Some multivariable master polynomials for permutations, set partitions, and perfect matchings, and their continued fractions[a]

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[a] Joint work with Alan Sokal (New York/London)

S-type and J-type continued fractions

If $(a_n)_{n\geq 0}$ is a sequence of combinatorial numbers or polynomials with $a_0 = 1$, it is often fruitful (total positivity of $(a_{i+j})_{i,j\geq 0}$, log-convexity, γ-positivity, moment sequence, ...) to seek to express its ordinary generating function (OGF) as a continued fraction of either Stieltjes (S) type,

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}},$$

or Jacobi type (J),

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \cdots}}},$$

Contraction formulae of an S-fraction to a J-fraction

$$\cfrac{1}{1-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{\cdots}}} = \cfrac{1}{1-\alpha_1 x - \cfrac{\alpha_1\alpha_2 x^2}{1-(\alpha_2+\alpha_3)x - \cfrac{\alpha_3\alpha_4 x^2}{\cdots}}}.$$

i.e.,

$$\begin{aligned}
\gamma_0 &= \alpha_1 \\
\gamma_n &= \alpha_{2n} + \alpha_{2n+1} \quad \text{for} \quad n \geq 1 \\
\beta_n &= \alpha_{2n-1}\alpha_{2n}.
\end{aligned}$$

Two approaches

This line of investigation, i.e. $(a_n) \mapsto (\alpha_n)$ (or $((\gamma_n), (\beta_n))$), goes back at least to Euler, but it gained impetus following Flajolet's seminal discovery that any S-type (resp. J-type) continued fraction can be interpreted combinatorially as a generating function of Dyck (resp. Motzkin) paths with suitable weights for each rise and fall (resp. each rise, fall and level step).

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Our approach will be (in part) to run this program in reverse: we start from a continued fraction in which the coefficients α (or γ and β) contain indeterminates in a nice pattern, and we attempt to find a combinatorial interpretation for the resulting polynomials a_n.

Euler's continued formulae

$$\sum_{n\geq 0} n!x^n = \cfrac{1}{1-\cfrac{1x}{1-\cfrac{1x}{1-\cfrac{2x}{1-\cfrac{2x}{\cdots}}}}}$$

$$= \cfrac{1}{1-x-\cfrac{1^2x^2}{1-3x-\cfrac{2^2x^2}{\cdots}}}$$

with coefficients $\alpha_{2k-1} = k$, $\alpha_{2k} = k$.

A naive generalization

Introduce the polynomials $P_n(x, y, u, v)$ by the following CF

$$\sum_{n\geq 0} P_n(x, y, u, v)t^n = \cfrac{1}{1 - \cfrac{x\,t}{1 - \cfrac{y\,t}{1 - \cfrac{(x+u)\,t}{1 - \cfrac{(y+v)\,t}{1 - \cdots}}}}}. \tag{1}$$

with coefficients

$$\alpha_{2k-1} = x + (k-1)u \quad \alpha_{2k} = y + (k-1)v. \tag{2}$$

Clearly $P_n(x, y, u, v)$ is a homogeneous polynomial of degree n and $P_n(1, 1, 1, 1) = n!$.

Record classification

Given a permutation $\mathfrak{S}_n$, an index $i \in [n]$ (or value $\sigma(i) \in [n]$) is called a

- *record* (rec) (or *left-to-right maximum*) if $\sigma(j) < \sigma(i)$ for all $j < i$;
- *antirecord* (arec) (or *right-to-left minimum*) if $\sigma(j) > \sigma(i)$ for all $j > i$;
- *exclusive record* (erec) if it is a record and not also an antirecord;
- *exclusive antirecord* (earec) if it is an antirecord and not also a record;
- *record-antirecord* (rar) if it is both a record and an antirecord;
- *neither-record-antirecord* (nrar) if it is neither a record nor an antirecord.

Cycle classification

We say that an index $i \in [n]$ is a

- *cycle peak* (cpeak) if $\sigma^{-1}(i) < i > \sigma(i)$;
- *cycle valley* (cval) if $\sigma^{-1}(i) > i < \sigma(i)$;
- *cycle double rise* (cdrise) if $\sigma^{-1}(i) < i < \sigma(i)$;
- *cycle double fall* (cdfall) if $\sigma^{-1}(i) > i > \sigma(i)$;
- *fixed point* (fix) if $\sigma^{-1}(i) = i = \sigma(i)$.

We denote the number of cycles, records, antirecords, ... in σ by cyc(σ), rec(σ), arec(σ), ..., respectively.
A rougher classification is that an index $i \in [n]$ (or value $\sigma(i)$) is an

- *excedance* (exc) if $\sigma(i) > i$;
- *anti-excedance* (aexc) if $\sigma < i$;
- *fixed point* (fix) if $\sigma = i$.

Two combinatorial interpretations

Theorem 1

The polynomials defined by the S-fraction have the combinatorial interpretations

$$P_n(x,y,u,v) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} u^{n-\mathrm{exc}(\sigma)-\mathrm{arec}(\sigma)} v^{\mathrm{exc}(\sigma)-\mathrm{erec}(\sigma)} \quad (3)$$

and

$$P_n(x,y,u,v) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{erec}(\sigma)} u^{n-\mathrm{exc}(\sigma)-\mathrm{cyc}(\sigma)} v^{\mathrm{exc}(\sigma)-\mathrm{erec}(\sigma)}. \quad (4)$$

N.B. The triple statistics (arec, erec, exc) and (cyc, erec, exc) are equidistributed on $\mathfrak{S}_n$.

Special cases (1)

- The Stirling cycle polynomials

$$P_n(x,1,1,1) = \sum_{k=0}^{n} S(n,k)x^k = x(x+1)\dots(x+n-1).$$

or their homogenized version

$$P_n(x,y,y,y) = \sum_{k=0}^{n} S(n,k)x^k y^{n-k} = x(x+y)\dots(x+(n-1)y).$$

- The Eulerian polynomials

$$P_n(1,y,1,y) = A_n(y) = \sum_{k=0}^{n} A(n,k)y^k$$

or

$$P_n(x,y,x,y) = A_n(x,y) = \sum_{k=0}^{n} A(n,k)x^{n-k}y^k.$$

Special cases (2): Dumont-Kreweras 1988

The record-antirecord permutation polynomials

$$P_n(a, b, 1, 1) = \sum_{\sigma \in \mathfrak{S}_n} a^{\mathrm{arec}(\sigma)} b^{\mathrm{erec}(\sigma)}$$

or

$$P_n(a, b, c, c) = \sum_{\sigma \in \mathfrak{S}_n} a^{\mathrm{arec}(\sigma)} b^{\mathrm{erec}(\sigma)} c^{n-\mathrm{arec}(\sigma)-\mathrm{erec}(\sigma)}.$$

Note that

$$\sum_{n=0}^{\infty} P_n(a, b, 1, 1) t^n = \frac{{}_2F_0(a, b; -|t)}{{}_2F_0(a, b-1; -|t)}.$$

Special cases (3)

The polynomials [sequence A145879/A202992]

$$P_n(x, x, u, u) = \sum_{k=0}^{n} T(n, k) x^{n-k} u^k$$

where $T(n, k)$ is the number of permutations in $\mathfrak{S}_n$ having exactly k indices that are the middle point of a pattern 321 (or 123). In particular $T(n, 0)$ is the number of 123-avoiding permutations, which equals the Catalan number C_n. So the polynomials interpolate between C_n and $n!$.

Special cases (4): Narayanan polynomials

$$
\begin{aligned}
P_n(x, y, 0, 0) &= \sum_{\sigma \in \mathfrak{S}_n(123)} x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} \\
&= \sum_{\sigma \in \mathfrak{S}_n(123)} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{erec}(\sigma)} \\
&= \sum_{k=0}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} x^k y^{n-k}.
\end{aligned}
$$

The cycle interpretation (with $y = 1$) was given by Parviainen (2007).

Record and cycle classifications

We have classified indices in a permutation according to their record status:
exlusive record, exclusive antirecord, record-antirecord or neither-record-antirecord
and aslo according to their cycle status:
cycle peak, cycle valley, cycle double rise, cycle double fall or fixed point.
Applying now both classifications simultaneously, we obtain 10 disjoint categories.

Record-cycle classifications: 10 classes

- ereccval: exclusive records that are also cycle valleys;
- erecdrise: exclusive records that are also cycle double rises;
- eareccpeak: exclusive antirecords that are also cycle peaks;
- eareccdfall: exclusive antirecords that are also cycle double falls;
- rar: record-antirecords (that are always fixed points);
- nrcpeak: neither-record-antirecords that are also cycle peakss;
- nrcval: neither-record-antirecords that are also cycle valleys;
- nrcdrise: neither-record-antirecords that are also cycle double falls;
- nrcfall: neither-record-antirecords that are also cycle falls;
- nrfix: neither-record-antirecords that are also fixed points.

First J-fraction

$$Q_n(x_1, x_2, y_1, y_2, z, u_1, u_2, v_1, v_2, w) =$$

$$\sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{earccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)}$$

$$\times\ u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

If i is a fixed point of σ, we define its level by

$$\mathrm{lev}(i, \sigma) := \#\{j < i : \sigma(j) > i\} = \#\{j > i : \sigma(j) < i\}.$$

Clearly, a fixed point i is a record-antirecord if its level is 0, and a neither-record-antirecord if its level is ≥ 1.

First master polynomial

Introduce indeterminates $\mathbf{w} = (w_\ell)_{\ell \geq 0}$ and write

$$\mathbf{w}^{\mathbf{fix}(\sigma)} := \prod_{\ell=0}^{\infty} w_\ell^{\mathrm{fix}(\sigma,\ell)} = \prod_{i \in \mathrm{Fix}(\sigma)} w_{\mathrm{lev}(i,\sigma)}.$$

The master polynomial encoding all these (now infinitely many) statistics is

$$Q_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, \mathbf{w}) =$$

$$\sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{earccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)}$$

$$\times\ u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} \mathbf{w}^{\mathbf{fix}(\sigma)}$$

Theorem 2 (First J-fraction for permutations)

The OGF of the polynomials Q_n has the J-type continued fraction

$$\sum_{n=0}^{\infty} Q_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, \mathbf{w}) t^n =$$

$$\cfrac{1}{1 - w_0 t - \cfrac{x_1 y_1 t^2}{1 - (x_2 + y_2 + w_1)t - \cfrac{(x_1 + u_1)(y_1 + v_1)t^2}{1 - \cdots}}},$$

with coefficients $\gamma_0 = w_0$*,*

$$\gamma_n = [x_2 + (n-1)u_2] + [y_2 + (n-1)v_2] + w_n \quad \textit{for} \quad n \geq 1$$
$$\beta_n = [x_1 + (n-1)u_1][y_1 + (n-1)v_1].$$

Second J-fraction

Define the polynomial

$$\hat{Q}_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, \mathbf{w}, \lambda) =$$

$$\sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{earccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)}$$

$$\times\, u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} \mathbf{w}^{\mathbf{fix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}.$$

Second J-fraction

Theorem 3 (Second J-fraction for permutations)

The OGF of the polynomials Q_n has the J-type continued fraction

$$\sum_{n=0}^{\infty} \hat{Q}_n(x_1, x_2, y_1, y_2, u_1, u_2, y_1, y_2, \mathbf{w}, \lambda) t^n =$$

$$\cfrac{1}{1 - \lambda w_0 t - \cfrac{\lambda x_1 y_1 t^2}{1 - (x_2 + y_2 + \lambda w_1)t - \cfrac{(\lambda+1)(x_1+u_1)t^2}{1 - \cdots}}},$$

with coefficients $\gamma_0 = \lambda w_0$,

$$\gamma_n = [x_2 + (n-1)u_2] + n y_2 + \lambda w_n \quad \textit{for} \quad n \geq 1$$
$$\beta_n = (\lambda + n - 1)[x_1 + (n-1)u_1] y_1.$$

Statistics on permutations (1)

Comparing Theorem 1 (1) with the first J-fraction the polynomial Q_n reduces to $P_n(x, y, u, v)$ if we set

$$x_1 = x_2 = x,\ y_1 = y_2 = y,\ w_0 = xz$$
$$u_1 = u_2 = w_\ell = 1\ (\ell \geq 1),\ v_1 = v_2 = v.$$

The weight function reduces to

$$w(\sigma) = x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} v^{\mathrm{exc}(\sigma)} z^{\mathrm{rar}(\sigma)}.$$

Comparing with Theorem 1 (2) with the second J-fraction the polynomial $\hat{Q}_n$ reduces to $P_n(x, y, u, v)$ if we set

$$x_1 = x_2 = y,\ u_1 = u_2 = v,\ w_0 = z$$
$$y_1 = y_2 = v_1 = v_2 = w_\ell = 1(\ell \geq 1),\ \lambda = x.$$

The weight function reduces to

$$\hat{w}(\sigma) = x^{\mathrm{cyc}(\sigma)} y^{\mathrm{earec}(\sigma)} v^{\mathrm{aexc}(\sigma)} z^{\mathrm{rar}(\sigma)}.$$

Statistics on permutations (2)

We have the following equidistribution:

$$(\mathrm{arec}, \mathrm{erec}, \mathrm{exc}, \mathrm{rar}) \sim (\mathrm{cyc}, \mathrm{erec}, \mathrm{exc}, \mathrm{rar}).$$

- Cori (2008) and Foata-Han (2009) : $(\mathrm{cyc}, \mathrm{arec}) \sim (\mathrm{rec}, \mathrm{arec})$ on $\mathfrak{S}_n$ and the distribution of $(\mathrm{cyc}, \mathrm{arec})$ is symmetric.
- Kim-Stanton (2015): $(\mathrm{rec}, \mathrm{arec}, \mathrm{rar})$ moments of associated Laguerre polynomials.
- Elizalde (2017): $(\mathrm{cyc}, \mathrm{fix}, \mathrm{aexc}, \mathrm{cdfall})$, which is $\sim (\mathrm{cyc}, \mathrm{fix}, \mathrm{exc}, \mathrm{cdrise})$ by $\sigma \mapsto \sigma^{-1}$.

Setting $v = 1$ and $z = y$ we have

$$\sum_{n=0}^{\infty} \left(\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{arec}(\sigma)} \right) t^n =$$

$$\cfrac{1}{1 - xy\, t - \cfrac{xy\, t}{1 - (x+y+1)\, t - \cfrac{(x+1)(y+1)\, t}{1 - \cdots}}}$$

with $\gamma_0 = xy$,

$$\begin{aligned} \gamma_n &= x + y + 2n - 1 \\ \beta_n &= (x + n - 1)(y + n - 1) \quad \text{for} \quad n \geq 1. \end{aligned}$$

p,q-generalizations of Euler's continued fractions

We define

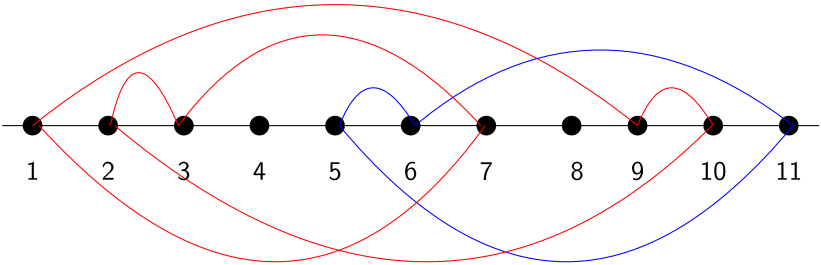
$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = \sum_{j=0}^{n-1} p^j q^{n-1-j}.$$

Foata-Zeilberger (1990), Biane (1993), De Mdicis-Viennot (1994), Simion-Stanton(1994, 1996), Clarke-Steingrimsson-Z. (1997), Randrianarivony (1998), Corteel (2007), ...
Let $[n; a]_{p,q} = ap^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}$. Then Randrianarivony (1998) :

$$\gamma_n = (a[n+1;\alpha]_{r,s} + b[n;\beta]_{t,u})x^n,$$
$$\beta_n = cd[n;\gamma]_{p,q}[n;\mu]_{v,w}x^{2n-1}.$$

Crossings and nestings

To the permutation $\pi = {\color{blue}(1,9,10,2,3,7)(4)(5,6,11)(8)} \in \mathfrak{S}_{11}$ we associate a pictorial representation as follows:



Upper and lower crossings and nestings

We say that a quadruple $i < j < k < l$ forms an

- *upper crossing* (ucross) if $k = \sigma(i)$ and $l = \sigma(j)$;
- *lower crossing* (lcross) if $i = \sigma(k)$ and $j = \sigma(l)$;
- *upper nesting* (unest) if $l = \sigma(i)$ and $k = \sigma(j)$;
- *lower nesting* (lnest) if $i = \sigma(l)$ and $j = \sigma(k)$.

We consider also some "degenerate" cases with $j = k$, by saying a triplet $i < j < k$ forms an

- *upper joining* (ujoin) if $j = \sigma(i)$ and $l = \sigma(j)$;
- *lower joining* (lcross) if $i = \sigma(j)$ and $j = \sigma(l)$;
- *upper pseudo-nesting* (upsnest) if $l = \sigma(i)$ and $j = \sigma(j)$;
- *lower pseudo-nesting* (lpsnest) if $i = \sigma(l)$ and $j = \sigma(j)$.

Refinement of crossing categories

We say that a quadruplet $i < j < k < l$ forms an

- *upper crossing of type cval* (ucrosscval) if $k = \sigma(i)$ and $l = \sigma(j)$ and $\sigma^{-1}(j) > j$;
- *upper crossing of type cdrise* (ucrosscdrise) if $k = \sigma(i)$ and $l = \sigma(j)$ and $\sigma^{-1}(j) < j$;
- *lower crossing of type cpeak* lcrosscpeak) if $l = \sigma(i)$ and $\sigma^{-1}(k) < k$;
- *lower crossing of type cdfall* (lcrosscdfall) if $i = \sigma(k)$ and $j = \sigma(l)$ and $\sigma^{-1}(k) > k$;

Refinement of nesting categories

We say that a quadruplet $i < j < k < l$ forms an

- *upper nesting of type cval* (unestcval) if $l = \sigma(i)$ and $k = \sigma(j)$ and $\sigma^{-1}(j) > j$;
- *upper nesting of type cdrise* (unestcdrise) if $l = \sigma(i)$ and $k = \sigma(j)$ and $\sigma^{-1}(j) < j$;
- *lower nesting of type cpeak* (lnestcdpeak) if $l = \sigma(i)$ and $j = \sigma(j)$ and $\sigma^{-1}(k) < k$;
- *lower nesting of type cdfall* (lnestcdfall) if $i = \sigma(l)$ and $j = \sigma(j)$ and $\sigma^{-1}(k) > k$.

First J-fraction for permutations 1

Define the polynomial

$$Q_n(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{p}, \mathbf{q}, s) := Q_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, \mathbf{w}, p_{+1}, p_{+2}, p_{+2}, p_{-1}, p_{-2}, q_{+1}, q_{+2}, q_{-1}, q_{-2}, s) =$$

$$\sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{earccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} \mathbf{w}^{\mathbf{fix}(\sigma)} \times p_{+1}^{\mathrm{ucrosscval}(\sigma)} p_{+2}^{\mathrm{ucrosscdrise}(\sigma)} p_{-1}^{\mathrm{lcrosscpeak}(\sigma)} p_{-2}^{\mathrm{lcrosscdfall}(\sigma)} \times q_{+1}^{\mathrm{unestcval}(\sigma)} q_{+2}^{\mathrm{unestcdrise}(\sigma)} q_{-1}^{\mathrm{lnestcpeak}(\sigma)} q_{-2}^{\mathrm{inestcdfall}(\sigma)} s^{\mathrm{psnest}(\sigma)}.$$

First J-fraction for permutations 2

$$\sum_{n=0}^{\infty} Q_n(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{p}, \mathbf{q}, s) t^n =$$

$$\cfrac{1}{1 - w_0 t - \cfrac{x_1 y_1 t^2}{1 - (x_2 + y_2 + s w_1) t - \cfrac{(p_1 x_1 + q_{-1} u_1)(p_{+1} y_1 + q_{+1} v_1) t^2}{1 - \cdots}}}$$

with coefficents $\gamma_0 = w_0$ and for $n \geq 1$,

$$\gamma_n = (p_{-2}^{n-1} x_2 + q_{-2} [n-1]_{p_{-2}, q_{-2}} u_2) + (p_{+2}^{n-1} y_2 + q_{+2} [n-1]_{p_{+2}, q_{+2}} v_2) + s^n w_n$$

$$\beta_n = (p_{-1}^{n-1} x_1 + q_{-1} [n-1]_{p_{-1}, q_{-1}} u_1)(p_{+1}^{n-1} y_1 + q_{+1} [n-1]_{p_{+1}, q_{+1}} v_1).$$

First master J-fraction (1)

Rather than counting the total numbers of nestings, we should instead count the number of upper (resp. lower) crossings or nestings that use a particular vertex j (resp. k) in second (resp. third) position, and then attribute weights to the vertex j (resp. k) depending on these values.

$$\begin{aligned} \mathrm{ucross}(j,\sigma) &= \#\{i < j < k < l :\ k = \sigma(i) \text{ and } l = \sigma(j)\} \\ \mathrm{unest}(j,\sigma) &= \#\{i < j < k < l :\ k = \sigma(j) \text{ and } l = \sigma(i)\} \\ \mathrm{lcross}(k,\sigma) &= \#\{i < j < k < l :\ i = \sigma(k) \text{ and } j = \sigma(l)\} \\ \mathrm{lnest}(k,\sigma) &= \#\{i < j < k < l :\ i = \sigma(l) \text{ and } j = \sigma(k)\}. \end{aligned}$$

First master J-fraction (2)

N.B. $\mathrm{ucross}(j,\sigma)$ and $\mathrm{unest}(j,\sigma)$ can be nonzero only when j is a cycle valley or a cycle double rise, while $\mathrm{lcross}(k,\sigma)$ and $\mathrm{lnest}(k,\sigma)$ can be nonzero only when k is a cycle peak or a cycle double fall. And obviously we have

$$\mathrm{ucrosscval}(\sigma) = \sum_{j \in \mathrm{cval}} \mathrm{ucross}(j,\sigma)$$

and analogously for the other seven crossing/nesting quantities.

First master J-fraction (3)

We now introduce five infinite families of indeterminates **a**, **b**, **c**, **d** where $\mathbf{x} = (x_{\ell,\ell'})_{\ell,\ell' \geq 0}$ and $\mathbf{w} = (w_\ell)_{\ell \geq 0}$, and define the polynomial

$$Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{w}) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i \in \mathrm{cval}} a_{\mathrm{ucross}(i,\sigma),\mathrm{unest}(i,\sigma)} \prod_{i \in \mathrm{cpeak}} b_{\mathrm{lcross}(i,\sigma),\mathrm{lnest}(i,\sigma)} \times \prod_{i \in \mathrm{cdfall}} c_{\mathrm{lcross}(i,\sigma),\mathrm{lnest}(i,\sigma)} \prod_{i \in \mathrm{cdrise}} d_{\mathrm{ucross}(i,\sigma),\mathrm{unest}(i,\sigma)} \prod_{i \in \mathrm{fix}} w_{\mathrm{lev}(i,\sigma)}$$

These polynomials then have a beautiful J-fraction.

First master J-fraction (4)

Theorem 4 (First master J-fraction for permutations)

The OGF of the polynomials $Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{w})$ has the J-type continued fraction

$$\sum_{n=0}^{\infty} Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{w}) t^n =$$

$$\cfrac{1}{1 - w_0 t - \cfrac{a_{00} b_{00} t^2}{1 - (c_{00} + d_{00} + w_1) t - \cfrac{(a_{00} + a_{10})(b_{01} + b_{10}) t^2}{1 - \cdots}}}$$

with coefficients $\gamma_n = c^*_{n-1} + d^*_{n-1} + w_n$ and $\beta_n = a^*_{n-1} b^*_{n-1}$, where $a^*_{n-1} := \sum_{\ell=0}^{n-1} a_{\ell,n-1-\ell} = a_{0,n-1} + a_{1,n-2} + \ldots + a_{n-1,0}$.

A remark on the inversion statistic

A inversion of a permutation $\sigma \in \mathfrak{S}_n$ is a pair $i, j \in [n]$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

Lemma 1 (Shin-Z. 2010)

We have

$$\begin{aligned} \mathsf{inv} &= cval + \mathsf{cdrise} + \mathsf{cdfall} + \mathsf{ucross} + \mathsf{lcross} \\ &\qquad + 2(\mathsf{unest} + \mathsf{lnest} + \mathsf{psnest}) \\ &= \mathsf{exc} + (\mathsf{ucross} + \mathsf{lcross} + \mathsf{ljoin}) + 2(\mathsf{unest} + \mathsf{lnest} + \mathsf{psnest}). \end{aligned}$$

p,q-generalization of the second J-fraction

We can also make a (p,q)-generalization of the second J-fraction involving cyc. Define the polynomial

$$\hat{Q}_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, \mathbf{w}, p_{+1}, p_{+2}, p_{-1}, p_{-2}, q_{+1}, q_{+2}, q_{-1}, q_{-2}, s, \lambda) =$$

$$\sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{earccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} \times$$

$$u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} \mathbf{w}^{\mathbf{fix}(\sigma)} \times$$

$$p_{+1}^{\mathrm{ucrosscval}(\sigma)} p_{+2}^{\mathrm{ucrosscdrise}(\sigma)} p_{-1}^{\mathrm{lcrosscpeak}(\sigma)} p_{-2}^{\mathrm{lcrosscdfall}(\sigma)} \times$$

$$q_{+1}^{\mathrm{unestcval}(\sigma)} q_{+2}^{\mathrm{unestcdrise}(\sigma)} q_{-1}^{\mathrm{lnestcpeak}(\sigma)} q_{-2}^{\mathrm{lnestcdfall}(\sigma)} s^{\mathrm{psnest}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}.$$

p,q-generalization of the second J-fraction

Theorem 5 (Second J-fraction for permutations)

$$\sum_{n=0}^{\infty} \hat{Q}_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, \mathbf{w},$$

$$p_{+1}, p_{+2}, p_{-1}, p_{-2}, q_{+1}, q_{+2}, q_{-1}, q_{-2}, s, \lambda)t^n =$$

$$\cfrac{1}{1 - \lambda w_0 t - \cfrac{\lambda x_1 y_1 t^2}{1 - (x_2 + y_2 + \lambda w_1)t - \cfrac{(\lambda+1)(x_1+u_1)t^2}{1 - \cdots}}},$$

with coefficients $\gamma_0 = \lambda w_0$,

$$\gamma_n = (p_{-2}^{n-1} x_2 + q_{-2}[n-1]_{p_{-2},q_{-2}} u_2) + n p_{+2}^{n-1} y_2 + \lambda s^n w_n \textit{ for } n \geq 1$$
$$\beta_n = (p_{-1}^{n-1} x_1 + q_{-1}[n-1]_{p_{-1},q_{-1}} u_1) p_{+1}^{n-1} y_1 (\lambda + n - 1).$$

Set partitions: S-fraction

The Bell number B_n is the number of partitions of an n-element set into nonempty blocks with $B_0 = 1$.

$$\sum_{n=0}^{\infty} B_n t^n = \cfrac{1}{1 - \cfrac{1\,t}{1 - \cfrac{1\,t}{1 - \cfrac{1\,t}{1 - \cfrac{2\,t}{1 - \cdots}}}}}$$

with coefficients $\alpha_{2k-1} = 1$, $\alpha_{2k} = k$.

$$\sum_{n=0}^{\infty} B_n(x,y,v)\, t^n = \cfrac{1}{1 - \cfrac{x\,t}{1 - \cfrac{y\,t}{1 - \cfrac{x\,t}{1 - \cfrac{(y+2v)\,t}{1 - \cdots}}}}}$$

with coefficients $\alpha_{2k-1} = x$, $\alpha_{2k} = y + (k-1)v$.
Clearly $B_n(x,y,v)$ is a homogeneous polynomial of degree n; it has three truly independent variables.

Theorem 6 (S-fraction for set)

The polynomials $B_n(x, y, v)$ have the combinatorial interpretation

$$B_n(x, y, v) = \sum_{\pi \in \Pi_n} x^{|\pi|} y^{\mathrm{erec}(\pi)} v^{n-|\pi|-\mathrm{erec}(\pi)}$$

where $|\pi|$ (resp. $\mathrm{erec}(\pi)$) denotes the number of blocks (resp. exclusive records) in π.

Given $\pi \in \Pi_n$, for $i \in [n]$, we define $\sigma'(i)$ to be the next-larger element after i in its block, if i is not the largest element in its block, and 0 otherwise. Then $\mathrm{erec}(\pi) := \mathrm{erec}(\sigma')$. For example, if $\pi = \{1, 5\} - \{2, 3, 7\} - \{4\} - \{6\}$, then $\sigma' = 5370000$.

Given a partition π of $[n]$, we say that an element $i \in [n]$ is

- an *opener* if it is the smallest element of a block of size ≥ 2;
- a *colser* if it is the largest element of a block of size ≥ 2;
- an *insider* if it is a non-opener non-closer element of a block of size ≥ 3;
- a *singleton* if it is the sole element of a block of size 1.

Clearly every element $i \in [n]$ belongs to precisely one of these four classes.

J-fraction

We can refine the polynomial $B_n(x, y, v)$ by distinguishing between singletons and blocks of size ≥ 2; in addition, we can distinguish between exclusive records that are openers and those that are insiders. Define

$$B_n(x_1, x_2, y_1, y_2, v) = \sum_{\pi \in \Pi_n} x_1^{m_1(\pi)} x_2^{m_{\geq 2}(\pi)} \times \\ y_1^{\mathrm{erecin}(\pi)} y_2^{\mathrm{erecop}(\pi)} v^{n-|\pi|-\mathrm{erec}(\pi)},$$

where $m_1(\pi)$ is the number of singletons in π, $m_{\geq 2}(\pi)$ is the number of non-singletons blocks, $\mathrm{erecin}(\pi)$ is the number of exclusive records that are insiders, and $\mathrm{erecop}(\pi)$ is the number of exclusive records that are openers.

Theorem 7 (J-fraction for set partitions)

$$\sum_{n=0}^{\infty} B_n(x_1, x_2, y_1, y_2, v) t^n =$$

$$\cfrac{1}{1 - x_1 t - \cfrac{x_2 y_2 t^2}{1 - (x_1 + y_1)t - \cfrac{x_2(y_2 + v)t^2}{1 - \cdots}}}$$

with coefficients $\gamma_0 = x_1$,

$$\gamma_n = x_1 + y_1 + (n-1)v \quad \textit{for } n \geq 1$$
$$\beta_n = x_2[y_2 + (n-1)v].$$

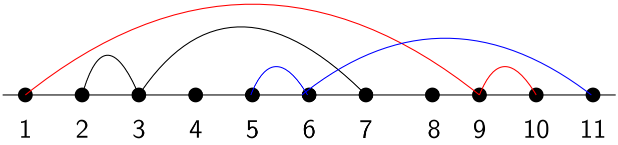
First $p, q-$generalization

Let $\pi = \{B_1, B_2, \ldots, B_k\}$ be a partition of $[n]$. We associate a graph $\mathcal{G}_\pi$ with vertex set $[n]$ such that i, j are joined by an edge if and only if they are consecutive elements within the same block. We then say that a quadruplet $i < j < k < l$ forms a

- *crossing* (cr) if $(i, k) \in \mathcal{G}_\pi$ and $(j, l) \in \mathcal{G}_\pi$;
- *nesting* (ne) if $(i, l) \in \mathcal{G}_\pi$ and $(j, k) \in \mathcal{G}_\pi$.

We also say that a triplet $i < k < l$ forms a

- *pseudo-nesting* (psne) if $(i, l) \in \mathcal{G}_\pi$.



$$\pi = \{\{1, 9, 10\},\ \{2, 3, 7\},\ \{4\},\ \{5, 6, 11\},\ \{8\}\}.$$

We now introduce a (p,q)-generalization of previous polynomial:

$$B_n(x_1,x_2,y_1,y_2,v,p,q,r) = \sum_{\pi\in\Pi_n} x_1^{m_1(\pi)} x_2^{m_{\geq 2}(\pi)} y_1^{\mathrm{erecin}(\pi)} y_2^{\mathrm{erecop}(\pi)} \times v^{n-|\pi|-\mathrm{erecop}(\pi)} p^{\mathrm{cr}(\pi)} q^{\mathrm{ne}(\pi)} r^{\mathrm{psne}(\pi)}.$$

Theorem 8

$$\sum_{n=0}^{\infty} B_n(x_1,x_2,y_1,y_2,v,p,q,r)t^n = \cfrac{1}{1-x_1t-\cfrac{x_2y_2t^2}{1-\cdots}}$$

with coefficients $\gamma_0 = x_1$,

$$\gamma = r^n x_1 + p^{n-1}y_1 + q[n-1]_{p,q}v \quad \textit{for } n \geq 1$$
$$\beta_n = x_2(p^{n-1}y_2 + q[n-1]_{p,q}v).$$

First master J-fraction

Rather that counting the *total* numbers of quadrauplets $i < j < k < l$ that form crossings or nestings, we should instead count the number of crossings or nestings that use a particular vertex k in third (or sometimes second) position, and then attribute weights to the vertex k depending on those values. We define

$$\begin{aligned}
\mathrm{cr}(k,\pi) &= \#\{i < j < k < l\colon\ (i,k) \in \mathcal{G}_\pi \text{ and } (j,l) \in \mathcal{G}_\pi\} \\
\mathrm{ne}(k,\pi) &= \#\{i < j < k < l\colon\ (i,l) \in \mathcal{G}_\pi \text{ and } (j,k) \in \mathcal{G}_\pi\} \\
\mathrm{psne}(k,\pi) &= \#\{i < k < l\colon\ k \text{ is a singleton and } (i,l) \in \mathcal{G}_\pi\}
\end{aligned}$$

First master J-fraction

Note that $\mathrm{cr}(\pi)$ and $\mathrm{ne}(k,\pi)$ can be nonzero only when k is either an insider or a closer; and we obviously have

$$\mathrm{cr}(\pi) = \sum_{k \in \mathrm{insiders} \cap \mathrm{closers}} \mathrm{cr}(k,\pi)$$

$$\mathrm{ne}(\pi) = \sum_{k \in \mathrm{insiders} \cap \mathrm{closers}} \mathrm{ne}(k,\pi)$$

$$\mathrm{psne}(\pi) = \sum_{k \in \mathrm{singletons}} \mathrm{psne}(k,\pi).$$

Finally we define

$$\mathrm{crne}'(\pi) = \#\{i < k < l :\ k \text{ is an opener and } (i,l) \in \mathcal{G}_\pi\}$$

which counts the number of times that the opener k occurs in second position in a crossing or nesting.

First master J-fraction

We now introduce four infinite families of indterminates $\mathbf{a} = (a_\ell)_{\ell \geq 0}$, $\mathbf{b} = (a_{\ell,\ell'})_{\ell,\ell' \geq 0}$, $\mathbf{c} = (c_{\ell,\ell'})_{\ell,\ell' \geq 0}$, $\mathbf{e} = (e_\ell)_{\ell \geq 0}$ and define the polynomials $B_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e})$ by

$$B_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}) = \sum_{\pi \in \Pi_n} \prod_{i \in \text{openers}} \mathbf{a}_{\text{crne}'(i,\pi)} \prod_{i \in \text{closers}} \mathbf{b}_{\text{cr}(i,\pi),\text{ne}(i,\pi)} \prod_{i \in \text{insiders}} \mathbf{c}_{\text{cr}(i,\pi),\text{ne}(i,\pi)} \prod_{i \in \text{singletons}} \mathbf{e}_{\text{psne}(i,\pi)}$$

Theorem 9 (Master J-fraction for set partitions)

The OGF of the polynomials $B_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e})$ has the J-type CF

$$\sum_{n=0}^{\infty} B_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}) t^n = \cfrac{1}{1 - e_0 t - \cfrac{a_0 b_{00} t^2}{1 - (c_{00} + e_1) t - \cfrac{a_1 (b_{01} + b_{10}) t^2}{1 - \cdots}}}$$

with coefficients

$$\gamma_n = \sum_{\ell=0}^{n-1} c_{\ell, n-1-\ell} + e_n, \quad \beta_n = a_{n-1} \sum_{\ell=0}^{n-1} b_{\ell, n-1-\ell}.$$

Perfect matchings

Euler:

$$\sum_{n=0}^{\infty}(2n-1)!!t^n = \cfrac{1}{1-\cfrac{1\,t}{1-\cfrac{2\,t}{1-\cfrac{3\,t}{1-\cdots}}}}$$

We introduce the polytnomials $M_n(x,y,u,v)$ by

$$\sum_{n=0}^{\infty} M_n(x,y,u,v)t^n = \cfrac{1}{1-\cfrac{x\,t}{1-\cfrac{(x+v)\,t}{1-\cfrac{(x+2u)\,t}{1-\cdots}}}}$$

with coefficients $\alpha_{2k-1} = x+(2k-2)u$, $\alpha_{2k} = y+(2k-1)v$

Master S-fraction

We can regard a perfect matching either as a special type of partition (namely, one in which all blocks are of size 2) or as a special type of permutation (namely, a fixed-point-free involution). We now introduce four infinite families of indterminates $\mathbf{a} = (a_\ell)_{\ell \geq 0}$, $\mathbf{b} = (a_{\ell,\ell'})_{\ell,\ell' \geq 0}$,and define the polynomials $M_n(\mathbf{a}, \mathbf{b})$ by

$$M_n(\mathbf{a}, \mathbf{b}) = \sum_{\pi \in \mathcal{M}_n} \prod_{i \in \text{openers}} \mathbf{a}_{\text{crne}'(i,\pi)} \prod_{i \in \text{closers}} \mathbf{b}_{\text{cr}(i,\pi),\text{ne}(i,\pi)}.$$

Of course, we have $M_n(\mathbf{a}, \mathbf{b}) = B_{2n}(\mathbf{a}, \mathbf{b}, \mathbf{0}, \mathbf{0})$.

Theorem 10 (Master S-fraction for perfect matchings)

The OGF of the polynomials $B_n(\mathbf{a},\mathbf{b})$ has the S-type CF

$$\sum_{n=0}^{\infty} M_n(\mathbf{a},\mathbf{b}) t^n = \cfrac{1}{1 - \cfrac{a_0 b_{00} t^2}{1 - \cfrac{a_1 (b_{01} + b_{10}) t^2}{1 - \cdots}}}$$

with coefficients $\alpha_n = a_{n-1} b^*_{n-1}$, where

$$b^*_{n-1} = \sum_{\ell=0}^{n-1} b_{\ell,n-1-\ell}.$$