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DISSERTATION

Titel der Dissertation

Random lattice walks in a Weyl chamber of type A or B and non-intersecting lattice paths

Verfasser

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Für Marlies und meine Eltern.
Vielen Dank für eure Unterstützung.

Preface

This thesis essentially consists of three research articles, referred to as Paper A, Paper B and Paper C. The latest version of these articles can be found on the arXiv (http://arxiv.org). For detailed links, see [10–12]. Some of the results contained in the Papers A, B and C have been presented at international conferences, and extended abstracts have been published in the corresponding conference proceedings (see [13, 14]). A detailed list can also be found at the end of this thesis.

I would like to thank my advisor Christian Krattenthaler for introducing me to this fascinating area of research and for his time for discussions as well as his advice. I would also like to thank him for financially supporting me through the National Research Network S9600 "Analytic Combinatorics and Probabilistic Number Theory", grant S9607-N21, of the Austrian Science Foundations FWF.

Thomas Feierl

Contents

Introduction

Lattice path models are important and well-studied structures in combinatorics and probability theory as well as in statistical mechanics. For example, they naturally appear in basic probabilistic and combinatorial settings such as coin tossing and ballot type problems (see, e.g., [15, Chaper III]). In general, a lattice walk is represented by a (finite) sequence of vertices of a given lattice. In this thesis we will be concerned with two related lattice path models, namely *non-intersecting lattice paths* and *lattice walks in a Weyl chamber*. The lattice underlying the first model is always assumed to be the acyclic and directed square grid.

A set of lattice paths is called *non-intersecting* if the vertex sets corresponding to any pair of different lattices walks in this set are disjoint. In statistical mechanics, non-intersecting lattice paths (or *vicious walkers*) are for example used to describe certain wetting and melting processes (see Fisher [16]), or to encode certain non-colliding particle systems (one may think of a one-dimensional discrete gas). Non-intersecting lattice paths are also in bijection with other important combinatorial objects such as plane partitions, integer partitions and Young tableaux (see, e.g., [20, 25, 33, 34]). Furthermore, non-intersecting lattice paths turned out to be a very useful tool for proving Schur function identities as well as identities for orthogonal and symplectic characters (see [18–20]). In [22, 30], the authors prove convergence results for certain (suitably rescaled) configurations of non-intersecting lattice paths, called *watermelons* and *stars*, to systems of non-colliding Brownian motions (see also [44]), which are closely related to Dyson's Brownian motion [9]. These limiting laws establish a close relation to random matrix theory (see, e.g., [37]), a fact that has been exhibited earlier, e.g., in [2, 3, 38]. More precisely, it is known that the distribution of the positions of random non-intersecting lattice paths at a certain time are related to the eigenvalue distribution in some random matrix ensembles. In particular, the top most path corresponds to the largest eigenvalue of the random matrix.

The second type of objects we study in this thesis are random lattice walks confined to the region $0 < x_1 < x_2 < \cdots < x_k$ (here, x_j refers to the j-th coordinate in $\mathbb{R}^d$). This region is identified with a Weyl chamber of type B. Special configurations of lattice walks in a Weyl chamber correspond, e.g., to certain non-intersecting lattice paths, namely vicious walkers in the lock step model or in the random turns model with wall restriction (see Fisher [16] and [Paper A, Section 7]) and k-non-crossing tangled diagrams (see [7]). Walks in a Weyl chamber also have an interpretation in terms of multiplicities of weights in tensor powers (see [24] and [42]).

In Paper A, we determine asymptotics for the number of lattice walks in a Weyl chamber of type B as the number of steps tends to infinity for a general class of steps. This class of steps is such that an exact expression for the total number of walks can be determined with the help of a reflection principle argument, that generalises a result by Gessel and Zeilberger [21] (see also Theorem 1 below). We want to stress that the admissible class of steps can be precisely described (see [Paper A, Lemma 3.5]), and is such that it also includes types of lattice walks that are not of the nearest neighbour type. As corollaries to our main results, we obtain asymptotics for vicious walks in the lock step model as well as in the random turns model, and also for k-non-crossing tangled diagrams (see [Paper A, Section 7]). Special cases of these results include asymptotics given in [33, 40], as well as precises asymptotics for the number of certain objects studied in [7, 23], where the authors could only determine the asymptotic growth order.

In Papers B and C, we study non-intersecting lattice paths on the $\mathbb{Z}$-lattice spanned by the set $\{(1,1),(1,-1)\}$ with steps from this set. In Paper B, the paths are confined to the upper half plane, and the horizontal axis plays the role of an impenetrable wall. In Paper C we study non-intersecting lattice paths without such a wall restriction. On these structures, we analyse the statistics "height" (the maximum ordinate of the top most path) and "range" (difference between maximum ordinate of top most path and minimum ordinate of bottom most path). Assuming the uniform probability measure on the set of configurations with a fixed number n of steps, we determine the limiting distribution of the random variables "height" and "range" as $n \to \infty$. Additionally, we determine first and second order asymptotics for all moments of the random variable "height". This research was motivated by computer experiments published in [5]. As a special case, the results in Paper B and Paper C include well-known results on the maximum and the range of Brownian excursions and Brownian bridges (see [4, 39]) as well as results in [17] on the height of pairs of non-intersecting paths. In [28, 41], the authors consider this model in the thermo-dynamical limit (i.e., non-intersecting Brownian motions instead of non-intersecting lattice paths) and re-derive the asymptotically dominant terms for some of the quantities given in Paper B and Paper C. The derivation of the results in [28] is very close to the approach taken in Paper B, and makes essential use of a result originally proven in Paper B (cf. [Paper B, Corollary 1] and [28, Lemma 3]).

All of the above mentioned results share an interesting characteristic: while it is relatively easy to set up exact counting formulas for the quantities in question (mainly because they can already be found in the literature), it is rather hard to obtain asymptotic results from these formulas. The main reason for this fact is, that all these exact formulas involve determinants that cannot be evaluated to a simple closed product form. As a consequence, we have to cope with a large number of cancellations of leading asymptotic terms that increases with the dimension of the matrix inside the determinant. The solution to this problem was one of the key steps in the proofs of the above mentioned results. For details, we directly refer to the research manuscripts A, B and C.

The rest of the introduction is organised as follows. In Section 1 we give precise definitions for walks in a Weyl chamber, and state a fundamental counting result for the total number of lattice walks in a Weyl chamber due to Gessel and Zeilberger [21] (see Theorem 1 below). This result relies on the so-called reflection principle, that we also present in this section in its most basic form.

In Section 2 we give definitions for non-intersecting lattice paths and state a theorem due to Lindström [36] and Gessel and Viennot [20] that (under certain conditions) gives a determinantal expression for the total number of non-intersecting lattice walks on a directed acyclic graph (see Theorem 2 below). As a special case, this result includes a discrete version of a formula by Karlin and McGregor [27]. We also illustrate the main idea of the proof of this theorem, which is closely related to the reflection principle argument of Section 1.

In Section 3 we explain the relation between lattice walks in a Weyl chamber of type A or B and non-intersecting lattice paths on the square grid or the halfed square grid, respectively, and derive exact formulas for the number of these objects from Theorem 1 and Theorem 2.

The last three sections of this introduction contain more detailed descriptions of the research papers A, B and C.

1. Walks in a Weyl chamber

A classical problem in combinatorics, a variant of the two candidate ballot problem, asks for the number of walks of length $2n$ with steps from the set $\{(1,1),(1,-1)\}$ from $(0,0)$ to $(2n,0)$ that do not go below the horizontal axis ($y=0$). André [1] gave a solution to this problem (and more general ones) utilising a reflection principle argument which we want to repeat now. As a first remark, we note that the number of "good walks" (those that do not go below the horizontal axis) is equal to the total number of walks minus the number of "bad walks" (walks that go below the horizontal axis). Now, the *basic reflection principle argument* goes as follows (see Figure 1 for an illustration). Imagine a typical "bad walk". If we reflect the initial part of this walk up to the first contact with the horizontal line $y=-1$ at this very line, we obtain a walk starting in $(0,-2)$ and ending at $(2n,0)$. This clearly sets up a bijection between the set of "bad walks" and the set of walks from $(0,-2)$ to $(2n,0)$. By basic combinatorial arguments, the cardinality of the latter set is seen to be equal to $\binom{2n}{n-1}$. Consequently, the number of "good paths" is given by (the total number of paths from $(0,0)$ to $(2n,0)$ is equal to $\binom{2n}{n}$)

$$\binom{2n}{n}-\binom{2n}{n-1}=\frac{1}{n+1}\binom{2n}{n}.$$

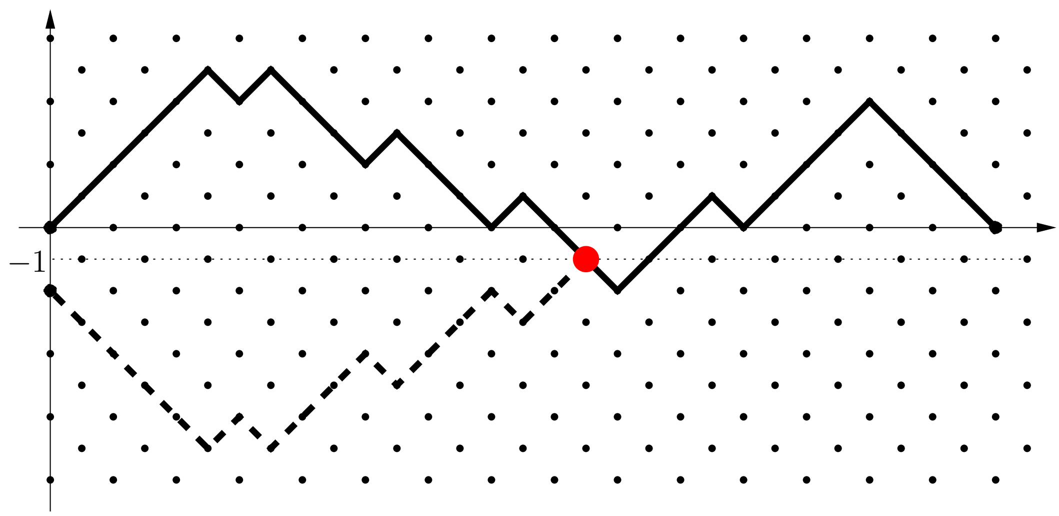

FIGURE 1. Illustration of the basic reflection principle. The solid path is a typical "bad walk" from $(0,0)$ to $(30,0)$. Reflecting the initial part of this walk up to the first contact with the horizontal line $y=-1$ at this line yields a walk from $(0,-2)$ to $(30,0)$.

Generalisations of this basic reflection principle to higher dimensions have been given in [45, 46], while q-analogues can be found in [31, 32]. Gessel and Zeilberger [21] formulated a general reflection principle argument for lattice walks confined to regions (Weyl chambers) associated with reflection groups.

In order to state Gessel and Zeilberger's theorem, we first need to present some notation concerning reflection groups. This presentation will be very brief. For details on reflection groups (or Coxeter groups) we refer the reader to Humphreys [26].

A *root system* is a finite set of vectors (the *roots*) of $\mathbb{R}^k$ satisfying the following properties: the set is invariant under reflection in any of the hyperplanes orthogonal to one of the roots; the difference of any root and its mirror image with respect to any such hyperplane is an integer multiple of the root corresponding to this hyperplane. The Coxeter group W (or Weyl group) associated with the root system is the set of linear transformations generated by the set of reflections in hyperplanes orthogonal to one of the roots. The connected components of the complement of the union of all hyperplanes are called *Weyl chambers*. For any element w of the Weyl group W and any simple system Δ of the root system, we define the *length* of w, denoted by $l(w)$, as the minimal number of terms needed to express w as a product of reflections σ_α, $\alpha \in \Delta$. The fundamental Weyl chamber C associated with Δ is defined as

$$C = \left\{ x \in \mathbb{R}^k \ : \ \text{for all } \alpha \in \Delta \text{ we have } \langle x, \alpha \rangle > 0 \right\},$$

where $\langle x, \alpha \rangle$ denotes the usual Euclidean scalar product on $\mathbb{R}^k$. Finally, by $\bar{C}$ we denote the closure of C in $\mathbb{R}^k$.

Let us now turn our attention to lattice walks in $\mathbb{R}^k$. Fix an arbitrary lattice $\mathcal{L}$ in $\mathbb{R}^k$. A *lattice walk of length* n is a sequence $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ of $n+1$ elements of $\mathcal{L}$. Alternatively, the lattice walk can be described by a starting point $\mathbf{x}_0$ together with a sequence $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of length n of *steps*. Clearly, we have the relation $\mathbf{v}_t = \mathbf{x}_t - \mathbf{x}_{t-1}$, $t = 1, 2, \ldots, n$. In the following, we will not distinguish between these two possible representations for lattice walks. Furthermore, the set of possible steps is always assumed to be a finite set.

We are interested in the number of lattice walks that are confined to the Weyl chamber C corresponding to an arbitrary Weyl group. Under certain assumptions about the lattice $\mathcal{L}$ and the set of allowed steps it is possible to give a nice formula for this number. This is a result due to Gessel and Zeilberger [21] and is the content of Theorem 1 below. The proof relies on a generalised reflection principle argument. Roughly speaking, the following theorem applies in cases where the lattice and the set of steps are such that it is not possible to exit $\bar{C}$ from a lattice point within C in a single step. As a consequence, every walk failing to stay within C contains a lattice point on the boundary of $\bar{C}$.

THEOREM 1 (see Gessel and Zeilberger [21, Theorem 1]). *Fix a Weyl group W and a simple system Δ with associated fundamental Weyl chamber C. Let $\mathcal{L}$ be a lattice in $\mathbb{R}^k$, and let $\mathcal{S}$ be a finite set of differences of points in $\mathcal{L}$, and assume that both sets, $\mathcal{L}$ and $\mathcal{S}$, are invariant under the action of the Weyl Group.*

If for any $\lambda \in C \cap \mathcal{L}$ and any $s \in \mathcal{S}$ we have $\lambda + s \in \bar{C} \cap \mathcal{L}$, then the total number of walks of length n from $a \in C \cap \mathcal{L}$ to $b \in C \cap \mathcal{L}$ with steps from the set $\mathcal{S}$ is equal to

$$\sum_{w \in W} (-1)^{l(w)} P_n(w(a) \to b),$$

where $P_n(a \to b)$ denotes the total number of lattice walks of length n from a to b.

2. Non-intersecting walks on an acyclic graph

Let $G = (V, E)$ be a directed acyclic graph with vertex set V and edge set E. A *walk on* G of length n is a sequence $(v_0, v_1, \ldots, v_n)$ of $n+1$ points such that for $j = 0, 1, \ldots, n-1$ we have $(v_j, v_{j+1}) \in E$ (that is, the walk moves along edges in G). Alternatively, we can represent this walk by the starting point v_0 together with the step sequence $(e_1, \ldots, e_n)$, where $e_j = (v_{j-1}, v_j) \in E$. Again, we will not distinguish between these two representations. The set of all walks from $a \in V$ to $b \in V$ will be denoted by $\{a \to b\}$.

Two walks on G are said to be *non-intersecting* if the sets of vertices corresponding to these walks are disjoint. We are interested in the total number of non-intersecting walks on G for given sets of starting points and end points. Under certain assumptions on the underlying graph and the sets of starting and end points, Gessel and Viennot [20, Corollary 3] as well as Lindström [36, Lemma 1] proved that this number is equal to a certain determinant. This enumeration result is the content of Theorem 2 below. For the special case when the graph is the two dimensional $\mathbb{Z}$-lattice spanned by the vectors $\{(1,1),(1,-1)\}$ (all edges are oriented from left to right), this theorem is essentially a discrete version of a theorem by Karlin and McGregor [27] for the transition density function of the *absorbing Brownian motion.*

Theorem 2 (Gessel and Viennot [20, Corollary 3], Lindström [36, Lemma 1]). *Let $G = (V, E)$ be a directed acyclic graph with vertex set V and edge set E. Fix two sets $\mathcal{A} = \{a_1, \ldots, a_k\}$ and $\mathcal{B} = \{b_1, \ldots, b_k\}$ of vertices, all of which are distinct.*

If the sets $\mathcal{A}$ and $\mathcal{B}$ are such that for all $i < j$ an all $k < l$ every path in $\{a_i \to b_l\}$ intersects every path in $\{a_j \to b_k\}$ in at least one vertex, then the total number of non-intersecting lattice paths, where the j-th path runs from a_j to b_j, is given by

$$\det_{1 \le i,j \le k} \left(P(a_i \to b_j) \right),$$

where $P(u \to v)$ denotes the cardinality of the set $\{u \to v\}$.

Proof (sketch). We want to sketch the proof for the case $k = 2$, where the graph G is the $\mathbb{Z}$-lattice spanned by the vectors $\{(1,1),(1,-1)\}$. In this case we have

$$\det_{1 \le i,j \le 2} \left(P(a_i \to b_j) \right) = P(a_1 \to b_1) P(a_2 \to b_2) - P(a_1 \to b_2) P(a_2 \to b_1).$$

Now, the basic idea is to show that the contribution of intersecting sets of walks in the difference on the right hand side is equal to zero. This is accomplished by the following bijection (for an illustration, see Figure 2). Consider a typical intersecting pair of walks from the set $\{a_1 \to b_1\} \times \{a_2 \to b_2\}$ (for an example, see Figure 2). Exchanging the initial parts of boths walks up to their first common vertex yields an element of $\{a_1 \to b_2\} \times \{a_2 \to b_1\}$. This shows that the contribution of intersecting pairs of walks to the difference on the right above is equal to zero. Consequently, the determinant is equal to the number of non-intersecting pairs of walks in the set $\{a_1 \to b_1\} \times \{a_2 \to b_2\}$.

The proof for the case $k > 2$ is pretty similar, but requires a bit more care and needs a refined notion of "first common vertex". We omit the details. □

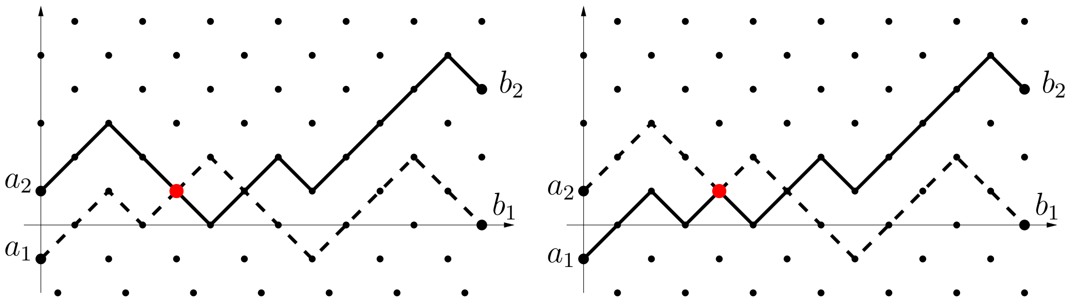

FIGURE 2. Illustration of the bijection on the set of intersecting pairs of walks for the case $k = 2$ and the $\mathbb{Z}$-lattice spanned by the vectors $\{(1,1),(1,-1)\}$ used in the proof of Theorem 2. The red point indicates the first common vertex of both walks.

3. Walks in a Weyl chamber of type A or B and non-intersecting lattice paths on the square grid

In the following, we denote by $\mathcal{L}$ the $\mathbb{Z}$-lattice spanned by the set of vectors $\mathcal{S} = \{(1,1),(1,-1)\}$, that is

$$\mathcal{L} = \left\{(i,j) \in \mathbb{Z}^2 \;:\; i \equiv j \mod 2\right\}.$$

Furthermore, we define

$$\mathcal{L}_{\geq 0} = \{(i,j) \in \mathcal{L} \;:\; j \geq 0\}.$$

By *non-intersecting lattice paths without wall restriction* and *non-intersecting lattice paths with wall restriction* we mean sets of non-intersecting walks with steps from the set $\mathcal{S}$ on either $\mathcal{L}$ or $\mathcal{L}_{\geq 0}$, respectively. Figure 3 (the picture on the right hand side) shows an example of a set of non-intersecting lattice paths with wall restriction. These non-intersecting lattice paths are in a close relationship with walks in a Weyl chamber of type A and type B, as will be shown in the following.

Let C_A denote the Weyl chamber of type A defined by

$$C_A = \left\{(x_1, \ldots, x_k) \in \mathbb{R}^k \;:\; x_1 < x_2 < \cdots < x_k\right\}.$$

The boundary of C_A is contained in the union of the hyperplanes (x_j refers to the j-th coordinate in $\mathbb{R}^k$)

$$x_i - x_j = 0, \qquad 1 \leq i < j \leq k.$$

The reflections in these hyperplanes form a generating set for the Weyl group of type A_{k-1}, which is isomporphic to the symmetric group S_k on k elements (see [26]). Finally, we let Λ be the $\mathbb{Z}$-lattice generated by the 2^k vectors $(\pm 1, \ldots, \pm 1)$. Now, let $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \Lambda^n$ denote a lattice walk on Λ of length n with steps of the form $(\pm 1, \ldots, \pm 1)$ that is confined to the Weyl chamber C_A. If we write $\mathbf{x}_j = (x_{j,1}, \ldots, x_{j,k})$, then we clearly have

$$x_{j,1} < x_{j,2} < \cdots < x_{j,k}, \qquad \text{for } j = 0, 1, \ldots n.$$

Consequently, the lattice paths $\Big((x_{0,m} - 1, 0), (x_{1,m} - 1, 1), \ldots, (x_{n,m} - 1, n)\Big)$, $m = 1, 2, \ldots, k$, form a set of non-intersecting lattice paths on $\mathcal{L}$ with steps from the set $\mathcal{S}$.

If, instead, we consider lattice walks of the same type confined to the Weyl chamber of type B given by

$$C_B = \left\{(x_1, \dots, x_k) \in \mathbb{R}^k \ : \ 0 < x_1 < x_2 < \cdots < x_k\right\},$$

then we obtain by the same procedure as above a set of non-intersecting lattice walks on $\mathcal{L}_{\geq 0}$ with steps from the set $\mathcal{S}$. We omit the details and refer to Figure 3 for an illustration of this correspondence in this latter case.

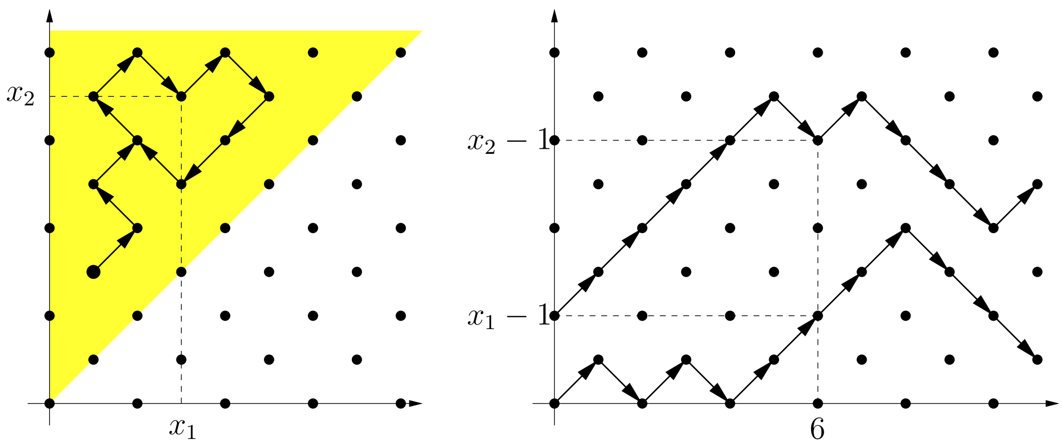

FIGURE 3. Illustration of the correspondence of lattice walks in a Weyl chamber of type B and non-intersecting lattice paths with wall restriction. The lower path in the right picture corresponds to the horizontal coordinate of the path in the left picture minus one, while the upper path in the right picture corresponds to the vertical coordinate of the walk in the left picture minus one. This correspondence is indicated for the position of the paths after six steps. The coloured region in the left picture indicates the Weyl chamber $0 < x_1 < x_2$. (The walk is restricted to the interior of this region).

The two correspondences described above show that there is some overlap of Theorems 1 and 2. In fact, since the Weyl group of type A_{k-1} is isomorphic to the symmetric group S_k on k elements (see [26]), Theorem 1 yields for the total number of lattice walks of length n on Λ with steps of the form $(\pm 1, \dots, \pm 1)$ from $\mathbf{a} \in \Lambda$ to $\mathbf{b} \in \Lambda$ (we assume $b_j - a_j \equiv n \mod 2$, $j = 1, \dots, k$) confined to C_A the expression

$$\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) P_n\Big((a_{\sigma(1)}, \dots, a_{\sigma(k)}) \to \mathbf{b}\Big).$$

Noting that

$$P_n\Big((a_{\sigma(1)}, \dots, a_{\sigma(k)}) \to (b_1, \dots, b_k)\Big) = \prod_{j=1}^{k} \binom{n}{(n - a_{\sigma(j)} + b_j)/2}$$

we see that the expression above is equal to

$$\det_{1 \leq j,m \leq k} \left(\binom{n}{(n - a_m + b_j)/2}\right).$$

This last expression is exactly what we would have obtained from Theorem 2 for the total number of sets of non-intersecting lattice paths on $\mathcal{L}$, where the j-th path runs from $(0, a_j)$ to (n, b_j).

Analogously, we see by Theorem 1 and 2, respectively, that the total number of walks of length n on Λ confined to C_B and the total number of sets of non-intersecting lattice paths on $\mathcal{L}_{\geq 0}$ are equal to

$$\sum_{\varepsilon_1,\ldots,\varepsilon_k\in\{-1,+1\}} \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) \prod_{j=1}^{k} \varepsilon_j \binom{n}{(n-\varepsilon_j a_{\sigma(j)}+b_j)/2} = \det_{1\leq j,m\leq k}\left(\binom{n}{(n+a_m-b_j)/2} - \binom{n}{(n-a_m-b_j)/2} \right).$$

4. Overview of Paper A

We consider lattice walks in $\mathbb{R}^k$ confined to the region $0 < x_1 < \cdots < x_k$, a Weyl chamber of type B, for a general set of steps. The allowed sets of steps are such that a generalised version of Theorem 1 gives the total number of such walks. For details, we refer to [Paper A, Lemma 2.1]. The main results of this work are [Paper A, Theorem 5.1] and [Paper A, Theorem 6.2], in which we, respectively, determine asymptotics for the number of such walks with either a fixed end point or a free end point as the number of steps tends to infinity.

As applications, we determine asymptotics for the number of so-called k-non-crossing tangled diagrams with and without isolated points. This solves a problem raised by Chen et al. [6], who could only determine the asymptotic growth order of these objects. The solution to this problem essentially relies on the generalisation of Theorem 1 mentioned above. Additionally, we determine asymptotics for vicious walks with wall restriction in the lock step model as well as in the random turns model for an arbitrary starting point and either an arbitrary end point or a free end point (for definitions, we directly refer to [Paper A, Section 7]). This completes a partial result in [23]. As special cases, these results also include asymptotics for certain configurations of vicious walks in the lock step model derived in [33, 40]. The asymptotics for the number vicious walks in the random turns model seem to be new.

The principle idea of the proofs for [Paper A, Theorem 5.1 and Theorem 6.2] is as follows. Starting from [Paper A, Lemma 2.1] (the generalised version of Theorem 1), we derive a contour integral representation for the number of walks of interest. Asymptotics for this integral can then be determined by means of saddlepoint techniques. The main difficulty here lies in the fact that we need to determine asymptotics for certain determinants. This problem is surmounted by means of a general technique that we develop in [Paper A, Section 4].

5. Overview of Paper B

Watermelons with wall are certain special configurations of non-intersecting lattice paths on $\mathcal{L}_{\geq 0}$ with specifically chosen starting and end points. More precisely, a k-watermelon with wall of length $2n$ is a set of non-intersecting lattice paths on $\mathcal{L}_{\geq 0}$, where the j-th paths starts at $(0, 2j-2)$ and ends at $(2n, 2j-2)$, $j = 1, 2, \ldots, k$. The *height* of a watermelon is defined as the maximum ordinate of its top most path (for an illustration, see [Paper B, Figure 1]).

Assuming the uniform probability measure on the set of k-watermelons with wall of length $2n$, we study the parameter "height" on this set and determine its asymptotic cumulative distribution function as well as asymptotics for the moments of this random variable. More precisely, we show that the s-th moment of the random variable "height" behaves like $s\kappa_s^{(p)} n^{s/2} - 3\binom{s}{2}\kappa_{s-1}^{(p)} n^{(s-1)/2} + O(n^{s/2-1} + n^{p/2-p^2}\log n)$ as the number $2n$ of steps tends to infinity, for some explicit constants κ_s. The precise statement of this result is the content of [Paper B, Theorem 1]. This generalises a well-known result by de Bruijn, Knuth and Rice [8] on the average height of planted plane trees (which, in fact, are in bijection 1-watermelons). Our result on the limiting distribution ([Paper B, Theorem 2]) contains as a special case a result in [39] on the height of trees.

The main problem to solve is the determination of asymptotics for certain harmonic sums involving determinants. The difficulty here lies again in the fact that, due to the determinants, there happens a large number of cancellation of asymptotically leading terms. One of the key problems in Paper B is to determine first surviving term of these harmonic sums (see [Paper B, Lemma 8]). Furthermore, the solution given in Paper B involves Mellin transform techniques, and requires the understanding of the analytic behaviour of certain multidimensional Dirichlet series (for details, we refer to [Paper B, Section 2]). This analysis is accomplished very much in the spirit of one of Riemann's methods for the analytic continuation of the Riemann zeta function (see, e.g., [43, Section 2.6]), and is based on a reciprocity relation for derivatives of one of Jacobi's theta functions (see [Paper B, Section 2]. This reciprocity relation generalises the well-known reciprocity law for Jacobi's theta functions (see, e.g., [35, Section 2.3]), and, to the author's best knowledge, seems to be new.

After distribution of the first version of Paper B, Schehr et al. [41] and Katori et al. [28, 29] re-derived the asymptotically dominant terms for some of the quantities studied in Paper B. The methods in [28] are very much in parallel to the approach taken in Paper B and rely essentially on [Paper B, Corollary 1] (see also [28, Lemma 3]).

6. Overview of Paper C

Watermelons without wall are non-intersecting lattice paths on $\mathcal{L}$ with specifically chosen starting and end points. More precisely, a k-watermelon without wall of length $2n$ is a set of non-intersecting lattice paths on $\mathcal{L}$, where the j-th paths starts at $(0, 2j-2)$ and ends at $(2n, 2j-2)$, $j = 1, 2, \dots, k$. As in the case of watermelons with wall, the "height" of a watermelon is defined as the maximum ordinate of its top most branch. A second interesting parameter that is also studied in Paper C is the "range" of a watermelon, which is the difference between the maximum ordinate of its top most branch and the minimum ordinate of its bottom most branch.

Assuming the uniform probability measure on the set of watermelons without wall of length $2n$, we study the random variables "height" and "range" on this set, and determine the cumulative distribution of the limiting laws as the number $2n$ of steps tends to infinity (see [Paper C, Theorem 1] and [Paper C, Theorem 3], respectively). Additionally, we derive asymptotics for the moments of the random variable "height", and, more precisely, show that for $s \geq 2$ the s-th moment of the random variable "height" behaves like $s\mu_s n^{s/2} + (s-1)(p-1-s/2)\mu_{s-1} n^{(s-1)/2} + O\left(n^{s/2-1}\right)$ as $n \to \infty$, for some explicit numbers μ_s. For $s = 1$, the second order term is equal to $p - 3/2$.

Exact expressions in terms of determinants for all quantities in question can easily be obtained from Theorem 2 (see also [Paper C, Lemma 1]). In order to derive asymptotics from these expressions we need to determine asymptotics for determinants involving binomial coefficients that cannot be evaluated in a closed form expression. This is accomplished with the help Stirling's approximation for the factorials together with techniques to determine asymptotics for the resulting determinants. Again, the main difficulty lies in the fact that there occurs a large number of cancellations of asymptotically leading terms, so that it is a non-trivial task to determine the first surviving term in the asymptotic expansion for these determinants. The final expression for the asymptotics of the moments of the random variable "height" is obtained by observing a non-obvious relation between two determinants (see [Paper C, Lemma 8]).

The limiting distribution for the random variable "height" has been re-derived by Schehr et al. [41]. Since, at first sight, their expression differs considerably from the one given in [Paper C, Theorem 1], we show how derive the expression given in [41] from our expression (see Remark 2).

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PAPER A

Asymptotics for the number of walks in a Weyl chamber of type B

ASYMPTOTICS FOR THE NUMBER OF WALKS IN A WEYL CHAMBER OF TYPE B

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Abstract. We consider lattice walks in $\mathbb{R}^k$ confined to the region $0 < x_1 < x_2 ... < x_k$ with fixed (but arbitrary) starting and end points. The walks are required to be "reflectable", that is, we assume that the number of paths can be counted using the reflection principle. The main results are asymptotic formulas for the total number of walks of length n with either a fixed or a free end point for a general class of walks as n tends to infinity. As applications, we find the asymptotics for the number of k-non-crossing tangled diagrams on the set $\{1, 2, ..., n\}$ as n tends to infinity, and asymptotics for the number of k-vicious walkers subject to a wall restriction in the random turns model as well as in the lock step model. Asymptotics for all of these objects were either known only for certain special cases, or have only been partially determined or were completely unknown.

1. Introduction

Lattice paths are well-studied objects in combinatorics as well as in probability theory. A typical problem that is often encountered is the determination of the number of lattice paths that stay within a certain fixed region. In many situations, this region can be identified with a Weyl chamber corresponding to some reflection group. In this paper, the region is a Weyl chamber of type B, and, more precisely, it is given by $0 < x_1 < \cdots < x_k$. (Here, x_j refers to the j-th coordinate in $\mathbb{R}^k$.)

Under certain assumptions on the set of allowed steps and on the underlying lattice, the total number of paths as described above can be counted using the *reflection principle* as formulated by Gessel and Zeilberger [9]. This reflection principle is a generalisation of a reflection argument, which is often attributed to André [1], to the context of general finite reflection groups (for details on reflection groups, see [14]).

A necessary and sufficient condition on the set of steps for ensuring the applicability of the reflection principle as formulated by Gessel and Zeilberger [9] has been given by Grabiner and Magyar [12]. In their paper, Grabiner and Magyar also stated a precise list of steps that satisfy these conditions.

In a recent paper that attracted the author's interest, and that was also the main initial motivation for this work, Chen et al. [5, Obervations 1 and 2] gave lattice path descriptions for combinatorial objects called *k-non-crossing tangled diagrams*. In their work, they determined the order of asymptotic growth of these objects, but they did not succeed in determining precise asymptotics. Interestingly, the sets of steps appearing in this description do not satisfy Grabiner and Magyar's condition. Nevertheless, a slightly generalised reflection principle turns out to be applicable because the steps can be interpreted as sequences of certain *atomic steps*,

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where these atomic steps satisfy Grabiner and Magyar's condition. In this manuscript, we state a generalised reflection principle that applies to walks consisting of steps that are sequences of such atomic steps (see Lemma 2.1 below).

Our main results are asymptotic formulas for the total number of walks as the number of steps tends to infinity that stay within the region $0 < x_1 < \cdots < x_k$, with either a fixed end point or a free end point (see Theorem 5.1 and Theorem 6.2, respectively). The starting point of our walks may be chosen anywhere within the allowed region. The proofs of the main results can be roughly summarised as follows. Using a generating function approach, we are able to express the number of walks that we are interested in as a certain coefficient in a specific Laurent polynomial. We then express this coefficient as a Cauchy integral and extract asymptotics with the help of saddle point techniques. Of course, there are some technical problems in between that we have to overcome. The most significant comes from the fact that we have to determine asymptotics for a determinant. The problem here is the large number of cancellations of asymptotically leading terms. It is surmounted by means of a general technique that is presented in Section 4. As a corollary to our main results, we obtain precise asymptotics for k-non-crossing tangled diagrams with and without isolated points (for details, see Section 7). Moreover, we find asymptotics for the number of vicious walks with a wall restriction in the lock step model as well as asymptotics for the number of vicious walks with a wall restriction in the random turns model. Special instances of our asymptotic formula for the total number of vicious walks in the lock step model have been established by Krattenthaler et al. [16, 17] and Rubey [21]. The growth order for the number of vicious walks in the lock step model with a free and point, and for the number of k-non-crossing tangled diagrams has been determined by Grabiner [11] and Chen et al. [4], respectively. To the author's best knowledge, the asymptotics for the number of vicious walks in the random turns model seem to be new.

In some sense, one of the achievements of the present work is that it shows how to overcome a technical difficulty put to the fore in [24]. In order to explain this remark, we recall that Tate and Zelditch [24] determined asymptotics of multiplicities of weights in tensor powers, which are related to reflectable lattice paths in a Weyl chamber (for details, we refer to [12, Theorem 2]). For the so-called *central limit region of irreducible multiplicities* (for definition, we directly refer to [24]) they did not manage to determine the asymptotic behaviour of these multiplicities, and, therefore, had to resort to a result of Biane [2, Théorème 2.2]. More precisely, although they were able to obtain the (indeed correct) dominant asymptotic term in a formal manner, they were not able to actually prove its validity by establishing a sufficient bound on the error term. For a detailed elaboration on this problem we refer to the paragraph after [24, Theorem 8]. The techniques applied in [24] are in fact quite similar to those applied in this manuscript (namely, the Weyl character formula/reflection principle and saddle point techniques). However, it is the above mentioned technique presented in Section 4 which forms the key to resolve the problem by providing sufficiently small error bounds in situations as the one described by Tate and Zelditch [24].

The paper is organised as follows. In the next section, we give the basic definitions and precise description of the lattice walk model underlying this work. We also state and prove a slightly generalised reflection principle (see Lemma 2.1 below) that can be used to count the number of lattice walks in our model. At the end of this section, we prove an exact integral formula for this number. In Section 3, we determine the possible step sets the walks in our model may consist of. Additionally, we state and prove some technical results. This allows us to give the proofs of our main results in a more accessible manner. Section 4 presents a factorisation technique for certain functions defined by determinants. These results are crucial to our proofs since they enable us to determine precise asymptotics for these functions. Our main results, namely asymptotics for total number of random walks with a fixed end point and with a free end point, are the content of Section 5 and Section 6, respectively. The last section presents applications of our main results, namely Theorem 5.1 and Theorem 6.2. Here we determine asymptotics for the number of vicious walks with a wall restriction in the lock step model as well as asymptotics for the number of vicious walks with a wall restriction in the random turns model. Furthermore, we determine precise asymptotics for the number of k-non-crossing tangled diagrams on the set $\{1, 2, \ldots, n\}$ as n tends to infinity. This generalises results by Krattenthaler et al. [16, 17] and Rubey [21]. Additionally, we provide precise asymptotic formulas for counting problems for which only the asymptotic growth order has been established. In particular, we give precise asymptotics for the total number of vicious walkers with wall restriction and free end point, as well as precise asymptotics for the number of k-non-crossing tangled diagrams with and without isolated points. (The growth order for the former objects has been established by Grabiner [11], whereas the growth order for the latter objects has been determined by Chen et al. [5].)

2. Reflectable walks of type B

The intention of this section is twofold. First, we give a precise description of the lattice walk model underlying this work, and state some basic results. Second, we derive an *exact integral formula* (see Lemma 2.3 below) for the generating function of lattice walks in this model with respect to a given weight.

Let us start with the presentation of the lattice path model. We will have two kind of steps: atomic steps and composite steps. *Atomic steps* are elements of $\mathbb{R}^k$. The set of all atomic steps in our model will always be denoted by $\mathcal{A}$. *Composite steps* are finite sequences of atomic steps. The set of composite steps in our model will be always be denoted by $\mathcal{S}$. Both sets, $\mathcal{A}$ and $\mathcal{S}$, are assumed to be finite sets. By $\mathcal{L}$ we denote the $\mathbb{Z}$-lattice spanned by the atomic step set $\mathcal{A}$.

The walks in our model are walks on the lattice $\mathcal{L}$ consisting of steps from the composite step set $\mathcal{S}$ that are confined to the region

$$\mathcal{W}^0 = \left\{(x_1, \ldots, x_k) \in \mathbb{R}^k \;:\; 0 < x_1 < \cdots < x_k\right\}.$$

For a given function $w : \mathcal{S} \to \mathbb{R}_+$, called the *weight function*, we define the weight of a walk with step sequence $(\mathbf{s}_1, \ldots, \mathbf{s}_n) \in \mathcal{S}^n$ by $\prod_{j=1}^n w(\mathbf{s}_j)$.

The generating function for all n-step paths from $\mathbf{u} \in \mathcal{L}$ to $\mathbf{v} \in \mathcal{L}$ with respect to the weight w will be denoted by $P_n(\mathbf{u} \to \mathbf{v})$, that is,

$$P_n(\mathbf{u} \to \mathbf{v}) = \sum_{\substack{\mathbf{s}_1, \ldots, \mathbf{s}_n \in \mathcal{S} \\ \mathbf{u} + \mathbf{s}_1 + \cdots + \mathbf{s}_n = \mathbf{v}}} \prod_{j=1}^n w(\mathbf{s}_j),$$

and the generating function of those paths of length n from $\mathbf{u}$ to $\mathbf{v}$ with respect to the weight w that stay within the region $\mathcal{W}^0$ will be denoted by $P_n^+(\mathbf{u} \to \mathbf{v})$.

The ultimate goal of this work is the derivation of an asymptotic formula for $P_n^+(\mathbf{u} \to \mathbf{v})$ as n tends to infinity for certain step sets $\mathcal{S}$ and certain weight functions w.

In the theory of reflection groups (or Coxeter groups), $\mathcal{W}^0$ is called a *Weyl chamber of type* B_k. By $\mathcal{W}$, we denote the closure of $\mathcal{W}^0$, viz.

$$\mathcal{W} = \left\{(x_1, \dots, x_k) \in \mathbb{R}^k \ : \ 0 \le x_1 \le \dots \le x_k\right\},$$

The boundary of $\mathcal{W}$ is contained in the union of the hyperplanes

$$x_i - x_j = 0 \quad \text{for } 1 \le i < j \le k, \qquad \text{and} \qquad x_1 = 0. \tag{2.1}$$

The set of reflections in these hyperplanes is a generating set for the finite reflection group of type B_k (see Humphreys [14]).

We would like to point out that all results presented in this section have analogues for all general finite or affine reflection groups. In order to keep this section as short and simple as possible, we restrict our presentation to the type B_k case. For the general results, we refer the interested reader to the corresponding literature. A good introduction to the theory of reflection groups can be found in the standard reference book by Humphreys [14].

The fundamental assumption underlying this manuscript is the applicability of a reflection principle argument to the problem of counting walks with n composite steps that stay within the region $\mathcal{W}^0$. Such a reflection principle has been proved by Gessel and Zeilberger [9] for lattice walks in Weyl chambers of arbitrary type that consist of steps from an atomic step set. We need to slightly extend their result for Weyl chambers of type B_k to walks consisting of steps from a composite step set. The precise result is stated in the following lemma, and is followed by a short sketch of its proof.

Lemma 2.1 (Reflection Principle)**.** *Let $\mathcal{A}$ be an atomic step set that is invariant under the reflection group generated by the reflections* (2.1)*, and such that for all $\mathbf{a} \in \mathcal{A}$ and all $\mathbf{u} \in \mathcal{W}^0 \cap \mathcal{L}$ we have $\mathbf{u} + \mathbf{a} \in \mathcal{W}$. By $\mathcal{S}$ we denote a composite step set over $\mathcal{A}$ such that for all $(\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathcal{S}$ we also have $(\rho(\mathbf{a}_1), \dots, \rho(\mathbf{a}_j), \mathbf{a}_{j+1}, \dots, \mathbf{a}_m) \in \mathcal{S}$ for all $j = 1, 2, \dots, m$ and all reflections ρ in the group generated by* (2.1)*. Finally, assume that the weight function $w : \mathcal{S} \to \mathbb{R}_+$ satisfies $w((\mathbf{a}_1, \dots, \mathbf{a}_m)) = w((\rho(\mathbf{a}_1), \dots, \rho(\mathbf{a}_j), \mathbf{a}_{j+1}, \dots, \mathbf{a}_m))$ for all j and ρ as before.*

Then, for all $\mathbf{u} = (u_1, \dots, u_k) \in \mathcal{W}^0 \cap \mathcal{L}$ and all $\mathbf{v} \in \mathcal{W}^0 \cap \mathcal{L}$, the generating function for all n-step walks with steps from the composite step set $\mathcal{S}$ with respect to the weight w that stay within $\mathcal{W}^0$ satisfies

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \varepsilon_1, \dots, \varepsilon_k \in \{-1,+1\}}} \left(\prod_{j=1}^k e_j\right) \operatorname{sgn}(\sigma) P_n\Big((\varepsilon_1 u_{\sigma(1)}, \dots, \varepsilon_k u_{\sigma(k)}) \to \mathbf{v}\Big), \tag{2.2}$$

where $\mathfrak{S}_k$ is the set of all permutations on $\{1, \dots, k\}$.

Proof (Sketch). The proof of this lemma is almost identical to the proof of the reflection principle for lattice walks consisting of atomic steps in [9]. The basic idea of the proof is the following. We set up an involution on the set of n-step walks starting in one of the points$(\rho(a_1), \dots, \rho(a_k))$, where ρ denotes an arbitrary reflection in the group generated by (2.1), to $\mathbf{v}$ that percolate or touch the boundary of $\mathcal{W}$. For a typical such walk we then show that the contributions of it and its image under this involution to the right hand side of (2.2) differ by sign only. This shows that the total contribution of n-step walks percolating or touching the boundary of $\mathcal{W}$ to the right hand side of (2.2) is equal to zero.

This involution is constructed with the help of the involution defined in the proof of [9, Theorem 1] as follows. Consider the walk starting in $(\rho(u_1),\dots,\rho(u_k))$ with step sequence

$$\big((\mathbf{a}_{1,1},\dots,\mathbf{a}_{1,m_j}),(\mathbf{a}_{2,1},\dots,\mathbf{a}_{2,m_2}),\cdots,(\mathbf{a}_{n,1},\dots,\mathbf{a}_{n,m_n})\big)\in\mathcal{S}^n,$$

where the $\mathbf{a}_{j,\ell}$ denote atomic steps. If we ignore all the inner brackets in the step sequence above, we can view this walk as a walk starting $(\rho(u_1),\dots,\rho(u_k))$ that consists of $(m_1+\cdots+m_n)$ atomic steps. To this walk, we can apply the involution of the proof of [9, Theorem 1].

For example, assume that the first contact of the walker with the boundary of $\mathcal{W}$ occurs right after the atomic step $\mathbf{a}_{j,\ell}$. Then, the image of this path under the involution is the path starting in $(\tau(\rho(u_1)),\dots,\tau(\rho(u_k)))$ with step sequence

$$\big((\tau(\mathbf{a}_{1,1}),\dots,\tau(\mathbf{a}_{1,m_1})),\cdots,(\tau(\mathbf{a}_{j,1}),\dots,\tau(\mathbf{a}_{j,\ell}),\mathbf{a}_{j,\ell+1},\dots,\mathbf{a}_{j,m_j}),\cdots,(\mathbf{a}_{n,1},\dots,\mathbf{a}_{n,m_n})\big),$$

for a specifically chosen reflection τ in one of the hyperplanes (2.1).

For a details, we refer the reader to the proof of [9, Theorem 1]. □

In view of this last lemma, the question that now arises is: what composite step sets $\mathcal{S}$ satisfy the conditions in Lemma 2.1? This question boils down the question: what atomic step sets $\mathcal{A}$ satisfy the conditions in Lemma 2.1? The answer to this latter question has been given by Grabiner and Magyar [12]. For type B, the result reads as follows.

Lemma 2.2 (Grabiner and Magyar [12])**.** *The atomic step set $\mathcal{A}\subset\mathbb{R}^k\setminus\{\mathbf{0}\}$ satisfies the conditions stated in Lemma 2.1 if and only if $\mathcal{A}$ is (up to rescaling) equal either to*

$$\big\{\pm\mathbf{e}^{(1)},\pm\mathbf{e}^{(2)},\dots,\pm\mathbf{e}^{(k)}\big\}\qquad\text{or to}\qquad\left\{\sum_{j=1}^{k}\varepsilon_j\mathbf{e}^{(j)}\ :\ \varepsilon_1,\dots,\varepsilon_k\in\{-1,+1\}\right\},$$

where $\big\{\mathbf{e}^{(1)},\dots,\mathbf{e}^{(k)}\big\}$ is the canonical basis in $\mathbb{R}^k$.

In this manuscript we will always assume that our lattice walk model satisfies all the requirements of Lemma 2.1. Therefore, we make the following assumption.

Assumption 2.1. From now on, we assume that the atomic step set $\mathcal{A}$ is equal to one of the two sets given in Lemma 2.2. Further, we assume that the composite step set $\mathcal{S}$ and the weight function $w:\mathcal{S}\to\mathbb{R}^k$ satisfy the conditions of Lemma 2.1.

The final objective in this section is an integral formula for $P_n^+(\mathbf{u}\to\mathbf{v})$. The result is stated in Lemma 2.3 below. Its derivation is based on a generating function approach.

In order to simplify the presentation, we apply the standard multi-index notation: If $\mathbf{z}=(z_1,\dots,z_k)$ is a vector of indeterminates and $\mathbf{a}=(a_1,\dots,a_k)\in\mathbb{Z}^k$, then we set $\mathbf{z}^{\mathbf{a}}:=z_1^{a_1}z_2^{a_2}\dots z_k^{a_k}$. Furthermore, if $F(\mathbf{z})$ is a series in $\mathbf{z}$, then we denote by $[\mathbf{z}^{\mathbf{a}}]F(\mathbf{z})$ the coefficient of the monomial $\mathbf{z}^{\mathbf{a}}$ in $F(\mathbf{z})$.

Now, we define the *atomic step generating function* $A(\mathbf{z})=A(z_1,\dots,z_k)$ associated with the atomic step set $\mathcal{A}$ by

$$A(z_1,\dots,z_k)=A(\mathbf{z})=\sum_{\mathbf{a}\in\mathcal{A}}\mathbf{z}^{\mathbf{a}}.$$

The *composite step generating function* associated with the composite step set $\mathcal{S}$ with respect to the weight w is defined by

$$S(z_1, \dots, z_k) = S(\mathbf{z}) = \sum_{\substack{m \geq 0 \\ (\mathbf{a}_1, \dots, \mathbf{a}_m) \in \mathcal{S}}} w\Big((\mathbf{a}_1, \dots, \mathbf{a}_m)\Big) \mathbf{z}^{\mathbf{a}_1 + \dots + \mathbf{a}_m}.$$

The generating function for the number of n-step paths with steps from the composite step set $\mathcal{S}$ that start in $\mathbf{u} \in \mathcal{L}$ and end in $\mathbf{v} \in \mathcal{L}$ with respect to the weight w can then be expressed as

$$P_n(\mathbf{u} \to \mathbf{v}) = \left[\mathbf{z}^{\mathbf{v}-\mathbf{u}}\right] S(\mathbf{z})^n. \tag{2.3}$$

We can now state and prove the main result of this section: the integral formula for $P_n^+(\mathbf{u} \to \mathbf{v})$.

Lemma 2.3. *Let $\mathcal{S}$ be a composite step set and let $w : \mathcal{S} \to \mathbb{R}_+$ be weight function, both satisfying Assumption 2.1. Furthermore, let $S(z_1, \dots, z_k)$ be the associated composite step generating function.*

Then the generating function $P_n^+(\mathbf{u} \to \mathbf{v})$ for the number of n-step paths from $\mathbf{u} \in \mathcal{W}^0 \cap \mathcal{L}$ to $\mathbf{v} \in \mathcal{W}^0 \cap \mathcal{L}$ that stay within $\mathcal{W}^0$ with steps from the composite step set S satisfies

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \frac{1}{(2\pi i)^k} \int \cdots \int_{|z_1| = \dots = |z_k| = \rho} \det_{1 \leq j,m \leq k} \left(z_j^{u_m} - z_j^{-u_m}\right) S(z_1, \dots, z_k)^n \left(\prod_{j=1}^k \frac{dz_j}{z_j^{v_j+1}}\right), \tag{2.4}$$

where $\rho > 0$.

Proof. The proof of this lemma relies on the reflection principle (Lemma 2.1) and Cauchy's integral formula.

Lemma 2.1 and Equation (2.3) together give us

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \sum_{\substack{\sigma \in \mathfrak{S}_k \\ (\varepsilon_1, \dots, \varepsilon_k) \in \{-1,+1\}^k}} \left(\prod_{j=1}^k \varepsilon_j\right) \operatorname{sgn}(\sigma) \left[z_1^{v_1 - \varepsilon_1 u_{\sigma(1)}} \dots z_k^{v_k - \varepsilon_k u_{\sigma(k)}}\right] S(z_1, \dots, z_k)^n,$$

and by Cauchy's integral formula, we have

$$\left[z_1^{v_1 - \varepsilon_1 u_{\sigma(1)}} \dots z_k^{v_k - \varepsilon_k u_{\sigma(k)}}\right] S(z_1, \dots, z_k)^n = \frac{1}{(2\pi i)^k} \int \cdots \int_{|z_1| = \dots = |z_k| = 1} S(z_1, \dots, z_k)^n \left(\prod_{j=1}^k \frac{dz_j}{z_j^{v_j - \varepsilon_k u_{\sigma(j)} + 1}}\right).$$

Now, substituting the right hand side of the last equation above for the corresponding term in the second to last equation, and interchanging summation and integration, we obtain the expression

$$\int \cdots \int_{|z_1| = \dots = |z_k| = 1} \frac{S(z_1, \dots, z_k)^n}{(2\pi i)^k} \left(\sum_{\substack{\sigma \in \mathfrak{S}_k \\ (\varepsilon_1, \dots, \varepsilon_k) \in \{-1,+1\}^k}} \operatorname{sgn}(\sigma) \left(\prod_{j=1}^k \varepsilon_j z_j^{\varepsilon_j u_{\sigma(j)}}\right)\right) \left(\prod_{j=1}^k \frac{dz_j}{z_j^{v_j+1}}\right).$$

The result now follows from this expression by noting that

$$\sum_{\substack{\sigma\in\mathfrak{S}_k \\ (\varepsilon_1,\dots,\varepsilon_k)\in\{-1,+1\}^k}} \left(\prod_{j=1}^k \varepsilon_j\right) \operatorname{sgn}(\sigma) \left(\prod_{j=1}^k z_j^{\varepsilon_j u_{\sigma(j)}}\right) = \det_{1\le j,m\le k}\left(z_j^{u_m} - z_j^{-u_m}\right).$$

□

We close this section with an alternative exact expression for the quantity $P_n^+(\mathbf{u}\to\mathbf{v})$.

Corollary 2.1. *Under the conditions of Lemma 2.3, the generating function $P_n^+(\mathbf{u}\to\mathbf{v})$ for the number of n-step paths from $\mathbf{u}\in\mathcal{W}^0\cap\mathcal{L}$ to $\mathbf{v}\in\mathcal{W}^0\cap\mathcal{L}$ that stay within $\mathcal{W}^0$ with steps from the composite step set S satisfies*

$$P_n^+(\mathbf{u}\to\mathbf{v}) = \frac{(-1)^k}{(2\pi i)^k k!} \times \int_{|z_1|=\dots=|z_k|=\rho}\cdots\int \det_{1\le j,m\le k}\left(z_j^{u_m} - z_j^{-u_m}\right) S(z_1,\dots,z_k)^n \det_{1\le j,m\le k}\left(z_j^{v_m}\right)\left(\prod_{j=1}^k \frac{dz_j}{z_j}\right),$$

where $\rho>0$.

Proof. The substitution $z_j\mapsto 1/z_j$, for $j=1,2,\dots,k$, transforms Equation (2.4) into

$$P_n^+(\mathbf{u}\to\mathbf{v}) = \frac{(-1)^k}{(2\pi i)^k}\int_{|z_1|=\dots=|z_k|=\rho}\cdots\int \det_{1\le j,m\le k}\left(z_j^{u_m} - z_j^{-u_m}\right) S(z_1,\dots,z_k)^n \left(\prod_{j=1}^k z_j^{v_j}\frac{dz_j}{z_j}\right).$$

Now, we make the following observation. If σ denotes an arbitrary permutation on the set $\{1,2,\dots,k\}$, then we have

$$\det_{1\le j,m\le k}\left(z_{\sigma(j)}^{u_m} - z_{\sigma(j)}^{-u_m}\right)\left(\prod_{j=1}^k z_{\sigma(j)}^{v_m}\right) = \det_{1\le j,m\le k}\left(z_j^{u_m} - z_j^{-u_m}\right)\left(\operatorname{sgn}(\sigma)\prod_{j=1}^k z_{\sigma(j)}^{v_m}\right),$$

which can be seen to be true by rearranging the rows of the determinant on the left hand side and taking into account the sign changes. This implies

$$P_n^+(\mathbf{u}\to\mathbf{v}) = \frac{(-1)^k}{(2\pi i)^k} \times \int_{|z_1|=\dots=|z_k|=\rho}\cdots\int \det_{1\le j,m\le k}\left(z_j^{u_m} - z_j^{-u_m}\right) S(z_1,\dots,z_k)^n \left(\operatorname{sgn}(\sigma)\prod_{j=1}^k z_{\sigma(j)}^{v_j}\frac{dz_j}{z_j}\right).$$

The claim is now proved upon summing the expression above over all $k!$ possible permutations and dividing the result by $k!$. □

3. Auxiliary results

In this section, we are going to derive some auxiliary results that we are going to use in the proof of our main results. In the first part of this section, we are going to have a closer look at composite step generating functions. At the end of this section, we present two rather technical results, that should be skipped at a first reading until they are used in the proof of our main result.

The proofs of Theorem 5.1 and Theorem 6.2 rely on some structural results for composite step generating functions $S(z_1,\dots,z_k)$ associated with composite step sets that satisfy Assumption 2.1 (the conditions of Lemma 2.1). These structural results are the content of the following lemmas.

A direct consequence of the classification of Grabiner and Magyar [12], presented in Lemma 2.2, is the following result on atomic step generating functions.

Lemma 3.1. *Let $\mathcal{A}$ be an atomic step set satisfying Assumption 2.1. Then the associated atomic step generating function $A(z_1,\dots,z_k)$ is equal either to*

$$\sum_{j=1}^{k}\left(z_j+\frac{1}{z_j}\right) \qquad \text{or to} \qquad \prod_{j=1}^{k}\left(z_j+\frac{1}{z_j}\right). \tag{3.5}$$

As a direct consequence of this last lemma, we obtain the following result.

Lemma 3.2. *Let $\mathcal{S}$ be composite step set over the atomic step set $\mathcal{A}$, and let $w:\mathcal{S}\to\mathbb{R}_+$ be a weight function. If $\mathcal{S}$, $\mathcal{A}$ and w satisfy Assumption 2.1, then there exists a polynomial $P(x)$ with non-negative coefficients such that either*

$$S(z_1,\dots,z_k)=P\left(\sum_{j=1}^{k}\left(z_j+\frac{1}{z_j}\right)\right) \qquad \text{or} \qquad S(z_1,\dots,z_k)=P\left(\prod_{j=1}^{k}\left(z_j+\frac{1}{z_j}\right)\right).$$

Proof. Let $A(z_1,\dots,z_k)$ denote the atomic step generating function corresponding to $\mathcal{A}$.

Our assumptions imply that if $(\mathbf{a}_1,\dots,\mathbf{a}_m)\in\mathcal{S}$, then we also have

$$(\rho(\mathbf{a}_1),\dots,\rho(\mathbf{a}_j),\mathbf{a}_{j+1},\dots,\mathbf{a}_m)\in\mathcal{S}, \qquad j=1,2,\dots,m$$

and all reflections ρ in the group generated by (2.1). This means that if the composite step set $\mathcal{S}$ contains a composite step consisting of m atomic steps, then $\mathcal{S}$ has to contain *all* composite steps consisting of m atomic steps. Also, our assumptions on w imply that the same weight is assigned to all composite steps consisting of the same number of atomic steps. Since the generating function for all composite steps consisting of m atomic steps is given by $A(z_1,\dots,z_k)^m$, we deduce that there exists a polynomial $P(x)$ with non-negative coefficients such that $S(z_1,\dots,z_k)=P(A(z_1,\dots,z_k))$. This fact, together with Lemma 3.1, proves the claim. ☐

Lemma 3.3. *Let $\mathcal{S}$ be a composite step set, and let $S(z_1,\dots,z_k)$ denote the associated composite step generating function. Further, let w be a weight function.*

If $\mathcal{S}$ and w satisfy Assumptions 2.1, then all maxima of the function $(\varphi_1,\dots,\varphi_k)\mapsto|S(e^{i\varphi_1},\dots,e^{i\varphi_k})|$ lie within the set $\{0,\pi\}^k$. The point $(\varphi_1,\dots,\varphi_k)=(0,\dots,0)$ is always a maximum.

Proof. From Lemma 3.2, we deduce that $S(e^{i\varphi_1},\dots,e^{i\varphi_k})$ is either equal to

$$P\left(2\sum_{j=1}^{k}\cos\varphi_j\right) \qquad \text{or to} \qquad P\left(2^k\prod_{j=1}^{k}\cos\varphi_j\right), \tag{3.6}$$

for some polynomial $P(x)$ with non-negative coefficients. Now, if $S(e^{i\varphi_1},\dots,e^{i\varphi_k})$ is equal to the expression on the left in (3.6), then the triangle inequality shows that

$$|S(e^{i\varphi_1},\dots,e^{i\varphi_k})| = \left|P\left(2\sum_{j=1}^{k}\cos(\varphi_j)\right)\right| \le P\left(2\sum_{j=1}^{k}|\cos(\varphi_j)|\right) \le S(1,\dots,1).$$

If $S(e^{i\varphi_1},\dots,e^{i\varphi_k})$ is equal to the expression on the right in (3.6), then similar arguments can be used to show the inequality $|S(e^{i\varphi_1},\dots,e^{i\varphi_k})| \le S(1,\dots,1)$ in this case. This inequality shows that $(0,\dots,0)$ is always a maximum of the function $(\varphi_1,\dots,\varphi_k)\mapsto|S(e^{i\varphi_1},\dots,e^{i\varphi_k})|$, and further, since $P(x)$ is monotonic increasing for $x>0$, that all points maximising this function lie within the set $\{0,\pi\}^k$. ☐

We end this section with two results of a rather technical nature.

Lemma 3.4. *Let $\mathcal{S}$ be an composite step set over the atomic step set $\mathcal{A}$, and assume that both sets, $\mathcal{A}$ and $\mathcal{S}$ satisfy Assumptions 2.1. The corresponding step generating functions are denoted by $S(z_1,\dots,z_k)$ and $A(z_1,\dots,z_k)$, respectively. Further, let $\mathbf{u}=(u_1,\dots,u_k)\in\mathcal{W}^0\cap\mathcal{L}$, $\mathbf{v}=(v_1,\dots,v_k)\in\mathcal{W}^0\cap\mathcal{L}$ and $n\in\mathbb{N}$ be such that $P_n^+(\mathbf{u}\to\mathbf{v})>0$.*

If $(\hat\varphi_1,\dots,\hat\varphi_k)\in\{0,\pi\}^k$ is maximum of the function $(\varphi_1,\dots,\varphi_k)\mapsto|S(e^{i\varphi_1},\dots,e^{i\varphi_k})|$, then, for any function $F(u,v)$, we have

$$S(e^{i\hat\varphi_1},\dots,e^{i\hat\varphi_k})^n \det_{1\le j,m\le k}\left((-1)^{(v_m+u_j)\hat\varphi_j/\pi}F(u_j,v_m)\right) = S(1,\dots,1)^n \det_{1\le j,m\le k}\Big(F(u_j,v_m)\Big).$$

Proof. For the sake of brevity, we say that a point in $\{0,\pi\}^k$ is a *maximal point*, if this point is a maximum of the function $(\varphi_1,\dots,\varphi_k)\mapsto|S(e^{i\varphi_1},\dots,e^{i\varphi_k})|$.

For $(\hat\varphi_1,\dots,\hat\varphi_k)=(0,\dots,0)$ the claim is obviously true. Now, recall that according to Lemma 3.2, we have $S(z_1,\dots,z_k)=P(A(z_1,\dots,z_k))$ for some polynomial $P(x)$ with non-negative coefficients. We proceed with a case-by-case analysis.

Let us first assume that $A(z_1,\dots,z_k)=\sum_{j=1}^k\left(z_j+\frac{1}{z_j}\right)$. If $|P(-x)|=P(x)$, then we have two maximal points, namely $(0,\dots,0)$ and $(\pi,\dots,\pi)$. If $P(-x)=P(x)$, then we know that each step in $\mathcal{S}$, viewed as a sequence of atomic steps, has even length. Analogously, we see that every step has odd length whenever $P(-x)=-P(x)$. Since, by assumption $P_n^+(\mathbf{u}\to\mathbf{v})>0$, we must have $\sum_{j=1}^k(v_j-u_j)\equiv n \mod 2$, which proves the claim in these two cases. If $P(x)$ is neither even nor odd, then $(0,\dots,0)$ is the only maximal point, and there is nothing to prove.

Let us now assume that $A(z_1,\dots,z_k)=\prod_{j=1}^k\left(z_j+\frac{1}{z_j}\right)$. In this case, the $\mathbb{Z}$-lattice $\mathcal{L}$ spanned by the atomic step set $\mathcal{A}$ is given by

$$\mathcal{L}=\left\{(c_1,\dots,c_k)\in\mathbb{Z}^k\ :\ c_1\equiv c_2\equiv\cdots\equiv c_k \mod 2\right\}.$$

Consequently, we have

$$v_m-u_j\equiv v_1-u_1 \mod 2 \qquad \text{for all } m \text{ and } j,$$

which implies

$$S(e^{i\hat{\varphi}_1},\dots,e^{i\hat{\varphi}_k})^n \det_{1\le j,m\le k}\left((-1)^{(v_m+u_j)\hat{\varphi}_j/\pi}F(u_j,v_m)\right)$$
$$= (-1)^{\frac{(u_1+v_1)}{\pi}\sum_{j=1}^{k}\hat{\varphi}_j} S(e^{i\hat{\varphi}_1},\dots,e^{i\hat{\varphi}_k})^n \det_{1\le j,m\le k}\left(F(u_j,v_m)\right).$$

Now, if $P(x)$ is even, then $v_1+u_1\equiv 0 \mod 2$, and the claim is proved. For $P(x)$ odd, we note that

$$S(e^{i\hat{\varphi}_1},\dots,e^{i\hat{\varphi}_k}) = (-1)^{\sum_{j=1}^{k}\hat{\varphi}_j/\pi}S(1,\dots,1).$$

The result now follows from the fact that in this case we must have $n\equiv v_1+u_1 \mod 2$.

If $P(x)$ is neither even or odd, then the set of maximal points is given by

$$\left\{(\hat{\varphi}_1,\dots,\hat{\varphi}_k)\in\{0,\pi\}^k \;:\; \left(\sum_{j=1}^{k}\hat{\varphi}_j/\pi\right)\equiv 0 \mod 2\right\},$$

and the claim follows upon noting that $S(e^{i\hat{\varphi}_1},\dots,e^{i\hat{\varphi}_k})>0$ for all maximal points. □

Lemma 3.5. *For any two real numbers $u,v\in\mathbb{R}$, we have*

$$\int_0^\infty \sin(u\vartheta)\sin(v\vartheta)e^{-\vartheta^2/2}d\vartheta = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(e^{-(u-v)^2/2}-e^{-(u+v)^2/2}\right).$$

Proof. Since, by definition we have

$$\sin(u\vartheta)\sin(v\vartheta) = \frac{1}{4}\left(e^{i(u-v)\vartheta}+e^{-i(u-v)\vartheta}-e^{i(u+v)\vartheta}-e^{-i(u+v)\vartheta}\right),$$

we see that the integral of interest is a sum of four integrals, all of which are of the form

$$\int_0^\infty e^{i\kappa\vartheta-\vartheta^2/2}d\vartheta = e^{-\kappa^2/2}\int_0^\infty e^{-(\vartheta-i\kappa)^2/2}d\vartheta.$$

By Cauchy's integral theorem, we know that

$$\left(\int_0^R+\int_R^{R-i\kappa}+\int_{R-i\kappa}^{-i\kappa}+\int_{-i\kappa}^{0}\right)e^{-z^2/2}dz=0$$

for any R and any κ. Letting R tend to $+\infty$ and rearranging the last equation, we obtain

$$\int_0^\infty e^{-(\vartheta-i\kappa)^2/2}d\vartheta = \int_0^\infty e^{-\vartheta^2/2}d\vartheta + i\int_0^\kappa e^{t^2/2}d\vartheta,$$

and further

$$\int_0^\infty\left(e^{-(\vartheta-i\kappa)^2/2}+e^{-(\vartheta+i\kappa)^2/2}\right)d\vartheta = 2\int_0^\infty e^{-\vartheta^2/2}d\vartheta=\sqrt{2\pi},$$

which proves the lemma. □

4. Determinants and asymptotics

Asymptotics for determinants are often hard to obtain, the reason being a typical large number of cancellations of asymptotically leading terms. In this section, we present a factorisation technique that allows one to represent certain functions in several complex variables defined by determinants as a product of two factors. One of these factors will always be a symmetric (Laurent) polynomial (this accounts for the cancellations of asymptotically leading terms mentioned before). The second factor is a determinant, the entries of which are certain contour integrals. The fundamental technique is illustrated in Lemma 4.1 below.

We want to stress that Lemma 4.1 should be seen as a general technique, not as a particular result. The main intention of this lemma is to give the reader an unblurred view at the technique. An application of Lemma 4.1 together with some remarks on asymptotics can be found right after the proof.

Let us now start with the illustration of our factorisation technique.

Lemma 4.1. *Let $A_m(x,y)$, $1 \le m \le k$, be analytic and one-valued for $(x,y) \in \mathcal{R} \times \mathcal{D} \subset \mathbb{C}^2$, where $\mathcal{D} \subset \mathbb{C}$ is some non empty set and $\mathcal{R} = \{x \in \mathbb{C} \ : \ r^* \le |x| < R^*\}$ for some $0 \le r^* < R^*$.*

Then, the function

$$\det_{1\le j,m\le k} \left(A_m(x_j,y_m)\right)$$

is analytic for $(x_1,\dots,x_k,y_1,\dots,y_k) \in \mathcal{R}^k \times \mathcal{D}^k$, and it satisfies

$$\det_{1\le j,m\le k} \left(A_m(x_j,y_m)\right) = \left(\prod_{1\le j<m\le k} (x_m - x_j)\right) \times \det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int\limits_{|\xi|=R} \frac{A_m(\xi,y_m)d\xi}{\prod\limits_{\ell=1}^{j}(\xi - x_\ell)} - \frac{1}{2\pi i}\int\limits_{|\xi|=r} \frac{A_m(\xi,y_m)d\xi}{\prod\limits_{\ell=1}^{j}(\xi - x_\ell)}\right),$$

where $r^* < r < \min_j |x_j| \le \max_j |x_j| < R < R^*$.

Proof. By Laurent's theorem, we have

$$\det_{1\le j,m\le k} \left(A_m(x_j,y_m)\right) = \det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int\limits_{|\xi|=R} \frac{A_m(\xi,y_m)d\xi}{\xi - x_j} - \frac{1}{2\pi i}\int\limits_{|\xi|=r} \frac{A_m(\xi,y_m)d\xi}{\xi - x_j}\right). \tag{4.7}$$

Now, short calculations show that for any $L \ge 0$ and all $n_1,\dots,n_L \in \{1,2,\dots,k\}$ we have

$$\int_{|\xi|=\rho_1} \frac{A_m(\xi,y_m)d\xi}{(\xi - x_j)\prod_{\ell=1}^{L}(\xi - x_{n_\ell})} - \int_{|\xi|=\rho_1} \frac{A_m(\xi,y_m)d\xi}{(\xi - x_j)\prod_{\ell=1}^{L}(\xi - x_{n_\ell})} = (x_m - x_j)\int_{|\xi|=\rho_1} \frac{A(\xi,y)d\xi}{(\xi - x_j)(\xi - x_m)\prod_{\ell=1}^{L}(\xi - x_{n_\ell})}.$$

Consequently, we can prove the claimed factorisation as follows. First, we subtract the first row of the determinant in (4.7) from all other rows. By the computations above we can then take the factor $(x_j - x_1)$ out of the j-th row of the determinant. In a second run, we subtract the second row from the rows $3, 4, \ldots, k$, and so on. In general, after subtracting row j from row ℓ we take the factor $(x_\ell - x_j)$ out of the determinant. $\square$

Example 4.1. Consider the function

$$\det_{1\le j,m\le k}\left(e^{x_j y_m}\right).$$

An application of Lemma 4.1 with $A(x, y) = e^{xy}$ immediately gives us the factorisation

$$\det_{1\le j,m\le k}\left(e^{x_j y_m}\right) = \left(\prod_{1\le j<m\le k}(x_m - x_j)\right) \det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int_{|\xi|=R}\frac{e^{\xi y_m}d\xi}{\prod_{\ell=1}^{j}(\xi - x_\ell)}\right),$$

where $R > \max_j |x_j|$. Note that the second contour integral occurring in the factorisation given in Lemma 4.1 is equal to zero because the function $A(x, y) = e^{xy}$ is an entire function.

Now we want to demonstrate how one can derive asymptotics for $\det_{1\le j,m\le k}\left(e^{x_j y_m}\right)$ as $x_1, \ldots, x_k \to 0$ from this factorisation. The geometric series expansion gives us

$$\begin{aligned}\frac{1}{2\pi i}\int_{|\xi|=R}\frac{e^{\xi y}d\xi}{\prod_{\ell=1}^{j}(\xi - x_\ell)} &= \frac{1}{2\pi i}\int_{|\xi|=R} e^{\xi y}\frac{d\xi}{\xi^j} + O\left(\sum_{j=1}^{k}|x_k|\right)\\ &= \frac{y^{j-1}}{(j-1)!} + O\left(\sum_{j=1}^{k}|x_k|\right)\end{aligned}$$

as $x_1, \ldots, x_k \to 0$. Consequently, we have

$$\begin{aligned}\det_{1\le j,m\le k}\left(e^{x_j y_m}\right) &= \left(\prod_{1\le j<m\le k}(x_m - x_j)\right)\left(\det_{1\le j,m\le k}\left(\frac{y_m^{j-1}}{(j-1)!}\right) + O\left(\sum_{j=1}^{k}|x_k|\right)\right)\\ &= \left(\prod_{1\le j<m\le k}(x_m - x_j)\right)\left(\left(\prod_{1\le j<m\le k}\frac{y_m - y_j}{m - j}\right) + O\left(\sum_{j=1}^{k}|x_j|\right)\right)\end{aligned}$$

as $x_1, \ldots, x_k \to \infty$.

This illustrates that the problem of establishing asymptotics for functions of the form $\det_{1\le j,m\le k}(A_m(x_j, y_m))$ can be reduced to an application of Lemma 4.1 and the extraction of certain coefficients of the functions $A_m(x, y)$.

If we would have considered the function $\det_{1\le j,m\le k}\left(e^{\xi^2 y}\right)$, $k > 1$, instead of $\det_{1\le j,m\le k}\left(e^{x_j y_m}\right)$ as in the example above, we would have got only the upper bound

$$\det_{1\le j,m\le k}\left(e^{x_j^2 y_m}\right) = O\left(\left(\prod_{1\le j<m\le k}(x_m - x_j)\right)\sum_{j=1}^{k}|x_j|\right)$$

as $x_1, \ldots, x_k \to 0$, because

$$\det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int_{|\xi|=R} e^{\xi^2 y_m}\frac{d\xi}{\xi^j}\right) = 0, \qquad k > 1.$$

The reason for this is that the function $A(x, y) = e^{x^2 y}$ satisfies the symmetry $A(-x, y) = A(x, y)$ which induces additional cancellations of asymptotically leading terms.

In order to obtain precise asymptotic formulas in cases where the functions $A_m(x,y)$ exhibit certain symmetries, we have to take into account these symmetries. This can easily be accomplished by a small modification to our factorisation technique presented in Lemma 4.1. In fact, the only thing we have to do is to modify the representation (4.7), the rest of our technique remains - mutatis mutandis - unchanged.

The following series of lemmas should illustrate these modifications to our factorisation method for some selected symmetry conditions, and should underline the general applicability of our factorisation method.

Lemma 4.2. *Let $A(x,y)$ be analytic for $(x,y)\in\mathcal{R}_1\times\mathcal{R}_2\subset\mathbb{C}^2$, where*

$$\mathcal{R}_j=\left\{x\in\mathbb{C}\ :\ |x|<R_j^*\right\},\qquad j=1,2,$$

for some $R_1^*,R_2^*>0$. Furthermore, assume that $A(x,y)=A(-x,y)=A(x,-y)$.

Then, the function

$$\det_{1\le j,m\le k}\left(A(x_j,y_m)\right)$$

is analytic for $(x_1,\dots,x_k,y_1,\dots,y_k)\in\mathcal{R}_1^k\times\mathcal{R}_2^k$, and it satisfies

$$\det_{1\le j,m\le k}\left(A(x_j,y_m)\right)=\left(\prod_{1\le j<m\le k}(x_m^2-x_j^2)(y_m^2-y_j^2)\right)\\ \times\det_{1\le j,m\le k}\left(\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=R_1\\|\eta|=R_2}}\frac{A(\xi,\eta)\xi\eta d\xi d\eta}{\left(\prod_{\ell=1}^{j}(\xi^2-x_\ell^2)\right)\left(\prod_{\ell=1}^{m}(\eta^2-y_m^2)\right)}\right),$$

where $\max_j|x_j|<R_1<R_1^*$ and $\max_j|y_j|<R_2<R_2^*$.

Proof (sketch). By Cauchy's theorem, we have

$$A(x,y)=\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=R_1\\|\eta|=R_2}}\frac{A(\xi,\eta)d\xi d\eta}{(\xi-x)(\eta-y)}.$$

By assumption, we have $4A(x,y)=A(x,y)+A(-x,-y)+A(-x,y)+A(x,-y)$. This equation together with the integral representation above implies

$$A(x,y)=\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=R_1\\|\eta|=R_2}}\frac{A(\xi,\eta)\xi\eta d\xi d\eta}{(\xi^2-x^2)(\eta^2-y^2)}.$$

Consequently we should replace Equation (4.7) in the proof of Lemma 4.1 with

$$\det_{1\le j,m\le k}\left(A(x_j,y_m)\right)=\det_{1\le j,m\le k}\left(\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=R_1\\|\eta|=R_2}}\frac{A(\xi,\eta)\xi\eta d\xi d\eta}{(\xi^2-x_j^2)(\eta^2-y_m^2)}\right).$$

Now we apply the same sequence of row operations as in the proof of Lemma 4.1 to the determinant on the right hand side above. After each of these row operations, we can take a factor of the form $(x_\ell^2 - x_j^2)$, $\ell < j$, out of the determinant.

Finally, we transpose the resulting determinant (which does not change its value) and apply the same sequence of row operations a second time. This yields successively factors of the form $(y_\ell^2 - y_j^2)$, and completes the proof of the lemma. □

Lemma 4.3. *Let $A(x,y)$ be analytic for $(x,y) \in \mathcal{R}_1 \times \mathcal{R}_2 \subset \mathbb{C}^2$, where*

$$\mathcal{R}_j = \left\{ x \in \mathbb{C} \ : \ |x| < R_j^* \right\}, \qquad j = 1,2,$$

for some $R_1^*, R_2^* > 0$. Furthermore, assume that $A(x,y) = -A(-x,y) = -A(x,-y)$.

Then, the function

$$\det_{1 \le j,m \le k} (A(x_j, y_m))$$

is analytic for $(x_1, \dots, x_k, y_1, \dots, y_k) \in \mathcal{R}_1^k \times \mathcal{R}_2^k$, and it satisfies

$$\det_{1 \le j,m \le k} (A(x_j, y_m)) = \left(\prod_{j=1}^k x_j y_j \right) \left(\prod_{1 \le j < m \le k} (x_m^2 - x_j^2)(y_m^2 - y_j^2) \right) \times \det_{1 \le j,m \le k} \left(\frac{1}{(2\pi i)^2} \int_{\substack{|\xi|=R_1 \\ |\eta|=R_2}} \frac{A(\xi,\eta) d\xi d\eta}{\left(\prod_{\ell=1}^{j} (\xi^2 - x_\ell^2) \right) \left(\prod_{\ell=1}^{m} (\eta^2 - y_m^2) \right)} \right),$$

where $\max_j |x_j| < R_1 < R_1^*$ *and* $\max_j |y_j| < R_2 < R_2^*$.

Proof (sketch). The present situation is very similar to the one considered in the last lemma. But now, our assumption implies $4A(x,y) = A(x,y) + A(-x,-y) - A(-x,y) - A(x,-y)$, which, together with Cauchy's theorem, yields

$$A(x,y) = \frac{xy}{(2\pi i)^2} \int_{\substack{|\xi|=R_1 \\ |\eta|=R_2}} \frac{A(\xi,\eta) d\xi d\eta}{(\xi^2 - x^2)(\eta^2 - y^2)}.$$

Consequently we should replace Equation (4.7) in the proof of Lemma 4.1 with

$$\det_{1 \le j,m \le k} (A(x_j, y_m)) = \left(\prod_{j=1}^k x_j y_j \right) \det_{1 \le j,m \le k} \left(\frac{1}{(2\pi i)^2} \int_{\substack{|\xi|=R_1 \\ |\eta|=R_2}} \frac{A(\xi,\eta) d\xi d\eta}{(\xi^2 - x_j^2)(\eta^2 - y_m^2)} \right).$$

Now the determinant on the right hand side is treated exactly as in the previous lemma. □

Lemma 4.4. *Let $A_m(x)$, $1 \le m \le k$, be analytic and one-valued in*

$$\mathcal{R} = \left\{ x \in \mathbb{C} \ : \ \frac{1}{R^*} < |x| < R^* \right\}, \qquad R^* > 1.$$

Furthermore, assume that $A_m(x) = -A_m(1/x)$, $m = 1, \dots, k$.

Then, the function

$$\det_{1\le j,m\le k}\left(A_m(x_j)\right)$$

is analytic for $(x_1,\dots,x_k)\in\mathcal{R}^k$*, and it satisfies*

$$\det_{1\le j,m\le k}\left(A_m(x_j)\right)=\left(\prod_{j=1}^{k}x_j\right)^{-k}\left(\prod_{1\le j<m\le k}(x_j-x_m)(1-x_jx_m)\right)\left(\prod_{j=1}^{k}(x_j^2-1)\right)$$
$$\times\det_{1\le j,m\le k}\left(\frac{1}{4\pi i}\int\limits_{|\xi|=R}\frac{A_m(\xi)\xi^{j-1}d\xi}{\prod_{\ell=1}^{j}(\xi-x_\ell)\left(\xi-\frac{1}{x_\ell}\right)}-\frac{1}{4\pi i}\int\limits_{|\xi|=\frac{1}{R}}\frac{A_m(\xi)\xi^{j-1}d\xi}{\prod_{\ell=1}^{j}(\xi-x_\ell)\left(\xi-\frac{1}{x_\ell}\right)}\right),$$

where $\frac{1}{R^*}<\frac{1}{R}<\min_j|x_j|\le\max_j|x_j|<R<R^*$.

Proof (sketch). Laurent's theorem together with our assumption $2A_m(x)=A_m(x)-A_m(1/x)$ implies

$$A_m(x)=\frac{1}{4\pi i}\left(x-\frac{1}{x}\right)\left(\int\limits_{|\xi|=R}\frac{A_m(\xi)}{(\xi-x)\left(\xi-\frac{1}{x}\right)}d\xi-\int\limits_{|\xi|=\frac{1}{R}}\frac{A_m(\xi)}{(\xi-x)\left(\xi-\frac{1}{x}\right)}d\xi\right).$$

Consequently, we should replace Equation (4.7) in the proof of Lemma 4.1 with

$$\det_{1\le j,m\le k}\left(A_m(x_j)\right)=\left(\prod_{j=1}^{k}\left(x_j-\frac{1}{x_j}\right)\right)$$
$$\times\det_{1\le j,m\le k}\left(\frac{1}{4\pi i}\left(\int\limits_{|\xi|=R}\frac{A_m(\xi)d\xi}{(\xi-x_j)\left(\xi-\frac{1}{x_j}\right)}-\int\limits_{|\xi|=\frac{1}{R}}\frac{A_m(\xi)d\xi}{(\xi-x_j)\left(\xi-\frac{1}{x_j}\right)}\right)\right).$$

Now, short computations show that for $\frac{1}{R}\le\rho\le R$ we have

$$\int\limits_{|\xi|=\rho}\frac{A_m(\xi)d\xi}{(\xi-x_\ell)\left(\xi-\frac{1}{x_\ell}\right)}-\int\limits_{|\xi|=\rho}\frac{A_m(\xi)d\xi}{(\xi-x_j)\left(\xi-\frac{1}{x_j}\right)}$$
$$=\frac{1}{x_\ell x_j}(x_j-x_\ell)(1-x_jx_\ell)\int\limits_{|\xi|=\rho}\frac{A_m(\xi)\xi d\xi}{(\xi-x_\ell)\left(\xi-\frac{1}{x_\ell}\right)(\xi-x_j)\left(\xi-\frac{1}{x_j}\right)}.$$

We can now apply the same series of row operations as in the proof of Lemma 4.1. The only difference here is that after each row operation we take a factor of the form $x_\ell^{-1}x_j^{-1}(x_j-x_\ell)(1-x_jx_\ell)$, $j<\ell$, out of the determinant. □

The rest of this section is devoted to some particular results that can be obtained by the above described technique. More precisely, we determine asymptotics for two determinants that will become important in subsequent sections. As illustrated in Example 4.1, asymptotics for determinants can be determined as follows. First, we factorise our determinants according to our technique. At this point it is important to take into account all the symmetries satisfied by the entries $A(x_j, y_m)$ of the determinant. Second, we apply the geometric series expansion. This gives us the coefficient of the asymptotically leading term as a determinant, the entries of which being certain coefficients of the functions $A(x_j, y_m)$. In both cases, this last determinant can then be evaluated into a closed form expression.

Lemma 4.5. *We have the asymptotics*

$$\det_{1\le j,m\le k}\left(e^{-(x_j-y_m)^2}-e^{-(x_j+y_m)^2}\right)=\left(\prod_{j=1}^{k}x_jy_j\right)\left(\prod_{1\le j<m\le k}(x_m^2-x_j^2)(y_m^2-y_j^2)\right)\\ \times\frac{2^{k^2+k}}{\prod_{j=1}^{k}(2j-1)!}\left(1+O\left(\sum_{j=1}^{k}(|x_j|^2+|y_j|^2)\right)\right)$$

as $x_1,\dots,x_k,y_1,\dots,y_k\to 0$.

Proof. The function $A(x,y)=e^{-(x-y)^2}-e^{-(x+y)^2}$ satisfies the requirements of Lemma 4.3. Therefore, we have

$$\det_{1\le j,m\le k}\left(e^{-(x_j-y_m)^2}-e^{-(x_j+y_m)^2}\right)=\left(\prod_{j=1}^{k}x_jy_j\right)\left(\prod_{1\le j<m\le k}(x_m^2-x_j^2)(y_m^2-y_j^2)\right)\\ \times\det_{1\le j,m\le k}\left(\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=1\\|\eta|=1}}\frac{A(\xi,\eta)d\xi d\eta}{\left(\prod\limits_{\ell=1}^{j}(\xi^2-x_\ell^2)\right)\left(\prod\limits_{\ell=1}^{m}(\eta^2-y_\ell^2)\right)}\right)$$

for $\max_j|x_j|,\max_j|y_j|<1$. Since

$$\frac{1}{(2\pi i)^2}\int_{\substack{|\xi|=1\\|\eta|=1}}\left(e^{-(\xi-\eta)^2}-e^{-(\xi+\eta)^2}\right)\frac{d\xi}{\xi^{2j}}\frac{d\eta}{\eta^{2m}}=\frac{2}{(j+m-1)!}\binom{2j+2m-2}{2j-1},$$

we deduce with the help of the geometric series expansion

$$\frac{1}{(2\pi i)^2}\int_{\substack{|\xi|=1\\|\eta|=1}}\frac{A(\xi,\eta)d\xi d\eta}{\left(\prod\limits_{\ell=1}^{j}(\xi^2-x_\ell^2)\right)\left(\prod\limits_{\ell=1}^{m}(\eta^2-y_\ell^2)\right)}\\ =\frac{2}{(j+m-1)!}\binom{2j+2m-2}{2j-1}\left(1+O\left(\sum_{j=1}^{k}(|x_j|^2+|y_j|^2)\right)\right).$$

Consequently, we have

$$\det_{1\le j,m\le k}\left(\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=1\\|\eta|=1}}\frac{A(\xi,\eta)d\xi d\eta}{\left(\prod_{\ell=1}^{j}(\xi^2-x_\ell^2)\right)\left(\prod_{\ell=1}^{m}(\eta^2-y_\ell^2)\right)}\right)$$
$$=\det_{1\le j,m\le k}\left(\frac{2}{(j+m-1)!}\binom{2j+2m-2}{2j-1}\right)\left(1+O\left(\sum_{j=1}^{k}(|x_j|^2+|y_j|^2)\right)\right).$$

The determinant on the right hand side can be evaluated into a closed form expression by taking some factors and applying [15, Lemma 3], which gives us

$$\det_{1\le j,m\le k}\left(\frac{2}{(j+m-1)!}\binom{2j+2m-2}{2j-1}\right)=\frac{2^{k^2+k}}{\prod_{j=1}^{k}(2j-1)!},$$

and completes the proof of the lemma. ☐

Lemma 4.6. *For all $u_1,\dots,u_k\in\mathbb{C}$ we have the asymptotics*

$$\det_{1\le j,m\le k}(\sin(u_m\varphi_j))=\left(\prod_{j=1}^{k}u_j\varphi_j\right)\left(\prod_{1\le j<m\le k}(u_m^2-u_j^2)(\varphi_m^2-\varphi_j^2)\right)\left(\prod_{j=1}^{k}\frac{(-1)^j}{(2j-1)!}\right)$$
$$\times\left(1+O\left(\sum_{j=1}^{k}|\varphi_j|^2\right)\right)$$

as $(\varphi_1,\dots,\varphi_k)\to(0,\dots,0)$.

Proof. An application of Lemma 4.3 shows that

$$\det_{1\le j,m\le k}(\sin(u_m\varphi_j))=\left(\prod_{j=1}^{k}u_j\varphi_j\right)\left(\prod_{1\le j<m\le k}(u_m^2-u_j^2)(\varphi_m^2-\varphi_j^2)\right)$$
$$\times\det_{1\le j,m\le k}\left(\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=R\\|\eta|=1}}\frac{\sin(\xi\eta)d\xi d\eta}{\left(\prod_{\ell=1}^{j}(\xi^2-u_\ell^2)\right)\left(\prod_{\ell=1}^{m}(\eta^2-\varphi_\ell^2)\right)}\right).$$

Since

$$\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=R\\|\eta|=1}}\sin(\xi\eta)\frac{d\xi}{\xi^{2j}}\frac{d\eta}{\eta^{2m}}=\begin{cases}\frac{(-1)^{j-1}}{(2j-1)!} & \text{if } j=m,\\ 0 & \text{else,}\end{cases}$$

we deduce with the help of the geometric series expansion that

$$\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=R\\|\eta|=1}}\frac{\sin(\xi\eta)d\xi d\eta}{\left(\prod_{\ell=1}^{j}(\xi^2-u_\ell^2)\right)\left(\prod_{\ell=1}^{m}(\eta^2-\varphi_\ell^2)\right)} \\ =\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=R\\|\eta|=1}}\sin(\xi\eta)\frac{d\xi}{\xi^{2j}}\frac{d\eta}{\eta^{2m}}\left(1+O\left(\sum_{j=1}^{k}|\varphi_j|^2\right)\right).$$

Consequently, we have

$$\det_{1\le j,m\le k}\left(\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=R\\|\eta|=1}}\frac{\sin(\xi\eta)d\xi d\eta}{\left(\prod_{\ell=1}^{j}(\xi^2-u_\ell^2)\right)\left(\prod_{\ell=1}^{m}(\eta^2-\varphi_\ell^2)\right)}\right) \\ =\det_{1\le j,m\le k}\left(\frac{1}{(2\pi i)^2}\int\limits_{\substack{|\xi|=R\\|\eta|=1}}\sin(\xi\eta)\frac{d\xi}{\xi^{2j}}\frac{d\eta}{\eta^{2m}}\right)\left(1+O\left(\sum_{j=1}^{k}|\varphi_j|^2\right)\right)$$

as $(\varphi_1,\dots,\varphi_k)\to(0,\dots,0)$. Finally, noting that, by the above calculations, the matrix inside the determinant on the right hand side is a diagonal matrix, we obtain the claimed result. □

5. Walks with a fixed end point

In this section, we are going to derive asymptotics for $P_n^+(\mathbf{u}\to\mathbf{v})$ as n tends to infinity (see Theorem 5.1 below). The asymptotics are derived by applying saddle point techniques to the integral representation (2.3) together with the techniques developed in Section 4.

Theorem 5.1. *Let $\mathcal{S}$ be a composite step set over the atomic step set $\mathcal{A}$, and let $w:\mathcal{S}\to\mathbb{R}_+$ be a weight function. By $\mathcal{L}$ we denote the $\mathbb{Z}$-lattice spanned by $\mathcal{A}$. The composite step generating function associated with $\mathcal{S}$ is denoted by $S(z_1,\dots,z_k)$. Finally, let $\mathcal{M}\subseteq\{0,\pi\}^k$ denote the set of points such that the function $(\varphi_1,\dots,\varphi_k)\mapsto|S(e^{i\varphi_1},\dots,e^{i\varphi_k})|$ attains a maximum value, and let $|\mathcal{M}|$ denote the cardinality of the set $\mathcal{M}$.*

If $\mathcal{A}$, $\mathcal{S}$ and w satisfy Assumption 2.1 and $S(1,\dots,1)>0$, then for any two points $\mathbf{u},\mathbf{v}\in\mathcal{W}^0\cap\mathcal{L}$ we have the asymptotic formula

$$P_n^+(\mathbf{u}\to\mathbf{v})=|\mathcal{M}|S(1,\dots,1)^n\left(\frac{2}{\pi}\right)^{k/2}\left(\frac{S(1,\dots,1)}{nS''(1,\dots,1)}\right)^{k^2+k/2} \\ \times\frac{\left(\prod_{1\le j<m\le k}(u_m^2-u_j^2)(v_m^2-v_j^2)\right)\left(\prod_{j=1}^{k}u_jv_j\right)}{\left(\prod_{j=1}^{k}(2j-1)!\right)}\left(1+O(n^{-1/4})\right) \tag{5.8}$$

as $n \to \infty$ in the set $\{n : P_n^+(\mathbf{u} \to \mathbf{v}) > 0\}$. Here, $S''(z_1, \ldots, z_k)$ denotes the second derivative of $S(z_1, \ldots, z_k)$ with respect to any of the z_j.

Proof. According to Lemma 2.3, we have to asymptotically analyse the integral

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \frac{1}{(2\pi i)^k} \int\limits_{|z_1|=\cdots=|z_k|=1} \det_{1\le j,m\le k}\left(z_j^{u_m} - z_j^{-u_m}\right) S(z_1, \ldots, z_k)^n \left(\prod_{j=1}^k z_j^{-v_j-1} dz_j\right)$$

as $n \to \infty$. The substitution $z_j = e^{i\varphi_j}$ gives

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \left(\frac{i}{\pi}\right)^k \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \det_{1\le j,m\le k}\Big(\sin(u_m\varphi_j)\Big) S\left(e^{i\varphi_1}, \ldots, e^{i\varphi_k}\right)^n \left(\prod_{j=1}^k e^{-iv_j\varphi_j} d\varphi_j\right). \tag{5.9}$$

For large n, the absolute value of the integral is governed by the factor $|S(e^{i\varphi_1}, \ldots, e^{i\varphi_k})|^n$. By Lemma 3.3, the set $\mathcal{M}$ of maximal points of $(\varphi_1, \ldots, \varphi_k) \mapsto |S(e^{i\varphi_i}, \ldots, e^{i\varphi_k})|$ is a subset of $\{0, \pi\}^k$. We are now going to prove that, for large n, the asymptotically dominant part of the integral is captured by small neighbourhoods around these maxima. Asymptotics for the integral can then be determined by saddle point techniques.

For notational convenience, we define the sets

$$\mathcal{U}_\varepsilon(\hat\varphi) = \left\{\varphi \in \mathbb{R}^k : |\hat\varphi - \varphi|_\infty < \varepsilon\right\}, \qquad \hat\varphi = (\hat\varphi_1, \ldots, \hat\varphi_k) \in \mathcal{M},$$

where $\varepsilon > 0$ and $|\cdot|_\infty$ denotes the maximum norm on $\mathbb{R}^k$. We claim that the dominant asymptotic term of $P_n^+(\mathbf{u} \to \mathbf{v})$ is captured by

$$\left(\frac{i}{\pi}\right)^k \sum_{\hat\varphi \in \mathcal{M}} \int \cdots \int_{\mathcal{U}_\varepsilon(\hat\varphi)} \det_{1\le j,m\le k}\Big(\sin(u_m\varphi_j)\Big) S\left(e^{i\varphi_1}, \ldots, e^{i\varphi_k}\right)^n \left(\prod_{j=1}^k e^{-iv_j\varphi_j} d\varphi_j\right), \tag{5.10}$$

where we choose $\varepsilon = \varepsilon(n) = n^{-5/12}$. This claim can be proved by means of the saddle point method: (1) Determine an asymptotically equivalent expression for (5.10) that is more convenient to work with; (2) Find a bound for the remaining part of the integral (5.9).

Let us start with task (1). Fix a point $\hat\varphi \in \mathcal{M}$ and consider the corresponding summand in the sum (5.10), viz.

$$\left(\frac{i}{\pi}\right)^k \int \cdots \int_{\mathcal{U}_\varepsilon(\hat\varphi)} \det_{1\le j,m\le k}\Big(\sin(u_m\varphi_j)\Big) S\left(e^{i\varphi_1}, \ldots, e^{i\varphi_k}\right)^n \left(\prod_{j=1}^k e^{-iv_j\varphi_j} d\varphi_j\right).$$

We can then transform this expression with the help of the substitution $\varphi_j \mapsto \varphi_j + \hat\varphi_j$, $j = 1, \ldots, k$, into

$$\left(\frac{i}{\pi}\right)^k \int \cdots \int_{\mathcal{U}_\varepsilon(\mathbf{0})} \det_{1\le j,m\le k}\Big(\sin(u_m(\varphi_j + \hat\varphi_j))\Big) S\left(e^{i(\varphi_1+\hat\varphi_1)}, \ldots, e^{i(\varphi_k+\hat\varphi_k)}\right)^n$$
$$\times \left(\prod_{j=1}^k e^{-iv_j(\varphi_j+\hat\varphi_j)} d\varphi_j\right).$$

Since $\hat{\varphi}_1, \dots, \hat{\varphi}_k \in \{0, \pi\}$, we know that the determinant in the expression above is an odd function of each the variables φ_j, $j = 1, 2, \dots, k$. On the other hand, we deduce from (3.6) that $S\left(e^{i(\varphi_1+\hat{\varphi}_1)}, \dots, e^{i(\varphi_k+\hat{\varphi}_k)}\right)$ is an even function of the variables φ_j. Consequently, we can further simplify our integral to

$$\left(\frac{2}{\pi}\right)^k \int_0^{n^{-5/12}} \cdots \int_0^{n^{-5/12}} \det_{1 \le j,m \le k} \Big(\sin(u_m(\varphi_j + \hat{\varphi}_j))\Big) S\left(e^{i(\varphi_1+\hat{\varphi}_1)}, \dots, e^{i(\varphi_k+\hat{\varphi}_k)}\right)^n \times \left(\prod_{j=1}^{k} \sin(v_j(\varphi_j + \hat{\varphi}_j)) d\varphi_j\right).$$

Incorporating the product of the sines into the determinant and noting that $(\hat{\varphi}_1, \dots, \hat{\varphi}_k) \in \{0, \pi\}^k$, we finally obtain the expression

$$\left(\frac{2}{\pi}\right)^k \int_0^{n^{-5/12}} \cdots \int_0^{n^{-5/12}} \det_{1 \le j,m \le k} \Big((-1)^{(u_m+v_j)\hat{\varphi}_j/\pi} \sin(u_m\varphi_j) \sin(v_j\varphi_j)\Big) \times S\left(e^{i(\varphi_1+\hat{\varphi}_1)}, \dots, e^{i(\varphi_k+\hat{\varphi}_k)}\right)^n \left(\prod_{j=1}^{k} d\varphi_j\right).$$

Asymptotics for this integral can now be determined by replacing the second part of the integrand with an appropriate Taylor series approximation around $(\varphi_1, \dots, \varphi_k) = (0, \dots, 0)$. Recall that, according to Lemma 3.2, there exists a polynomial $P(x)$ with non-negative coefficients such that either

$$S(z_1, \dots, z_k) = P\left(\sum_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right)\right) \qquad \text{or} \qquad S(z_1, \dots, z_k) = P\left(\prod_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right)\right).$$

For $\varphi \in \mathcal{U}_{n^{-5/12}}(\mathbf{0})$ we have the Taylor series approximation

$$S\left(e^{i(\varphi_1+\hat{\varphi}_1)}, \dots, e^{i(\varphi_k+\hat{\varphi}_k)}\right) = S(e^{i\hat{\varphi}_1}, \dots, e^{i\hat{\varphi}_k}) \exp\left(-\Lambda \sum_{j=1}^{k} \frac{\varphi_j^2}{2}\right) \left(1 + O\left(n^{-5/4}\right)\right) \tag{5.11}$$

as $n \to \infty$, where $\Lambda = \frac{S''(1,\dots,1)}{S(1,\dots,1)}$ and $S''(z_1, \dots, z_k) = \frac{\partial^2}{\partial z_1^2} S(z_1, \dots, z_k)$. Short calculations show that either

$$\Lambda = 2\frac{P'(2k)}{P(2k)} > 0 \qquad \text{or} \qquad \Lambda = 2^k \frac{P'(2^k)}{P(2^k)} > 0,$$

corresponding to the two possible cases for $S(z_1, \dots, z_k)$ given in Lemma 3.2. Here, $P'(x)$ is the derivative of $P(x)$ with respect to x.

Substituting the Taylor approximation (5.11) for the corresponding term in the integral above, we obtain the asymptotic expression

$$\left(\frac{2}{\pi}\right)^k S(e^{i\hat{\varphi}_1},\dots,e^{i\hat{\varphi}_k})^n \det_{1\le j,m\le k}\left((-1)^{(u_m+v_j)\hat{\varphi}_j/\pi}\int_0^{n^{-5/12}} \sin(u_m\vartheta)\sin(v_j\vartheta)e^{-n\Lambda\vartheta^2/2}d\vartheta\right) \times\left(1+O(n^{-1/4})\right)$$

as $n\to\infty$.

From now on we assume that $\mathbf{u},\mathbf{v}\in\mathcal{W}^0\cap\mathcal{L}$ and $n\in\mathbb{N}$ are such that $P_n^+(\mathbf{u}\to\mathbf{v})>0$. Then, according to Lemma 3.4, the asymptotic expression above is equal to

$$\left(\frac{2}{\pi}\right)^k S(1,\dots,1)^n \det_{1\le j,m\le k}\left(\int_0^{n^{-5/12}} \sin(u_m\vartheta)\sin(v_j\vartheta)e^{-n\Lambda\vartheta^2/2}d\vartheta\right)\left(1+O(n^{-1/4})\right)$$

as $n\to\infty$ in the set $\{n : P_n^+(\mathbf{u}\to\mathbf{v})>0\}$. This shows that Expression (5.10) is asymptotically equal to $|\mathcal{M}|$ times this last expression as n tends to infinity.

The second step of the saddle point method is to establish a bound for the remaining part of the integral (5.9), viz.

$$\left(\frac{2}{\pi}\right)^k \int\cdots\int_{[0,2\pi)^k\setminus\mathcal{U}_\varepsilon(\mathcal{M})} \det_{1\le j,m\le k}\Big(\sin(u_m\varphi_j)\Big) S\left(e^{i\varphi_1},\dots,e^{i\varphi_k}\right)^n\left(\prod_{j=1}^k e^{-iv_j\varphi_j}d\varphi_j\right),$$

where $\mathcal{U}_\varepsilon(\mathcal{M})=\bigcup_{\hat{\varphi}\in\mathcal{M}}\mathcal{U}_\varepsilon(\hat{\varphi})$ and $\varepsilon=\varepsilon(n)=n^{-5/12}$. Since $\mathcal{M}$ is the set of maximal points of the function $(\varphi_1,\dots,\varphi_k)\mapsto|S(e^{i\varphi_1},\dots,e^{i\varphi_k})|$, we see that (at least for n large enough) the maximum of this function on the set $[0,2\pi]^k\setminus\mathcal{U}_\varepsilon(\mathcal{M})$ is attained somewhere on the boundary of one of the sets $\mathcal{U}_\varepsilon(\hat{\varphi})$, $\hat{\varphi}\in\mathcal{M}$. Let $\psi\in[0,2\pi]^k\setminus\mathcal{U}_\varepsilon(\mathcal{M})$ be one such maximal point. Since the Expansion (5.11) is valid for ψ, we immediately obtain the upper bound

$$\left(\frac{2}{\pi}\right)^k\left|S\left(e^{i\varphi_1},\dots,e^{i\varphi_k}\right)\right|\le\left|S\left(e^{i\psi_1},\dots,e^{i\psi_k}\right)\right|=S(1,\dots,1)^{n-C_1n^{1/6}+O\left(n^{-1/4}\right)}$$

as $n\to\infty$ for some constant $C_1>0$. This gives us the bound

$$\int\cdots\int_{[0,2\pi)^k\setminus\mathcal{U}_\varepsilon(\mathcal{M})} \det_{1\le j,m\le k}\Big(\sin(u_m\varphi_j)\Big) S\left(e^{i\varphi_1},\dots,e^{i\varphi_k}\right)^n\left(\prod_{j=1}^k e^{-iv_j\varphi_j}d\varphi_j\right) = O\left(S(1,\dots,1)^{n-C_1n^{1/6}}\right)$$

as $n\to\infty$.

Consequently, we see that, if $\mathbf{u},\mathbf{v}\in\mathcal{L}$ and $n\in\mathbb{N}$ are chosen such that $P_n^+(\mathbf{u}\to\mathbf{v})>0$, then we have

$$P_n^+(\mathbf{u}\to\mathbf{v}) = |\mathcal{M}|S(1,\dots,1)^n \det_{1\le j,m\le k}\left(\frac{2}{\pi}\int_0^{n^{-5/12}} \sin(u_m\vartheta)\sin(v_j\vartheta)e^{-n\Lambda\vartheta^2/2}d\vartheta\right)\times\left(1+O(n^{-1/4})\right) + O\left(S(1,\dots,1)^{n-C_1n^{1/6}}\right)$$

as $n\to\infty$ in the set $\{n\ :\ P_n^+(\mathbf{u}\to\mathbf{v})>0\}$, where $|\mathcal{M}|$ denotes the cardinality of the set $\mathcal{M}$.

Let us now have a closer look at the determinant

$$\det_{1\le j,m\le k}\left(\frac{2}{\pi}\int_0^{n^{-5/12}} \sin(u_m\vartheta)\sin(v_j\vartheta)e^{-n\Lambda\vartheta^2/2}d\vartheta\right). \tag{5.12}$$

We need to determine the asymptotic behaviour of this determinant. This task will be accomplished with the help of Lemma 4.5, for which we have to have a closer look at the entries of the determinant.

The change of variables $\vartheta\mapsto\vartheta/\sqrt{n\Lambda}$ and the simple bound $\int_L^\infty e^{-\alpha^2}d\alpha = O\left(e^{-L^2}\right)$ gives us

$$\frac{2}{\pi}\int_0^{n^{-5/12}} \sin(u\vartheta)\sin(v\vartheta)e^{-n\Lambda\vartheta^2/2}d\vartheta = \frac{2}{\pi\sqrt{n\Lambda}}\int_0^\infty \sin\left(\frac{u\vartheta}{\sqrt{n\Lambda}}\right)\sin\left(\frac{v\vartheta}{\sqrt{n\Lambda}}\right)e^{-\vartheta^2/2}d\vartheta + O\left(e^{-\Lambda n^{1/3}}\right)$$

as $n\to\infty$, and Lemma 3.5 yields

$$\frac{2}{\pi\sqrt{n\Lambda}}\int_0^\infty \sin\left(\frac{u\vartheta}{\sqrt{n\Lambda}}\right)\sin\left(\frac{v\vartheta}{\sqrt{n\Lambda}}\right)e^{-\vartheta^2/2}d\vartheta = \frac{1}{\sqrt{2\pi n\Lambda}}\left(e^{-(u-v)^2/(2n\Lambda)} - e^{-(u+v)^2/(2n\Lambda)}\right).$$

Consequently, our determinant (5.12) satisfies the asymptotics

$$\det_{1\le j,m\le k}\left(\frac{2}{\pi}\int_0^{n^{-5/12}} \sin(u_m\vartheta)\sin(v_j\vartheta)e^{-n\Lambda\vartheta^2/2}d\vartheta\right) = (2\pi n\Lambda)^{-k/2}\det_{1\le j,m\le k}\left(A\left(\frac{v_j}{\sqrt{2n\Lambda}},\frac{u_m}{\sqrt{2n\Lambda}}\right)\right) + O\left(e^{-\Lambda n^{1/3}}\right)$$

as $n \to \infty$, where $A(x,y) = e^{-(x-y)^2} - e^{-(x+y)^2}$. Asymptotics for the determinant on the right hand side are given in Lemma 4.5, viz.

$$\det_{1\le j,m\le k}\left(e^{-(x_j-y_m)^2} - e^{-(x_j+y_m)^2}\right)$$
$$= \left(\prod_{j=1}^{k} x_j y_j\right)\left(\prod_{1\le j<m\le k}(x_m^2-x_j^2)(y_m^2-y_j^2)\right)\frac{2^{k^2+k}}{\prod_{j=1}^{k}(2j-1)!}\left(1+O\left(n^{-1}\right)\right).$$

This completes the proof of the theorem. □

6. Walks with a free end point

In this section, we are interested in the generating function $P_n^+(\mathbf{u})$ for walks starting in $\mathbf{u}$ consisting of n steps that are confined to the region $\mathcal{W}^0$. This quantity can be written as the sum

$$P_n^+(\mathbf{u}) = \sum_{\mathbf{v}\in\mathcal{W}^0} P_n^+(\mathbf{u}\to\mathbf{v}),$$

where $P_n^+(\mathbf{u}\to\mathbf{v})$ denotes the generating functions for walks from $\mathbf{u}$ to $\mathbf{v}$ consisting of n steps that are confined to the region $\mathcal{W}^0$. This sum is in fact a finite sum, because there is only a finite number of points in $\mathcal{W}^0$ that are reachable from $\mathbf{u}$ in n steps. In order to find a nice expression for $P_n^+(\mathbf{u})$ that is amenable to asymptotic methods, we proceed as follows. First, we substitute the integral expression from Lemma 2.3 for $P_n^+(\mathbf{u}\to\mathbf{v})$ in the sum above. In a second step, we interchange summation and integration. This yields a sum that can be evaluated with the help of a known identity relating Schur functions and odd orthogonal characters (see Lemma 6.1 below). The resulting expression can then be asymptotically evaluated by means of saddle point techniques and the techniques from Section 4.

Lemma 6.1 (see, e.g., Macdonald [18, I.5]). *For any integer $c>0$, we have the identity*

$$\sum_{0\le\lambda_1\le\cdots\le\lambda_k\le 2c}\frac{\det_{1\le j,m\le k}\left(z_j^{\lambda_m+m-1}\right)}{\det_{1\le j,m\le k}\left(z_j^{m-1}\right)} = \frac{\det_{1\le j,m\le k}\left(z_j^{2c+m-1/2} - z_j^{-(m-1/2)}\right)}{\det_{1\le j,m\le k}\left(z_j^{m-1/2} - z_j^{-(m-1/2)}\right)}. \tag{6.13}$$

Remark 6.1. Equation (6.13) is well-known in representation theory as well as in the theory of Young tableaux, but is usually given in a different form, for which we first need some notation.

For $\nu=(\nu_1,\dots,\nu_k)$, $\nu_1\ge\cdots\ge\nu_k\ge 0$, define the *Schur function* $s_\nu(z_1,\dots,z_k)$ by

$$s_\nu(z_1,\dots,z_k) = \frac{\det_{1\le j,m\le k}\left(z_j^{\nu_m+k-m}\right)}{\det_{1\le j,m\le k}\left(z_j^{k-m}\right)},$$

and further define for any k-tuple $\mu=(\mu_1,\dots,\mu_k)$ of integers or half-integers the *odd orthogonal character* $\mathrm{so}_\mu(z_1^{\pm},\dots,z_k^{\pm},1)$ by

$$\mathrm{so}_\mu(z_1^{\pm},\dots,z_k^{\pm},1) = \frac{\det_{1\le j,m\le k}\left(z_j^{\mu_m+k-m+1/2} - z_j^{-(\mu_m+k-m+1/2)}\right)}{\det_{1\le j,m\le k}\left(z_j^{k-m+1/2} - z_j^{-(k-m+/2)}\right)}.$$

For details on Schur functions and odd orthogonal characters, we refer the reader to [8]. Combinatorial interpretations of Schur functions and odd orthogonal characters can be found in [18] and [7, 20, 23], respectively.

With the above notation at hand, we may rewrite Equation (6.13) as

$$\sum_{2c\geq\nu_1\geq\cdots\geq\nu_1\geq 0} s_{(\nu_1,\dots,\nu_k)}(z_1,\dots,z_k) = \left(\prod_{j=1}^{k} z_j\right)^{c} \mathrm{so}_{(c,\dots,c)}(z_1^{\pm},\dots,z_k^{\pm},1).$$

Proofs for this identity have been given by, e.g., Gordon [10], Macdonald [18, I.5, Example 16] and Stembridge [22, Corollary 7.4(a)]. An elementary proof of Lemma 6.1 based on induction has been given by Bressoud [3, Proof of Lemma 4.5].

For a much more detailed account on this identity, we refer to [16, Proof of Theorem 2].

Theorems 6.1 and 6.2 below also rely on two results which we are going to summarise in the following lemmas.

Lemma 6.2 (see Krattenthaler [15, Lemma 2]). *We have the determinant evaluations*

$$\det_{1\leq j,m\leq k}\left(z_j^{m}-z_j^{-m}\right) = \left(\prod_{j=1}^{k} z_j\right)^{-k}\left(\prod_{1\leq j<m\leq k}(z_j-z_m)(1-z_jz_m)\right)\left(\prod_{j=1}^{k}(z_j^2-1)\right)$$

$$\det_{1\leq j,m\leq k}\left(z_j^{m-1/2}-z_j^{-(m-1/2)}\right) = \left(\prod_{j=1}^{k} z_j\right)^{-k+1/2}\left(\prod_{1\leq j<m\leq k}(z_j-z_m)(1-z_jz_m)\right)\left(\prod_{j=1}^{k}(z_j-1)\right).$$

Lemma 6.3. *For any non-negative integers $u_1,\dots,u_m$, the function*

$$\frac{\det_{1\leq j,m\leq k}\left(x_j^{u_m}-x_j^{-u_m}\right)}{\det_{1\leq j,m\leq k}\left(x_j^{m}-x_j^{-m}\right)}$$

is a Laurent polynomial in the complex variables $x_1,\dots,x_k$.

We note that the quantity considered in this last lemma is known in the literature as a *symplectic character*. For details on symplectic characters we refer to [8].

Theorem 6.1. *Let $\mathcal{S}$ be a composite step set over the atomic step set $\mathcal{A}$. By $\mathcal{L}$ we denote the $\mathbb{Z}$-lattice spanned by $\mathcal{A}$. The composite step generating function associated with $\mathcal{S}$ is denoted by $S(z_1,\dots,z_k)$.*

If $\mathcal{A},\mathcal{S}$ satisfy Assumption 2.1, then for any point $\mathbf{u}=(u_1,\dots,u_k)\in\mathcal{W}^0\cap\mathcal{L}$ we have the exact formula

$$P_n^{+}(\mathbf{u}) = \frac{(2\pi)^{-k}}{k!}\int\cdots\int_{|z_1|=\cdots=|z_k|=\rho} \det_{1\leq j,m\leq k}(z_j^{m})\det_{1\leq j,m\leq k}(z_j^{-m})\frac{\det_{1\leq j,m\leq k}\left(z_j^{u_m}-z_j^{-u_m}\right)}{\det_{1\leq j,m\leq k}\left(z_j^{m}-z_j^{-m}\right)} \times S(z_1,\dots,z_k)^n\left(\prod_{j=1}^{k}\frac{(z_j+1)dz_j}{z_j}\right), \tag{6.14}$$

where $\rho>0$.

Proof. We start from the exact expression for $P_n^+(\mathbf{u}\to\mathbf{v})$ as given by Corollary 2.1, viz.

$$P_n^+(\mathbf{u}\to\mathbf{v}) = \frac{(-1)^k}{(2\pi i)^k k!}\int\cdots\int_{|z_1|=\cdots=|z_k|=\rho} \det_{1\le j,m\le k}\left(z_j^{u_m}-z_j^{-u_m}\right) S(z_1,\dots,z_k)^n \det_{1\le j,m\le k}\left(z_j^{v_m}\right)\left(\prod_{j=1}^k \frac{dz_j}{z_j}\right),$$

where we choose $0<\rho<1$. We want to sum this expression over all $\mathbf{v}\in\mathcal{W}^0$. This will be accomplished in two steps. First, we sum the expression above over all $\mathbf{v}=(v_1,\dots,v_k)\in\mathcal{W}^0$ such that $v_k\le 2c+k$ for some fixed c. Second, we let c tend to infinity.

This yields

$$\sum_{0<v_1<\cdots<v_k\le 2c+k} P_n^+(\mathbf{u}\to\mathbf{v}) = \frac{(-1)^k}{(2\pi i)^k k!}\int\cdots\int_{|z_1|=\cdots=|z_k|=\rho} \det_{1\le j,m\le k}\left(z_j^{u_m}-z_j^{-u_m}\right) S(z_1,\dots,z_k)^n \times\left(\sum_{0<v_1<\cdots<v_k\le 2c+k}\det_{1\le j,m\le k}\left(z_j^{v_m}\right)\right)\left(\prod_{j=1}^k\frac{dz_j}{z_j}\right).$$

Setting $\lambda_m=v_m-m$ in Lemma 6.1, we obtain

$$\sum_{0<v_1<\cdots<v_k\le 2c+k}\det_{1\le j,m\le k}\left(z_j^{v_m}\right)=\det_{1\le j,m\le k}\left(z_j^m\right)\frac{\det_{1\le j,m\le k}\left(z_j^{2c+m-1/2}-z_j^{-(m-1/2)}\right)}{\det_{1\le j,m\le k}\left(z_j^{m-1/2}-z_j^{-(m-1/2)}\right)}.$$

Now, since $|z_j|=\rho<1$, we can let c tend to infinity, and obtain

$$\sum_{0<v_1<\cdots<\lambda_k}\det_{1\le j,m\le k}\left(z_j^{v_m}\right)=\det_{1\le j,m\le k}\left(z_j^m\right)\frac{\det_{1\le j,m\le k}\left(-z_j^{-(m-1/2)}\right)}{\det_{1\le j,m\le k}\left(z_j^{m-1/2}-z_j^{-(m-1/2)}\right)}$$
$$=(-1)^k\left(\prod_{j=1}^k z_j\right)^{1/2}\frac{\det_{1\le j,m\le k}\left(z_j^m\right)\det_{1\le j,m\le k}\left(z_j^{-m}\right)}{\det_{1\le j,m\le k}\left(z_j^{m-1/2}-z_j^{-(m-1/2)}\right)}.$$

Finally, we deduce from Lemma 6.2 that

$$\det_{1\le j,m\le k}\left(z_j^{m-1/2}-z_j^{-(m-1/2)}\right)=\left(\prod_{j=1}^k\frac{\sqrt{z_j}}{z_j+1}\right)\det_{1\le j,m\le k}\left(z_j^m-z_j^{-m}\right),$$

which proves Equation (6.14) for $0<\rho<1$.

By Lemma 6.3, the factor

$$\frac{\det_{1\le j,m\le k}\left(z_j^{u_m}-z_j^{-u_m}\right)}{\det_{1\le j,m\le k}\left(z_j^m-z_j^{-m}\right)}$$

is a Laurent polynomial. Hence, by Cauchy's theorem, the value of the integral (6.14) for $1 \leq \rho < \infty$ is the same as for $0 < \rho < 1$. This proves the theorem. ☐

Theorem 6.2. *Let $\mathcal{S}$ be a composite step set over the atomic step set $\mathcal{A}$. By $\mathcal{L}$ we denote the $\mathbb{Z}$-lattice spanned by $\mathcal{A}$. The composite step generating function associated with $\mathcal{S}$ is denoted by $S(z_1, \dots, z_k)$.*

If $\mathcal{A}, \mathcal{S}$ satisfy Assumption 2.1 and $S(1, \dots, 1) > 1$, then we have for any point $\mathbf{u} = (u_1, \dots, u_k) \in \mathcal{W}^0 \cap \mathcal{L}$ the asymptotic formula

$$(6.15)\quad P_n^+(\mathbf{u}) = S(1, \dots, 1)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{S(1, \dots, 1)}{nS''(1, \dots, 1)}\right)^{k^2/2} \times \left(\prod_{j=1}^{k} \frac{u_j (j-1)!}{(2j-1)!}\right) \left(\prod_{1 \leq j < m \leq k} (u_m^2 - u_j^2)\right) \left(1 + O\left(n^{-1/4}\right)\right)$$

as $n \to \infty$. Here, $S''(1, \dots, 1)$ denotes the second derivative of $S(z_1, \dots, z_k)$ with respect to any of the z_j.

Remark 6.2. For the special case $\mathcal{S} \cong \mathcal{A}$ (i.e., $\mathcal{S}$ and $\mathcal{A}$ are isomorphic), the order of the asymptotic growth of $P_n^+(\mathbf{u})$ has already been determined by Grabiner [11, Theorem 1]. There, Grabiner gives the asymptotic growth order of the number of walks with a free end point in a Weyl chamber for any of the classical Weyl groups as the number of steps tends to infinity, but his method does not allow to determine the coefficient of the asymptotically dominant term.

Proof of Theorem 6.2. We prove the claim with the help of a saddle point approach applied to Equation (6.14). Choosing $\rho = 1$ in (6.14), substituting $z_j = e^{i\varphi_j}$, $j = 1, 2, \dots, k$, and applying Vandermonde's determinant evaluation twice we obtain

$$(6.16)\quad P_n^+(\mathbf{u}) = \frac{1}{(2\pi)^k k!} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left(\prod_{1 \leq j < m \leq k} \left|e^{i\varphi_m} - e^{i\varphi_j}\right|^2\right) \frac{\det_{1 \leq j,m \leq k} (\sin(u_m \varphi_j))}{\det_{1 \leq j,m \leq k} (\sin(m \varphi_j))} \times S\left(e^{i\varphi_1}, \dots, e^{i\varphi_k}\right)^n \left(\prod_{j=1}^{k} \left(1 + e^{i\varphi_j}\right) d\varphi_j\right).$$

For large n, the absolute value of the integral above is mainly governed by the factor $S(e^{i\varphi_1}, \dots, e^{i\varphi_k})^n$. We therefore expect that the main contribution to the integral on the right hand side above comes from small neighbourhoods around the maxima of the function $(\varphi_1, \dots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \dots, e^{i\varphi_k})|$ on the torus $|z_1| = \dots = |z_k| = 1$, and that this dominant part can again be determined using a saddle point approach.

In Lemma 3.3, we have shown that the set of maxima of the function $(\varphi_1, \dots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \dots, e^{i\varphi_k})|$ on the torus $|z_1| = \dots = |z_k| = 1$ is a subset of the set $\{0, \pi\}^k$, and $(0, \dots, 0)$ is always a maximum. It will turn out that the maxima different from $(0, \dots, 0)$ do not contribute to the leading asymptotic term of $P_n^+(\mathbf{u})$. Hence, the asymptotic behaviour of $P_n^+(\mathbf{u})$ is captured by a small neighbourhood around $(0, \dots, 0)$. The reason for this, as we will see below, is the factor $\prod_{j=1}^{k}(1 + e^{i\varphi_j})$ of the integrand.

We proceed with a precise statement of our claim: the integral in (6.16) above is asymptotically equal to

$$\int_{-\varepsilon}^{\varepsilon}\cdots\int_{-\varepsilon}^{\varepsilon}\left(\prod_{1\le j<m\le k}\left|e^{i\varphi_m}-e^{i\varphi_j}\right|^2\right)\frac{\det_{1\le j,m\le k}(\sin(u_m\varphi_j))}{\det_{1\le j,m\le k}(\sin(m\varphi_j))} \times S(e^{i\varphi_1},\dots,e^{i\varphi_k})^n\left(\prod_{j=1}^{k}(1+e^{i\varphi_j})d\varphi_j\right), \tag{6.17}$$

as $n\to\infty$, where we choose $\varepsilon=\varepsilon(n)=n^{-5/12}$.

We are going to prove this claim by means of a saddle point approach: (1) Determine an asymptotically equivalent expression for (6.17) that is more convenient to work with. (2) Find a bound for the remaining part of the integral in (6.16).

Let us start with task (1). We have already seen in the proof of Theorem 5.1 (see Equation (5.11)) that for $|\varphi_j|\le n^{-5/12}$, $j=1,2,\dots,k$, we have the expansion

$$S\left(e^{i\varphi_1},\dots,e^{i\varphi_k}\right)=S(1,\dots,1)\exp\left(-\Lambda\sum_{j=1}^{k}\frac{\varphi_j^2}{2}\right)\left(1+O\left(n^{-5/4}\right)\right)$$

as $n\to\infty$, where $\Lambda=\frac{S''(1,\dots,1)}{S(1,\dots,1)}>0$ and $S''(z_1,\dots,z_k)=\frac{\partial^2}{\partial z_1^2}S(z_1,\dots,z_k)$. Further, we have the expansions

$$\prod_{1\le j<m\le k}\left|e^{i\varphi_m}-e^{i\varphi_j}\right|^2=\left(\prod_{1\le j<m\le k}(\varphi_m-\varphi_j)^2\right)+O\left(n^{-\binom{k}{2}-5/12}\right)$$

and

$$\prod_{j=1}^{k}\left(1+e^{i\varphi_j}\right)=2^k+O\left(n^{-5/12}\right)$$

as $n\to\infty$.

Finally, Lemma 4.6 gives us

$$\frac{\det_{1\le j,m\le k}(\sin(u_m\varphi_j))}{\det_{1\le j,m\le k}(\sin(m\varphi_j))}=\left(\prod_{1\le j<m\le k}\frac{u_m^2-u_j^2}{m^2-j^2}\right)\left(\prod_{j=1}^{k}\frac{u_j}{j}\right)\left(1+O\left(n^{-5/6}\right)\right)$$

as $n\to\infty$. Therefore, the integral (6.17) is asymptotically equal to

$$S(1,\dots,1)^n\left(1+O\left(n^{-1/4}\right)\right)\left(\prod_{j=1}^{k}\frac{2u_j}{(2j-1)!}\right)\left(\prod_{1\le j<m\le k}(u_m^2-u_j^2)\right) \times\int_{-\varepsilon}^{\varepsilon}\cdots\int_{-\varepsilon}^{\varepsilon}\left(\prod_{1\le j<m\le k}(\varphi_m-\varphi_j)^2\right)\exp\left(-n\Lambda\sum\frac{\varphi_j^2}{2}\right)\left(\prod_{j=1}^{k}d\varphi_j\right)$$

as $n \to \infty$. Now, the substitution $\varphi_j \mapsto \varphi_j/\sqrt{2/(n\Lambda)}$ transforms this last integral into

$$\left(\frac{2}{n\Lambda}\right)^{k^2/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\prod_{1 \le j < m \le k} (\varphi_m - \varphi_j)^2 \right) e^{-\sum_{j=1}^{k} \varphi_j^2} \prod_{j=1}^{k} d\varphi_j + O\left(e^{-\Lambda n^{1/6}/2}\right).$$

This resulting integral is a *Selberg integral*, and it is well known (see, e.g., [19, p. 321]), that

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\prod_{1 \le j < m \le k} (\varphi_m - \varphi_j)^2 \right) e^{-\sum_{j=1}^{k} \varphi_j^2} \prod_{j=1}^{k} d\varphi_j = \frac{(2\pi)^{k/2}}{2^{k^2/2}} \prod_{j=1}^{k} j!.$$

This shows that the integral (6.17) is asymptotically equal to

$$S(1,\dots,1)^n \left(1 + O\left(n^{-1/4}\right)\right) \left(\frac{2}{n\Lambda}\right)^{k^2/2} \frac{(2\pi)^{k/2}}{2^{k^2/2}} \left(\prod_{j=1}^{k} \frac{2u_j j!}{(2j-1)!} \right) \left(\prod_{1 \le j < m \le k} (u_m^2 - u_j^2) \right),$$

which completes task (1).

We now turn towards task (2) of the saddle point approach: finding a bound for the remaining part of the integral. For the sake of convenience, we adopt the notation of the proof of Theorem 5.1: by $\mathcal{M}$, we denote the set of maximal points of the function $(\varphi_1,\dots,\varphi_k) \mapsto |S(e^{i\varphi_1},\dots,e^{i\varphi_k})|$, and we define the sets

$$\mathcal{U}_\varepsilon(\hat\varphi) = \left\{ \varphi \in \mathbb{R}^k \ : \ |\hat\varphi - \varphi|_\infty < \varepsilon \right\}, \qquad \hat\varphi = (\hat\varphi_1,\dots,\hat\varphi_k) \in \mathcal{M},$$

and $\mathcal{U}_\varepsilon^c(\hat\varphi) = [0,2\pi) \setminus \mathcal{U}_\varepsilon(\hat\varphi)$ as well as $\mathcal{U}_\varepsilon(\mathcal{M}) = \bigcup_{\hat\varphi \in \mathcal{M}} \mathcal{U}_\varepsilon(\hat\varphi)$.

Analogous to the reasoning in the proof of Theorem 5.1, we see that

$$\int \dots \int_{\mathcal{U}_\varepsilon^c(\mathcal{M})} \left(\prod_{1 \le j < m \le k} \left| e^{i\varphi_m} - e^{i\varphi_j} \right|^2 \right) \frac{\det_{1 \le j,m \le k} (\sin(u_m \varphi_j))}{\det_{1 \le j,m \le k} (\sin(m\varphi_j))} S\left(e^{i\varphi_1},\dots,e^{i\varphi_k}\right)^n \left(\prod_{j=1}^{k} \left(1 + e^{i\varphi_j}\right) d\varphi_j \right)$$

is $O\left(S(1,\dots,1)^{n - C_1 n^{1/6}}\right)$ for some constant $C_1 > 0$ as $n \to \infty$.

It remains to establish bounds for the (finitely many) integrals

$$\int \dots \int_{\mathcal{U}_\varepsilon(\hat\varphi)} \left(\prod_{1 \le j < m \le k} \left| e^{i\varphi_m} - e^{i\varphi_j} \right|^2 \right) \frac{\det_{1 \le j,m \le k} (\sin(u_m \varphi_j))}{\det_{1 \le j,m \le k} (\sin(m\varphi_j))} S\left(e^{i\varphi_1},\dots,e^{i\varphi_k}\right)^n \left(\prod_{j=1}^{k} \left(1 + e^{i\varphi_j}\right) d\varphi_j \right),$$

where $\hat\varphi$ ranges over $\mathcal{M} \setminus \{(0,\dots,0)\}$. If $(\hat\varphi_1,\dots,\hat\varphi_k) \neq (0,\dots,0)$, $\hat\varphi_r = \pi$, say, then we have $1 + e^{i(\hat\varphi_r + \vartheta_r)} = O(\vartheta_r)$, and consequently,

$$\prod_{j=1}^{k} \left(1 + e^{i(\hat\varphi_j + \vartheta_j)}\right) = O\left(n^{-5/12}\right)$$

for $|(\vartheta_1,\dots,\vartheta_k)|_\infty < \varepsilon = n^{-5/12}$ as $n\to\infty$. Hence, we obtain

$$\int_{\mathcal{U}_\varepsilon(\hat\varphi)}\!\!\cdots\int \left(\prod_{1\le j<m\le k}\left|e^{i\varphi_m}-e^{i\varphi_j}\right|^2\right)\frac{\det\limits_{1\le j,m\le k}(\sin(u_m\varphi_j))}{\det\limits_{1\le j,m\le k}(\sin(m\varphi_j))} S\left(e^{i\varphi_1},\dots,e^{i\varphi_k}\right)^n\left(\prod_{j=1}^k\left(1+e^{i\varphi_j}\right)d\varphi_j\right) = O\left(n^{-5/12-k^2/2}S(1,\dots,1)^n\right)$$

for $\hat\varphi=(\hat\varphi_1,\dots,\hat\varphi_k)\in\mathcal{M}\setminus\{(0,\dots,0)\}$ as $n\to\infty$. This finally gives us the bound

$$\int_{\mathcal{U}^c_\varepsilon(\mathbf{0})}\!\!\cdots\int \left(\prod_{1\le j<m\le k}\left|e^{i\varphi_m}-e^{i\varphi_j}\right|^2\right)\frac{\det\limits_{1\le j,m\le k}(\sin(u_m\varphi_j))}{\det\limits_{1\le j,m\le k}(\sin(m\varphi_j))} S\left(e^{i\varphi_1},\dots,e^{i\varphi_k}\right)^n\left(\prod_{j=1}^k\left(1+e^{i\varphi_j}\right)d\varphi_j\right) = O\left(n^{-5/12-k^2/2}S(1,\dots,1)^n\right)$$

as $n\to\infty$, and completes the proof of the theorem. □

7. Applications

The rest of this manuscript is entirely devoted to applications of Theorem 5.1 and Theorem 6.2.

Special cases of some of the results presented in the following subsections have already been derived earlier by other authors. Also, we can give precise answers to some questions to which only partial results were known. In these cases, we provide the reader with pointers to the original literature. Some other results (in particular, Corollaries 7.1, 7.3, 7.5 and 7.6) in this section seem, to the author's best knowledge, to be new.

7.1. **Lock step model of vicious walkers with wall restriction.** In general, the vicious walkers model is concerned with k random walkers on a d-dimensional lattice. In the lock step model, at each time step all of the walkers move one step in any of the allowed directions, such that at no time any two random walkers share the same lattice point. This model was defined by Fisher [6] as a model for wetting and melting processes.

In this subsection, we consider a *two dimensional lock step model of vicious walkers with wall restriction*, which we briefly describe now. The only allowed steps are $(1,1)$ and $(1,-1)$, and the lattice is the $\mathbb{Z}$-lattice spanned by these two vectors. Fix two vectors $\mathbf{u},\mathbf{v}\in\mathbb{Z}^k$ such that $0<u_1<u_2<\cdots<u_k$ and $u_i\equiv u_j \mod 2$ for $1\le i<j\le k$, and analogously for $\mathbf{v}$. For $1\le j\le k$, the j-th walker starts at $(0,u_j-1)$ and, after n steps, ends at the point (n,v_j-1) in a way such that at no time the walker moves below the horizontal axis ("the wall") or shares a lattice point with another walker.

Certain configurations of the two dimensional vicious walkers model, such as *watermelons* and *stars* consisting of k vicious walkers with or without the presence of an impenetrable walls, have been fully analysed by Guttmann et al. [13] and Krattenthaler et al. [16, 17]. In their papers, they prove exact as well as asymptotic results for the total number of these configurations.

The results in this subsection include asymptotics for the total number of vicious walkers configurations with an arbitrary (but fixed) starting point having either an arbitrary (but fixed) end point or a free end point (see Corollary 7.1 and Corollary 7.3, respectively). Special cases of these asymptotics have been derived earlier by Krattenthaler et al. [16, 17] and Rubey [21]. For further links to the literature concerning this model, we refer to the references given in the papers mentioned before.

The two dimensional lock step model of vicious walkers as described above can easily be reformulated as a model of lattice paths in a Weyl chamber of type B as follows: at each time, the positions of the walkers are encoded by a k-dimensional vector, where the j-th coordinate records the current second coordinate (the *height*) of the j-th walker. Clearly, if $(c_1, \dots, c_k) \in \mathbb{Z}^k$ is such a vector encoding the heights of our walkers at a certain point in time, then we necessarily have $0 \le c_1 < c_2 < \cdots < c_k$ and $c_i \equiv c_j \mod 2$ for $1 \le i < j \le k$. Hence, each realisation of the lock step model with k vicious walkers, where the j-th walker starts at $(0, u_j - 1)$ and ends at $(n, v_j - 1)$, naturally corresponds to a lattice path in

$$\left\{(x_1, x_2, \dots, x_k) \in \mathbb{Z}^k \ : \ 0 < x_1 < \cdots < x_k \text{ and } x_i \equiv x_j \mod 2 \text{ for } 1 \le i < j \le k\right\}$$

that starts at $\mathbf{u} = (u_1, \dots, u_k)$ and ends at $\mathbf{v} = (v_1, \dots, v_k)$. (Note the shift by $+1$.) The atomic step set is given by

$$\mathcal{A} = \left\{\sum_{j=1}^{k} \varepsilon_j \mathbf{e}^{(j)} \ : \ \varepsilon_1, \dots, \varepsilon_k \in \{-1, +1\}\right\},$$

and the composite step set $\mathcal{S}$ is set of all sequences of length one of elements in $\mathcal{A}$. This means, that in the present case there is only a formal difference between the atomic steps and composite steps. Both sets, $\mathcal{A}$ and $\mathcal{S}$ satisfy Assumption 2.1 (the conditions of Lemma 2.1). Consequently, asymptotics for this model can be obtained from Theorem 5.1 and Theorem 6.2.

The composite step generating function associated with $\mathcal{S}$ is

$$S(z_1, \dots, z_k) = \prod_{j=1}^{k} \left(z_j + \frac{1}{z_j}\right),$$

and the set $\mathcal{M} \subseteq \{0, \pi\}^k$ of points maximising the function $(\varphi_1, \dots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \dots, e^{i\varphi_k})|$ is given by $\mathcal{M} = \{0, \pi\}^k$. Hence, we have $|\mathcal{M}| = 2^k$, and after short calculations we find $S(1, \dots, 1) = S''(1, \dots, 1) = 2^k$. As a consequence of Theorem 5.1, we obtain the following result.

Corollary 7.1. *The number of vicious walkers of length n with k walkers that start at $(0, u_1 - 1), \dots, (0, u_k - 1)$ and end at $(n, v_1 - 1), \dots, (n, v_k - 1)$ (we assume that $u_1 + v_1 \equiv n \mod 2$) is asymptotically equal to*

$$2^{nk+3k/2} \pi^{-k/2} n^{-k^2-k/2} \frac{\left(\prod_{1 \le j < m \le k} (v_m^2 - v_j^2)(u_m^2 - u_j^2)\right) \left(\prod_{j=1}^{k} v_j u_j\right)}{\left(\prod_{j=1}^{k} (2j-1)!\right)}$$

as $n \to \infty$.

The special case $u_j = 2a_j + 1$, $j = 1, \dots, k$, of the corollary above implicitly appears in Rubey [21, Proof of Theorem 4.1, Chapter 2]. Other special instances of Corollary 7.1 can be found in [16, Theorem 15]. For example, let us consider the so-called *k-watermelon configuration.* In this case, the walkers start at $(0,0), (0,2), \dots, (0, 2k-2)$ and, after $2n$ steps, end at

$(2n,0),(2n,2),\dots,(2n,2k-2)$. Hence, setting $u_j=v_j=2j-1$, $1\le j\le k$, as well as replacing n with $2n$ in the asymptotics above, we obtain the following corollary.

Corollary 7.2 (see Krattenthaler et al. [16, Theorem 15]). *The number of k-watermelon configurations of length $2n$ is asymptotically equal to*

$$4^{kn}2^{k^2-k}\pi^{-k/2}n^{-k^2-k/2}\left(\prod_{j=1}^{k}(2j-1)!\right),\qquad n\to\infty.$$

Asymptotics for the number of walkers with a free end point can be derived from Theorem 6.2.

Corollary 7.3. *The number of vicious walkers of length n that start at $(0,u_1-1),\dots,(0,u_k-1)$, $0<u_1<\cdots<u_k$, $u_j\equiv u_\ell \mod 2$, is asymptotically equal to*

$$2^{nk+k/2}\pi^{-k/2}n^{-k^2/2}\left(\prod_{j=1}^{k}\frac{u_j(j-1)!}{(2j-1)!}\right)\left(\prod_{1\le j<m\le k}(u_m^2-u_j^2)\right),\qquad n\to\infty.$$

Setting $u_j=2a_j+1$, $j=1,\dots,k$ in the corollary above, we obtain as a special case [21, Theorem 4.1, Chapter 2].

The set of *k-star configurations* consists of all possible vicious walks with the starting points $(0,0),(0,2),\dots,(0,2k-2)$. Hence, setting $u_j=2j-1$, $j=1,\dots,k$, in the corollary above, we obtain the following result.

Corollary 7.4 (see Krattenthaler et al. [16, Theorem 8]). *The number of k-star configurations of length n is asymptotically equal to*

$$2^{nk+k^2-k/2}\pi^{-k/2}n^{-k^2/2}\prod_{j=1}^{k}(j-1)!,\qquad n\to\infty.$$

7.2. **Random turns model of vicious walkers with wall restriction.** This model is quite similar to the lock step model of vicious walkers. The difference here is, that at each time step exactly one walker is allowed to move (all the other walkers have to stay in place).

We consider the random turns model with k vicious walkers. Again, at no time any two of the walkers may share a lattice point, and none of them is allowed to go below the horizontal axis. Now, fix two points $\mathbf{u},\mathbf{v}\in\mathbb{Z}^k\cap\mathcal{W}^0$, and assume that for $1\le j\le k$, the j-th walker starts at $(0,u_j-1)$ and, after n steps, ends at (n,v_j-1). In an analogous manner as in the previous subsection, we interpret this as a lattice walk of length n in $\mathbb{Z}^k\cap\mathcal{W}^0$ that starts at $\mathbf{u}$ and ends at $\mathbf{v}$. Here, the underlying lattice is given by $\mathcal{L}=\mathbb{Z}^k$ and the atomic step set is seen to be

$$\mathcal{S}=\left\{\pm\mathbf{e}^{(1)},\pm\mathbf{e}^{(2)},\dots,\pm\mathbf{e}^{(k)}\right\}.$$

The composite step set is, as in the last subsection, the set of all sequences of length one of elements in $\mathcal{A}$. Since both sets, $\mathcal{S}$ and $\mathcal{A}$, satisfy Assumption 2.1, we may obtain asymptotics by means of Theorem 5.1 and Theorem 6.2.

From the description of $\mathcal{S}$ above it is seen that the associated composite step generating function is given by

$$S(z_1,\dots,z_k)=A(z_1,\dots,z_k)=\sum_{j=1}^{k}\left(z_j+\frac{1}{z_j}\right).$$

Short calculations give us $S(1,\dots,1) = 2k$ and $S''(1,\dots,1) = 2$. Furthermore, it is easily checked that the set of maximal points is given by $\mathcal{M} = \{(0,\dots,0),(\pi,\dots,\pi)\}$, which implies $|\mathcal{M}| = 2$. Consequently, according to Theorem 5.1, we have the following result.

Corollary 7.5. *The number of k vicious walkers in the random turns model, where the j-th walker starts at $(0, u_j - 1)$ and, after n steps ends at $(n, v_j - 1)$, is asymptotically equal to*

$$2(2k)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{k}{n}\right)^{k^2+k/2} \frac{\left(\prod_{1\le j<m\le k} (v_m^2 - v_j^2)(u_m^2 - u_j^2)\right)\left(\prod_{j=1}^{k} v_j u_j\right)}{\left(\prod_{j=1}^{k}(2j-1)!\right)}, \qquad n \to \infty.$$

Asymptotics for the number of vicious walks starting in $(0, u_j - 1)$, $j = 1,\dots,k$, with a free end point can be determined with the help of Theorem 6.2.

Corollary 7.6. *The number of k-vicious walkers in the random turns model, where the j-th walker starts at $(0, u_j - 1)$, of length n is asymptotically equal to*

$$(2k)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{k}{n}\right)^{k^2} \left(\prod_{j=1}^{k} \frac{u_j (j-1)!}{(2j-1)!}\right) \left(\prod_{1\le j<m\le k} (u_m^2 - u_j^2)\right), \qquad n \to \infty.$$

7.3. **k-non-crossing tangled diagrams with isolated points.** Tangled diagrams are certain special embeddings of graphs over the vertex set $\{1, 2, \dots, n\}$ and vertex degrees of at most two. More precisely, the vertices are arranged in increasing order on a horizontal line, and all edges are drawn above this horizontal line with a particular notion of crossings and nestings. Instead of giving an in-depth presentation of tangled diagrams we refer to the papers [4, 5] for details, and quote the following crucial observation by Chen et al. [5, Observation 2, page 3]:

> "The number of k-non-crossing tangled diagrams over $\{1, 2, \dots, n\}$ (allowing isolated points), equals the number of simple lattice walks in $x_1 \ge x_2 \ge \cdots \ge x_{k-1} \ge 0$, from the origin back to the origin, taking n days, where at each day the walker can either feel lazy and stay in place, or make one unit step in any (legal) direction, or else feel energetic and make any two consecutive steps (chosen randomly)."

In order to simplify the presentation, we replace k with $k + 1$, and determine asymptotics for the number of $(k + 1)$-non-crossing tangled diagrams. A simple change of the lattice path description given above shows the applicability of Theorem 5.1 to this problem. We proceed with a precise description. Consider a typical walk of the type described in the quotation above, and let $\left((c_1^{(m)}, \dots, c_k^{(m)})\right)_{m=0,\dots,n}$ be the sequence of lattice points visited during the walk. Then, the sequence $\left((c_k^{(m)} + 1, c_{k-1}^{(m)} + 2, \dots, c_1^{(m)} + k)\right)_{m=0,\dots,n}$ is sequence of lattice points visited by a walker starting and ending in $(1, 2, \dots, k)$ that is confined to the region $0 < x_1 < x_2 < \cdots < x_k$ with the same step set as described in the quotation above. This clearly defines a bijection between walks of the type described in the quotation above and walks confined to the region $0 < x_1 < \cdots < x_k$ starting and ending in $\mathbf{u} = (1, 2, \dots, k)$ with the same set of steps.

As a consequence, we see that the number of $(k + 1)$-non-crossing tangled diagrams with isolated points on the set $\{1, 2, \dots, n\}$ is equal to the number of walks starting and ending in

$\mathbf{u}$ that are confined to the region $0 < x_1 < \cdots < x_k$ and consist of composite steps from the set

$$\mathcal{S} = \{\mathbf{0}\} \cup \mathcal{A} \cup \mathcal{A} \times \mathcal{A},$$

where the atomic step set $\mathcal{A}$ is given by

$$\mathcal{A} = \left\{\pm\mathbf{e}^{(1)}, \pm\mathbf{e}^{(2)}, \dots, \pm\mathbf{e}^{(k)}\right\}.$$

The step sets $\mathcal{A}$ and $\mathcal{S}$ are seen to satisfy the assumptions of Theorem 5.1, and, therefore, may be used to obtain asymptotics for $P_n^+(\mathbf{u} \to \mathbf{u})$.

According to the definition of the composite step set $\mathcal{S}$, the composite step generating function $S(z_1, \dots, z_k)$ is given by

$$S(z_1, \dots, z_k) = 1 + \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j}\right) + \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j}\right)^2.$$

Short calculations show that $S(1, \dots, 1) = 1 + 2k + 4k^2$ and $S''(1, \dots, 1) = 2 + 8k$, and it is easily seen that $(0, \dots, 0)$ is the only point of maximal modulus of $S(z_1, \dots, z_k)$ on the torus $|z_1| = \cdots = |z_k| = 1$. Consequently, Theorem 5.1 gives us asymptotics for the number of $(k+1)$-non-crossing tangled diagrams.

Corollary 7.7. *The total number of $(k+1)$-non-crossing tangled diagrams is asymptotically equal to*

$$P_n^+(\mathbf{u} \to \mathbf{u}) \sim (1 + 2k + 4k^2)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{1 + 2k + 4k^2}{n(2 + 8k)}\right)^{k^2 + k/2} \left(\prod_{j=1}^{k} (2j-1)!\right), \qquad n \to \infty.$$

7.4. **k-non-crossing tangled diagrams without isolated points.** Consider a tangled diagram as defined in the previous example. A vertex of this tangled diagram is called *isolated*, if and only if its vertex degree is zero, that is, the vertex is isolated in the graph theoretical sense.

Again, for the sake of convenience, we shift k by one, and consider $(k+1)$-non-crossing tangled diagrams without isolated points. In an analogous manner as in the previous section, these diagrams can be bijectively mapped onto a set of lattice paths (see [5, Observation 1, p.3]) in the region $0 < x_1 < \cdots < x_k$ that start and end in $\mathbf{u} = (1, 2, \dots, k)$. The only difference to the situation described in the last example is the fact, that now the walker is not allowed to stay in place. Hence, the composite step set $\mathcal{S}$ is now given by

$$\mathcal{S} = \mathcal{A} \cup \mathcal{A} \times \mathcal{A}.$$

The atomic step set $\mathcal{A}$ remains unchanged.

According to the definition of $\mathcal{S}$, the composite step generating function is now given by

$$S(z_1, \dots, z_k) = \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j}\right) + \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j}\right)^2,$$

so that $S(1, \dots, 1) = 2k + 4k^2$ and $S''(1, \dots, 1) = 2 + 8k$, as well as $\mathcal{M} = \{(0, \dots, 0)\}$. Asymptotics for the number of $(k+1)$-non-crossing tangled diagrams without isolated points can now easily be determined with the help of Theorem 5.1.

Corollary 7.8. *The total number of $(k+1)$-non-crossing tangled diagrams without isolated points is asymptotically equal to*

$$P_n^+(\mathbf{u}\to\mathbf{u}) \sim (2k+4k^2)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{2k+4k^2}{n(2+8k)}\right)^{k^2+k/2} \left(\prod_{j=1}^{k}(2j-1)!\right), \qquad n\to\infty.$$

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PAPER B

The height of watermelons with wall

THE HEIGHT OF WATERMELONS WITH WALL

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Abstract. We derive asymptotics for the moments as well as the weak limit of the height distribution of watermelons with p branches with wall. This generalises a famous result of de Bruijn, Knuth and Rice [4] on the average height of planted plane trees, and results by Fulmek [9] and Katori et al. [15] on the expected value, respectively higher moments, of the height distribution of watermelons with two branches.

The asymptotics for the moments depend on the analytic behaviour of certain multidimensional Dirichlet series. In order to obtain this information we prove a reciprocity relation satisfied by the derivatives of one of Jacobi's theta functions, which generalises the well known reciprocity law for Jacobi's theta functions.

1. Introduction

The model of *vicious walkers* was introduced by Fisher [7]. He gave a number of applications in physics, such as modelling wetting and melting processes. In general, the model of vicious walkers is concerned with p random walkers on a d-dimensional lattice. In the lock step model, at each time step all of the walkers move one step in any of the allowed directions, such that at no time any two random walkers share the same lattice point.

A configuration that attracted much interest amongst mathematical physicists and combinatorialists is the *watermelon configuration*, which is a special case of the two dimensional vicious walker model. See Figure 1 for an example of a watermelon, where, for the moment, the broken line labelled 13 should be ignored. This configuration can be studied with or without presence of an impenetrable wall, and with or without deviation. We proceed with a description of *p-watermelons of length* $2n$ *with wall* (without deviation), which is the model underlying this paper. Consider the lattice in $\mathbb{R}^2$ spanned by the two vectors $(1,1)$ and $(1,-1)$. At time zero the walkers are located at the points $(0,0),(0,2),\ldots,(0,2p-2)$. The allowed directions for the walkers are given by the vectors $(1,1)$ and $(1,-1)$. Further, the horizontal line $y=0$ is an impenetrable wall, that is, no walker is allowed to cross the x-axis. The walkers may now simultaneously move one step in one of the allowed directions, but such that at no time two walkers share the same place. Additionally we demand that after $2n$ steps all walkers are located at $(2n,0),(2n,2),\ldots,(2n,2p-2)$.

Tracing the paths of the vicious walkers through the lattice we obtain a set of non-intersecting lattice paths with steps in the set $\{(1,1),(1,-1)\}$. In the case of watermelons without deviation, the i-th lattice path, also called i-th branch, starts at $(0,2i)$ and ends at $(2n,2i)$. Further, it is seen that the bottom most path is a Dyck path, so that the 1-watermelons with wall correspond to Dyck paths.

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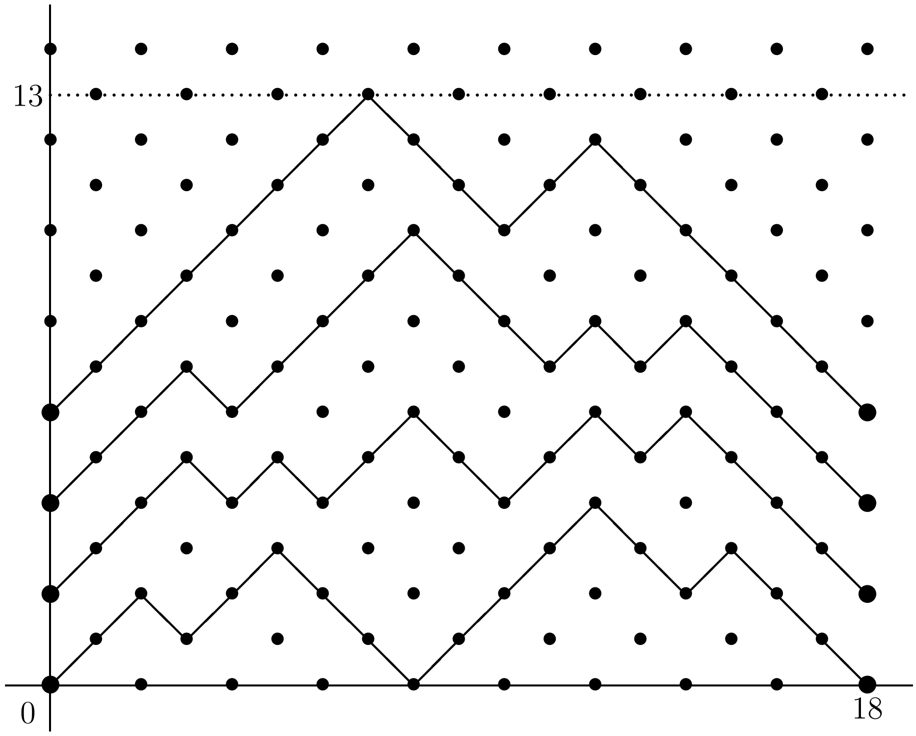

FIGURE 1. Example of a 4-watermelon of length 18 with wall and height 13

Since Fisher's introduction [7] of the vicious walkers model numerous papers on this subject have appeared. While early results mostly analyse vicious walkers in a continuum limit, there are nowadays many results for certain configurations directly based on the lattice path description above. For example, Guttmann, Owczarek and Viennot [13] related the star and watermelon configurations to the theory of Young tableaux and integer partitions, and re-derived results for the total number of stars and watermelons without wall. Later, Krattenthaler, Guttmann and Viennot [18] proved new, exact as well as asymptotic, results for the number of certain vicious walkers with wall. Recently, Krattenthaler [17] analysed the number of contacts of the bottom most walker in the case of watermelons with wall, continuing earlier work by Brak, Essam and Owczarek [22].

In 2003, Bonichon and Mosbah [2] presented an algorithm for uniform random generation of watermelons, which is based on the counting results by Krattenthaler, Guttmann and Viennot [18] (see Theorems 1 and 6 therein). Amongst other things they used their generator for obtaining experimental results on the *height of watermelons*. Here, the height of a watermelon is defined as the smallest number h such that the upper most branch does not cross the horizontal line $y = h$. See Figure 1 for an example with four branches and height 13.

As already mentioned, watermelons with one branch are simply Dyck paths. It is well-known that these are in bijection with planted plane trees, and that under this bijection the height of a Dyck path corresponds to the height of the corresponding tree. The asymptotic behaviour of the average height of planted plane trees was determined by de Bruijn, Knuth and Rice [4], that is, they solved the average height problem for 1-watermelons with wall. Recently, Fulmek [9] extended their reasoning and determined the asymptotic behaviour of the average height of 2-watermelons with wall. Katori, Izumi and Kobayashi [15] considered the diffusion scaling limit of 2-watermelons, and obtained the leading asymptotic term for all moments of the height distribution as well as a central limit theorem. The limiting process of p-watermelons has been investigated by Gillet [12]. He succeeded in proving the convergence of (properly scaled) watermelons to certain limiting processes, which he characterised by stochastic differential equations.

In this paper we rigorously analyse the height of p-watermelons of length $2n$ with wall, and obtain asymptotics for all moments of the height distribution as $n \to \infty$ as well as a central limit theorem. In particular, we show that the s-th moment behaves like $s\kappa_s^{(p)} n^{s/2} - 3\binom{s}{2}\kappa_{s-1}^{(p)} n^{(s-1)/2} + O(n^{s/2-1} + n^{p/2-p^2}\log n)$ as $n \to \infty$ for some explicit numbers $\kappa_s^{(p)}$, see Theorem 1 at the end of Section 3. The nature of our result explains the somewhat inconclusive predictions in [2]. To be more specific, Bonichon and Mosbah [2] predicted, based on numerical experiments, that $\kappa_1^{(p)} \approx \sqrt{(1.67p - 0.06)}$. Although it does not seem unlikely that the constant $\kappa_1^{(p)}$, as given in Theorem 1, behaves like $\sqrt{p}$ as $p \to \infty$, a rigorous proof is still lacking and work in progress.

The proof of our result can be summarised as follows. As a first step, we represent the total number of watermelons and the number with height restriction in terms of certain determinants (see Lemma 4), the entries being sums of binomial coefficients. From these determinants we then obtain an exact expression for the s-th moment of the height distribution. After normalisation we may apply Stirling's formula and obtain an expression that can be asymptotically evaluated using Mellin transform techniques (see Lemma 7). This kind of approach goes back to de Bruijn, Knuth and Rice [4]. Fulmek [9] adopted their approach for the asymptotic analysis of 2-watermelons. The new objects which arise here (and, in general, when extending this approach to the asymptotic analysis of p-watermelons) are certain multidimensional Dirichlet series (instead of Riemann's zeta function as in [4]). An additional complication with which one has to cope is the increasing number of cancellations of leading asymptotic terms that one encounters in the calculations while the number p of branches becomes bigger. Thus, while a brute force approach will eventually produce a result for any fixed p (this is, in essence, what Fulmek [9] and Katori et al. [15] do for $p = 2$), the main difficulty that we have to overcome in order to arrive at an asymptotic result for *arbitrary* p is to trace the roots of these cancellations. We accomplish this with the help of Lemma 8. It allows us to exactly pin down which cancellations take place and to extract explicit formulas for the first two terms which survive the cancellations. The multidimensional Dirichlet series which arise in our analysis are the subject of the subsequent section. What we need is information on their poles. This information is obtained with the help of a relation that generalises the reciprocity law for Jacobi's theta functions (see Equation (3)), that is proved in Proposition 1. We note that our definition of these Dirichlet series differs slightly from Fulmek's definition, which makes the analysis somewhat easier. These Dirichlet series that we encounter are related to so-called twisted multivariate zeta functions, studied, e.g., by de Crisenoy [5] and de Crisenoy and Essouabri [6]. However, their results cannot be used since they do not apply to our multidimensional Dirichlet series, which are explicitly excluded in these two papers. They can also be found within a class of multidimensional Dirichlet series studied by Cassou-Noguès [3]. In principle we could apply her results to our Dirichlet series and would obtain information on the poles of the analytic continuation of these series. But this would be cumbersome, and in our case it is more straightforward to obtain this information using the generalised reciprocity relation (see end of Section 2), which we are going to need in the proof of Theorem 1 anyway.

Small modifications immediately yield analogous results for p-watermelons with a horizontal wall positioned at some negative integer. Also, the analysis of the height distribution of watermelons *without wall* can be accomplished in a completely analogous fashion.

The paper is organised as follows. The second section contains information on the analytic character of certain multidimensional Dirichlet series that is crucial for the proof of our main result. The third section contains the main result, see Theorem 1 at the end of that section. The techniques applied in that section are then used to obtain a central limit law, see Theorem 2 at the end of this paper.

We close this section by fixing some notation. Vectors are denoted using bold face letters and are assumed to be p-dimensional row vectors. Further, we make use of the 1-norm and the 2-norm of vectors, viz. $|\mathbf{w}|_1 = w_0 + \cdots + w_{p-1}$ and $|\mathbf{w}|_2^2 = w_0^2 + \cdots w_{p-1}^2$. Finally, we define $\mathbf{v}^{\mathbf{w}} = v_0^{w_0} \ldots v_{p-1}^{w_{p-1}}$. The relation $\mathbf{v} \geq \mathbf{w}$ is to be understood component-wise.

2. Some multidimensional Dirichlet series

In this section we study the multidimensional Dirichlet series

$$Z_{\mathbf{a}}(z) = \sum_{\mathbf{m} \neq \mathbf{0}} \frac{m_0^{a_0} \ldots m_{p-1}^{a_{p-1}}}{(m_0^2 + \cdots + m_{p-1}^2)^z} = \sum_{\mathbf{m} \neq \mathbf{0}} \frac{\mathbf{m}^{\mathbf{a}}}{|\mathbf{m}|_2^{2z}},$$

where $\mathbf{m} = (m_0, \ldots, m_{p-1})$ ranges over $\mathbb{Z}^p \setminus \{0\}$, for $\mathbf{a} = (a_0, \ldots, a_{p-1}) \in \mathbb{Z}^p$, $\mathbf{a} \geq \mathbf{0}$. Our goal is to establish the analytic continuation of $Z_{\mathbf{a}}(z)$ to a meromorphic function and the determination of its poles. Also, we need information on the growth of $Z_{\mathbf{a}}(z)$ as $|z| \to \infty$ in some vertical strip.

It follows from the definition that $Z_{a_0,\ldots,a_{p-1}}(z) = Z_{a_{\sigma(0)},\ldots,a_{\sigma(p-1)}}(z)$ for every permutation $\sigma \in S_p$. If $p = 1$ then

$$Z_a(z) = 2\,[a \text{ even}]\,\zeta(2z - a),$$

where [Statement] is Iverson's notation, that is

$$[\text{Statement}] = \begin{cases} 1 & \text{if 'Statement' is true,} \\ 0 & \text{otherwise.} \end{cases}$$

If a_{p-1} is odd, the definition shows that $Z_{a_0,\ldots,a_{p-2},a_{p-1}}(z) = 0$. Consequently, we may assume that the parameters $a_0, \ldots, a_{p-1}$ are even.

The analytic continuation of $Z_{2\mathbf{a}}(z)$ is accomplished very much in the spirit of one of Riemann's methods for $\zeta(z)$ (see, e.g., [25, Section 2.6]). In fact we have

$$(1) \qquad \frac{(2\pi i)^{2|\mathbf{a}|_1}}{\pi^z} Z_{2\mathbf{a}}(z)\Gamma(z) = \int_0^\infty t^{z-1} \left(\left(\prod_{j=0}^{p-1} \vartheta_{2a_j}(t) \right) - [\mathbf{a} = \mathbf{0}] \right) dt,$$

where $\vartheta_a(t) = \theta_a(0, it)$ and where

$$\theta_a(x, y) = \frac{\partial^a}{\partial x^a} \theta(x, y) = \sum_{n=-\infty}^{\infty} (2\pi i n)^a e^{2\pi i (xn + n^2 y/2)}, \qquad \Im(y) > 0,$$

is the a-th derivative with respect to x of $\theta(x,y) = \sum_n e^{2\pi i(xn+n^2y/2)}$, a variant of one of Jacobi's theta functions. Here, Equation (1) is obtained by substitution of Euler's integral for the gamma function, viz. $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, and the series definition for $Z_{2\mathbf{a}}$ on the left hand side of the equation above followed by interchanging summation and integration as well as a change of variables in the integral.

We are now going to extract information on the poles of $Z_{2\mathbf{a}}(z)$ from the integral (1). This task is accomplished with the help of a generalised reciprocity relation (see Corollary 1), which is a consequence of the following two results, stated in Lemma 1 and Proposition 1. This relation generalises Jacobi's reciprocity law for $\theta(x,y)$, and is proved following along the lines of the proof of the reciprocity law in [19, Section 2.3].

Lemma 1. *Let $(f_a(x,y))_{a\geq 0}$ be a sequence of functions which are entire with respect to x for every fixed y with $\Im(y) > 0$. If $(f_a(x,y))_{a\geq 0}$ satisfies the conditions*

(i) $f_a(x+1,y) = f_a(x,y)$

(ii) $f_a(x-y,y) = e^{2\pi i(x-y/2)} \sum_{k=0}^{a} \binom{a}{k} f_k(x,y)$

then we have

$$f_a(x,y) = \sum_{k=0}^{a} \binom{a}{k} \frac{c_0^{(k)}(y)}{(2\pi i)^{a-k}} \theta_{a-k}(x,y), \tag{2}$$

where

$$c_0^{(k)}(y) = \int_0^1 f_k(x,y)dx$$

is the constant term in the Fourier expansion of $f_k(x,y)$ as a function in x.

Proof. Condition (i) implies the convergent Fourier expansion ($f_a(x,y)$ being understood as a function of x)

$$f_a(x,y) = \sum_n c_n^{(a)}(y) e^{2\pi i(xn+n^2y/2)}$$

for $a \geq 0$ which shows that

$$e^{-2\pi i(x-y/2)} f_a(x-y,y) = \sum_n c_{n+1}^{(a)}(y) e^{2\pi i(xn+n^2y/2)}.$$

Now, this last equation and Condition (ii) together imply the recursion

$$c_{n+1}^{(a)}(y) = \sum_{k=0}^{a} \binom{a}{k} c_n^{(k)}(y),$$

which yields

$$c_n^{(a)}(y) = \sum_{k=0}^{a} \binom{a}{k} n^{a-k} c_0^{(k)}(y).$$

This proves the lemma. □

Proposition 1. *We have*

$$\sum_{k=0}^{\lfloor\frac{a}{2}\rfloor}\binom{a}{2k}\frac{(2k)!}{k!}\pi^k\left(\frac{y}{i}\right)^{a-k+1/2}\theta_{a-2k}(x,y) \tag{3}$$
$$= e^{-i\pi x^2/y}\sum_{k=0}^{a}\binom{a}{k}(-x)^k i^{k-a}(2\pi)^k\theta_{a-k}\left(\frac{x}{y},-\frac{1}{y}\right).$$

Proof. We prove the claim by applying Lemma 1 to the functions

$$f_a(x,y)=\sum_n\left(-\frac{x+n}{y}\right)^a e^{-i\pi(x+n)^2/y}, \qquad a\geq 0.$$

Condition (i) of Lemma 1 is clearly satisfied by $f_a(x,y)$. For Condition (ii) we calculate

$$\begin{aligned} f_a(x-y,y) &= \sum_n\left(1-\frac{x+n}{y}\right)^a e^{-i\pi(x+n-y)^2/y}\\ &= e^{2\pi i(x-y/2)}\sum_{k=0}^{a}\binom{a}{k}\sum_n\left(-\frac{x+n}{y}\right)^k e^{-i\pi(x+n)^2/y}. \end{aligned}$$

It remains to determine the coefficients $c_0^{(a)}(y)$ of Lemma 1. Short calculations show that

$$c_0^{(a)}(y)=\int_0^1 f_a(x,y)dx = 2\,[a \text{ even}]\int_0^\infty\left(\frac{x}{y}\right)^a e^{-i\pi x^2/y}dx.$$

In particular we have for $a=0$

$$c_0^{(0)}(y)=\int_{-\infty}^{\infty}e^{-i\pi x^2/y}dx=\sqrt{\frac{y}{i}}.$$

Note that the evaluation of the integral above is true for $y=it$ for some $t>0$ and analytic continuation then proves the correctness for general y with $\Im(y)>0$. If $a>0$ then integration by parts yields the recursion

$$c_0^{(2a)}(y)=2\int_0^\infty\left(\frac{x}{y}\right)^{2a}e^{-i\pi x^2/y}dx=\frac{2a-1}{i\pi y}\int_0^\infty\left(\frac{x}{y}\right)^{2a-2}e^{-i\pi x^2/y}dx=\frac{2a-1}{2i\pi y}c_0^{(2a-2)}(y),$$

and we obtain

$$c_0^{(a)}(y)=\begin{cases}0 & \text{if } a \text{ is odd}\\ \frac{a!}{(4\pi i y)^{a/2}(a/2)!}\sqrt{\frac{y}{i}} & \text{if } a \text{ is even.}\end{cases}$$

Hence, by Lemma 1 we have

$$f_a(x,y)=\sum_{k=0}^{\lfloor\frac{a}{2}\rfloor}\binom{a}{2k}\frac{(2k)!}{k!}\frac{\pi^k}{(2\pi i)^a}\left(\frac{y}{i}\right)^{-k+1/2}\theta_{a-2k}(x,y).$$

On the other hand, expanding the binomial term shows that

$$f_a(x,y) = (-y)^{-a} e^{-i\pi x^2/y} \sum_{k=0}^{a} \binom{a}{k} x^k \sum_n n^{a-k} e^{-2\pi i (xn + n^2/y)}$$
$$= y^{-a} e^{-i\pi x^2/y} \sum_{k=0}^{a} \binom{a}{k} (-x)^k (2\pi i)^{k-a} \theta_{a-k}\left(\frac{x}{y}, -\frac{1}{y}\right).$$

The last two representations for $f_a(x,y)$ prove the lemma. □

Putting $a = 0$ in Equation (3), we obtain the reciprocity law for Jacobi's theta functions in the form

$$\sqrt{\frac{y}{i}}\theta(x,y) = e^{-i\pi x^2/y}\theta\left(\frac{x}{y}, -\frac{1}{y}\right).$$

Corollary 1. *The functions $\vartheta_a(y) = \theta_a(0, iy)$, $a \geq 0$, satisfy the relation*

$$\vartheta_a(y) = i^a \sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} \binom{a}{2k} \frac{(2k)!}{k!} \pi^k \left(\frac{1}{y}\right)^{a-k+1/2} \vartheta_{a-2k}\left(\frac{1}{y}\right), \qquad y > 0. \tag{4}$$

Proof. The corollary follows from Equation (3) upon setting $x = 0$ and replacing y by i/y. □

We can now prove the main result of this section.

Lemma 2. *The function $Z_{2\mathbf{a}}(z)$ can be analytically continued to a meromorphic function having a single pole of order 1 at $z = \frac{p}{2} + |\mathbf{a}|_1$ with residue*

$$\operatorname*{Res}_{z=\frac{p}{2}+|\mathbf{a}|_1} Z_{2\mathbf{a}}(z) = \frac{\pi^{p/2}}{\Gamma\left(\frac{p}{2} + |\mathbf{a}|_1\right)} \left(\prod_{i=0}^{p-1} \frac{(2a_i)!}{4^{a_i} a_i!}\right). \tag{5}$$

Furthermore, we have the representation

$$Z_{2\mathbf{a}}(z) = -\frac{[\mathbf{a}=\mathbf{0}]\pi^z}{\Gamma(z+1)} + \frac{\pi^{z-|\mathbf{a}|_1}}{\Gamma(z)} \frac{\prod_{i=0}^{p-1} \frac{(2a_i)!}{4^{a_i} a_i!}}{z - \frac{p}{2} - |\mathbf{a}|_1}$$
$$+ \frac{\pi^{z-2|\mathbf{a}|_1}}{(-4)^{|\mathbf{a}|_1}\Gamma(z)} \int_1^\infty t^{z-1} \left(\left(\prod_{j=0}^{p-1} \vartheta_{2a_j}(t)\right) - [\mathbf{a}=\mathbf{0}]\right) dt$$
$$+ \frac{\pi^{z-2|\mathbf{a}|_1}}{(-4)^{|\mathbf{a}|_1}\Gamma(z)} \int_1^\infty t^{-z-1} \left(\left(\prod_{j=0}^{p-1} \vartheta_{2a_j}\left(\frac{1}{t}\right)\right) - (-\pi)^{|\mathbf{a}|_1} \left(\prod_{i=0}^{p-1} \frac{(2a_i)!}{a_i!}\right) t^{p/2+|\mathbf{a}|_1}\right) dt,$$

where the two integrals above define entire functions with respect to z. For any non-negative integer k we have

$$Z_{2\mathbf{a}}(-k) = \begin{cases} -1 & \text{if } \mathbf{a} = \mathbf{0} \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider again Equation (1), viz.

$$\frac{(2\pi i)^{2|\mathbf{a}|_1}}{\pi^z} Z_{2\mathbf{a}}(z)\Gamma(z) = \int_0^\infty t^{z-1}\left(\left(\prod_{j=0}^{p-1}\vartheta_{2a_j}(t)\right) - [\mathbf{a}=\mathbf{0}]\right) dt.$$

We split the integral above into two parts, one over $[0,1]$ and one over $[1,\infty)$. The second integral is seen to define an entire function with respect to z. We consider the first integral

$$\int_0^1 t^{z-1}\left(\left(\prod_{j=0}^{p-1}\vartheta_{2a_j}(t)\right) - [\mathbf{a}=\mathbf{0}]\right) dt = -\frac{[\mathbf{a}=\mathbf{0}]}{z} + \int_0^1 t^{z-1}\left(\prod_{j=0}^{p-1}\vartheta_{2a_j}(t)\right) dt.$$

By virtue of (4) we obtain

$$\begin{aligned}\left(\prod_{j=0}^{p-1}\vartheta_{2a_j}(t)\right) &- (-\pi)^{|\mathbf{a}|_1}\left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{a_j!}\right) t^{-p/2-|\mathbf{a}|_1} \\ &= (-1)^{|\mathbf{a}|_1}\left(\sum_{\substack{\mathbf{0}\le\mathbf{k}\le\mathbf{a}\\ \mathbf{k}\ne\mathbf{a}}}\left(\prod_{j=0}^{p-1}\binom{2a_j}{2k_j}\frac{(2k_j)!}{k_j!}\pi^{k_j}t^{k_j-2a_j-1/2}\vartheta_{2a_j-2k_j}\left(\frac{1}{t}\right)\right)\right) \\ &\qquad + (-\pi)^{|\mathbf{a}|_1}\left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{a_j!}\right) t^{-p/2-|\mathbf{a}|_1}\left(\vartheta\left(\frac{1}{t}\right)^p - 1\right).\end{aligned}$$

Now, since for $\mathbf{a}\ne\mathbf{0}$ the integrals

$$\int_0^1 t^{z-1}\left(\vartheta\left(\frac{1}{t}\right)^p - 1\right) dt \qquad\text{and}\qquad \int_0^1 t^{z-1}\left(\prod_{j=1}^{p}\vartheta_{2a_j}\left(\frac{1}{t}\right)\right) dt$$

define entire functions with respect to z we see that

$$\int_0^1 t^{z-1}\left(\left(\prod_{j=1}^{p}\vartheta_{2a_j}(t)\right) - (-\pi)^{|\mathbf{a}|_1}\left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{a_j!}\right) t^{-p/2-|\mathbf{a}|_1}\right) dt$$

defines an entire function with respect to z, too.

Combining all the parts and noting that

$$(-\pi)^{|\mathbf{a}|_1}\left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{a_j!}\right)\int_0^1 t^{z-1-p/2-|\mathbf{a}|_1} dt = \frac{(-\pi)^{|\mathbf{a}|_1}\prod_{j=0}^{p-1}\frac{(2a_j)!}{a_j!}}{z-\frac{p}{2}-|\mathbf{a}|_1}$$

we obtain the representation for $Z_{2\mathbf{a}}$ claimed in the lemma. The evaluations at the non-positive integers immediately follow from this representation. □

We close this section with a result on the growth of $Z_{2\mathbf{a}}(\sigma+it)$ as $|t|\to\infty$.

Lemma 3. *For $\sigma\in\mathbb{R}$ fixed we have the estimate*

$$Z_{2\mathbf{a}}(\sigma+it) = O\left(e^{\varepsilon|t|}\right), \qquad |t|\to\infty, \tag{6}$$

for any $\varepsilon>0$.

Proof. Mellin transform asymptotics show that

$$\vartheta_{2a}(t) = \frac{(-4\pi^2)^a}{\pi^{a+1/2}} \frac{\Gamma(a+1/2)}{t^{a+1/2}} + O\left(t^M\right), \qquad t \to 0,$$
$$\vartheta_{2a}(t) = [a=0] + O\left(t^{-M}\right), \qquad t \to \infty,$$

for any $M > 0$. Consequently, we have for $\mathbf{a} \in \mathbb{N}^p$ the asymptotics

$$\left(\prod_{i=0}^{p-1} \vartheta_{2a_i}(t)\right) - [\mathbf{a}=\mathbf{0}] = \frac{(-4\pi)^{|\mathbf{a}|_1}}{t^{|\mathbf{a}|_1+p/2}} \frac{\prod_{j=0}^{p-1} \Gamma(a_j+1/2)}{\pi^{p/2}} - [\mathbf{a}=\mathbf{0}] + O\left(t^M\right), \qquad t \to 0,$$
$$\left(\prod_{i=0}^{p-1} \vartheta_{2a_i}(t)\right) - [\mathbf{a}=\mathbf{0}] = O\left(t^{-M}\right), \qquad t \to \infty,$$

for any $M > 0$. Now, by [8, Proposition 5] we see that the Mellin transform of $\left(\prod_{i=0}^{p-1} \vartheta_{2\mathbf{a}}(t)\right) - [\mathbf{a}=\mathbf{0}]$, viz.

$$f^*_{2\mathbf{a}}(z) = \frac{(2\pi i)^{2|\mathbf{a}|_1}}{\pi^z} Z_{2\mathbf{a}}(z)\Gamma(z),$$

satisfies

$$f^*_{2\mathbf{a}}(\sigma + it) = O\left(e^{-(\pi/2-\varepsilon)|t|}\right), \qquad |t| \to \infty,$$

for any $\varepsilon > 0$ and σ in any closed subinterval of $(|\mathbf{a}|_1 + p/2, \infty)$, which can be extended to any closed subinterval of $(-\infty, \infty)$ (see the proof of [8, Prop. 4] for details). The result now follows from the behaviour of the gamma function along vertical lines, viz.

$$\Gamma(\sigma+it) \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}, \qquad |t| \to \infty.$$

□

3. The moments of the height distribution

We denote by $M^{(p)}_{2n,h}$ the number of p-watermelons with wall with length $2n$ and height strictly smaller than h. Further, we write $M^{(p)}_{2n}$ for the total number of p-watermelons with length $2n$. Note that $M^{(p)}_{2n} = M^{(p)}_{2n,h}$ for $h \geq n+2p-1$ and $M^{(p)}_{2n,h} = 0$ for $h < 2p$.

Now, let $\mathfrak{W}^{(p)}_n$ denote the set of p-watermelons of length $2n$, and let $\mathbb{P}$ denote the uniform probability measures on these sets, and let $H_{n,p}$ denote the random variable "height" on the probability space $\left(\mathfrak{W}^{(p)}_n, 2^{\mathfrak{W}^{(p)}_n}, \mathbb{P}\right)$.

The goal of this section is to obtain an asymptotic expression for the s-th moment $\mathbb{E}H^s_{n,p}$, where $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$, of this random variable as the length of the watermelons tends to infinity. Clearly, we have

$$\mathbb{E}H^s_{n,p} = \frac{1}{M^{(p)}_{2n}} \sum_{h\geq 1} (h^s - (h-1)^s)\left(M^{(p)}_{2n} - M^{(p)}_{2n,h}\right), \qquad s \geq 1. \tag{7}$$

For determining the asymptotics of $\mathbb{E}H^s_{n,p}$ we proceed as follows. First, we find expressions in terms of determinants for the quantities $M^{(p)}_{2n,h}$ and $M^{(p)}_{2n}$. This is accomplished by an application of a theorem by Lindström–Gessel–Viennot, respectively of a theorem by Gessel and Zeilberger. Second, we obtain asymptotics for

$$M_{2n,h}^{(p)} \qquad \text{and} \qquad \sum_{h\geq 1} (h^s-(h-1)^s)\left(M_{2n}^{(p)}-M_{2n,h}^{(p)}\right). \tag{8}$$

The asymptotics for $\mathbb{E}H_{n,p}^s$ are then easily established. The main result is stated in Theorem 1 at the end of this section.

We start with exact expressions for $M_{2n,h}^{(p)}$ and $M_{2n}^{(p)}$.

Lemma 4. *We have*

$$M_{2n}^{(p)} = \det_{0\leq i,j<p}\left(\binom{2n}{n+i-j}-\binom{2n}{n-1-i-j}\right), \tag{9}$$

and for $h \geq 0$ we have

$$M_{2n,h}^{(p)} = \det_{0\leq i,j<p}\left(\sum_{m\in\mathbb{Z}}\left(\binom{2n}{n+m(h+1)+i-j}-\binom{2n}{n+m(h+1)-1-i-j}\right)\right). \tag{10}$$

Proof (Sketch). For $h \geq 2p$ both equations follow from a theorem by Lindström–Gessel–Viennot (see [10, Corollary 3] or [20, Lemma 1]), respectively from a theorem of Gessel and Zeilberger [11]. To be more specific, Equation (9) follows from the type C_p case of the main theorem in [11], while Equation (10) follows from the type $\tilde{C}_p$ case.

The reader should observe that the entries of the determinant (9) are the numbers of lattice paths from $(0, 2i)$ to $(2n, 2j)$ that do not cross the x-axis. On the other hand, the entries of the determinant (10) are the numbers of lattice paths from $(0, 2i)$ to $(2n, 2j)$ that do not cross the x-axis and have height smaller than h. These sums are obtained by a repeated reflection principle (see, e.g., Mohanty [21, p.6]).

For $0 \leq h < 2p$ the identity

$$\begin{aligned}\sum_{m\in\mathbb{Z}}&\left(\binom{2n}{n+m(h+1)+i-j}-\binom{2n}{n+m(h+1)-1-i-j}\right)\\ &= -\sum_{m\in\mathbb{Z}}\left(\binom{2n}{n+m(h+1)+(h-i)-j}-\binom{2n}{n+m(h+1)-1-(h-i)-j}\right)\end{aligned}$$

shows that the right hand side of (10) is equal to zero, since for $h = 2i$ the i-th row of the determinant is equal to zero, and for $h = 2i+1$ we see that the i-th and $(i+1)$-th row of the determinant only differ by sign and thus are linear dependent. □

We now turn towards the problem of determining asymptotics for the expressions (8). Asymptotics for the total number of watermelons are easily established since the determinant in (9) admits a simple closed form. The result is stated in the following lemma.

Lemma 5. *We have*

$$M_{2n}^{(p)} = 4^{\binom{p}{2}}\left(\prod_{i=0}^{p-1}(2i+1)!\right)\binom{2n}{n}^p n^{-p^2}\left(1+O\left(n^{-1}\right)\right)$$

as $n \to \infty$.

Proof. The determinant (9) can be evaluated in closed form, e.g., by means of [16, Theorem 30], and is in fact given by

$$M_{2n}^{(p)} = \prod_{j=0}^{p-1} \frac{\binom{2n+2j}{n}}{\binom{n+2j+1}{n}} = \binom{2n}{n}^p \left(\prod_{j=0}^{p-1}(2j+1)!\right) \left(\prod_{j=0}^{p-1} \frac{(2n+2j)\dots(2n+1)}{(n+2j+1)(n+2j)^2\dots(n+1)^2}\right).$$

This proves, upon determining asymptotics for the right-most product, the result as stated in the lemma.

For a comprehensive discussion and references of this counting problem we refer to [18, Section 4]. ☐

Asymptotics for the second part of (8) are much harder to obtain. As a first step we note that

$$\begin{aligned} &\sum_{h\geq 1} (h^s - (h-1)^s) \left(M_{2n}^{(p)} - M_{2n,h}^{(p)}\right) \qquad (11) \\ &= \sum_{h\geq 1} ((h-1)^s - h^s) \sum_{\mathbf{m}\neq \mathbf{0}} \det_{0\leq i,j<p} \left(\binom{2n}{n+m_i(h+1)+i-j} - \binom{2n}{n+m_i(h+1)-1-i-j} \right) \end{aligned}$$

by (9) and (10), where the inner sum ranges over $\mathbb{Z}^p \setminus \{\mathbf{0}\}$.

For determining asymptotics for (11) we closely follow the proof of de Bruijn, Knuth and Rice [4] (in our case we have to overcome some additional difficulties). For sake of convenience, we give a short plan of the proof. As a first step we factor $\binom{2n}{n}$ out of each row of the determinant on the right-hand side of (10). We then replace the quotients of binomial coefficients by its (sufficiently accurate) asymptotic series expansion, which is determined with the help of Stirling's asymptotic series for the factorials (see Lemma 6). This shows that the asymptotic series expansion for (11) can be expressed in terms of products of derivatives of Jacobi's theta functions we considered in the last section. With the help of the Mellin transforms and the results of the last section we are able to derive asymptotics for these functions (see Lemma 7). In Lemma 8 we finally obtain the desired asymptotics for (11).

We start with the asymptotic series expansion for the quotients of binomial coefficients mentioned above.

Lemma 6. *For $|m-z| \leq n^{5/8}$ and $N > 1$ we have the asymptotic expansion*

$$\begin{aligned} &\frac{\binom{2n}{n+m-z}}{\binom{2n}{n}} \qquad (12) \\ &= e^{-m^2/n} \left(\sum_{u=0}^{4N+1} \left(-\frac{z}{\sqrt{n}}\right)^u \left(\phi_u\left(\frac{m}{\sqrt{n}}\right) + \sum_{l=1}^{3N+1} \sum_{k=0}^{u-1} \sum_{r=1}^{2l} \frac{F_{r,l}}{n^l} \binom{2r}{u-k} \phi_k\left(\frac{m}{\sqrt{n}}\right) \left(\frac{m}{\sqrt{n}}\right)^{2r+k-u} \right)\right) \\ &\qquad + O\left(n^{-1-2N} e^{-m^2/n}\right) \end{aligned}$$

as $n \to \infty$. Here, the $F_{r,l}$ are some constants the explicit form of which is of no importance in the sequel, and $(-1)^k k! \phi_k(w)$ is the k-th Hermite polynomial, that is

$$\phi_k(z) = \sum_{m\geq 0} \frac{(-1)^m}{m!} \binom{m}{k-m} (2z)^{2m-k}, \qquad k \geq 0. \qquad (13)$$

Proof. For sake of convenience, set $x = (m-z)/n$. With the help of Stirling's asymptotic series for the factorials we see that for x sufficiently small, $|x| < \frac{1}{2}$, say, we have

$$\log \frac{\binom{2n}{n+m-z}}{\binom{2n}{n}} = \left(n + \frac{1}{2}\right) \log \frac{1}{1-x^2} - nx \log \frac{1+x}{1-x} + \sum_{k=1}^{N} \frac{B_{2k} n^{1-2k}}{2k(2k-1)} \left(2 - (1+x)^{1-2k} - (1-x)^{1-2k}\right) + O\left(n^{-1-2N}\right)$$

for all fixed $N > 0$ as $n \to \infty$. Here, B_k denotes the k-th Bernoulli number defined via $\sum_{k\geq 0} B_k t^k/k! = t/(e^t - 1)$.

For the range $|x| \leq n^{-1/4}$ we further obtain by Taylor series expansion and some simplifications the expression

$$\log \frac{\binom{2n}{n+m-z}}{\binom{2n}{n}} = -n \left(\sum_{r=1}^{4N+3} \frac{x^{2r}}{r(2r-1)}\right) + \frac{1}{2} \left(\sum_{r=1}^{4N+1} \frac{x^{2r}}{r}\right) - \left(\sum_{r=1}^{4N-1} \left(\sum_{k=1}^{N} \frac{B_{2k} n^{1-2k}}{k(2k-1)} \binom{-2k+1}{r}\right) x^{2r}\right) + O\left(n^{-1-2N}\right).$$

Further restricting ourselves to the range $|x| \leq n^{-3/8}$ we obtain, upon taking the exponential of both sides of the expression above and another Taylor series expansion, the asymptotic series expansion

$$\frac{\binom{2n}{n+m-z}}{\binom{2n}{n}} = e^{-nx^2} \left(1 + \sum_{r=1}^{4N+3} \left(\sum_{l=-\lfloor r/2 \rfloor}^{2N-\lfloor 3r/4 \rfloor} F_{r,l+r} n^{-l}\right) x^{2r} + O\left(n^{-1-2N}\right)\right)$$

for some constants $F_{r,l}$.

Now, if $N > 1$ we obtain upon interchanging the two sums on the right-hand side above, replacing x with its defining expression $(m-z)/n$ and simple rearrangements the expression

$$\frac{\binom{2n}{n+m-z}}{\binom{2n}{n}} = e^{-(m-z)^2/n} \left(1 + \sum_{l=1}^{3N+1} n^{-l} \sum_{r=1}^{2l} F_{r,l} \left(\frac{m-z}{\sqrt{n}}\right)^{2r} + O\left(n^{-1-2N}\right)\right)$$

Finally, expanding $e^{-(m-z)^2/n}$ in the expression above in the form

$$e^{-(m-z)^2/n} = e^{-m^2/n} \sum_{k\geq 0} \phi_k \left(\frac{m}{\sqrt{n}}\right) \left(-\frac{z}{\sqrt{n}}\right)^k,$$

and collecting powers of z, we obtain the result. Here, the $\phi_k(m/\sqrt{n})$ represent certain polynomials the explicit form of which is given in the lemma. □

We mentioned before, that the non-normalised s-th moment (11) is a linear combination of certain functions related to products of the functions $\vartheta_{2a}(t)$, $a \geq 0$, considered in the last section. In the next lemma we obtain asymptotics for these functions with the help of the Mellin transform and the results proved in the last section.

Lemma 7. *For* $\mathbf{a} \in \mathbb{Z}^p$*,* $\mathbf{a} \geq \mathbf{0}$*, and* $k \in \mathbb{N}$ *define the function*

$$g_{k,\mathbf{a}}(n) = \sum_{h\geq 1}(h+1)^k \sum_{\substack{\mathbf{m}\in(h+1)\mathbb{Z}^p\\ \mathbf{m}\neq\mathbf{0}}} e^{-|\mathbf{m}|_2^2/n}\left(\frac{\mathbf{m}}{\sqrt{n}}\right)^{2\mathbf{a}}. \tag{14}$$

For any fixed $M > 0$ *we have the asymptotics*

$$g_{k,\mathbf{a}}(n) = \left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{4^{a_j}a_j!}\right)\Omega_k(n) + \omega_{k,\mathbf{a}}n^{(k+1)/2} + \left(1 - B_{k+1}\frac{(-1)^k}{k+1}\right)[\mathbf{a}=\mathbf{0}] + O\left(n^{-M}\right) \tag{15}$$

as $n \to \infty$*, where*

$$\Omega_k(n) = (n\pi)^{p/2} \times \begin{cases}\gamma - 1 + \log\sqrt{n} & \text{if } p = k+1,\\ \zeta(p-k) - 1 & \text{else,}\end{cases}$$

and

$$\omega_{k,\mathbf{a}} = \frac{1}{2}\times\begin{cases}\lim\limits_{z\to p/2}\left(Z_{2\mathbf{a}}(z+|\mathbf{a}|_1)\Gamma(z+|\mathbf{a}|_1) - \left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{4^{a_j}a_j!}\right)\frac{\pi^{p/2}}{z-\frac{p}{2}}\right) & \text{if } p = k+1,\\ \Gamma\left(\frac{k+1}{2}+|\mathbf{a}|_1\right)Z_{2\mathbf{a}}\left(\frac{k+1}{2}+|\mathbf{a}|_1\right) & \text{else.}\end{cases}$$

Here, $\gamma = 0.5772\ldots$ *is the Euler-Mascheroni constant.*

Proof. First, note that the function $g_{k,\mathbf{a}}(n)$ can be written in terms of derivatives of theta functions, viz.

$$g_{k,\mathbf{a}}(n) = (-4\pi)^{-|\mathbf{a}|_1}\sum_{h\geq 1}(h+1)^k\left(\frac{(h+1)^2}{n\pi}\right)^{|\mathbf{a}|_1}\left(\prod_{j=0}^{p-1}\vartheta_{2a_j}\left(\frac{(h+1)^2}{n\pi}\right) - [\mathbf{a}=\mathbf{0}]\right).$$

Now, by the harmonic sum rule and Equation (1), the Mellin transform of $g_{k,\mathbf{a}}(n)$ for $\Re(z) > \frac{1}{2}\max\{p, k+1\}$ is seen to be

$$g^*_{k,\mathbf{a}}(z) = \int_0^\infty g_{k,\mathbf{a}}(x^{-1})x^{z-1}dx = (\zeta(2z-k)-1)\,\Gamma(z+|\mathbf{a}|_1)Z_{2\mathbf{a}}(z+|\mathbf{a}|_1).$$

Consequently, the function $g_{k,\mathbf{a}}(n)$ can be represented with the help of the inverse Mellin transform by the contour integral

$$g_{k,\mathbf{a}}(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g^*_{k,\mathbf{a}}(z)n^z dz, \qquad c > \frac{1}{2}\max\{p, k+1\}.$$

Asymptotics are now being obtained by pushing the line of integration to the left and taking into account the residues of the poles of the integrand.

From the well-known analytic behaviour of the gamma and the zeta function (see, e.g., [26]) and the analytic behaviour of $Z_{2\mathbf{a}}(z)$ as given by Lemma 2 we infer that the integrand $g^*_{k,\mathbf{a}}(z)n^z$ has potential poles at $z = p/2$, $z = (k+1)/2$ and $z = -|\mathbf{a}|_1 - m$ for $m \in \mathbb{N}$. For $p \neq k+1$ all poles are of order one. Furthermore, the residues are given by

$$\operatorname*{Res}_{z=p/2} g^*_{k,\mathbf{a}}(z)n^z = (\zeta(p-k)-1)\left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{4^{a_j}a_j!}\right)(n\pi)^{p/2}$$

$$\operatorname*{Res}_{z=(k+1)/2} g^*_{k,\mathbf{a}}(z)n^z = \frac{1}{2}\Gamma\left(\frac{k+1}{2}+|\mathbf{a}|_1\right)Z_{2\mathbf{a}}\left(\frac{k+1}{2}+|\mathbf{a}|_1\right)n^{(k+1)/2}$$

$$\operatorname*{Res}_{z=-|\mathbf{a}|_1-m} g^*_{k,\mathbf{a}}(z)n^z = -\left(B_{k+1}\frac{(-1)^k}{k+1}-1\right)[\mathbf{a}=\mathbf{0} \text{ and } m=0],$$

where B_l denotes the l-th Bernoulli number defined via $\sum_{l\geq 0} B_l t^l/l! = t/(e^t-1)$.

In the case $p=k+1$, the only difference is the pole at $z=p/2$, which is now a pole of order two. By Lemma 2 we know that

$$\operatorname*{Res}_{z=p/2} Z_{2\mathbf{a}}(z+|\mathbf{a}|_1)\Gamma(z+|\mathbf{a}|_1) = \left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{4^{a_j}a_j!}\right)\pi^{p/2},$$

and consequently, we have

$$\operatorname*{Res}_{z=p/2} g^*_{k,\mathbf{a}}(z)n^z = \left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{4^{a_j}a_j!}\right)\left(\gamma-1+\log\sqrt{n}\right)(\pi n)^{p/2}$$
$$+\frac{n^{p/2}}{2}\lim_{z\to p/2}\left(Z_{2\mathbf{a}}(z+|\mathbf{a}|_1)\Gamma(z+|\mathbf{a}|_1)-\left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{4^{a_j}a_j!}\right)\frac{\pi^{p/2}}{z-\frac{p}{2}}\right).$$

Note that the limit above is equal to the constant term in the Laurent expansion of $Z_{2\mathbf{a}}(z+|\mathbf{a}|_1)\Gamma(z+|\mathbf{a}|_1)$ around its pole $z=p/2$.

For completing the proof we have to show the admissibility of the displacement of the contour of integration above. But this follows by well known estimates for the gamma and the zeta function along vertical lines in the complex plane together with Lemma 3. See [8] for details. □

This last lemma finally enables us to determine the asymptotics for the non normalised s-th moment (11).

Lemma 8. *We have the asymptotics*

$$\sum_{h\geq 1}(h^s-(h-1)^s)\left(M^{(p)}_{2n}-M^{(p)}_{2n,h}\right) = 2^{-p}\binom{2n}{n}^p n^{-p^2}$$
$$\times\left(s\lambda_s n^{s/2}-3\binom{s}{2}\lambda_{s-1}n^{(s-1)/2}-\frac{3}{2}2^{p^2}\left(\prod_{i=0}^{p-1}(2i+1)!\right)+O\left(n^{s/2-1}+n^{p/2-p^2}\log n\right)\right)$$

as $n\to\infty$, where

$$\lambda_k = -\sum_{\mathbf{a}\geq\mathbf{0}}(-4)^{|\mathbf{a}|_1}\det_{0\leq i,j<p}\left(\frac{(2i+2j+2)!}{(i+j+1-a_i)!\,(2a_i)!}\right)\omega_{k-1,\mathbf{a}}, \qquad k>0,$$

with $\omega_{k-1,\mathbf{a}}$ being defined in Lemma 7.

Proof. Substituting the determinant expressions (9) and (10) for $M_{2n}^{(p)}$ and $M_{2n,h}^{(p)}$ we see that

$$\sum_{h\geq 1}(h^s-(h-1)^s)\left(M_{2n}^{(p)}-M_{2n,h}^{(p)}\right)$$
$$=\sum_{h\geq 1}((h-1)^s-h^s)\sum_{\mathbf{m}\neq\mathbf{0}}\det_{0\leq i,j<p}\left(\binom{2n}{n+m_i(h+1)+i-j}-\binom{2n}{n+m_i(h+1)-1-i-j}\right).$$

Instead of determining asymptotics for the right-hand side expression above directly we consider the more general quantity

$$D_n(\mathbf{x},\mathbf{y},z)$$
$$=\sum_{h\geq 1}((h-1)^s-h^s)\sum_{\substack{\mathbf{m}\in(h+1)\mathbb{Z}^p\\ \mathbf{m}\neq\mathbf{0}}}\det_{0\leq i,j<p}\left(\binom{2n}{n+m_i+x_i-y_j}-\binom{2n}{n+m_i-z-x_i-y_j}\right).$$

Now, we factor $\binom{2n}{n}$ out of each row of the determinant above, and restrict the sum above to those $(p+1)$-tuples $(h,m_0,\ldots,m_{p-1})$ such that for $i=0,\ldots,p-1$ we have $|(h+1)m_i|\leq n^{1/2+\varepsilon}$ for some fixed ε satisfying $0<\varepsilon\leq 1/8$. Since, by Stirling's formula, we have

$$\frac{\binom{2n}{n+\alpha}}{\binom{2n}{n}}=O\left(e^{-n^{2\varepsilon}}\right),\qquad n\to\infty,$$

whenever $|\alpha|\geq n^{1/2+\varepsilon}$, we see that the sum of all terms failing to satisfy the condition above is $O\left(n^{-M}\right)$ for all $M>0$ and, therefore, is negligible.

In the remaining sum we replace all quotients of binomial coefficients by their asymptotic series expansion as given by Lemma 6. Having done so, we extend the range of summation to $\mathbb{N}\times(\mathbb{Z}^p-\{\mathbf{0}\})$. This adds some additional terms, their sum being exponentially small and, therefore, again negligible. This technique of truncating the (exponentially small) tail of the exact sum, replacing the addends by their asymptotic expansion and finally adding a new (exponentially small) tail to the resulting sum has also been applied by de Bruijn, Knuth and Rice [4].

This procedure yields, upon noticing some cancellations due to summation over $\mathbf{m}$ which eliminates all odd powers of m_i for $i=0,\ldots,p-1$, for arbitrary $N>0$ the expression

$$P_N(\mathbf{x},\mathbf{y},z)\tag{16}$$
$$=\sum_{h\geq 1}\mathfrak{h}_s(h)\sum_{\substack{\mathbf{m}\in(h+1)\mathbb{Z}^p\\ \mathbf{m}\neq\mathbf{0}}}e^{-|\mathbf{m}|_2^2/n}\det_{0\leq i,j<p}\left(\sum_{u=0}^{2N}\left(\frac{(y_j-x_i)^{2u}-(z+x_i+y_j)^{2u}}{n^u}\right)T_{2u;N}\left(\frac{m_i}{\sqrt{n}},n\right)\right),$$

where $\mathfrak{h}_s(h)=(h-1)^s-h^s$ and

$$T_{u;N}(w,n)=\phi_u(w)+\sum_{l=1}^{3N+1}n^{-l}\sum_{k=0}^{u-1}\sum_{r=1}^{2l}F_{r,l}\binom{2r}{u-k}\phi_k(w)w^{2r+k-u},\tag{17}$$

such that

$$D_n(\mathbf{x},\mathbf{y},z)=\binom{2n}{n}^p\left(P_N(\mathbf{x},\mathbf{y},z)+O\left(n^{-2N-1}G_{s,\mathbf{0}}(n)\right)\right),\qquad n\to\infty.\tag{18}$$

Here, the functions $\phi_k(w)$ are defined by (13), and

$$G_{s,\mathbf{a}}(n) = \sum_{h\geq 1} ((h-1)^s - h^s) \sum_{\substack{\mathbf{m}\in(h+1)\mathbb{Z}^p \\ \mathbf{m}\neq \mathbf{0}}} \left(\frac{\mathbf{m}}{\sqrt{n}}\right)^{2\mathbf{a}} e^{-|\mathbf{m}|_2^2/n}. \tag{19}$$

Clearly, $P_N(\mathbf{x},\mathbf{y},z)$ is a polynomial with respect to the variables $x_0,\dots,x_{p-1},y_0,\dots,y_{p-1},z$. Furthermore, expanding the determinants and interchanging summations in (16) reveals that $P_N(\mathbf{x},\mathbf{y},z)$ is of the form

$$P_N(\mathbf{x},\mathbf{y},z) = \sum_{\substack{\mathbf{i},\mathbf{j},\mathbf{l}\geq \mathbf{0} \\ |\mathbf{i}|_1+|\mathbf{j}|_1+|\mathbf{l}|_1 \text{ even}}} \frac{\mathbf{x}^{\mathbf{i}}\mathbf{y}^{\mathbf{j}} z^{|\mathbf{l}|_1}}{n^{(|\mathbf{i}|_1+|\mathbf{j}|_1+|\mathbf{l}|_1)/2}} \left(\sum_{\mathbf{a}\geq\mathbf{0}} q_{\mathbf{a},\mathbf{i},\mathbf{j},\mathbf{l}}\left(n^{-1}\right) G_{s,\mathbf{a}}(n)\right) \tag{20}$$

for some polynomials $q_{\mathbf{a},\mathbf{i},\mathbf{j},\mathbf{l}}(n^{-1})$ in n^{-1}. Noting that $[w^{2a}]T_{2u+1;N}(w,n)=0$ for all a by (17) we obtain upon extracting the corresponding coefficients in (16) the explicit representation

$$q_{\mathbf{a},\mathbf{v},\mathbf{w},\mathbf{l}}(n^{-1}) = \det_{0\leq i,j<p}\left(\binom{v_i+w_j+l_i}{v_i,\,w_j,\,l_i}\left((-1)^{v_i}[l_i=0]-1\right)\left[w^{2a_i}\right] T_{v_i+w_j+l_i;N}(w,n)\right), \tag{21}$$

and short calculations show that

$$\begin{aligned}&\left[w^{2a}\right] T_{2u;N}(w,n)\\ &\quad= \frac{(-1)^{u+a}}{(u+a)!}\binom{u+a}{2a}4^a + \sum_{l=1}^{3N+1}\sum_{r=1}^{2l}\frac{F_{r,l}}{n^l}\sum_{k=0}^{2u-1}\frac{(-1)^{u+a-r}}{(u+a-r)!}\binom{u+a-r}{2a+2u-2r-k}2^{2a+2u-2r-k}.\end{aligned}$$

By expanding $(h-1)^s-h^s$ in powers of $(h+1)$ in (19) and interchanging summations we see that

$$G_{s,\mathbf{a}}(n) = \sum_{k=0}^{s-1}\binom{s}{k}(2^{s-k}-1)(-1)^{s-k} g_{k,\mathbf{a}}(n), \tag{22}$$

where the functions $g_{k,\mathbf{a}}(n)$ are defined in Lemma 7.

Thus, we are led to consider sums of the form

$$\sum_{\mathbf{a}\geq\mathbf{0}} q_{\mathbf{a},\mathbf{v},\mathbf{w},\mathbf{l}}(n^{-1}) g_{k,\mathbf{a}}(n).$$

Now, we replace $g_{k,\mathbf{a}}(n)$ by its asymptotic expansion (15), viz.

$$g_{k,\mathbf{a}}(n) = \left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{4^{a_j}a_j!}\right)\Omega_k(n) + \omega_{k,\mathbf{a}} n^{(k+1)/2} + [\mathbf{a}=\mathbf{0}]\left(1-B_{k+1}\frac{(-1)^k}{k+1}\right) + O(n^{-M})$$

as $n\to\infty$ for all $M>0$. The quantities $\Omega_k(n)$ and $\omega_{k,\mathbf{a}}$ have already been defined in Lemma 7.

The multi-linearity of the determinant in (21) then shows that

$$\sum_{\mathbf{a}\geq\mathbf{0}} q_{\mathbf{a},\mathbf{v},\mathbf{w},\mathbf{l}}(n^{-1})\left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{4^{a_j}a_j!}\right)\Omega_k(n)$$
$$=\Omega_k(n)\det_{0\leq i,j<p}\left(\binom{v_i+w_j+l_i}{v_i,w_j,l_i}\left([l_i=0](-1)^{v_i}-1\right)\sum_{a\geq 0}\frac{(2a)!}{4^a a!}\left[w^{2a_i}\right]T_{v_i+w_j+l_i;N}(w,n)\right).$$

The sum inside the determinant is further seen to be

$$\sum_{a\geq 0}\frac{(2a)!}{4^a a!}\left[w^{2a}\right]T_{2u}(w,n)$$
$$=\sum_{l=1}^{3N+1}\sum_{r=1}^{2l}\frac{F_{r,l}}{n^l}\sum_{k=2(u-r)}^{2u-1}\frac{(-1)^{u-r}}{2^{2r+k-2u}}\frac{(2r+k-2u)!}{(k+r-u)!}\sum_{a\geq 0}(-1)^a\binom{k+r-u}{a}\binom{2a}{2r+k-2u}.$$

By the Chu-Vandermonde summation formula we obtain for the innermost sum above

$$\sum_{a\geq 0}(-1)^a\binom{k+r-u}{a}\binom{2a}{2r+k-2u}$$
$$={}_2F_1\left[\begin{matrix}-\lfloor k/2\rfloor,\frac{1}{2}+r-u+\lceil k/2\rceil\\ \frac{1}{2}+\lceil k/2\rceil-\lfloor k/2\rfloor\end{matrix};1\right]=\frac{\Gamma\left(\frac{1}{2}+\lceil k/2\rceil-\lfloor k/2\rfloor\right)}{\Gamma\left(\frac{1}{2}+\lceil k/2\rceil\right)}\frac{\Gamma(u-r)}{\Gamma(u-r-\lfloor k/2\rfloor)},$$

and, from the fact that $\lfloor k/2\rfloor\geq u-r$ on the right-hand side of the second to last equation above, we conclude that all terms having $r<u$ vanish, since in these cases this last sum evaluates to zero. But this shows that

$$\sum_{a\geq 0}\frac{(2a)!}{4^a a!}\left[w^{2a}\right]T_{2u}(w,n)=O\left(n^{-u/2}\right),\qquad n\to\infty,$$

and we infer that

$$\sum_{\mathbf{a}\geq\mathbf{0}} q_{\mathbf{a},\mathbf{v},\mathbf{w},\mathbf{l}}(n^{-1})\left(\prod_{j=0}^{p-1}\frac{(2a_j)!}{4^{a_j}a_j!}\right)\Omega_k(n)=O\left(\Omega_k(n)n^{-(|\mathbf{v}|_1+|\mathbf{w}|_1+|\mathbf{l}|_1)/2}\right),\qquad n\to\infty,$$

and further, noting that $\Omega_k(n)=O(n^{p/2}\log n)$ as $n\to\infty$,

$$\sum_{\mathbf{a}\geq\mathbf{0}} q_{\mathbf{a},\mathbf{v},\mathbf{w},\mathbf{l}}(n^{-1})g_{k,\mathbf{a}}(n)=\left(\sum_{\mathbf{a}\geq\mathbf{0}}\omega_{k,\mathbf{a}}q_{\mathbf{a},\mathbf{v},\mathbf{w},\mathbf{l}}(n^{-1})\right)n^{(k+1)/2}$$
$$+q_{\mathbf{0},\mathbf{v},\mathbf{w},\mathbf{l}}(n^{-1})\left(1-B_{k+1}\frac{(-1)^k}{k+1}\right)+O\left(n^{(p-|\mathbf{v}|_1-|\mathbf{w}|_1-|\mathbf{l}|_1)/2}\log n\right)\tag{23}$$

as $n\to\infty$.

Now, lets turn back to Equation (16). Since the determinants involved in the definition of $P_N(\mathbf{x},\mathbf{y},z)$ vanish whenever $x_i=x_j$ or $y_i=y_j$ for some $i\neq j$ or $x_i=-z-x_j$ or $y_i=-z-y_j$

for some i and j, we conclude that $P_N(\mathbf{x},\mathbf{y},z)$, which is a polynomial with respect to the variables $\mathbf{x}$, $\mathbf{y}$ and z, is divisible by

$$\left(\prod_{0\le i<j<p}(x_i-x_j)(y_i-y_j)\right)\left(\prod_{0\le i\le j<p}(z+x_i+x_j)(z+y_i+y_j)\right).$$

Consequently, all monomials of $P_N(\mathbf{x},\mathbf{y},z)$ have total degree $\ge 2p^2$. Furthermore, we see that

$$P_N(\mathbf{x},\mathbf{y},z)=\left(\prod_{0\le i<j<p}(x_i-x_j)(y_i-y_j)\right)\left(\prod_{0\le i\le j<p}(z+x_i+x_j)(z+y_i+y_j)\right)\times n^{-p^2}C(n)\left(1+O\left(n^{-1}\right)\right) \tag{24}$$

for some unknown function $C(n)$ as $n\to\infty$. This function $C(n)$ can be determined by comparing the coefficient of $\prod_{i=0}^{p-1}x_i^{2i+1}y_i^{2i+1}$ in (20) and (24). In this way we obtain

$$n^{-p^2}\sum_{\mathbf{a}\ge\mathbf{0}}q_{\mathbf{a},\mathbf{J},\mathbf{J},\mathbf{0}}(n^{-1})G_{s,\mathbf{a}}(n)=4^pn^{-p^2}C(n)\left(1+O(n^{-1})\right),\qquad n\to\infty, \tag{25}$$

where $\mathbf{J}=(1,3,\dots,2p-1)$. Since

$$G_{s,\mathbf{a}}(n)=-sg_{s-1,\mathbf{a}}(n)+3\binom{s}{2}g_{s-2,\mathbf{a}}(n)+O(g_{s-3,\mathbf{a}}(n)),\qquad n\to\infty, \tag{26}$$

by (22), we see by (23) that

$$\sum_{\mathbf{a}\ge\mathbf{0}}q_{\mathbf{a},\mathbf{J},\mathbf{J},\mathbf{0}}(n^{-1})G_{s,\mathbf{a}}(n)=-\sum_{\mathbf{a}\ge\mathbf{0}}q_{\mathbf{a},\mathbf{J},\mathbf{J},\mathbf{0}}(n^{-1})\left(sg_{s-1,\mathbf{a}}(n)-3\binom{s}{2}g_{s-2,\mathbf{a}}(n)\right)+O\left(n^{s/2-1}+n^{p/2-p^2}\log n\right)$$

as $n\to\infty$. Noting that

$$q_{\mathbf{a},\mathbf{J},\mathbf{J},\mathbf{0}}(n^{-1})=\frac{2^p(-4)^{|\mathbf{a}|_1}}{\left(\prod_{i=0}^{p-1}(2i+1)!\right)^2}\det_{0\le i,j<p}\left(\frac{(2i+2j+2)!}{(i+j+1-a_i)!\,(2a_i)!}\right)+O(n^{-1}),\qquad n\to\infty,$$

we further see by (23) that

$$\sum_{\mathbf{a}\ge\mathbf{0}}q_{\mathbf{a},\mathbf{J},\mathbf{J},\mathbf{0}}(n^{-1})G_{s,\mathbf{a}}(n)=\frac{2^p}{\left(\prod_{i=0}^{p-1}(2i+1)!\right)^2}\left(s\lambda_sn^{s/2}-3\binom{s}{2}\lambda_{s-1}n^{(s-1)/2}+\lambda_0\right)+O\left(n^{s/2-1}+n^{p/2-p^2}\log n\right)$$

as $n\to\infty$, where

$$\lambda_k=-\sum_{\mathbf{a}\ge\mathbf{0}}(-4)^{|\mathbf{a}|_1}\det_{0\le i,j<p}\left(\frac{(2i+2j+2)!}{(i+j+1-a_i)!\,(2a_i)!}\right)\omega_{k-1,\mathbf{a}},\qquad k>0,$$

and

$$\lambda_0=-\frac{3}{2}\det_{0\le i,j<p}\left(\frac{(2i+2j+2)!}{(i+j+1)!}\right).$$

Here, the constant λ_0 is of interest only in the case $s=1$ (it can be absorbed into the O-term otherwise), and comes from the asymptotic expansion of $g_{1,\mathbf{0}}(n)$.

Now, with the help of Equation (25) we can determine asymptotics for the function $C(n)$, which gives us asymptotics for $P_N(\mathbf{x},\mathbf{y},z)$ by Equation (24), and finally also asymptotics for $D_N(\mathbf{x},\mathbf{y},z)$ by Equation (18).

The proof is now completed upon specialising to $x_i=y_i=i$ for $i=0,\dots,p-1$ and $z=1$ in the asymptotics for $D_N(\mathbf{x},\mathbf{y},z)$. For sake of convenience we finally note the identities

$$\left(\prod_{0\le i<j<p}(i-j)^2\right)\left(\prod_{0\le i\le j<p}(1+i+j)^2\right)=\prod_{i=0}^{p-1}(2i+1)!^2,$$

$$\det_{0\le i,j<p}\left(\frac{(2i+2j+2)!}{(i+j+1)!}\right)=2^{p^2}\prod_{i=0}^{p-1}(2i+1)!.$$

The second identity can be proved by means of standard determinant evaluation techniques (see [16] for details). ☐

Finally, we can state and prove the main result of this paper.

Theorem 1. *Set $M_p=2^{p^2}\prod_{i=0}^{p-1}(2i+1)!$ and $T_p(t)=\det\limits_{0\le i,j<p}(\vartheta_{2i+2j+2}(t))$. For $s\in\mathbb{N}$, the s-th moment of the height distribution of p-watermelons with wall satisfies*

$$\mathbb{E}H_{n,p}^s=s\kappa_s^{(p)}n^{s/2}-3\binom{s}{2}\kappa_{s-1}^{(p)}n^{(s-1)/2}-\frac{3}{2}+O\left(n^{s/2-1}+n^{p/2-p^2}\log n\right) \tag{27}$$

as $n\to\infty$, where

$$\kappa_s^{(p)}=\frac{\pi^{s/2}}{2}\int_0^\infty t^{-1-s/2}\left(1-\frac{t^{p^2+p/2}}{(-\pi)^{p^2}}\frac{T_p(t)}{M_p}\right)dt,\qquad s>0.$$

Proof. Replacing $M_{2n}^{(p)}$ and the sum in Equation (7) with their asymptotic expansions as given by Lemma 5 and Lemma 8 we see that

$$\mathbb{E}H_{n,p}^s=s\kappa_s^{(p)}n^{s/2}-3\binom{s}{2}\kappa_{s-1}^{(p)}n^{(s-1)/2}-\frac{3}{2}+O\left(n^{s/2-1}+n^{p/2-p^2}\log n\right),\qquad n\to\infty,$$

where, for $k>0$,

$$\kappa_k^{(p)}=-\frac{1}{M_p}\sum_{\mathbf{a}\ge\mathbf{0}}(-4)^{|\mathbf{a}|_1}\det_{0\le i,j<p}\left(\frac{(2i+2j+2)!}{(i+j+1-a_i)!\,(2a_j)!}\right)\omega_{k-1,\mathbf{a}}.$$

The quantity $\omega_{k-1,\mathbf{a}}$ has already been defined in Lemma 7.

In order to prove the integral representation for $\kappa_s^{(p)}$ when $s\ne p$, where we have $\omega_{k-1,\mathbf{a}}=\frac{1}{2}\Gamma\left(\frac{k}{2}+|\mathbf{a}|_1\right)Z_{2\mathbf{a}}\left(\frac{k}{2}+|\mathbf{a}|_1\right)$, we consider the more general expression

$$\begin{aligned}\kappa^{(p)}(z)&=-\frac{1}{2M_p}\sum_{\mathbf{a}\ge\mathbf{0}}\det_{0\le i,j<p}\left(\frac{(2i+2j+2)!(-4)^{a_i}}{(i+j+1-a_i)!(2a_i)!}\right)\Gamma\left(z+|\mathbf{a}|_1\right)Z_{2\mathbf{a}}\left(z+|\mathbf{a}|_1\right)\\&=-\frac{\pi^z}{2M_p}\int_0^\infty t^{z-1}\left(\det\left(\sum_{a\ge0}\frac{(2i+2j+2)!(t/\pi)^a}{(i+j+1-a)!(2a)!}\vartheta_{2a}(t)\right)-M_p\right)dt\end{aligned}$$

TABLE 1. This table gives numerical approximations for $s\kappa_s^{(p)}$ for small values of s and p. The quantity $s\kappa_s^{(p)}$ is the coefficient of the dominant part of the asymptotics for the s-th moment of the height of p-watermelons (see Theorem 1). The calculations have been carried out using the integral representation for $\kappa_s^{(p)}$ as given in Theorem 1. The results shown here conform well with with numerical results obtained by Fulmek [9] and Katori et al. [15].

$s\kappa_s^{(p)}$	$s=1$	$s=2$	$s=3$	$s=4$
$p=1$	$\sqrt{\pi}$	$3.289\ldots$	$6.391\ldots$	$12.987\ldots$
$p=2$	$2.577\ldots$	$6.790\ldots$	$18.282\ldots$	$50.306\ldots$
$p=3$	$3.207\ldots$	$10.429\ldots$	$34.371\ldots$	$114.817\ldots$
$p=4$	$3.742\ldots$	$14.141\ldots$	$53.939\ldots$	$207.712\ldots$
$p=5$	$4.215\ldots$	$17.898\ldots$	$76.536\ldots$	$329.655\ldots$

for $\Re z$ sufficiently large. Here, the second line is a direct consequence of Equation (1). The reciprocity relation (4) followed by the change of variables $t \mapsto t^{-1}$ then shows that

$$\kappa^{(p)}(z) = \frac{\pi^z}{2} \int_0^\infty t^{-z-1} \left(1 - \frac{t^{p^2+p/2}}{(-\pi)^{p^2}} \frac{T_p(t)}{M_p}\right) dt.$$

Asymptotics for $\vartheta_{2a}(t)$ for $t \to 0$ and $t \to \infty$ as given in the proof of Lemma 3 then show that this last integral is convergent for $\Re z > 0$. The representation for $s \neq p$ is now proved upon observing that, by definition, we have $\kappa^{(p)}\left(\frac{s}{2}\right) = \kappa_s^{(p)}$.

Now, consider the case $s = p$. Here, we have

$$\omega_{p-1,\mathbf{a}} = \frac{1}{2} \lim_{z \to p/2} \left(Z_{2\mathbf{a}}\left(z + |\mathbf{a}|_1\right) \Gamma(z + |\mathbf{a}|_1) - \frac{\pi^{p/2} \left(\prod_{i=0}^{p-1} \frac{(2a_i)!}{4^{a_i} a_i!}\right)}{\left(z - \frac{p}{2}\right)} \right),$$

which shows that, as in the other case,

$$-\frac{1}{M_p} \sum_{\mathbf{a} \geq \mathbf{0}} \det_{0 \leq i,j < p} \left(\frac{(2i+2j+2)!(-4)^{a_i}}{(i+j+1-a_i)!(2a_i)!} \right) \omega_{p-1,\mathbf{a}} = \lim_{z \to p/2} \kappa^{(p)}(z) = \kappa^{(p)}\left(\frac{p}{2}\right).$$

In this last calculation, we have, after interchanging the order of the limit and the sum, applied the results obtained in the case $s \neq p$. This proves the theorem. □

Some numerical approximations for the coefficient of the dominant term of the asymptotics proved in Theorem 1 are shown in Table 3. But our last theorem does not only give the dominant term of the asymptotics of the s-th moment of the height distribution of p-watermelons but also the second order term. So, for example, we obtain the more precise asymptotics

$$\mathbb{E}H_{n,1} = \sqrt{\pi n} - \frac{3}{2} + O\left(n^{-1/2} \log n\right), \qquad n \to \infty,$$
$$\mathbb{E}H_{n,2} = 2.577\ldots\sqrt{n} - \frac{3}{2} + O\left(n^{-1/2}\right), \qquad n \to \infty,$$
$$\mathbb{E}H_{n,2}^2 = 6.790\ldots n - 3.866\ldots\sqrt{n} + O(1), \qquad n \to \infty.$$

Remark 1. It can be shown that Theorem 1 is even valid for $s \in \mathbb{C}$, $\Re(s) > 0$. The proof of this more general result is the same as for our theorem except for two small changes which we are going to address now.

In the proof of Lemma 8 we defined the functions $G_{s,\mathbf{a}}(n)$ (see Equation (19)). For $s \in \mathbb{N}$ the asymptotics (26) for $G_{s,\mathbf{a}}(n)$ were easily found by the expansion (22). This is not possible for $s \in \mathbb{C} \setminus \mathbb{N}$. In order to prove the asymptotics (26) in that case we note that (see Equation (19))

$$(h-1)^s - h^s = (h+1)^s \left(\left(1 - \frac{2}{h+1}\right)^s - \left(1 - \frac{1}{h+1}\right)^s \right).$$

The term for $h = 1$ in (19) is seen to be negligible due to summation over $\mathbf{a} \geq \mathbf{0}$ (see the discussion of the function $\Omega_k(n)$ following Equation (22) in the proof of Lemma 8). For $h \geq 2$, we can use the binomial series expansion in the expression above and finally obtain the asymptotics (26).

The second change concerns Lemma 7, which has to be generalised to $k \in \mathbb{C}$. But this makes no difficulties.

4. A central limit law

We are going to derive asymptotics for the cumulative distribution function of the random variable "height" on the set of p-watermelons with length $2n$ with wall, i.e.,

$$F_n(h) = \mathbb{P}\left\{H_{n,p} \leq h\right\} = \frac{M^{(p)}_{2n,h+1}}{M^{(p)}_{2n}}$$

for the range $h + 2 = t\sqrt{n}$, where $t \in (0, \infty)$.

Theorem 2. *For $t \in (0, \infty)$ fixed, the random variable $H_{n,p}$ on the set of p-watermelons of length $2n$ with wall satisfies*

$$\mathbb{P}\left\{\frac{H_{n,p}+2}{\sqrt{n}} \leq t\right\} = \frac{\pi^{p/2} t^{-2p^2-p}}{(-2)^{p^2} \prod_{i=0}^{p-1}(2i+1)!} \det_{0\leq i,j<p}\left(\vartheta_{2i+2j+2}\left(\frac{\pi}{t^2}\right)\right) + O\left(\frac{1}{nt}\right) \tag{28}$$

as $n \to \infty$, where the constant implied by the O-term is independent of t.

Proof. The result can be proved in pretty much the same way as Theorem 1. Therefore, we only give a rather brief account of the proof, and refer to Lemma 8 for the details.

Instead of the exact expression (10) for $M^{(p)}_{2n,h}$ we consider the more general quantity

$$F_n(h; \mathbf{x}, \mathbf{y}, z) = \binom{2n}{n}^p \det_{0\leq i,j<p}\left(\sum_{m\in(h+2)\mathbb{Z}} \frac{\binom{2n}{n+m+x_i-y_j}}{\binom{2n}{n}} - \frac{\binom{2n}{n+m-z-x_i-y_j}}{\binom{2n}{n}}\right). \tag{29}$$

Again, we find the polynomial

$$Q_N(\mathbf{x}, \mathbf{y}, z) = \det_{0\leq i,j<p}\left(\sum_{u=0}^{2N} \frac{(y_j - x_i)^{2u} - (z + x_i + y_j)^{2u}}{n^u} \sum_{m\in(h+2)\mathbb{Z}} T_{2u;N}\left(\frac{m}{\sqrt{n}}, n\right) e^{-m^2/n}\right),$$

such that

$$F_n(h;\mathbf{x},\mathbf{y},z)=\binom{2n}{n}^p\left(Q_N(\mathbf{x},\mathbf{y},z)+O\left(n^{-2N-1}\sum_{m\in(h+2)\mathbb{Z}}e^{-m^2/n}\right)\right),$$

where N can be chosen arbitrarily large and $T_{2u;N}$ being defined by (17). The polynomial $Q_N(\mathbf{x},\mathbf{y},z)$ is seen to be divisible by

$$\left(\prod_{0\le i<j<p}(x_i-x_j)(y_i-y_j)\right)\left(\prod_{0\le i\le j<p}(z+x_i+x_j)(z+y_i+y_j)\right)$$

since the determinant in the definition of $Q_N(\mathbf{x},\mathbf{y},z)$ vanishes whenever $x_i=x_j$ or $y_i=y_j$ for some $i\neq j$ or $x_i=-z-x_j$ or $y_i=-z-y_j$ for some i and j. Hence,

$$\begin{aligned}&Q_N(\mathbf{x},\mathbf{y},z)\\&\quad=\left(\prod_{0\le i<j<p}(x_i-x_j)(y_i-y_j)\right)\left(\prod_{0\le i\le j<p}(z+x_i+x_j)(z+y_i+y_j)\right)C(t)\left(1+O(n^{-1})\right)\end{aligned}$$

as $n\to\infty$ for some unknown constant $C(t)$. Now, we are going to determine asymptotics for $C(t)$ as $n\to\infty$. This task can be accomplished by comparing the coefficients of the monomial $\prod_{0\le i<p}x_i^{2i+j}y_i^{2i+1}$ in the expression above and the defining expression for $Q_N(\mathbf{x},\mathbf{y},z)$. We obtain

$$4^pC(t)\left(1+O(n^{-1})\right)=(-2)^pn^{-p^2}\det_{0\le i,j<p}\left(\binom{2i+2j+2}{2i+1}\sum_{m\in\mathbb{Z}}T_{2i+2j+2;N}\left(mt,n\right)e^{-(mt)^2}\right).$$

Recalling the definition of the functions $T_{2a;N}(w,n)$ (see Equation (17)), our attention is drawn to sums of the form

$$(2a)!\sum_{m=-\infty}^{\infty}\phi_{2a}(mt)e^{-(mt)^2},\qquad a\in\mathbb{N},$$

where the polynomials ϕ_{2a} are defined by (13).

Now, rewriting the reciprocity relation (4) as

$$\vartheta_{2a}\left(\frac{1}{y}\right)=y^{a+1/2}\pi^a\sum_{m=-\infty}^{\infty}(2a)!\phi_{2a}\left(m\sqrt{\pi y}\right)e^{-m^2\pi y},$$

we see that

$$(2a)!\sum_{m=-\infty}^{\infty}\phi_{2a}(mt)e^{-(mt)^2}=\frac{\sqrt{\pi}}{t^{2a+1}}\vartheta_{2a}\left(\frac{\pi}{t^2}\right).$$

From the asymptotics for $\vartheta_{2a}(t)$ as given in the proof of Lemma 3 we deduce that

$$\begin{aligned}\frac{\sqrt{\pi}}{t^{2a+1}}\vartheta_{2a}\left(\frac{\pi}{t^2}\right)&=\text{const}+O(t^{-M}),\qquad t\to\infty,\\ \frac{\sqrt{\pi}}{t^{2a+1}}\vartheta_{2a}\left(\frac{\pi}{t^2}\right)&=[a=0]\frac{\sqrt{\pi}}{t}+O\left(t^{-M}\right),\qquad t\to0,\end{aligned}$$

for all $M > 0$. Consequently, we obtain for $h+2 = t\sqrt{n}$, where t is fixed,

$$\sum_{m\in\mathbb{Z}} T_{2a;N}\left(mt,n\right)e^{-(mt)^2} = \frac{\sqrt{\pi}}{t^{2a+1}}\vartheta_{2a}\left(\frac{\pi}{t^2}\right) + O\left(\frac{1}{nt}\right), \qquad n\to\infty.$$

Note that the constant implied by the O-term can be chosen independent of t.

The theorem is now proved upon substituting these asymptotics for our sums appearing in the expression for $C(t)$ above, taking out some factors, specialising to $x_j = y_j = 1$, and dividing by $M_{2n}^{(p)}$. □

Remark 2. The asymptotic cumulative distribution function of the random variable "height" as given in Theorem 2 has been re-derived by two groups since the first version of this manuscript was distributed. Since their expressions differ from the one given by Equation (28) we want to give some comments on the equivalence of these three (more or less) different expressions.

The expression found by Katori et al. [14] can easily be obtained by an application of the reciprocity relation (4) to Equation (28), and therefore, is not essentially different from the one given here.

Schehr et al. [24] expressed the cumulative distribution function of the height as a multiple sum, which can also be easily derived from Theorem 2. By definition, we have $\vartheta_{2a}(t) = \sum_{n=-\infty}^{\infty}(-4\pi^2n^2)^a e^{-n^2\pi t}$. Consequently, the determinant in Equation (28) is equivalent to

$$\det_{0\le i,j<p}\left(\vartheta_{2i+2j+2}\left(\frac{\pi}{t^2}\right)\right) = \sum_{n_0,\dots,n_{p-1}\in\mathbb{Z}} e^{-\sum_{j=0}^{p-1}(n_j\pi/t)^2}\det_{0\le i,j<p}\left((-4n_j^2\pi^2)^{i+j+1}\right).$$

The determinant on the right hand side is of Vandermonde type and, therefore, can be evaluated to a closed form expression. Consequently, the expression above is equal to

$$(-\pi^2)^{p^2}2^{2p^2+p}\sum_{n_0,\dots,n_{p-1}\in\mathbb{N}}\left(\prod_{j=0}^{p-1}n_j^{2j+2}\right)\left(\prod_{0\le i<j<p}(n_j^2-n_i^2)\right)e^{-\sum_{j=0}^{p-1}(n_j\pi/t)^2}.$$

Arranging the summation variables in ascending order further gives

$$(-\pi^2)^{p^2}2^{2p^2+p}\sum_{1\le n_0<\dots<n_{p-1}}\left(\prod_{0\le i<j<p}(n_j^2-n_i^2)\right)\left(\sum_{\sigma\in\mathfrak{S}_p}\operatorname{sgn}(\sigma)\prod_{j=0}^{p-1}n_{\sigma(j)}^{2j+2}e^{-(n_{\sigma(j)}\pi/t)^2}\right),$$

where $\mathfrak{S}_p$ denotes the set of permutations on the set $\{0,1,\dots,p-1\}$. Note that the permutation sign $\operatorname{sgn}(\sigma)$ exactly cancel the sign introduced by rearrangment of the product $\prod_{0\le i<j<}(n_j^2-n_i^2)$ that took place in this last step. The alternating sum on the right hand side is again identified with a Vandermonde type determinant. Consequently, we obtain the final expression

$$(-\pi^2)^{p^2}2^{2p^2+p}\sum_{1\le n_0<\dots<n_{p-1}}\left(\prod_{0\le i<j<p}(n_j^2-n_i^2)\right)^2\left(\prod_{j=0}^{p-1}n_j^2e^{-(n_j\pi/t)^2}\right).$$

Substituting this expression for the determinant in Equation (28) we arrive at

$$\frac{2^{p^2+p}\pi^{2p^2+p/2}t^{-2p^2-p}}{\prod_{i=0}^{p-1}(2i+1)!}\sum_{1\le n_0<\cdots<n_{p-1}}\left(\prod_{0\le i<j<p}(n_j^2-n_i^2)\right)^2\left(\prod_{j=0}^{p-1}n_j^2e^{-(n_j\pi/t)^2}\right),$$

which is Schehr's expression for the cumulative distribution function.

For the special case $p=1$ we obtain the well known central limit law first proved by Rényi and Szekeres [23], viz.

$$\mathbb{P}\left\{\frac{H_{n,1}}{\sqrt{n}}\le t\right\}\to\sum_{m\in\mathbb{Z}}\left(1-2(mt)^2\right)e^{-(mt)^2},\qquad n\to\infty. \tag{30}$$

This limiting distribution is known to be the distribution function of $\sqrt{2}\max_{0\le x\le 1}e(x)$, where $e(x)$ denotes the standard Brownian excursion of duration 1. For details and references we refer to the survey paper by Biane, Pitman and Yor [1], in which the authors consider probability laws related to Brownian motion, Riemann's zeta function and Jacobi's theta functions.

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PAPER C

The height and range of watermelons without wall

THE HEIGHT AND RANGE OF WATERMELONS WITHOUT WALL

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Abstract. We determine the weak limit of the distribution of the random variables "height" and "range" on the set of p-watermelons without wall restriction as the number of steps tends to infinity. Additionally, we provide asymptotics for the moments of the random variable "height".

1. Introduction

The model of *vicious walkers* was originally introduced by Fisher [6] as a model for wetting and melting processes. In general, the vicious walkers model is concerned with p random walkers on a d-dimensional lattice. In the lock step model, at each time step all of the walkers move one step in one of the allowed directions, such that at no time any two random walkers share the same lattice point.

A configuration that attracted much interest amongst mathematical physicists and combinatorialists is the *watermelon configuration*[1], which is the model underlying this paper (see Figure 1 for an example). This configuration can be studied with or without the presence of an impenetrable wall. By tracing the paths of the vicious walkers through the lattice we can identify the (probabilistic) vicious walkers model with certain sets of non-intersecting lattice paths. It is exactly this equivalent point of view that we adopt in this paper. We proceed with a precise definition. A *p-watermelon of length* $2n$ is a set of p lattice paths in $\mathbb{Z}^2$ satisfying the following conditions:

- the i-th path starts at position $(0, 2i)$ and ends at $(2n, 2i)$, $i = 0, 1, \dots, p-1$,
- the paths consist of steps from the set $\{(1, 1), (1, -1)\}$ only and
- the paths are *non-intersecting*, that is, at no time any two path share the same lattice point.

An example of a 4-watermelon of length 16 is shown in Figure 1 (for the moment, the dashed lines and the labels should be ignored).

Since its introduction, the vicious walkers model has been studied in numerous papers. While early results mostly analyse the vicious walkers model in the continuum limit, there are nowadays many results for certain configurations directly based on the lattice path description given above. With the increasing number of results it became clear that vicious walkers are very important objects in mathematical areas far beyond its original scope. For example, Guttmann, Owczarek and Viennot [10] related the star and watermelon configurations to the theory of Young tableaux and integer partitions. Later, Krattenthaler, Guttmann and Viennot [16]

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[1] This term comes from the resemblance of large configurations to the colour patterns of certain watermelons (see [4, Figure 1(b)]).

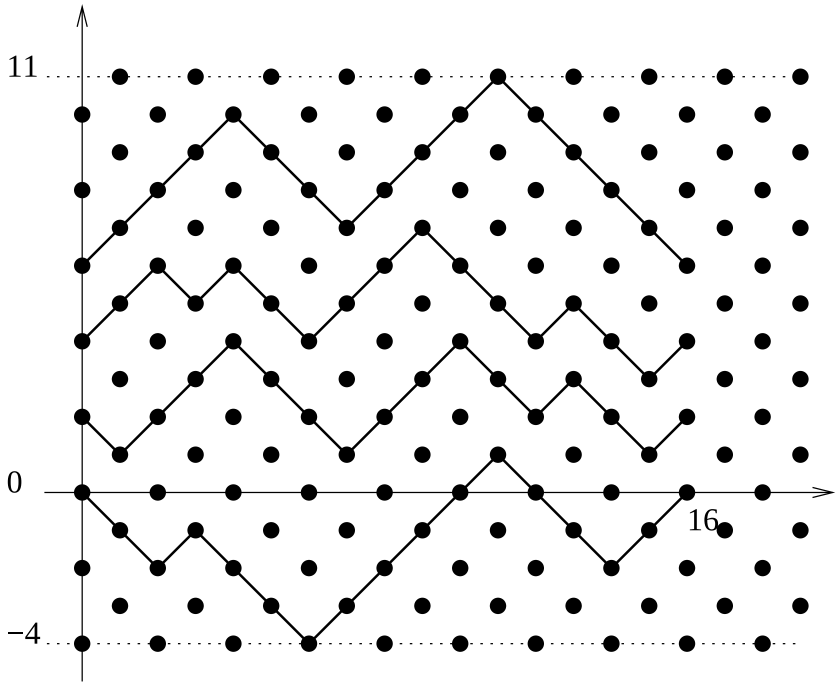

FIGURE 1. Example of a 4-watermelon of length 16 without wall, height 11, depth 4 and range 15

proved new, exact as well as asymptotic, results for the number of certain configurations of vicious walkers.

The vicious walkers model is also very closely related to random matrix theory, as can be seen from articles by, e.g., Baik [1], Johansson [11] and Nagao and Forrester [18]. More recently, Katori and Tanemura [13] and Gillet [9] studied the diffusion scaling limit of certain configurations of vicious walkers, namely stars and watermelons, respectively.

In 2003, Bonichon and Mosbah [2] presented an algorithm for uniform random generation of watermelons, which relies on the counting results by Krattenthaler, Guttmann and Viennot [16]. Amongst other things, Bonichon and Mosbah studied the parameter height on the set of watermelons (with and without wall).

In this paper we rigorously analyse the following two parameters on the set of p-watermelons:

- The *height* of a watermelon is the maximum ordinate reached by its top most branch.
- The *range* of a watermelon is the difference of the maximum of its top most branch and the minimum of its bottom most branch (the *depth* of the watermelon).

The 4-watermelon depicted in Figure 1 has the height 11 and the range $11 + 4 = 15$.

Katori et. al. [12] and Schehr et. al. [19] studied the parameter "height" in the continuous limit, and recovered the leading terms for some of the asymptotics proved in this manuscript and in [5]. Additionally, Schehr et. al. gave some arguments concerning the behaviour of the parameter "height" as the number of walkers tends to infinity.

Now, consider the set $\mathfrak{m}_n^{(p)}$ of p-watermelons of length $2n$, endowed with the uniform probability measure. We can then speak of the random variables "height", denoted by $H_{n,p}$, and "range", denoted by $R_{n,p}$, on this set. We determine the weak limits of $H_{n,p}$ and $R_{n,p}$ as the number n of steps tends to infinity (see Theorem 1 and Theorem 3, respectively). Additionally, we determine asymptotics for the moments of $H_{n,p}$. More precisely, we prove that the s-th moment of the random variable "height" behaves like $\kappa_s n^{s/2} + \tau_s n^{(s-1)/2} + O\left(n^{s/2-1}\right)$ for some explicit numbers κ_s and τ_s, see Theorem 2.

Techniques similar to those applied in this paper can also be used to analyse the random variable height on the set of p-watermelons under the presence of an impenetrable wall. For details we refer to [5].

The paper is organised as follows. The next section contains some well known results that are needed in the subsequent sections. In Section 3 we consider the random variable "height", and we determine the weak limit as well as asymptotics for all moments. In the last section, we determine the weak limit of the random variable "range".

2. Preliminaries

In this section we collect several results which will be needed in the two subsequent sections. All these results are either well known in the literature and/or can easily be derived by means of standard techniques. We, therefore, remain very brief, give only a few comments on the proofs and in each case refer to the corresponding literature for details.

We start with an exact enumeration result for the total number of watermelons confined to a horizontal strip. (Recall, that the depth of a watermelon is the minimum ordinate of its bottom most branch.)

Lemma 1. *The number $m^{(p)}_{n,h,k}$ of p-watermelons without wall, length $2n$, height $< h$ and depth $> -k$ is given by*

$$m^{(p)}_{n,h,k} = \det_{0\le i,j<p}\left(\sum_{\ell\in\mathbb{Z}}\left(\binom{2n}{n+\ell(h+k)+i-j} - \binom{2n}{n+\ell(h+k)+h-i-j}\right)\right).$$

The total number $m^{(p)}_n$ of p-watermelons is given by

$$m^{(p)}_n = \det_{0\le i,j<p}\left(\binom{2n}{n+i-j}\right).$$

This lemma follows immediately from the well-known Lindström–Gessel–Viennot formula (see [8, Corollary 3] or [17, Lemma 1]), together with an iterated reflection principle.

Remark 1. Since any p-watermelon without wall and length $2n$ has depth $> -n-1$, we see that the number of watermelons with height $< h$ and no restriction on the depth is given by $m^{(p)}_{n,h,n+1}$. For the sake of convenience, this quantity will also be denoted by $m^{(p)}_{n,h}$. In this special case, the determinantal expression above simplifies to

$$m^{(p)}_{n,h} = \det_{0\le i,j<p}\left(\binom{2n}{n+i-j} - \binom{2n}{n+h-i-j}\right).$$

Lemma 2. *We have*

$$m^{(p)}_n = \left(\frac{2}{n}\right)^{\binom{p}{2}}\binom{2n}{n}^p\left(\prod_{i=0}^{p-1} i!\right)\left(1+O(n^{-1})\right)$$

as $n \to \infty$.

Proof (Sketch). The result is established from the closed form expression for $m_n^{(p)}$, viz

$$m_n^{(p)} = \det_{0\le i,j<p}\left(\binom{2n}{n+i-j}\right) = \binom{2n}{n}^p \left(\prod_{i=0}^{p-1} i!\frac{(2n+i)!}{(2n)!}\left(\frac{n!}{(n+i)!}\right)^2\right).$$

For details on the evaluation of this (and many more) determinant, we refer to [15]. □

Lemma 3. *For $|m-z|\le n^{5/8}$, z bounded, and arbitrary $N>1$ we have the asymptotic expansion*

$$(1)\quad \frac{\binom{2n}{n+m-z}}{\binom{2n}{n}} = e^{-m^2/n}\sum_{u=0}^{4N+1}\left(\frac{z}{\sqrt{n}}\right)^u \frac{1}{u!}H_u\left(\frac{m}{\sqrt{n}}\right)$$
$$+ e^{-m^2/n}\sum_{u=0}^{4N+1}\left(\frac{z}{\sqrt{n}}\right)^u \sum_{l=1}^{3N+1} n^{-l}\sum_{k=0}^{u-1}\sum_{r=1}^{2l} F_{r,l}\binom{2r}{u-k}\frac{(-1)^{u-k}}{k!}H_k\left(\frac{m}{\sqrt{n}}\right)\left(\frac{m}{\sqrt{n}}\right)^{2r+k-u}$$
$$+O\left(e^{-m^2/n}n^{-1-2N}\right)$$

as $n\to\infty$. Here, the $F_{r,l}$ are some constants the explicit form of which is of no importance in the sequel, and $H_k(z)$ denotes the k-th Hermite polynomial, that is,

$$(2)\quad \frac{H_k(z)}{k!} = \sum_{m\ge 0}\frac{(-1)^{k-m}}{(k-m)!}\frac{(2z)^{2m-k}}{(2m-k)!},\qquad k\ge 0.$$

The lemma above follows from Stirling's approximation for the factorials. For a detailed proof we refer to [5, Lemma 6].

3. Height

In this section we derive asymptotics for the distribution as well as for the moments of the random variable $H_{n,p}$. As mentioned before, the number of p-watermelons with length $2n$ and height $< h$ is given by $m_{n,h}^{(p)} = m_{n,h,n+1}^{(p)}$. Consequently, we have for the distribution of $H_{n,p}$

$$(3)\quad \mathbb{P}\left\{H_{n,p}+1\le h\right\} = \frac{m_{n,h}^{(p)}}{m_n^{(p)}}.$$

Theorem 1. *For each fixed $t\in(0,\infty)$ we have the asymptotics*

$$(4)\quad \mathbb{P}\left\{\frac{H_{n,p}+1}{\sqrt{n}}\le t\right\} = \frac{2^{-\binom{p}{2}}}{\prod_{j=0}^{p-1} j!}\det_{0\le i,j<p}\left((-1)^i H_{i+j}(0) - H_{i+j}(t)\,e^{-t^2}\right) + O\left(n^{-1/2}e^{-t^2}\right)$$

as $n\to\infty$, where $H_a(x)$ denotes the a-th Hermite polynomial.

Proof. Set $\mathbf{x}=(x_0,\dots,x_{p-1})$ and $\mathbf{y}=(y_0,\dots,y_{p-1})$, and consider the more general quantity

$$m_{n,h}^{(p)}(\mathbf{x},\mathbf{y}) = \det_{0\le i,j<p}\left(\binom{2n}{n+x_i-y_j} - \binom{2n}{n+h-x_i-y_j}\right).$$

Factoring $\binom{2n}{n}$ out of each row of the determinant above and replacing each entry with its asymptotic expansions as given in Lemma 3, we find the asymptotics

$$m_{n,h}^{(p)}(\mathbf{x},\mathbf{y}) = \binom{2n}{n}^p \left(D_N(\mathbf{x},\mathbf{y}) + O\left(e^{-h^2/n} n^{-1-2N}\right)\right), \qquad n\to\infty,$$

where

$$D_N(\mathbf{x},\mathbf{y}) = \det_{0\le i,j<p}\left(\sum_{u=0}^{4N+1}\left(\left(\frac{y_j-x_i}{\sqrt{n}}\right)^u T_{u;N}(0,n) - \left(\frac{y_j+x_i}{\sqrt{n}}\right)^u T_{u;N}(h,n)\right)\right)$$

and $N>0$ is an arbitrary integer. Here, $T_{u;N}(h,n)$ is given by (see Lemma 3)

$$T_{u;N}(h,n) = e^{-h^2/n} \times\left(\frac{H_u(h/\sqrt{n})}{u!} + \sum_{l=1}^{3N+1} n^{-l} \sum_{k=0}^{u-1}\sum_{r=1}^{2l} F_{r,l}\binom{2r}{u-k}\frac{(-1)^{u-k}}{k!} H_k\left(\frac{h}{\sqrt{n}}\right)\left(\frac{h}{\sqrt{n}}\right)^{2r+k-u}\right).$$

The quantity $D_N(\mathbf{x},\mathbf{y})$ is seen to be polynomial in the x_i's and y_j's. This polynomial is divisible by the factors (x_j-x_i) and (y_j-y_i) for $0\le i<j<p$, for if $x_j=x_i$ then the j-th and the i-th row are equal and, therefore, the determinant is zero (if $y_j=y_i$ then the j-th and i-th column are equal). Hence,

$$D_N(\mathbf{x},\mathbf{y}) = n^{-\binom{p}{2}} \frac{\prod_{0\le i<j<p}(x_j-x_i)(y_j-y_i)}{\prod_{0\le j<p} j!^2}\chi\left(\frac{h}{\sqrt{n}}\right)\left(1+O(n^{-1/2}e^{-h^2/n})\right), \qquad n\to\infty.$$

Here, the error term is determined by noting that every power of x_j and y_j entails a factor of $n^{-1/2}$, as can be seen from the definition of $D_N(\mathbf{x},\mathbf{y})$ above. The unknown coefficient $\chi(n,h)$ can now be determined by comparing coefficients on both sides of the equation above. Comparing the coefficients of $\prod_{j=0}^{p-1} x_j^j y_j^j$, we obtain (after some simplifications) the equation

$$\det_{0\le i,j<p}\left((-1)^i H_{i+j}(0) - H_{i+j}\left(\frac{h}{\sqrt{n}}\right)e^{-h^2/n}\right) = \chi\left(\frac{h}{\sqrt{n}}\right).$$

If we specialise by setting $x_j=y_j=j$, then we see that

$$m_{n,h}^{(p)} = n^{-\binom{p}{2}}\binom{2n}{n}^p \times \det_{0\le i,j<p}\left((-1)^i H_{i+j}(0) - H_{i+j}\left(\frac{h}{\sqrt{n}}\right)e^{-h^2/n}\right) + O\left(\binom{2n}{n}^{-p} n^{-\binom{p}{2}-1/2} e^{-h^2/n}\right).$$

Setting $h=t\sqrt{n}$ and replacing $m_{2n}^{(p)}$ with its asymptotic equivalent as given by Lemma 2, we obtain the result. □

Remark 2. After distribution of the first version of this manuscript, Schehr et al. [19] published an article in which they (amongst other things) determined the distribution function of the random variable "height" on the set of watermelons in the continuous limit, and thus, recovered the asymptotically leading term in Equation (4). Since, at first sight, the expression for the cummulative distribution function for $\sqrt{2}H_p = \lim_{n\to\infty} n^{-1/2}H_{n,p}$ given in [19] looks quite different from our expression, we want to show how Schehr's expression can easily be derived from Equation (4).

The only ingredients needed for our derivation is the identity

$$\cos(x+y)+\cos(x-y)=2\cos(x)\cos(y)$$

and the well-known (see, e.g., Szegö [20]) integral representation for the Hermite polynomials, viz.

$$e^{-t^2}H_k(t)=\frac{2^{k+1}}{\sqrt{\pi}}\int_0^\infty e^{-x^2}x^k\cos\left(2xt+\frac{k}{2}\pi\right)dx.$$

Substituting the integrals above for the corresponding terms in

$$F_p(t)=\frac{2^{-\binom{p}{2}}}{\prod_{j=0}^{p-1}j!}\det_{0\le i,j<p}\left((-1)^i H_{i+j}(0)-H_{i+j}(t)\,e^{-t^2}\right)$$

and taking the integrals as well as some factors out of the determinant, we obtain

$$F_p(t)=\frac{2^{\binom{p+1}{2}}}{\pi^{p/2}\prod_{j=0}^{p-1}j!}\int_0^\infty\cdots\int_0^\infty e^{-u_0^2-\cdots-u_{p-1}^2}\times\det_{0\le i,j<p}\left(u_i^{i+j}\left(\cos\left(\frac{j-i}{2}\pi\right)-\cos\left(2tu_i+\frac{i+j}{2}\pi\right)\right)\right)du_0\cdots du_{p-1}.$$

The determinant inside the integral above can be rewritten as (using the cos-identity mentioned above)

$$\sum_{\sigma\in\mathfrak{S}_p}\operatorname{sgn}(\sigma)\left(\prod_{j=0}^{p-1}u_j^{\sigma(j)+j}\left(\cos\left(\frac{\sigma(j)-j}{2}\pi\right)+\cos\left(2tu_j+\frac{\sigma(j)+j+2}{2}\pi\right)\right)\right)$$
$$=2^p\sum_{\sigma\in\mathfrak{S}_p}\operatorname{sgn}(\sigma)\left(\prod_{j=0}^{p-1}u_j^{\sigma(j)+j}\cos\left(tu_j+\frac{\sigma(j)+1}{2}\pi\right)\cos\left(tu_j+\frac{j+1}{2}\pi\right)\right)$$

where $\mathfrak{S}_p$ denotes the set of permutations on the set $\{0,1,\cdots,p-1\}$. This can further be rewritten as

$$\frac{2^p}{p!}\sum_{\sigma,\tau\in\mathfrak{S}_p}\operatorname{sgn}(\sigma)\left(\prod_{j=0}^{p-1}u_{\tau(j)}^{\sigma(j)+\tau(j)}\cos\left(tu_{\tau(j)}+\frac{\sigma(j)+1}{2}\pi\right)\cos\left(tu_{\tau(j)}+\frac{\tau(j)+1}{2}\pi\right)\right)$$
$$=\frac{2^p}{p!}\left(\det_{0\le i,j<p}\left(u_i^j\cos\left(tu_i+\frac{j+1}{2}\pi\right)\right)\right)^2.$$

This last equality can most easily be seen by replacing σ with $\sigma\circ\tau$ on the left hand side of the equation above.

Substituting this last expression for the determinant involved in the integral representation of $F_p(t)$ above followed by the change of variables $u_j \mapsto u_j/(t\sqrt{2})$, $j = 0, 1, \cdots, p-1$ gives us

$$F_p\left(t\sqrt{2}\right) = \frac{2^{2p}}{t^{p^2}(2\pi)^{p/2}\prod_{j=0}^{p} j!}\int_0^{\infty}\cdots\int_0^{\infty} e^{-(u_0^2+\cdots+u_{p-1}^2)/(2t^2)} \times \left(\det_{0\le i,j<p}\left(u_i^j \cos\left(u_i + \frac{j+1}{2}\pi\right)\right)\right)^2 du_0\cdots du_{p-1}.$$

This is Schehr's expression for the cummulative distribution function of the random variable $\sqrt{2}H_p$.

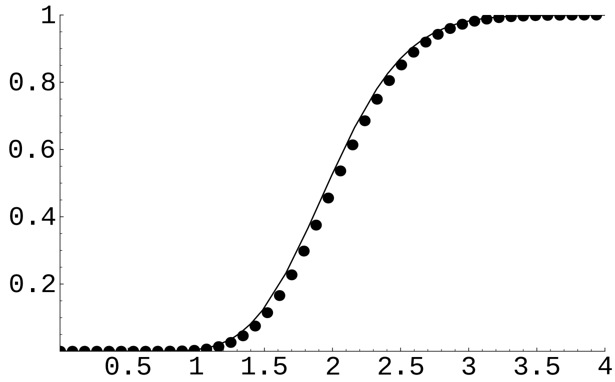

FIGURE 2. Comparison of the c.d.f. of the random variable "height" on the set of 3−watermelons of length 500 without wall (dotted curve) and the limiting distribution as given by Theorem 1.

Let us now turn our attention to the moments of the distribution of $H_{n,p}$. Clearly, we have for $s \in \mathbb{N}$,

$$\mathbb{E}\left(H_{n,p}^s\right) = \sum_{h\ge 1} h^s \frac{m_{n,h+1}^{(p)} - m_{n,h}^{(p)}}{m_n^{(p)}} = \sum_{h\ge 1}\left(h^s - (h-1)^s\right)\frac{m_n^{(p)} - m_{n,h}^{(p)}}{m_n^{(p)}}. \tag{5}$$

The dominant terms of the asymptotics for the moments are going to be expressed by linear combinations of certain infinite exponential sums. Asymptotics for these sums are to be determined now.

Lemma 4. *For $\nu \ge 0$ and $\mu > 0$ define*

$$f_{\nu,\mu}(n) = \sum_{h\ge 1} h^{\nu} e^{-\mu h^2/n}.$$

This sum admits the asymptotic series expansion

$$f_{\nu,\mu}(n) \approx \frac{1}{2}\Gamma\left(\frac{\nu+1}{2}\right)\left(\frac{n}{\mu}\right)^{(\nu+1)/2} + \sum_{m\ge 0}\left(\frac{\mu}{n}\right)^m \frac{(-1)^{\nu+m}B_{2m+\nu+1}}{(2m+\nu+1)!m!},$$

as $n \to \infty$, where Γ denotes the gamma function and B_m is the m-th Bernoulli number defined via $\sum_{j\ge 0} B_j t^j/j! = t/(e^t - 1)$.

Proof (Sketch). Asymptotics for sums of this form can often be obtain by means of Mellin transform techniques. For a detailed overview of Mellin transforms, harmonic sums and asymptotics, we refer to [7].

We proceed with a sketch of the proof. The inverse Mellin transform gives

$$\begin{aligned} f_{\nu,\mu}(n) = \sum_{h\geq 1} h^\nu e^{-\mu h^2/n} &= \sum_{h\geq 1} \frac{h^\nu}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(z)\left(\frac{\mu h^2}{n}\right)^{-z} dz \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(z)\left(\frac{\mu}{n}\right)^{-z}\zeta(2z-\nu)dz. \end{aligned}$$

The integrand has simple poles at $z = (\nu+1)/2$ and $z = 0, -1, -2, \ldots$ corresponding to the poles of the zeta and the gamma function, respectively. The result is now obtained by pushing the line of integration to the left and taking into account the residues.

For the sake of convenience, we mention the evaluations

$$\begin{aligned} \operatorname*{Res}_{z=-m} \Gamma(z) &= \frac{(-1)^m}{m!}, \qquad m = 0, 1, 2, \ldots, \\ \operatorname*{Res}_{z=1} \zeta(z) &= 1 \\ \zeta(-m) &= B_{m+1}\frac{(-1)^m}{m+1}, \qquad m = 0, 1, 2, \ldots, \end{aligned}$$

where B_m denotes the m-th Bernoulli number defined via $\sum_{j\geq 0} B_j t^j/j! = t/(e^t-1)$. □

The rest of this section is devoted to the proof of Theorem 2 below, which gives the final expression for the asymptotics of the moments. In order to present the proof of this theorem in a clear fashion we split it into a series of lemmas. For a more detailed overview of the proof, we refer directly to the proof of Theorem 2.

As a first step, we prove in Lemma 5 a preliminary asymptotic expression for the moments of the height distribution. The presented compact form of the asymptotics makes use of certain linear operators that are going to be defined now.

Definition 1. Let Ξ_1 and Ξ_0 denote the linear operators defined by

$$\begin{aligned} \Xi_1\left(h^\nu e^{-\mu h^2}\right) &= \frac{1}{2}\Gamma\left(\frac{\nu+1}{2}\right)\left(\frac{1}{\mu}\right)^{(\nu+1)/2} \\ \Xi_0\left(h^\nu e^{-\mu h^2}\right) &= (-1)^\nu \frac{B_{\nu+1}}{(\nu+1)!}, \end{aligned}$$

where B_k denotes the k-th Bernoulli number.

By Lemma 4 we have

$$f_{\nu,\mu}(n) = \Xi_1\left(h^\nu e^{-\mu h^2}\right) n^{(\nu+1)/2} + \Xi_0\left(h^\nu e^{-\mu h^2}\right) + O(n^{-1}), \qquad n\to\infty,$$

so that Ξ_1 and Ξ_0 yield the coefficients of the first two terms in the asymptotic expansion of $f_{\nu,\mu}(n)$.

The preliminary expression for the asymptotics of the moments can now be proven in pretty much the same way as in Theorem 1.

Lemma 5. *For $s \in \mathbb{N}$, $s \geq 1$, the s-th moment of the random variable "height" satisfies the asymptotics*

$$\mathbb{E}\left(H_{n,p}^s\right) = s\Xi_1\left(\kappa_p h^{s-1}\right) n^{s/2} - \Xi_1\left(\binom{s}{2}\kappa_p h^{s-2} + \tau_p h^{s-1}\right) n^{(s-1)/2} + \Xi_0(\kappa_p) + O\left(n^{s/2-1}\right) \tag{6}$$

as $n \to \infty$, where

$$\kappa_p = 1 - \frac{2^{-\binom{p}{2}}}{\prod_{0\leq j<p} j!} \det_{0\leq i,j<p}\left((-1)^i H_{i+j}(0) - H_{i+j}(h)\, e^{-h^2}\right)$$

and

$$\tau_p = (p-1)\frac{2^{-\binom{p}{2}}}{\prod_{0\leq j<p} j!} \det_{0\leq i,j<p}\left(\begin{cases}(-1)^i H_{i+j}(0) - H_{i+j}(h)\, e^{-h^2} & \text{if } i < p-1\\ (-1)^p H_{p+j}(0) - H_{p+j}(h)\, e^{-h^2} & \text{if } i = p-1\end{cases}\right).$$

Here, $H_k(z)$ denotes the k-th Hermite polynomial.

Proof. Recall the exact expression for the s-th moment of the random variable "height" (see Equation (5)),

$$\mathbb{E}\left(H_{n,p}^s\right) = \sum_{h=1}^{n+2p-2} \left(h^s - (h-1)^s\right)\frac{m_n^{(p)} - m_{n,h}^{(p)}}{m_n^{(p)}}. \tag{7}$$

Asymptotics for this quantity can be obtained in pretty much the same way as Theorem 1. Compared to the problem of determining asymptotics for (3), the main difference now is the summation over h.

We consider the more general quantity

$$m_n^{(p)}(\mathbf{x},\mathbf{y}) - m_{n,h}^{(p)}(\mathbf{x},\mathbf{y}) = \det_{0\leq i,j<p}\left(\binom{2n}{n+x_i-y_j}\right) - \det_{0\leq i,j<p}\left(\binom{2n}{n+x_i-y_j} - \binom{2n}{n+h-x_i-y_j}\right),$$

where $\mathbf{x} = (x_0, \dots, x_{p-1})$ and $\mathbf{y} = (y_0, \dots, y_{p-1})$. As a first step, we pull $\binom{2n}{n}$ out of each row of the determinants above. Now, we restrict the range of summation in (7) to $1 \leq h \leq n^{1/2+\varepsilon}$ for some $\varepsilon > 0$. This truncation is justified by Stirling's formula, which shows that

$$\frac{\binom{2n}{n+\alpha}}{\binom{2n}{n}} = O\left(e^{-n^{2\varepsilon}}\right), \qquad n \to \infty,$$

whenever $|\alpha| \geq n^{1/2+\varepsilon}$. This implies that the total contribution of all summands in (7) satisfying $h > n^{1/2+\varepsilon}$ is exponentially small as $n \to \infty$ and, therefore, negligible. In all the remaining summands we replace all the quotients of binomial coefficients with their asymptotic expansions as given in Lemma 3. Finally, we re-extend the range of summation to $h \geq 1$, which, again, introduces an exponentially small error term. This gives the asymptotics

$$\mathbb{E}\left(H_{n,p}^s\right) = \sum_{h\geq 1}\left(\left(h^s - (h-1)^s\right)\left(\frac{\binom{2n}{n}^p}{m_n^{(p)}} D_N(\mathbf{e},\mathbf{e}) + O\left(e^{-h^2/n} n^{\binom{p}{2}-1-2N}\right)\right)\right)$$

as $n \to \infty$, where $\mathbf{e} = (0, 1, \dots, p-1)$. Here, the structure of the error term is a consequence of Lemma 2, and the quantity $D_N(\mathbf{x}, \mathbf{y})$ is defined by

$$(8) \quad D_N(\mathbf{x}, \mathbf{y}) = \det_{0 \le i,j < p} \left(\sum_{u=0}^{4N+1} \left(\left(\frac{y_j - x_i}{\sqrt{n}} \right)^u T_{u;N}(0,n) \right) \right) - \det_{0 \le i,j < p} \left(\sum_{u=0}^{4N+1} \left(\left(\frac{y_j - x_i}{\sqrt{n}} \right)^u T_{u;N}(0,n) - \left(\frac{y_j + x_i}{\sqrt{n}} \right)^u T_{u;N}(h,n) \right) \right),$$

where $N > 0$ is an arbitrary integer and

$$T_{u;N}(h,n) = e^{-h^2/n} \times \left(\frac{H_u(h/\sqrt{n})}{u!} + \sum_{l=1}^{3N+1} n^{-l} \sum_{k=0}^{u-1} \sum_{r=1}^{2l} F_{r,l} \binom{2r}{u-k} \frac{H_k\left(h/\sqrt{n}\right)}{k!} \left(-\frac{h}{\sqrt{n}} \right)^{2r+k-u} \right).$$

As a consequence of Lemma 4, we see (after expanding the term $(h-1)^s$) that

$$\sum_{h \ge 1} \left((h^s - (h-1)^s) \, O\left(e^{-h^2/n} n^{\binom{p}{2} - 1 - 2N} \right) \right) = O\left(n^{\binom{p}{2} - 2N + (s-1)/2} \right),$$

which is negligible for sufficiently large N. Hence, we have the asymptotics

$$\mathbb{E}\left(H_{n,p}^s \right) = \frac{\binom{2n}{n}^p}{m_n^{(p)}} \sum_{h \ge 1} \Big((h^s - (h-1)^s) \, D_N(\mathbf{e}, \mathbf{e}) \Big) + O\left(n^{\binom{p}{2} - 2N + (s-1)/2} \right), \qquad n \to \infty.$$

It remains to determine the part of $D_N(\mathbf{x}, \mathbf{y})$ that gives the dominant contribution to the asymptotics above. First, we note that $D_N(\mathbf{x}, \mathbf{y})$ is a polynomial in the x_i's and y_i's. Obviously, $D_N(\mathbf{x}, \mathbf{y})$ is equal to zero whenever $x_i = x_j$ or $y_i = y_j$ for some $i \neq j$, for if $x_i = x_j$ $(y_i = y_j)$ then the i-th and j-th rows (columns) of the determinants involved in the definition of $D_N(\mathbf{x}, \mathbf{y})$ are equal, and, therefore, the determinants are equal to zero. This implies that $D_N(\mathbf{x}, \mathbf{y})$ is of the form

$$D_N(\mathbf{x}, \mathbf{y}) = n^{-\binom{p}{2}} \frac{\prod\limits_{0 \le i < j < p} (x_j - x_i)(y_j - y_i)}{\prod\limits_{0 \le j < p} j!^2} \times \left(\chi(n,h) + \sum_{j=0}^{p-1} \left(\xi_j(n,h) \frac{x_j}{\sqrt{n}} + \eta_j(n,h) \frac{y_j}{\sqrt{n}} \right) + O\left(n^{-1} e^{-h^2/n} \right) \right)$$

as $n \to \infty$. By comparing coefficients of $\prod_{j=0}^{p-1} x_j^j y_j^j$ on both sides of the equation above, we have already seen (see Theorem 1) that

$$\chi(n,h) = \det_{0 \le i,j < p} \left((-1)^i H_{i+j}(0) \right) - \det_{0 \le i,j < p} \left((-1)^i H_{i+j}(0) - H_{i+j}\left(\frac{h}{\sqrt{n}} \right) e^{-h^2/n} \right).$$

Analogously we can determine $\xi_k(n,h)$. By comparing the coefficients of $x_k \prod_{j=0}^{p-1} x_j^j y_j^j$ on both sides of the equation above we obtain the equations

$$0 = \xi_k(n,h) - \xi_{k+1}(n,h), \qquad k < p-1,$$

and

$$\xi_{p-1}(n,h) = -\frac{1}{p}\det_{0\le i,j<p}\left(\begin{cases}(-1)^i H_{i+j}(0) - H_{i+j}\left(\frac{h}{\sqrt{n}}\right)e^{-h^2/n} & \text{if } i<p-1\\ (-1)^p H_{p+j}(0) - H_{p+j}\left(\frac{h}{\sqrt{n}}\right)e^{-h^2/n} & \text{if } i=p-1\end{cases}\right).$$

Note, that the coefficient of $x_k\prod_{j=0}^{p-1}x_j^jy_j^j$ in the first determinant of (8) is equal to zero, which is easily seen to be true for $k<p-1$, and for $k=p-1$ this is seen to be true by a series of column and row operations that yield a new matrix consisting of two non-square blocks. Similar expressions (with i and j interchanged) can be found for the $\eta_k(n,h)$, $0\le k<p$.

Noting that $H_{i+j}(0)$ is non-zero if and only if $i+j$ is even we deduce that $(-1)^iH_{i+j}(0) = (-1)^jH_{i+j}(0)$, which implies

$$\xi_{p-1}(n,h) = \eta_{p-1}(n,h),$$

and also

$$\det_{0\le i,j<p}\left((-1)^iH_{i+j}(0)\right) = \det_{0\le i,j<p}\left((-1)^{(i+j)/2}H_{i+j}(0)\right) = 2^{\binom{p}{2}}\prod_{j=0}^{p-1}j!.$$

Here, the last equality has been proven in Lemma 6.

If we specialise to $x_j=y_j=j$, $0\le j<p$, then we obtain

$$D_N(\mathbf{e},\mathbf{e}) = n^{-\binom{p}{2}}\left(\chi(n,h) + 2\binom{p}{2}\xi_{p-1}(n,h)n^{-1/2}\right)\left(1+O\left(n^{-1}e^{-h^2/n}\right)\right), \qquad n\to\infty,$$

where $\mathbf{e}=(0,1,\dots,p-1)$.

Choosing N large enough and expanding the term $h^s-(h-1)^s$ in the asymptotics for $\mathbb{E}\left(H_{n,p}^s\right)$ above, we obtain with the help of Lemma 4 the asymptotics

$$\mathbb{E}\left(H_{n,p}^s\right) = \frac{\binom{2n}{n}^p}{m_n^{(p)}}\sum_{h\ge1}\left(sh^{s-1}-\binom{s}{2}h^{s-2}\right)D_N(\mathbf{e},\mathbf{e}) + O\left(n^{s/2-1}\right), \qquad n\to\infty,$$

and replacing $D_N(\mathbf{e},\mathbf{e})$ with its asymptotic expansion as given above proves the lemma. ☐

Lemma 6. *Let $H_k(x)$ denote the k-th Hermite polynomial as defined by Equation* (2). *We have the determinant evaluation*

$$\det_{0\le i,j<p}\left((-1)^{(i+j)/2}H_{i+j}(0)\right) = 2^{\binom{p}{2}}\prod_{j=0}^{p-1}j!. \tag{9}$$

Proof. The determinant under consideration is a Hankel determinant. Therefore, we can hope to evaluate it with the help of orthogonal polynomials (for details see [15, Section 2.7]). It is well known (see, e.g., [20, page 105]) that for $k\in\mathbb{N}$ we have

$$H_{2k+1}(0)=0 \qquad \text{and} \qquad H_{2k}(0) = (-1)^k\frac{(2k)!}{k!}.$$

Consequently, we obtain

$$\det_{0\le i,j<p}\left((-1)^{(i+j)/2}H_{i+j}(0)\right) = 2^{\binom{p}{2}}\det_{0\le i,j<p}\left(\frac{1+(-1)^{i+j}}{2}\frac{2^{(i+j)/2}}{\sqrt{\pi}}\Gamma\left(\frac{i+j+1}{2}\right)\right).$$

The (i,j)-th entry of the determinant on the right hand side above is seen to be precisely the $(i+j)$-th moment with respect to the Gaussian weight $w(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ on $\mathbb{R}$, that is,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x^k e^{-x^2/2}dx=\frac{1+(-1)^k}{2}\frac{2^{k/2}}{\sqrt{\pi}}\Gamma\left(\frac{k+1}{2}\right),\qquad k=0,1,2\dots$$

The family of monic orthogonal polynomials associated with the weight $w(x)$ is given by

$$2^{-k/2}H_k\left(\frac{x}{\sqrt{2}}\right),\qquad k=0,1,2,\dots \tag{10}$$

where $H_k(x)$ denotes the k-th Hermite polynomial as defined by Equation (2). The three term recursion relation for the orthogonal polynomials (10) is seen to be (cf. [20, p.105])

$$2^{-(k+1)/2}H_{k+1}\left(\frac{x}{\sqrt{2}}\right)=x2^{-k/2}H_k\left(\frac{x}{\sqrt{2}}\right)-k2^{-(k-1)/2}H_{k-1}\left(\frac{x}{\sqrt{2}}\right),\qquad k=1,2,\dots,$$

with the initial values $H_0\left(\frac{x}{\sqrt{2}}\right)=1$ and $2^{-1/2}H_1\left(\frac{x}{\sqrt{2}}\right)=x$. Now, an application of [15, Theorem 11]) shows that

$$\det_{0\le i,j<p}\left(\frac{1+(-1)^{i+j}}{2}\frac{2^{(i+j)/2}}{\sqrt{\pi}}\Gamma\left(\frac{i+j+1}{2}\right)\right)=\prod_{j=0}^{p-1}j!,$$

which proves the claim. $\square$

Lemma 7. *Let $\mu>0$ denote a real number. The operator Ξ_1 from Definition 1 satisfies the relation*

$$\Xi_1\left(\frac{d}{dh}\left(h^\nu e^{-\mu h^2}\right)\right)=\begin{cases}-1 & \text{if } \nu=0\\ 0 & \text{if } \nu>0.\end{cases} \tag{11}$$

Proof. For $\nu=0$ the claim follows immediately from the definition of the operator Ξ_1. For $\nu>0$ we calculate

$$\Xi_1\left(h^{\nu+1}e^{-\mu h^2}\right)=\frac{\nu}{2\mu}\Xi_1\left(h^{\nu-1}e^{-\mu h^2}\right),$$

from which the claims follows upon multiplying by 2μ and rearranging the terms. $\square$

The next result is not obvious at all, and, on the contrary, is a quite surprising fact.

Lemma 8. *Let κ_p and τ_p denote the determinants defined in Lemma 5. We have the relation*

$$(p-1)\frac{d}{dh}\kappa_p=\tau_p,\qquad p\ge 1. \tag{12}$$

Proof. For the sake of convenience we set

$$C=2^{-\binom{p}{2}}\left(\prod_{j=0}^{p-1}j!\right)^{-1}.$$

The derivative of a $p \times p$ determinant is the sum of p determinants, where the j-th addend is equal to the original determinant with the j-th row replaced by its derivative. Hence,

$$\frac{d}{dh}\kappa_p = C\left(\sum_{j=0}^{p-2} M_j\right) + CM_{p-1},$$

where

$$M_i = \det\left(\begin{pmatrix} \mathfrak{H}_{0,0} & \cdots & \mathfrak{H}_{0,p-1} \\ \vdots & \ddots & \vdots \\ \mathfrak{H}_{i-1,0} & \cdots & \mathfrak{H}_{i-1,p-1} \\ -H_{i+1}(h)e^{-h^2} & \cdots & -H_{i+p}(h)e^{-h^2} \\ \mathfrak{H}_{i+1,0} & \cdots & \mathfrak{H}_{i+1,p-1} \\ \vdots & \ddots & \vdots \\ \mathfrak{H}_{p-1,0} & \cdots & \mathfrak{H}_{p-1,p-1} \end{pmatrix}\right),$$

where $\mathfrak{H}_{i,j} = (-1)^i H_{i+j}(0) - H_{i+j}(h)e^{-h^2}$. We want to mention that $(p-1)CM_{p-1}$ is equal to the expression for τ_p except for the constant terms in the last row of the determinant.

For $0 \leq i < p-1$ the quantity M_i can also be represented by the expression

$$M_i = \det\left(\begin{pmatrix} \mathfrak{H}_{0,0} & \cdots & \mathfrak{H}_{0,p-1} \\ \vdots & \ddots & \vdots \\ \mathfrak{H}_{i-1,0} & \cdots & \mathfrak{H}_{i-1,p-1} \\ \mathfrak{H}_{i+1} & \cdots & \mathfrak{H}_{i+p} \\ (-1)^{i+1}H_{i+1}(0) & \cdots & (-1)^{i+1}H_{i+p}(0) \\ \mathfrak{H}_{i+2,0} & \cdots & \mathfrak{H}_{i+2,p-1} \\ \vdots & \ddots & \vdots \\ \mathfrak{H}_{p-1,0} & \cdots & \mathfrak{H}_{p-1,p-1} \end{pmatrix}\right), \qquad 0 \leq i < p-1,$$

which is more convenient to work with.

The Laplace expansion for determinants with respect to the row $j+1$, $0 \leq j < p-1$, gives

$$M_j = \sum_{k=0}^{p-1} (-1)^{j+1} H_{j+1+k}(0) M_{j,k}, \qquad 0 \leq j < p-1,$$

where $M_{j,k}$ denotes the minor of M_j obtained by removing row $j+1$ and column k, i.e.,

$$M_{j,k} = \det\left(\begin{pmatrix} \mathfrak{H}_{0,0} & \cdots & \mathfrak{H}_{0,k-1} & \mathfrak{H}_{0,k+1} & \cdots & \mathfrak{H}_{0,p-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathfrak{H}_{i-1,0} & \cdots & \mathfrak{H}_{i-1,k-1} & \mathfrak{H}_{i-1,k+1} & \cdots & \mathfrak{H}_{i-1,p-1} \\ \mathfrak{H}_{i+1,0} & \cdots & \mathfrak{H}_{i+1,k-1} & \mathfrak{H}_{i+1,k+1} & \cdots & \mathfrak{H}_{i+1,p-1} \\ \mathfrak{H}_{i+2,0} & \cdots & \mathfrak{H}_{i+2,k-1} & \mathfrak{H}_{i+2,k+1} & \cdots & \mathfrak{H}_{i+2,p-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathfrak{H}_{p-1,0} & \cdots & \mathfrak{H}_{p-1,k-1} & \mathfrak{H}_{p-1,k+1} & \cdots & \mathfrak{H}_{p-1,p-1} \end{pmatrix}\right).$$

Now, consider the sum

$$\sum_{j=0}^{p-2} M_j = \left(\sum_{j=0}^{p-2}\sum_{k=0}^{p-2}(-1)^{j+1}H_{j+1+k}(0)M_{j,k}\right) + \sum_{j=0}^{p-2}(-1)^{j+1}H_{j+p}(0)M_{j,p-1}.$$

The first sum on the right hand side in fact is equal to zero as is going to be shown now. First, note that

$$M_{j,k} = M_{k,j}$$

since the matrices involved are transposes of each other. Recalling that $H_k(0)$ is non zero if and only if k is an even number we deduce that

$$(-1)^{j+1}H_{j+1+k}(0)M_{j,k} = -(-1)^{k+1}H_{k+1+j}(0)M_{k,j},$$

and both expressions correspond to different addends of the double sum above ($j+1+k$ has to be even). This shows that the value of the double sum is indeed equal to zero.

For the second sum we have

$$\begin{aligned}\sum_{j=0}^{p-2}(-1)^{j+1}H_{j+p}(0)M_{j,p-1} &= -\sum_{j=0}^{p-2}(-1)^{p}H_{j+p}(0)M_{p-1,j}\\ &= \det_{0\le k,l<p}\left(\begin{cases}(-1)^k H_{k+l}(0) - H_{k+l}(h)e^{-h^2} & \text{if } k<p-1\\ (-1)^p H_{p+l} & \text{if } k=p-1\end{cases}\right),\end{aligned}$$

which proves the lemma. ☐

We are now able to to state and prove the final expression for the asymptotics of the moments.

Theorem 2. *The expected value of the random variable $H_{n,p}$ satisfies the asymptotics*

$$\mathbb{E}\left(H_{n,p}\right) = \Xi_1\left(\kappa_p\right)\sqrt{n} + p - \frac{3}{2} + O\left(n^{-1/2}\right), \qquad n\to\infty, \tag{13}$$

and for $s\in\mathbb{N}$, $s\ge 2$, we have the asymptotics

$$\mathbb{E}\left(H_{n,p}^s\right) = s\Xi_1(\kappa_p h^{s-1})n^{s/2} + (s-1)\left(p-1-\frac{s}{2}\right)\Xi_1\left(\kappa_p h^{s-2}\right)n^{(s-1)/2} + O\left(n^{s/2-1}\right) \tag{14}$$

as $n\to\infty$. Here, κ_p is defined by

$$\kappa_p = 1 - \frac{2^{-\binom{p}{2}}}{\prod_{0\le j<p} j!}\det_{0\le i,j<p}\left((-1)^i H_{i+j}(0) - H_{i+j}\left(h\right)e^{-h^2}\right),$$

where $H_k(z)$ denotes the k-th Hermite polynomial.

Proof. As a first step we need to establish some simple facts concerning the quantity κ_p. To be more precise, we have to show that κ_p is an even function with respect to h that has no constant term, i.e., is of the form

$$\kappa_p = \sum_{k=0}^{K}\sum_{m=1}^{M}\lambda_{k,m}h^{2k}e^{-mh^2}$$

for some numbers K, M and some constants $\lambda_{k,m}$.

It is obvious from the definition of the Hermite polynomials (see Equation (2)) that the k-th Hermite polynomial is an even (odd) polynomial whenever k is even (odd). This also implies the equality $(-1)^i H_{i+j}(0) = (-1)^j H_{i+j}(0)$. Now, replacing h by $-h$ in the definition of κ_p, factoring $(-1)^i$ out of the i-th row and $(-1)^j$ out of the j-th row we see that the expression remains unaltered. Hence, κ_p is an even function of h. The constant term of κ_p is seen to be equal to

$$1 - \frac{2^{-\binom{p}{2}}}{\prod_{j=0}^{p-1} j!} \det_{0\le i,j<p} \left((-1)^i H_{i+j}(0)\right) = 1 - \frac{2^{-\binom{p}{2}}}{\prod_{j=0}^{p-1} j!} \det_{0\le i,j<p} \left((-1)^{(i+j)/2} H_{i+j}(0)\right) = 0,$$

where the last equality is a consequence of Lemma 6. This proves the claimed form of κ_p.

We are now going to prove the asymptotics (14). Therefore, we assume that $s > 1$. The properties of κ_p established above together with Lemma 7 imply the equation

$$\Xi_1\left(\frac{d}{dh}\left(\kappa_p h^{s-1}\right)\right) = 0,$$

and the product rule for the derivative together with Lemma 8 show that

$$\Xi_1\left(\tau_p h^{s-1}\right) = -(s-1)(p-1)\Xi_1\left(\kappa_p h^{s-2}\right).$$

The asymptotics (14) is now obtained from the asymptotics (6) upon noting that the Ξ_0-term is negligible for $s \ge 2$.

Finally, we prove the asymptotics (13) and, therefore, assume $s = 1$. For the sake of simplicity we set

$$C = 2^{-\binom{p}{2}} \left(\prod_{j=0}^{p-1} j!\right)^{-1}.$$

From Lemma 8 and Lemma 7 we deduce that

$$\Xi_1(\tau_p) = (p-1)\Xi_1\left(\frac{d}{dh}\kappa_p\right) = -(p-1)\Xi_1\left(C\frac{d}{dh}\chi(h)\right),$$

where

$$\chi(h) = \det_{0\le i,j<p}\left((-1)^i H_{i+j}(0) - H_{i+j}(0)e^{-h^2}\right).$$

This last determinant can be evaluated to a closed form expression with the help of Lemma 6. Factoring $1-(-1)^j e^{-h^2}$ out of each column of the determinant we see that

$$\begin{aligned}\chi(h) &= \left(\prod_{j=0}^{p-1}\left(1-(-1)^j e^{-h^2}\right)\right) \det_{0\le i,j<p}\left((-1)^{(i+j)/2} H_{i+j}(0)\right)\\ &= \frac{1}{C}\left(1-e^{-2h^2}\right)^{\lfloor p/2\rfloor}\left(1-e^{-h^2}\right)^{\lceil p/2\rceil - \lfloor p/2\rfloor}.\end{aligned}$$

Now, an application of Lemma 7 shows that

$$\Xi_1\left(\frac{d}{dh}\chi(h)\right) = -1,$$

which implies

$$\Xi_1(\tau_p) = 1-p.$$

TABLE 1. This table gives the coefficient of the dominant asymptotic term of $\mathbb{E}H_{n,p}^s$ as $n \to \infty$ for small values of s and p (see Theorem 2).

$s\kappa_s^{(p)}$	$s=1$	$s=2$	$s=3$
$p=1$	$\frac{1}{2}\sqrt{\pi} = 0.88\dots$	1	$\frac{3}{4}\sqrt{\pi} = 1.32\dots$
$p=2$	$\frac{2+\sqrt{2}}{4}\sqrt{\pi} = 1.51\dots$	$\frac{5}{2}$	$\frac{3(12+\sqrt{2})}{16}\sqrt{\pi} = 4.45\dots$
$p=3$	$\frac{72+45\sqrt{2}-16\sqrt{3}}{96}\sqrt{\pi} = 1.99\dots$	$\frac{25}{6}$	$\frac{1584+315\sqrt{2}-32\sqrt{3}}{385}\sqrt{\pi} = 9.11\dots$
$p=4$	$\frac{10368+17091\sqrt{2}-3776\sqrt{3}}{20736}\sqrt{\pi} = 2.39\dots$	$\frac{1915}{324}$	$\frac{520992+165969\sqrt{2}-29824\sqrt{3}}{82944}\sqrt{\pi} = 15.04\dots$

The last step of the proof is the evaluation of the quantity $\Xi_0(\kappa_p)$. Recalling that κ_p is an even function with respect to h as well as the fact that all odd Bernoulli numbers except for B_1 are zero, i.e., $B_{2\nu+1} = 0$, $\nu \geq 1$, we deduce the equation

$$\Xi_0(\kappa_p) = \Xi_0\left(1 - C\chi(h)\right) = \Xi_0\left(1 - \left(1 - e^{-2h^2}\right)^{\lfloor p/2\rfloor}\left(1 - e^{-h^2}\right)^{\lceil p/2\rceil - \lfloor p/2\rfloor}\right).$$

The definition of Ξ_0 reveals that $\Xi_0\left(h^\nu e^{-\mu h^2}\right)$ is independent of μ. Consequently, we see that

$$\Xi_0(\kappa_p) = B_1 = -\frac{1}{2}.$$

This proves the asymptotics (13) and completes the proof of the theorem. □

Table 1 shows the constant of the dominant asymptotic term as $n \to \infty$ for the s-th moment of the height distribution for small values of s and p.

4. RANGE

We determine the asymptotics for $n \to \infty$ of

$$\mathbb{P}\left\{R_{n,p} \leq r\right\} = \frac{1}{m_n^{(p)}} \sum_{h=2p-2}^{r} \left(m_{n,h+1,r-h+1}^{(p)} - m_{n,h,r-h+1}^{(p)}\right). \tag{15}$$

Note that $m_{n,h+1,r-h+1}^{(p)} - m_{n,h,r-h+1}^{(p)}$ is the number of watermelons with height exactly h and range $\leq r$.

Theorem 3. *For each fixed $t \in (0,\infty)$ we have the asymptotics*

$$\mathbb{P}\left\{\frac{R_{n,p}+1}{\sqrt{n}} \leq t\right\} \to \frac{2^{-\binom{p}{2}}}{\prod_{i=0}^{p-1} i!} \int_0^t \left(\frac{d}{dz} T_p(z,w)\Big|_{z=t}\right) dw, \qquad n \to \infty, \tag{16}$$

where

$$T_p(z,w) = \det_{0\leq i,j<p}\left((-1)^i\left(\sum_{\ell\in\mathbb{Z}} H_{i+j}(\ell z)e^{-(\ell z)^2}\right) - \left(\sum_{\ell\in\mathbb{Z}} H_{i+j}\left(\ell z + w\right)e^{-(\ell z+w)^2}\right)\right).$$

Here, H_a denotes the a-th Hermite polynomial.

Proof. Since $m^{(p)}_{n,2p-2,k} = 0$ for any k, Equation (15) can be rewritten as

$$\mathbb{P}\left\{R_{n,p} \le r\right\} = \frac{m^{(p)}_{n,r+1,1}}{m^{(p)}_{n}} + \frac{1}{m^{(p)}_{n}} \sum_{h=2p-1}^{r} \left(m^{(p)}_{n,h,r-h+2} - m^{(p)}_{n,h,r-h+1}\right).$$

The first term on the right-hand side is negligible. To see this, we note that $m_{n,r+1,1}$ is equal to the number of p-watermelons with wall and height $\le r$, which is of order $\binom{2n}{n}^p n^{-p^2}$ as $n \to \infty$ (see [5] for details), whereas $m^{(p)}_n$ is of order $\binom{2n}{n}^p n^{-\binom{p}{2}}$ (see Lemma 2).

Asymptotics for the sum on the right-hand side can now be established in a fashion analogous to the proof of Theorem 1. A more detailed presentation of these techniques can also be found in [5, Theorem 2]. We find the asymptotics

$$\mathbb{P}\left\{R_{n,p} \le r\right\} \sim \frac{\binom{2n}{n}^p n^{-\binom{p}{2}}}{m^{(p)}_{n}} \sum_{h=2p-1}^{r} \left(T_p\left(\frac{r+2}{\sqrt{n}}, \frac{h}{\sqrt{n}}\right) - T_p\left(\frac{r+1}{\sqrt{n}}, \frac{h}{\sqrt{n}}\right)\right)$$

as $n \to \infty$, where

$$T_p(t,w) = \det_{0\le i,j<p}\left((-1)^i\left(\sum_{\ell\in\mathbb{Z}} H_{i+j}(\ell t)e^{-(\ell t)^2}\right) - \left(\sum_{\ell\in\mathbb{Z}} H_{i+j}\left(\ell t + w\right)e^{-(\ell t+w)^2}\right)\right).$$

Now, Taylor series expansion shows that

$$T_p\left(\frac{r+2}{\sqrt{n}}, \frac{h}{\sqrt{n}}\right) - T_p\left(\frac{r+1}{\sqrt{n}}, \frac{h}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}}T'_p\left(\frac{r+1}{\sqrt{n}}, \frac{h}{\sqrt{n}}\right) + O\left(n^{-1}\right), \qquad n \to \infty,$$

where T' denotes the derivative of T with respect to its first argument. Setting $r+1 = t\sqrt{n}$ we see that

$$\begin{aligned}\sum_{h=2p-1}^{r} \left(T_p\left(\frac{r+2}{\sqrt{n}}, \frac{h}{\sqrt{n}}\right) - T_p\left(\frac{r+1}{\sqrt{n}}, \frac{h}{\sqrt{n}}\right)\right) &\sim \sum_{h=2p-1}^{r} \frac{1}{\sqrt{n}}T'_p\left(\frac{r+1}{\sqrt{n}}, \frac{h}{\sqrt{n}}\right)\\ &\to \int_0^t T'\left(t,w\right)dw\end{aligned}$$

as $n \to \infty$.

□

Remark 3. For the special case $p = 1$ we recover a well-known fact originally proven by Chung [3] and Kennedy [14]. Namely, the equality of the distributions of the height of Brownian excursions and the range of Brownian bridges. This result also follows from a more general relation between excursions an bridges proved by Vervaat [21].

In fact, for $p = 1$ we have

$$\left.\frac{d}{dz}T_1(z,w)\right|_{z=t} = -\sum_{\ell\in\mathbb{Z}} 2\ell^2 t e^{-(\ell t)^2} + 2\sum_{\ell\in\mathbb{Z}} \ell(\ell t + w)e^{-(\ell t+w)^2},$$

which shows that

$$\mathbb{P}\left\{\frac{R_{n,1}+1}{\sqrt{n}} \le t\right\} \to \sum_{\ell\in\mathbb{Z}} \left(1 - 2(\ell t)^2\right)e^{-(\ell t)^2}, \qquad n \to \infty,$$

by Theorem 3. This shows that the distribution of the range of 1-watermelons without wall weakly converges to the limiting distribution of the height of 1-watermelons with wall restriction (see [5]).

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit zwei eng verwandten Modellen: Gitterpfaden in einer Weylkammer vom Typ B und nichtüberschneidenden Gitterpfaden im ganzzahligen Gitter aufgespannt durch die Vektoren $\{(1,1),(1,-1)\}$ mit Schritten aus dieser Menge. Diese Gitterpfadmodelle sind unter anderem von zentraler Bedeutung in der Kombinatorik und der statistischen Mechanik. In der statistischen Mechanik dienen diese Modelle der Beschreibung bestimmter nicht-kollidierender Teilchen-Systeme. Die Bedeutung von Gitterpfadmodellen in der Kombinatorik ist teilweise begründet durch ihre interessanten kombinatorischen Eigenschaften, vor allem aber auch durch die engen Beziehungen zu zahlreichen zentralen kombinatorischen Objekten wie z.B. Integer Partitions, Plane Partitions und Young Tableaux.

Im ersten Teil dieser Arbeit werden asymptotische Formeln für die Anzahl von Gitterpfaden in einer Weylkammer vom Typ B für eine allgemeine Klasse von Schritten hergeleitet. Die Klasse der zulässigen Schritte wird hierbei durch die Forderung der "Reflektierbarkeit" der resultierenden Pfade beschränkt. Spezialfälle dieser asymptotischen Formel lösen in der Literatur aufgeworfene Probleme und liefern bekannte Resultate für zweidimensionale Vicious Walkers Modelle und sogenannte k-non-crossing tangled diagrams.

Im zweiten Teil werden die Zufallsvariablen "Höhe" und "Ausdehnung" auf der Menge aller nichtüberschneidenden Gitterpfade mit n Schritten sowie auf der Teilmenge all jener auf die obere Halbebene beschränkten nichtüberschneidenden Gitterpfade mit n Schritten studiert. Unter der Annahme einer Gleichverteilung auf diesen Mengen wird die asymptotische Verteilung beider Zufallsvariablen bestimmt. Weiters werden die ersten beiden Terme der asymptotischen Entwicklung aller Momente der Zufallsvariable "Höhe" ermittelt. Dies löst ein in der Literatur aufgeworfenes Problem, und verallgemeinert ein bekanntes Resultat über die Höhe ebener Wurzelbäume.

Die in dieser Arbeit gelösten Probleme haben eine interessante Eigenschaft gemein. Während man relativ leicht exakte Abzählformeln für die betrachteten Größen aufstellen kann, da bereits entsprechende Resultate (siehe Theorem 1 und Theorem 2 in der Einleitung) in der Literatur vorhanden sind, ist es hingegen schwierig aus diesen exakten Formeln das asymptotische Verhalten der interessierenden Größen abzulesen. Der Hauptgrund hierfür ist die Tatsache, dass es sich bei den exakten Abzählformeln im Wesentlichen um Determinanten bzw. alternierende Summen handelt, welche keine einfache Darstellung als Produkt besitzen. In der asymptotischen Analyse tritt daher eine große Anzahl von Auslöschungen von asymptotisch führenden Termen auf. Die genaue Bestimmung der Anzahl dieser Auslöschungen stellt einen der wesentlichen Schritte in der vorliegenden Arbeit dar.

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Research Publications and Preprints

1. Thomas Feierl. Asymptotics for the number of walks in a Weyl chamber of type B. Preprint, available at arXiv:math.CO/0906.4642.
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3. Thomas Feierl. The height of watermelons with wall. Preprint, available at arXiv:math.CO/0802.2691.

4. Thomas Feierl. The height and range of watermelons without wall (extended abstract). In *Proceedings of the IWOCA 2009*, Lecture Notes in Computer Science. Springer-Verlag, 2009. to appear.
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