberger

CONJ G-R-Z '83 For any $d \geqslant 4$, the following function is non-negative:

$$\frac{1}{1-(x_1+x_2+\ldots+x_d)+d!x_1x_2\cdots x_d}$$

THM G-R-Z Suffices to prove that the diagonal coefficients are non-negative.

A conjecture of Gillis, Reznick and Zeilberger

CONJ G-R-Z '83 For any $d \geqslant 4$, the following function is non-negative:

$$\frac{1}{1-(x_1+x_2+\ldots+x_d)+d!x_1x_2\cdots x_d}$$

THM G-R-Z Suffices to prove that the diagonal coefficients are non-negative.

proof "omitted due to its length" □

A conjecture of Gillis, Reznick and Zeilberger

CONJ G-R-Z '83 For any $d \geqslant 4$, the following function is non-negative:

$$\frac{1}{1 - (x_1 + x_2 + \ldots + x_d) + d! x_1 x_2 \cdots x_d}$$

THM G-R-Z Suffices to prove that the diagonal coefficients are non-negative.

proof "omitted due to its length" □

- False for $d = 2, 3$.
- Kauers proved that diagonal is non-negative for $d = 4, 5, 6$.

A conjecture of Gillis, Reznick and Zeilberger

CONJ G-R-Z '83 For any $d \geqslant 4$, the following function is non-negative:

$$\frac{1}{1-(x_1+x_2+\ldots+x_d)+d!x_1x_2\cdots x_d}$$

THM G-R-Z Suffices to prove that the diagonal coefficients are non-negative.

proof "omitted due to its length" □

- False for $d=2,3$.
- Kauers proved that diagonal is non-negative for $d=4,5,6$.
- With c in place of $d!$, the coefficient of $x_1\cdots x_d$ is $d!-c$.

A conjecture of Gillis, Reznick and Zeilberger

CONJ G-R-Z '83 For any $d \geqslant 4$, the following function is non-negative:

$$\frac{1}{1-(x_1+x_2+\ldots+x_d)+d!x_1x_2\cdots x_d}$$

THM G-R-Z Suffices to prove that the diagonal coefficients are non-negative.

proof "omitted due to its length" □

- False for $d = 2, 3$.
- Kauers proved that diagonal is non-negative for $d = 4, 5, 6$.
- With c in place of $d!$, the coefficient of $x_1 \cdots x_d$ is $d! - c$.
- Diagonal coefficients eventually positive if $c < (d-1)^{d-1}$?

  Multivariate asymptotics?

Positivity vs diagonal positivity

- Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with p a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of F implied by the positivity of its diagonal?

Positivity vs diagonal positivity

- Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with p a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of F implied by the positivity of its diagonal?

EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

Positivity vs diagonal positivity

- Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with p a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of F implied by the positivity of its diagonal?

EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

Q SZ '15 F positive $\iff$ diagonal of F, and $F|_{x_d=0}$ are positive?

Positivity vs diagonal positivity

- Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with p a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of F implied by the positivity of its diagonal?

EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

Q SZ '15 F positive $\iff$ diagonal of F, and $F|_{x_d=0}$ are positive?

THM S-Zudilin 2015

$$F(x, y) = \frac{1}{1 + c_1(x + y) + c_2 xy} \quad \text{is positive}$$

$\iff$ diagonal of F, and $F|_{y=0}$ are positive

- $d = 3$: also yes, if the previous conjecture on $h_{a,b}$ is true.

Application: Szegő's rational function, once more

- Recall Szegő's rational function

$$S(x,y,z) = \frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}.$$

$S(2x,2y,2z)$ has diagonal coefficients

$$s_n = \sum_{k=0}^{n} (-27)^{n-k} 2^{2k-n} \frac{(3k)!}{k!^3} \binom{k}{n-k},$$

Application: Szegő's rational function, once more

- Recall Szegő's rational function

$$S(x,y,z)=\frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}.$$

$S(2x,2y,2z)$ has diagonal coefficients

$$s_n=\sum_{k=0}^{n}(-27)^{n-k}2^{2k-n}\frac{(3k)!}{k!^3}\binom{k}{n-k},$$

whose generating function is

$$y(z)={}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\middle|27z(2-27z)\right).$$

Application: Szegő's rational function, once more

- Recall Szegő's rational function

$$S(x,y,z)=\frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}.$$

$S(2x,2y,2z)$ has diagonal coefficients

$$s_n=\sum_{k=0}^{n}(-27)^{n-k}2^{2k-n}\frac{(3k)!}{k!^3}\binom{k}{n-k},$$

whose generating function is

$$y(z)={}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\middle|27z(2-27z)\right).$$

- Ramanujan's cubic transformation

$${}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\middle|1-\left(\frac{1-x}{1+2x}\right)^3\right)=(1+2x)\,{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\middle|x^3\right),$$

Application: Szegő's rational function, once more

- Recall Szegő's rational function

$$S(x,y,z)=\frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}.$$

$S(2x,2y,2z)$ has diagonal coefficients

$$s_n=\sum_{k=0}^{n}(-27)^{n-k}2^{2k-n}\frac{(3k)!}{k!^3}\binom{k}{n-k},$$

whose generating function is

$$y(z)={}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\right|27z(2-27z)\right).$$

- Ramanujan's cubic transformation

$${}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\right|1-\left(\frac{1-x}{1+2x}\right)^3\right)=(1+2x){}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\right|x^3\right),$$

puts this in the form

$$y(z)=(1+2x(z)){}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\right|x(z)^3\right),$$

where the algebraic $x(z)=c_1z+c_2z^2+\ldots$ has positive coefficients.

CONJ Kauers 2007 The following rational function is positive:

$$\frac{1}{1-(x+y+z+w)+\frac{64}{27}(yzw+xzw+xyw+xyz)}.$$

- The diagonal is positive. S–Zudilin '15
  (apply CAD to recurrence of order 3 and degree 6)
- The rational function obtained from setting $w=0$ is positive.

Application: A conjecture of Kauers

CONJ Kauers 2007 The following rational function is positive:

$$\frac{1}{1-(x+y+z+w)+\frac{64}{27}(yzw+xzw+xyw+xyz)}.$$

- The diagonal is positive. S–Zudilin '15
  (apply CAD to recurrence of order 3 and degree 6)
- The rational function obtained from setting $w=0$ is positive.
  (because $64/27<4$)

Application: Another conjecture of Kauers and Zeilberger

CONJ Kauers-Zeilberger 2008

The following rational function is positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

- Would imply conjectured positivity of Lewy–Askey rational function

$$\frac{1}{1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}.$$

Recent proof of non-negativity by Scott and Sokal, 2013

CONJ Kauers-Zeilberger 2008

The following rational function is positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

- Would imply conjectured positivity of Lewy–Askey rational function

$$\frac{1}{1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}.$$

Recent proof of non-negativity by Scott and Sokal, 2013

PROP S-Zudilin 2015

The Kauers–Zeilberger function has diagonal coefficients

$$d_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

Arithmetically interesting diagonals

Remarkably, several further rational functions on the boundary of positivity have **Apéry-like** diagonals:

EG $\dfrac{1}{1-(x+y+z)+4xyz}$ has diagonal coefficients $\displaystyle\sum_{k=0}^{n}\binom{n}{k}^3$.

- Next, time permitting: congruences stronger than Gauss for these

Arithmetically interesting diagonals

Remarkably, several further rational functions on the boundary of positivity have **Apéry-like** diagonals:

EG $$\frac{1}{1-(x+y+z)+4xyz}$$ has diagonal coefficients $$\sum_{k=0}^{n} \binom{n}{k}^3.$$

EG Koornwinder's rational function

$$\frac{1}{1-(x+y+z+w)+4e_3(x,y,z,w)-16xyzw}$$

has diagonal coefficients $$\sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2.$$

Using a positivity preserving operator, implies positivity of

$$1/e_3(1-x,1-y,1-z,1-w)$$

- Next, time permitting: congruences stronger than Gauss for these

III

Apéry-like sequences

Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).$$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

THM (Apéry '78) $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. $\square$

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q **Beukers, Zagier** Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q Beukers, Zagier: Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- Essentially, only 14 tuples (a, b, c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions (with generating functions

$${}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, \alpha, 1-\alpha \\ 1, 1\end{matrix}\right| 4C_\alpha z\right), \qquad \frac{1}{1 - C_\alpha z}\,{}_2F_1\left(\left.\begin{matrix}\alpha, 1-\alpha \\ 1\end{matrix}\right| \frac{-C_\alpha z}{1 - C_\alpha z}\right)^2,$$

    with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4 \cdot 3^3$)
  - 6 sporadic solutions
- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Supercongruences for Apéry numbers

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geqslant 5$,

$$A(p) \equiv 5 \pmod{p^3}.$$

Supercongruences for Apéry numbers

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geqslant 5$,
$$A(p) \equiv 5 \pmod{p^3}.$$
- Gessel (1982) proved that $A(mp) \equiv A(m) \pmod{p^3}$.

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geqslant 5$,
$$A(p) \equiv 5 \pmod{p^3}.$$
- Gessel (1982) proved that $A(mp) \equiv A(m) \pmod{p^3}$.

THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geqslant 5$,

$$A(p) \equiv 5 \pmod{p^3}.$$

- Gessel (1982) proved that $A(mp) \equiv A(m) \pmod{p^3}$.

THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

EG For primes p, simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

Supercongruences for Apéry numbers

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geqslant 5$,
$$A(p) \equiv 5 \pmod{p^3}.$$
- Gessel (1982) proved that $A(mp) \equiv A(m) \pmod{p^3}$.

THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$
$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

EG

Mathematica 7 miscomputes $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ for $n > 5500$.

$A(5 \cdot 11^3) = 12488301$...about 2000 digits...about 8000 digits...795652125

Weirdly, with this wrong value, one still has
$$A(5 \cdot 11^3) \equiv A(5 \cdot 11^2) \pmod{11^6}.$$

Supercongruences for Apéry-like numbers

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \qquad (\operatorname{mod} p^{3r})$$

hold for all Apéry-like numbers.

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '87-'88
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '16
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open; modulo p^3 Amdeberhan–Tauraso '16
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	Osburn–Sahu–S '16
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	open

THM
S 2014

Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

Multivariate supercongruences

THM S 2014

Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

- $\sum_{n \geqslant 0} a(n) x^n = F(x) \quad \Longrightarrow \quad \sum_{n \geqslant 0} a(pn) x^{pn} = \frac{1}{p} \sum_{k=0}^{p-1} F(\zeta_p^k x) \qquad \zeta_p = e^{2\pi i/p}$
- Hence, both $A(\boldsymbol{n} p^r)$ and $A(\boldsymbol{n} p^{r-1})$ have rational generating function.
  The proof, however, relies on an explicit binomial sum for the coefficients.

Multivariate supercongruences

THM
S 2014

Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

- By MacMahon's Master Theorem,

$$A(\boldsymbol{n}) = \sum_{k \in \mathbb{Z}} \binom{n_1}{k} \binom{n_3}{k} \binom{n_1 + n_2 - k}{n_1} \binom{n_3 + n_4 - k}{n_3}.$$

Multivariate supercongruences

THM
S 2014

Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4) - x_1x_2x_3x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- By MacMahon's Master Theorem,

$$A(\boldsymbol{n}) = \sum_{k \in \mathbb{Z}} \binom{n_1}{k} \binom{n_3}{k} \binom{n_1+n_2-k}{n_1} \binom{n_3+n_4-k}{n_3}.$$

- Because $A(n-1) = A(-n,-n,-n,-n)$, we also find

$$A(mp^r - 1) \equiv A(mp^{r-1} - 1) \pmod{p^{3r}}.$$

Beukers '85

More conjectural multivariate supercongruences

- Exhaustive search by Alin Bostan and Bruno Salvy:

  $1/(1-p(x,y,z,w))$ with $p(x,y,z,w)$ a sum of distinct monomials; Apéry numbers as diagonal

$$\frac{1}{1-(x+y+xy)(z+w+zw)}$$

$$\frac{1}{1-(1+w)(z+xy+yz+zx+xyz)}$$

$$\frac{1}{1-(y+z+xy+xz+zw+xyw+xyzw)}$$

$$\frac{1}{1-(y+z+xz+wz+xyw+xzw+xyzw)}$$

$$\frac{1}{1-(z+xy+yz+xw+xyw+yzw+xyzw)}$$

$$\frac{1}{1-(z+(x+y)(z+w)+xyz+xyzw)}$$

More conjectural multivariate supercongruences

- Exhaustive search by Alin Bostan and Bruno Salvy:

$1/(1-p(x,y,z,w))$ with $p(x,y,z,w)$ a sum of distinct monomials; Apéry numbers as diagonal

$$\frac{1}{1-(x+y+xy)(z+w+zw)}$$
$$\frac{1}{1-(1+w)(z+xy+yz+zx+xyz)}$$
$$\frac{1}{1-(y+z+xy+xz+zw+xyw+xyzw)}$$
$$\frac{1}{1-(y+z+xz+wz+xyw+xzw+xyzw)}$$
$$\frac{1}{1-(z+xy+yz+xw+xyw+yzw+xyzw)}$$
$$\frac{1}{1-(z+(x+y)(z+w)+xyz+xyzw)}$$

CONJ
S 2014
The coefficients $B(\boldsymbol{n})$ of each of these satisfy, for $p \geqslant 5$,

$$B(\boldsymbol{n}p^r) \equiv B(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

An infinite family of rational functions

THM
S 2014

Let $\lambda \in \mathbb{Z}_{>0}^{\ell}$ with $d = \lambda_1 + \ldots + \lambda_\ell$. Define $A_\lambda(\boldsymbol{n})$ by

$$\frac{1}{\prod_{1\leqslant j\leqslant \ell}\left[1 - \sum_{1\leqslant r\leqslant \lambda_j} x_{\lambda_1+\ldots+\lambda_{j-1}+r}\right] - x_1 x_2 \cdots x_d} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^d} A_\lambda(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- If $\ell \geqslant 2$, then, for all primes p,

$$A_\lambda(\boldsymbol{n}p^r) \equiv A_\lambda(\boldsymbol{n}p^{r-1}) \pmod{p^{2r}}.$$

- If $\ell \geqslant 2$ and $\max(\lambda_1, \ldots, \lambda_\ell) \leqslant 2$, then, for primes $p \geqslant 5$,

$$A_\lambda(\boldsymbol{n}p^r) \equiv A_\lambda(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

EG

$\lambda = (2,2)$

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4) - x_1x_2x_3x_4}$$

$\lambda = (2,1)$

$$\frac{1}{(1-x_1-x_2)(1-x_3) - x_1x_2x_3}$$

Further examples

EG

$$\frac{1}{(1-x_1-x_2)(1-x_3)-x_1x_2x_3}$$

has as diagonal the Apéry-like numbers, associated with $\zeta(2)$,

$$B(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}.$$

EG

$$\frac{1}{(1-x_1)(1-x_2)\cdots(1-x_d)-x_1x_2\cdots x_d}$$

has as diagonal the numbers $d=3$: Franel, $d=4$: Yang–Zudilin

$$Y_d(n)=\sum_{k=0}^{n}\binom{n}{k}^d.$$

- In each case, we obtain supercongruences generalizing results of Coster (1988) and Chan–Cooper–Sica (2010).

CONJ
S 2014

The coefficients $Z(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}^4_{\geqslant 0}} Z(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$Z(\boldsymbol{n}p^r) \equiv Z(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- Here, the diagonal coefficients are the **Almkvist–Zudilin numbers**

$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3},$$

for which the univariate congruences are still open.

- Which rational functions have the **Gauss property**?

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^r}$$

When are these necessarily combinations of $\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det\left(\frac{\partial f_j}{\partial x_i}\right)$?

- Which rational functions are **positive**?

When is diagonal, plus lower-dimensional, positivity sufficient?

- Can we establish all **supercongruences** via rational functions?

$$\frac{1}{1-(x+y+z)+4xyz}, \quad \frac{1}{1-(x+y+z+w)+27xyzw}$$

- Is there a rational function in three variables with the $\zeta(3)$-Apéry numbers as diagonal? As Alin showed us, the GF is transcendental, so two variables is impossible.

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

F. Beukers, M. Houben, A. Straub
Gauss congruences for rational functions in several variables
Preprint, 2017. arXiv:1710.00423

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
Journal of Approximation Theory (special issue dedicated to Richard Askey), Vol. 195, 2015, p. 57-69

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Algebra & Number Theory, Vol. 8, Nr. 8, 2014, p. 1985-2008

A. Straub
Positivity of Szegö's rational function
Advances in Applied Mathematics, Vol. 41, Issue 2, Aug 2008, p. 255-264