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# DIPLOMARBEIT

Titel der Diplomarbeit

## Combinatorial Approaches to the Asymmetric Simple Exclusion Process

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# Abstract

The asymmetric simple exclusion process (ASEP) is an important model from statistical mechanics. It describes a system of particles hopping left and right on a one-dimensional lattice of $n$ sites. At both boundaries particles may be injected and ejected, and the particles are subject to a strong exclusion rule which admits at most one particle per site at each time. The ASEP is regarded as a primitive model of kinetics of polymerization or traffic flow, and recently a connection to orthogonal polynomials was noted too.

Furthermore, it has been observed that the unique stationary distribution $P_n$ has remarkable connections to combinatorics, which is the starting point of this text. Here, we provide an overview about the work that was done investigating the combinatorics that lie behind the process. We start by showing how the stationary distribution $P_n$ can be expressed in terms of a simple matrix formulation, the so-called *Matrix-Ansatz*. We shall use this matrix formulation to establish a connection between certain lattice paths and states of a restricted version of the ASEP, where particles are only allowed to travel in one direction. We can then express the stationary probabilities $P_n$ in terms of lattice paths, e.g., *One up paths* or *bicolored Motzkin paths*. Enumeration of these paths yields the stationary distribution of the process.

One can generalize these results by making use of a fairly new class of combinatorial objects, namely *permutation tableaux* or *alternative tableaux*. With the help of these tableaux, we can handle more general cases as with lattice paths, like the partially ASEP, where particles travel to both sides, but can only enter at one boundary and leave at the opposite boundary, or the symmetric ASEP in which particles travel left and right at same rate and enter or leave the system to both sides. Again, we rely on the Matrix Ansatz, and, additionally, consider the combinatorics on permutation tableaux to obtain results concerning the stationary distributions of these processes.

## Zusammenfassung

Der "asymmetric simple exclusion process", (ASEP), ist ein wichtiges Modell in der statistischen Mechanik. Es beschreibt Teilchen, die entlang eines eindimensionalen Weges mit $n$ Plätzen nach links oder rechts wandern können. Die Teilchen können am Rand des Systems eintreten oder es eben dort verlassen. Außerdem unterliegen sie einer Exklusionsregel; auf Grund starker Wechselwirkungen darf sich zu jedem Zeitpunkt höchstens ein Teilchen an jeder Stelle aufhalten.

Der ASEP gilt, unter anderem, als ein einfaches Modell für den Vorgang der Polymerisation oder für den Verkehrsfluss. Des Weiteren wurde auch eine Verbindung zu Orthogonalpolynomen entdeckt.

Aus kombinatorischer Sicht ist der Prozess vor allem interessant, weil die eindeutige stationäre Verteilung $P_n$ enge Verbindungen zu der Kombinatorik aufweist. Diese Beobachtung ist der Ausgangspunkt dieser Arbeit, in der verschiedene kombinatorische Zugänge zu dem ASEP präsentiert werden. Eingangs wird gezeigt, dass man die stationäre Verteilung $P_n$ mit Hilfe einer simplen Matrixformel ausdrücken kann, mittels des sogenannten *Matrix Ansatzes*. Durch Interpretation von Matrizen, die diesem Ansatz genügen, kann man eine Verbindung zwischen den Zuständen des Prozesses und Gitterpfaden herstellen. Die Resultate werden für einen Spezialfall des ASEP gezeigt, bei dem sich Teilchen nur in eine Richtung fortbewegen können. Die stationäre Verteilung kann man dann mittels Abzählen von Gitterpfaden, nämlich von *One up Pfaden* oder von *zweifärbigen Motzkin Pfaden*, berechnen. Diese Herangehensweise kann man verallgemeinern, und zwar mittels relativ neuer kombinatorischer Objekte, sogenannter *Permutation Tableaux* und *Alternative Tableaux*. Untersucht man die Struktur und Eigenschaften dieser Tableaux, kann man, wiederum mit Hilfe des Matrix Ansatzes, Ausdrücke für die stationäre Verteilung der folgenden beiden Spezialfälle des ASEP finden: Für den Fall, dass Teilchen in beide Richtungen wandern, aber nur an der einen Seite eintreten und an der anderen Seite das System verlassen können, und für den recht allgemeinen Fall, dass Teilchen in beliebige Richtungen wandern können, dies aber mit den selben Wahrscheinlichkeiten tun.

# Contents

CHAPTER 0

# Introduction

The asymmetric simple exclusion process (ASEP) is a model from statistical mechanics that has been studied extensively.

Although quite simple, it exhibits some interesting behavior, like boundary-induced phase transitions, spontaneous symmetry breaking and phase separation, and especially the stationary properties have attracted much attention. It is regarded as a primitive model of kinetics of biopolymerization [**16**] or traffic flow [**1**], for formation of shocks [**10**] and also appears in enzyme kinetics [**21**]. Recently, a connection to orthogonal polynomials was noted as well [**20, 26**].

The ASEP describes a model of particles traveling on a one-dimensional lattice path of $n$ sites with a strong exclusion interaction, allowing at most one particle per site. In an infinite small time interval $dt$, particles may hop to the right with probability $pdt$, or to the left with probability $qdt$. Furthermore, new particles may be injected on or removed from the left-most site with probability $\alpha dt$ and $\gamma dt$, respectively, and at the right-most site with probability $\delta dt$ and $\beta dt$, respectively – see Figure 0.1 for an informal illustration. In the long time limit this Markov chain reaches a unique stationary distribution (as shown in Chapter 2).

These steady state probabilities can be expressed using a matrix formulation as noted in [**9**]: for each state, the steady state probability $P_n$ is given by $\frac{f_n}{Z_n}$, where $f_n$ are unnormalized weights given through a product of matrices (which satisfy some simple algebraic relations) and $Z_n$, the *partition function*, is a multiplicative constant (independent of the state). This approach is known as the *Matrix Ansatz* and has become a standard technique for investigations of the stationary properties of the ASEP. It also yields an easy way of calculating the *n-point (correlation) function* which indicates how particles influence each other at different sites.

Our main interest shall be to describe stationary properties of the ASEP combinatorially. We shall find combinatorial interpretations for the unnormalized weights $f_n$ as well as for the partition function $Z_n$ and the $n$-point function, and, the other way round, derive these quantities through combinatorial methods. Depending on the choice of parameters, we use a range of well known objects (e.g. *Dyck paths*) and some fairly new objects (e.g. *alternative tableaux*).

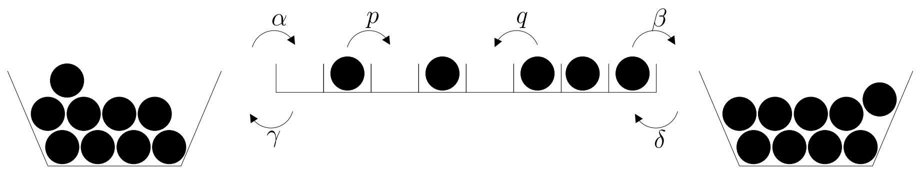



FIGURE 0.1. An informal illustration of the ASEP.

We will mainly rely on the Matrix Ansatz. Other combinatorial approaches bypassing the Matrix Ansatz include the considerations of new Markov chains which are linked to the ASEP in the sense that a walk on them is indistinguishable from a walk on the ASEP. In [**11**] this Markov chain reveals a second row of particles traveling backwards while in [**5**] the Markov chain is defined on *permutation tableaux* which are in bijection with permutations.

Other interesting results which are based on the Matrix Ansatz can be found, e.g., in [**26**]. There, an eigenvector to one of the matrices whose entries are related to orthogonal polynomials is found. By orthogonality relation and exploitation of the eigenvector property, the partition function can then be expressed in terms of integrals.

Finally, in [**14**] the model is placed in the context of Markov chain theory.

The structure of this text is as follows. The first chapter, Chapter 0, is devoted to the precise definition of the asymmetric simple exclusion process (ASEP) and to notation.

We then shall consider the Matrix Ansatz Theorem in Chapter 2 as it builds the basis for the results that are to follow. This Matrix Ansatz Theorem (Theorem 2.1) provides a simple way to calculate the long time limit probability $P_n$ of finding the ASEP in a fixed state. The theorem states that the unnormalized weights $f_n$ which differ from $P_n$ by a constant can be expressed as the product of matrices for which some simply algebraic equations hold. The constant that normalizes the weights $f_n$ is given by $1/Z_n$, where $Z_n$ is the so-called *partition function.* We shall later note that the validity of the Matrix Ansatz Theorem is not only limited to matrices, but since we mainly deal with matrices in the subsequent chapters, we investigate their form and size in Section 2.2 – we shall see that, in general, they are infinite-dimensional and triangular.

In Chapter 3 we start to investigate the combinatorics that lies behind the ASEP by first considering the case where particles can only hop to the right and do so at the same rate at which they enter and leave the system. We call this the totally asymmetric simple exclusion process (TASEP) with maximal regime flow. We first consider the partition function $Z_n$ and show that it can be expressed by enumerating certain lattice paths, namely *One up paths* (Lemma 3.5) and we derive that $Z_n = C_{n+1}$, where $C_{n+1}$ is a Catalan number (Corollary 3.8). To do so we interpret a matrix representation for the Matrix Ansatz in terms of *transfer matrices*, see Section 3.1. Moreover, we shall see that each unnormalized weight $f_n$ corresponds to a set of One up paths for which restrictions concerning the position of up and down steps hold (Lemma 3.9). Through usage of different bijections we shall see that $f_n$ can also be expressed in terms of pairs of *non-intersecting paths* (Proposition 3.13), or in terms of a determinant (Theorem 3.16). Additionally, we derive a formula to calculate the probability of finding a fixed number of particles in the TASEP with maximal flow regime (Proposition 3.17). For the proofs we use, among others, the Lindström-Gessel-Viennot Theorem (Theorem A.1). In Section 3.5 we shall see that in the case where we additionally allow particles to hop to the left, one can, by the same methods as before, express $P_n$, the steady state probabilites, in terms of lattice paths, more precisely, in terms of *Motzkin paths.* We close the chapter by giving a short overview on how similar results can be obtained by applying the methods described before to the TASEP with general parameters (Section 3.7).

The subsequent chapter, Chapter 4, is dedicated to a more general case of the ASEP: this time we allow particles to hop left or right and to enter the system at the left-hand and leave it at the right-hand side, the partially asymmetric simple exclusion process (PASEP). We shall connect the ASEP and its states to *permutation tableaux*, a certain class of tableaux with $0-1$-fillings (Definition 4.1). We show that for the PASEP the unnormalized weights $f_n$ are obtained through considerations of certain statistics of permutation tableaux (Theorem 4.3). The method is similar to the one in Chapter 3. Again, we present a representation for the Matrix Ansatz, but this time we see that the entries in the matrices correspond to a generating function for permutation tableaux according to certain statistics (see Section 4.2). (This is actually how the representation was found, through considerations of combinatorics on permutation tableaux, see [**6**], p. 296).

In Chapter 5 we then consider so-called *alternative tableaux*. These are tableaux with partial filling consisting of $\uparrow$ and $\leftarrow$, they stand in bijection with permutation tableaux and were first introduced in [**27**]. Whereas in the case of permutation tableaux the statistics involved eventually seem a little artificial, the results from Chapter 4 can be reformulated in a somewhat more catchy way through alternative tableaux. Before doing so, we shall present the bijection between permutation tableaux and alternative tableaux in Section 5.1. We shall formulate the bijection by means of two algorithms $\varphi, \psi$, one sending permutation tableaux to alternative tableaux and and the other doing the reverse operation. To this end we examine the *type* of entries in the fillings of permutation tableaux and alternative tableaux and work out the details of the results of X. Viennot in [**27**] explicitly.

In the last chapter, Chapter 6, we present results concerning the case of the ASEP where particles hop left and right at the same rate but might enter and leave the system at different rates on both sides (symmetric ASEP). Since we allow two more parameters to be different from 0 (namely the rates of particles entering at the right and leaving to the left) we will have to generalize the tableaux used so far to reflect this. We do so by introducing a labeling along the right-hand border of the tableaux. This leads to the definition of *bordered permutation tableaux* and *bordered alternative tableaux* in Section 6.1. With the help of these objects we can again express the unnormalized weights $f_n$ as well as the partition function $Z_n$ in terms of tableaux and their statistics (Theorems 6.4, 6.5). The proofs are given in Section 6.2. Finally, in Section 6.3, we present a second proof of the results in Section 6.1. Again, we modify the definition of the tableaux a little and use *decorated bordered permutation tableaux*. The special feature in this section is that we see that the Matrix Ansatz, although suggested by the name, is not only limited to matrices. We define some linear operators on the infinite-dimensional vector space whose basis is labeled by decorated bordered permutation tableaux and show that for them the Matrix Ansatz holds. We then show that the operators posses the desired combinatorial properties and hence imply the result.

The Appendix is devoted to the proof of the Lindström-Gessel-Viennot Theorem and some results which are implied by this theorem, and which are used in the main text.

CHAPTER 1

# Definitions and Notation

## 1.1. The Asymmetric Simple Exclusion Process – ASEP

In physics literature (see, e.g., ([**14**], p. 261) or ([**9**], Section 11.1.)) the one-dimensional *asymmetric simple exclusion process* (ASEP) is defined as a continuous-time Markov process modeling left and right hopping particles on a one-dimensional lattice of n sites, each of the sites being either empty or occupied by a particle. Due to an exclusion rule a given site is at most occupied by one particle. Particles in the system try to move on the lattice path but have a unit exponential holding time. Once a particle gets active it has the following options:

- it hops to the next site at its right at rate $p$ if this next site is empty (for particles on site $1 \leq i \leq n-1$)
- it hops to the next site at its left at rate $q$ if this next site is empty (for particles on site $2 \leq i \leq n$)

If the site to which the particle is aiming to hop to is occupied, then the jump is suppressed due to the exclusion rule. Through rescaling time we can always set $p = 1$. Furthermore, a particle

- is injected at site $i = 1$ at rate $\alpha$ if the site is empty,
- is removed from site $i = 1$ at rate $\gamma$,
- is injected at site $i = n$ at rate $\delta$ if the site is empty,
- is removed from site $i = n$ at rate $\beta$,

where $\alpha, \beta, \gamma, \delta \geq 0$. We denote a state of the ASEP with $n$ sites in two different ways: either as a word in $\{\circ, \bullet\}^n$ (where $\circ$ means that the site is empty and $\bullet$ that the site is occupied) or as $\tau = (\tau_1, \ldots, \tau_n)$ where $\tau_i \in \{0, 1\}$ for $i = 1, \ldots, n$. The first notation already suggests interpreting the ASEP as a system where not only particles and empty sites travel, but rather black particles and white particles. This point of view also exhibits a symmetry within the ASEP, e.g. in the case of $p = \gamma = \delta = 0$ black particles travel to the right, while white particles travel to the left – see Figure 1.1. This black particle/white particle symmetry (or particle/hole symmetry) of the model can be made explicit as follows: if black particles travel *right* at rate $p$, then white particles travel *left* at the same rate. In the case where all parameters are non-zero the following symmetry between black and white particles can be noted:

$$\begin{aligned} &\textit{sites} : i \leftrightarrow n+1-i \qquad\qquad i = 1, \ldots, n \\ &\textit{rates} : q \leftrightarrow -1,\ \alpha \leftrightarrow q^{-1}\delta,\ \beta \leftrightarrow q^{-1}\gamma,\ \gamma \leftrightarrow q^{-1}\beta,\ \delta \leftrightarrow q^{-1}\alpha \end{aligned}$$

To confirm this claim, note that the white particles move to the left at rate $q$ and to the right at rate 1. If we want a white particle to travel "forward" (which now is to the left) at rate 1, then we have to rescale time by factor $q^{-1}$. So, for white particles we

| Notation | Parameters free to choose | Fixed parameters |
|---|---|---|
| ASEP | $q, \alpha, \beta, \gamma, \delta$ | |
| TASEP | $\alpha, \beta$ | $q = 0, \gamma = \delta = 0$ |
| PASEP | $q, \alpha, \beta$ | $\gamma = \delta = 0$ |
| symmetric ASEP | $\alpha, \beta, \gamma, \delta$ | $q = 1$ |

TABLE 1. Some special cases of the ASEP.

obtain the parameters states above. Hence, for the general ASEP it suffices to consider the case where $0 \leq q \leq 1$.

Depending on the choice of the probability $q$ (and the other parameters), physics literature knows a lot of different special cases of the ASEP and a range of different names for it. We will use the following notation. The simplest case is the case where particles can only enter at the left-hand side and move to the right until they leave the system – this is the case where only $\alpha, \beta \geq 0$. We call this the TASEP (totally asymmetric simple exclusion process). A special case of the TASEP is the choice $\alpha = \beta = 1$, the *maximal flow regime*. If not only $\alpha, \beta \geq 0$ but, additionally, $q \neq 0$ we call this the PASEP (partially asymmetric simple exclusion process). The case where all parameters on the boundaries are chosen freely but $q = 1$ is called the symmetric ASEP. Table 1.1 provides a compact overview.

REMARK. It might seem strange to introduce a "*symmetric* case" of the *asymmetric* simple exclusion process, but it is more convenient to stick with the same name of the system, ASEP, all the time instead of changing it to, e.g. SSEP (symmetric simple exclusion process).

## 1.2. ASEP as a Discrete-Time Markov Chain

One can also describe the ASEP in terms of a discrete-time Markov chain, as it was done in ([**6**], Definition 2.2.). Through considerations of points in time at which an event occurs one can generally derive a discrete version of a time-continuous Markov chain.

DEFINITION 1.1. Let $\alpha', \beta', \gamma', \delta'$ and $p, q'$ be constants such that $\alpha', \beta', \gamma', \delta' \geq 0$, and $0 \leq pq' \leq p \leq 1$. Let $B_n$ be the set of all $2^n$ words in the language $\{\circ, \bullet\}$. The ASEP is the Markov chain on $B_n$ with transition probabilities:

- If $X = A\bullet, \circ B$ and $Y = A\circ, \bullet B$ then $P_{X,Y} = \frac{1}{2p(n+1)}$ (particle hops right) and $P_{Y,X} = \frac{q'}{2p(n+1)}$ (particle hops left)

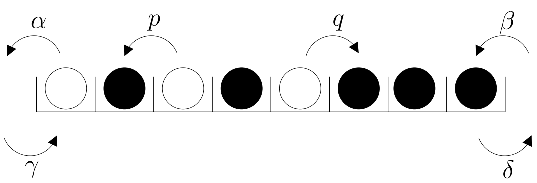



FIGURE 1.1. The symmetry of black and white particles within the ASEP – compare to Figure 0.1.

- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha'}{2p(n+1)}$ (particle enters from the left)
- If $X = B\bullet$ and $Y = B\circ$ then $P_{X,Y} = \frac{\beta'}{2p(n+1)}$ (particle exits to the right)
- If $X = \bullet B$ and $Y = \circ B$ then $P_{X,Y} = \frac{\gamma'}{2p(n+1)}$ (particle exits to the left)
- If $X = B\circ$ and $Y = B\bullet$ then $P_{X,Y} = \frac{\delta'}{2p(n+1)}$ (particle enters from the right)
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $PX, X = 1 - \sum_{X \neq Y} P_{X,Y}$.

We indicate the equivalence of the two chains following the arguments of ([**11**], Section 1.2.): Another way to describe the time-continuous ASEP is the following: we do not focus on the particles but on the walls between the particles. Each of the walls is considered independently (more precisely, each wall between a ordered pair of sites, hence there are two walls between any two sites). We think of them as being "asleep" and having unit exponential holding times before waking up and eventually triggering a move. The move consists in exchanging a particle and an empty site on the left-hand and the right-hand side of the wall and occurs with the transition rates of the ASEP. (This point of view is derived through coupling as in ([**14**], p. 215). Now we only consider the points in time where a wall wakes up. The index of the next wall to awake is then uniformly distributed among the total number of walls $\{0, 1, \ldots, 2n+2\}$ ([**13**], Satz 16 (c)) and the probability of a transition occurring is given by the rates of the ASEP. Setting $\alpha' = \alpha/p$, $\beta' = \beta/p$ etc. (this is the analogon to the rescaling in the continuous version) we arrive at Definition 1.1. Furthermore, these considerations imply that the ASEP and its discrete-time Markov chain replica have the same stationary distribution.

## 1.3. Generalizations of the ASEP

There also exist some generalizations like the 3-ASEP where three types of particle travel within the system. A "strong-type" particle can not only hop to empty sites but force a neighboring "weak-type" particle to change places, while the "weak-type" particle can only hop to the left or to the right if the neighboring site is empty. Interpreting empty sites as white particles we obtain the remaining third type. See, e.g., ([**11**], Section 4.1.).

CHAPTER 2

# The Matrix Ansatz

The Matrix Ansatz for the ASEP was introduced by Derrida *et al.* [**9**] and provides a very useful tool in further investigations. It reduces the problem of calculating the unique stationary distribution to finding matrices $D, E, V, W$ for which some simple algebraic equations hold. Actually, $D, E, V, W$ do not even need to be matrices (although most of the time we will consider a matrix representation) but they can be something else, e.g., operators between vector spaces. For now we focus on matrices, but we also give an example for the representation in terms of operators (see Section 6.3).

Remark. In this chapter we omit the assumption $p = 1$.

We are striving to find an expression for the quantity $P_n(\tau_1, \dots, \tau_n)$, the steady state probability of finding the ASEP in the state $\tau = (\tau_1, \dots, \tau_n)$ in the long time limit. It turns out to be more convenient to consider unnormalized weights $f_n(\tau_1, \dots, \tau_n)$, which are equal to $P_n(\tau_1, \dots, \tau_n)$ up to a multiplicative constant:

$$P_n(\tau_1, \dots, \tau_n) = \frac{f_n(\tau_1, \dots, \tau_n)}{Z_n} \tag{2.1}$$

with

$$Z_n = \sum_{(\tau_1,\dots,\tau_n)\in\{0,1\}^n} f_n(\tau_1, \dots, \tau_n) \tag{2.2}$$

being the so-called **partition function**.

Theorem 2.1 ([**9**], Section 2). *Suppose that $D$ and $E$ are matrices, $V$ a column vector, and $W$ a row vector such that the following equations hold:*

$$pDE - qED = D + E \tag{2.3}$$

$$(\beta D - \delta E)V = V \tag{2.4}$$

$$W(\alpha E - \gamma D) = W \tag{2.5}$$

*Then, the unnormalized steady state probability $f_n(\tau_1, \dots, \tau_n)$ of the ASEP being in state $\tau = (\tau_1, \dots, \tau_n) \in \{0,1\}^n$ is given by*

$$f_n(\tau_1, \dots, \tau_n) = W\left(\prod_{i=1}^{n} (\tau_i D + (1 - \tau_i)E)\right) V, \tag{2.6}$$

*where we assume that $W(\prod_{i=1}^n (\tau_i D + (1-\tau_i)E))V$ has the same sign for all possible states $\tau \in \{0,1\}^n$ and that $f_n$ exists (i.e. that the matrix product is well-defined).*

So, it turns out that the unnormalized weights $f_n(\tau_1, \dots, \tau_n)$ can be expressed as the product of $n$ matrices $D$ or $E$ with the matrix $D$ at position $i$ if site $i$ is occupied ($\tau_i = 1$) and the matrix $E$ at position $i$ if site $i$ is empty. Consider the following, simple example.

Example. Consider the $TASEP$ with 3 sites where we set $p = 1$, $\alpha = \beta = 1/2$ and the remaining parameters equal 0; that is, particles travel to the right at rate 1, and enter the system at the left and leave it to the right, both at rate 1/2. Consider the state $(\circ, \bullet, \circ)$. We set $D = 1/\beta = 2$, $E = 1/\alpha = 2$ and $W = V = 1$. Then, the Matrix Ansatz Equations (2.3)– (2.5) are satisfied. Hence the unnormalized steady state probability of finding the TASEP in the state $\tau = (0, 1, 0)$ is given by

$$f_n = EDE = \frac{1}{\alpha}\frac{1}{\beta}\frac{1}{\alpha} = \frac{1}{\alpha^2\beta}.$$

Another consequence of Theorem 2.1 is that the partition function $Z_n$ can be written as

$$Z_n = W(D + E)^n V. \tag{2.7}$$

Since, by definition, $Z_n$ equals the sum of the unnormalized weights $f_n$, we have

$$Z_n = \sum_{(\tau_1,\dots,\tau_n)\in\{0,1\}^n} f_n(\tau_1, \dots, \tau_n) \tag{2.8}$$

$$= \sum_{(\tau_1,\dots,\tau_n)\in\{0,1\}^n} W\left(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E)\right)V \tag{2.9}$$

$$= W \sum_{(\tau_1,\dots,\tau_n)\in\{0,1\}^n} \left(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E)\right)V = W(D+E)^n V. \tag{2.10}$$

If we find $D, E, V, W$ such that the Matrix Ansatz equations hold, we can easily calculate the steady state probabilities since then the Matrix Ansatz Theorem immediately provides the unnormalized weights $f_n$ as well as the partition function $Z_n$.

Remark. Although the assumption that all quantities $W(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))V$ have the same sign might appear very restrictive in Theorem 2.1, it actually is not. In general, it is feasible to drop this restriction using some Markov chain theory, see ([**14**], p. 263, Proof of Theorem 3.1.). But for all solutions $D, E, V, W$ to the Matrix Ansatz which we provide later on — we will refer to them as representations most of the time — the assumption holds simply by definition of $D, E, V, W$. Also by definition of these matrices we will ensure that the matrix product in $f_n$ is well-defined.

## 2.1. Proof of the Matrix Ansatz Theorem

Proof of Theorem 2.1. Consider the ASEP with $n$ sites and note that it is a continuous Markov chain with $2^n$ states – each of the $n$ sites can either be empty or occupied. Let $Q \in \mathbb{R}^{2^n \times 2^n}$ be the transition rate matrix of the Markov chain, that is

$$Q_{i,j} = \begin{cases} \text{the transition rate from state } i \text{ into state } j & i \neq j \\ -\sum_{j\neq i} Q_{i,j} & i = j \end{cases} \tag{2.11}$$

The ASEP, as a Markov chain, is *irreducible* (that is, one can reach any state out of any other state – in our case most of the time passing through several other states but that does not matter) and *recurrent* (that is, the probability of returning to a given state $i$ after having started in this state is equal to 1 – for our Markov chain there is no reason why it should be less likely to return into a given state). Therefore, there exists a unitary stationary distribution which we will denote by $\mathbb{P}$ ([**18**], Theorem 3.5.2).

Markov chain theory provides the following to find this unitary stationary distribution: if for a vector

$$P = (P_1, \ldots, P_{2^n}) \in \mathbb{R}^{2^n}$$

with $P_i \geq 0$ and $\sum_{i=1}^{2^n} P_i = 1$ the equation

$$P * Q = 0 \tag{2.12}$$

holds, then $P$ is the stationary distribution $\mathbb{P}$ ([**18**], Theorem 3.5.5). We will show that the unnormalized steady state probability is given by (2.6) and that $\mathbb{P}$ can hence be obtained by (2.1).

Since each entry of $P$ gives us the steady state probability of the state $\tau$, we denote the entries of the vector $P$ by $P_n(\tau_1, \ldots, \tau_n)$ with $\tau = (\tau_1, \ldots, \tau_n)$ being a state of the ASEP.

Solving the matrix equation in (2.12), we obtain a system of $2^n$ equations. Number all possible states of the ASEP by $i = 1, \ldots, 2^n$ and let them be represented by $S_i$. Fix a state $(\tau_1, \ldots, \tau_n) =: S_k$. Then the $k$-th equation reads

$$P_n(S_1) * Q_{1,k} + \ldots + P_n(S_{2^n}) * Q_{2^n,k} = 0 \tag{2.13}$$

where $Q_{i,k}$ is the transition rate from state $S_i$ into state $S_k$. If Equation (2.13) holds for $k = 1, 2, \ldots 2^k$, then $P_n$ is the steady state distribution.

Let us take a closer look at the transition rates, the entries of the matrix $Q$. We find a lot of 0 entries in $Q$, since in each row there are at most $n+2$ entries different from 0. (This can be seen by using one of the points of view in Chapter 1.2. If the ASEP has $n$ sites, then there are $n+1$ walls between these $n$ slots. Any of these walls can become active and might trigger a move, so there are up to $n+1$ possible new states – the exact number depends on the number of particles in a given state. Additionally, in the diagonal of the matrix we find one more (negative) entry that corresponds to the rate of staying in the given state.) For investigations, we split the transition matrix $Q$ up into "smaller pieces" and rewrite Equation(s) (2.13). We use the following notation. Define

$$h_1 := \begin{pmatrix} -\alpha & \alpha \\ \gamma & -\gamma \end{pmatrix}. \tag{2.14}$$

This matrix represents the transition rates due to particles entering or leaving at the left-hand boundary side. We will not label the entries of the matrix by $\{1; 2\}$ as usually but by $\{0; 1\}$. This is a very intuitive labeling, since e.g. $(h_1)_{0;1} = \alpha$ gives the transition-rate out of the state $(0, \tau_2, \ldots, \tau_n)$ into the state $(1, \tau_2, \ldots, \tau_n)$ (this being a particle entering at the left-hand side) whereas $(h_1)_{1;0} = \gamma$ gives the transition-rate out of the state $(1, \tau_2, \ldots, \tau_n)$ into the $(0, \tau_2, \ldots, \tau_n)$ (particle leaving at the left-hand side). Entries in the diagonal of $h_1$ are analogous to the entries in the diagonal of the transition rate matrix $Q$.

Similarly we define $h_n$ to be the matrix whose entries represent the transition rates due to particles leaving or entering from the right-hand boundary site:

$$h_n := \begin{pmatrix} -\delta & \delta \\ \beta & -\beta \end{pmatrix}. \tag{2.15}$$

Again label the entries by $\{0;1\}$. The transitions that occur due to a particle hopping between a pair of non-boundary sites $i, i+1$ are either

from $(\tau_1, \dots, \tau_{i-1}, 1, 0, \tau_{i+2}, \dots, \tau_n)$ to $(\tau_1, \dots, \tau_{i-1}, 0, 1, \tau_{i+2}, \dots, \tau_n)$ at rate $p$;

from $(\tau_1, \dots, \tau_{i-1}, 0, 1, \tau_{i+2}, \dots, \tau_n)$ to $(\tau_1, \dots, \tau_{i-1}, 1, 0, \tau_{i+2}, \dots, \tau_n)$ at rate $q$..

Therefore, we define

$$h := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & q & 0 \\ 0 & p & -p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.16}$$

this time choosing the labeling to be $\{0,0;0,1;1,0;1,1\}$.

We now rewrite and solve Equation (2.13) (which is just one representative for the $2^n$ equations). We will denote the left-hand side of (2.13) by $\frac{d}{dt}P_n(\tau_1, \dots, \tau_n)$. This is a common notation in the literature ([**9**], p. 1495) (and intuitively proposes that the distribution $P_n(\tau_1, \dots, \tau_n)$ is stationary). If

$$\frac{d}{dt}P_n(\tau_1, \dots, \tau_n) = 0 \tag{2.17}$$

then (2.13) is satisfied. Using (2.14) – (2.16) we can rewrite (2.13) as follows:

$$\begin{aligned} \frac{d}{dt}P_n(\tau_1, \dots, \tau_n) = & \sum_{\sigma_1=0,1} (h_1)_{\sigma_1;\tau_1} P_n(\sigma_1, \tau_2, \dots, \tau_n) \\ & + \sum_{i=1}^{n-1} \sum_{\sigma_i, \sigma_{i+1}} (h)_{\sigma_i, \sigma_{i+1}; \tau_i, \tau_{i+1}} P_n(\tau_1, \dots, \sigma_i, \sigma_{i+1}, \dots, \tau_n) \\ & + \sum_{\sigma_n=0,1} (h_n)_{\sigma_n;\tau_n} P_n(\tau_1, \dots, \tau_{n-1}, \sigma_n) \end{aligned} \tag{2.18}$$

We postpone showing that (2.13) equals (2.18) until the end of this proof. Note that in the term

$$\sum_{\sigma_i, \sigma_{i+1}} (h)_{\sigma_i, \sigma_{i+1}; \tau_i, \tau_{i+1}} P_n(\tau_1, \dots, \sigma_i, \sigma_{i+1}, \dots, \tau_n)$$

the sum is over $(\sigma_i, \sigma_{i+1}) \in \{0,1\}^2$. Furthermore note that $\tau_1$ and $\tau_n$ do not necessarily appear in the term (although they are written down explicitly). The notation rather means that $\sigma_i$ and $\sigma_{i+1}$ are injected on the $i$-th and $(i+1)$-th position (this could also be the first position) whereas the remaining positions are "filled up" by the corresponding $\tau_j$.

Now, suppose that the following algebraic equations hold for $D, E, V, W$, and some $\chi \in \mathbb{R}$:

$$pDE - qED = \chi(D + E) \tag{2.19}$$

$$(\beta D - \delta E)V = \chi V \tag{2.20}$$

$$W(\alpha E - \gamma D) = \chi W \tag{2.21}$$

Additionally, set

$$x_0 = -a\chi, \qquad x_1 = a\chi, \tag{2.22}$$

where $a \in \mathbb{R}$ will be chosen later on.

Recall that our aim is find a conditions for the probabilities $P(\tau_1,\dots,\tau_n)$ such that Equation (2.18) equals 0. If we can show that for the probabilities the following equations hold

$$\sum_{\sigma_1}(h_1)_{\sigma_1;\tau_1}P_n(\sigma_1,\tau_2,\dots,\tau_n)=x_{\tau_1}P_{n-1}(\tau_2,\dots,\tau_n), \tag{2.23}$$

$$\begin{aligned}\sum_{\sigma_i,\sigma_{i+1}}(h)_{\sigma_i,\sigma_{i+1};\tau_i,\tau_{i+1}}&P_n(\tau_1,\dots,\sigma_i,\sigma_{i+1},\dots,\tau_n)\\ &=-x_{\tau_i}P_{n-1}(\tau_1,\dots,\tau_{i-1},\tau_{i+1},\dots,\tau_n)\\ &\quad+x_{\tau_{i+1}}P_{n-1}(\tau_1,\dots,\tau_i,\tau_{i+2},\dots,\tau_n),\end{aligned} \tag{2.24}$$

$$\sum_{\sigma_n}(h_n)_{\sigma_n;\tau_n}P_n(\tau_1,\dots,\tau_{n-1},\sigma_n)=-x_{\tau_n}P_{n-1}(\tau_1,\dots,\tau_{n-1}), \tag{2.25}$$

then the $P_n$ which satisfy the recursions in (2.23) – (2.25) are automatically in a steady state, since upon substituting (2.23) – (2.25) into (2.18), the coefficients of $x_{\tau_i}$ cancel and the sums add up to zero, giving $\frac{d}{dt}P_n=0$. Let us examine Equations (2.23) – (2.25). We start with Equation (2.23) and set $\tau_1=0$.

$$\begin{aligned}-\alpha P_n(0,\tau_2,\dots,\tau_n)+\delta P_n(1,\tau_2,\dots,\tau_n)&=x_0P_{n-1}(\tau_2,\dots,\tau_n)\\ -\alpha\frac{f_n(0,\tau_2,\dots,\tau_n)}{Z_n}+\delta\frac{f_n(1,\tau_2,\dots,\tau_n)}{Z_n}&=x_0\frac{f_{n-1}(\tau_2,\dots,\tau_n)}{Z_{n-1}}\end{aligned} \tag{2.26}$$

where we replaced $P_n$ by the unnormalized weights $f_n$ and the partition function $Z_n$. Now we substitute $f_n$ by their expressions (2.6):

$$f_n(0,\tau_2,\dots,\tau_n)=W\left(E\prod_{i=2}^{n}(\tau_iD+(1-\tau_i)E)\right)V, \tag{2.27}$$

$$f_n(1,\tau_2,\dots,\tau_n)=W\left(D\prod_{i=2}^{n}(\tau_iD+(1-\tau_i)E)\right)V, \tag{2.28}$$

$$f_{n-1}(\tau_2,\dots,\tau_n)=W\left(\prod_{i=2}^{n}(\tau_iD+(1-\tau_i)E)\right)V. \tag{2.29}$$

Plugging (2.27) – (2.29) into (2.26) we obtain

$$-\alpha\frac{WE\mathcal{A}V}{Z_n}+\delta\frac{WD\mathcal{A}V}{Z_n}=x_0\frac{W\mathcal{A}V}{Z_{n-1}} \tag{2.30}$$

where we used the abbreviation

$$\mathcal{A}=\prod_{i=2}^{n}(\tau_iD+(1-\tau_i)E). \tag{2.31}$$

Rearranging of (2.30) yields

$$-\alpha WE\mathcal{A}V+\delta WD\mathcal{A}V=x_0\frac{Z_n}{Z_{n-1}}W\mathcal{A}V. \tag{2.32}$$

If we now set

$$a = \frac{Z_{n-1}}{Z_n} \tag{2.33}$$

in (2.22), we can rewrite (2.32) as

$$-\alpha WE\mathcal{A}V + \delta WD\mathcal{A}V = -\chi W\mathcal{A}V. \tag{2.34}$$

Analogously we consider (2.23) in the case of $\tau_1 = 1$:

$$\begin{aligned}
\alpha P_n(0,\tau_2,\ldots,\tau_n) - \delta P_n(1,\tau_2,\ldots,\tau_n) &= x_1 P_{n-1}(\tau_2,\ldots,\tau_n)\\
\alpha\frac{f_n(0,\tau_2,\ldots,\tau_n)}{Z_n} - \delta\frac{f_n(1,\tau_2,\ldots,\tau_n)}{Z_n} &= x_1\frac{f_{n-1}(\tau_2,\ldots,\tau_n)}{Z_{n-1}}\\
\alpha f_n(0,\tau_2,\ldots,\tau_n) - \delta f_n(1,\tau_2,\ldots,\tau_n) &= x_1\frac{Z_n}{Z_{n-1}} f_{n-1}(\tau_2,\ldots,\tau_n).
\end{aligned}$$

Using (2.33) together with (2.22) gives

$$\alpha f_n(0,\tau_2,\ldots,\tau_n) - \delta f_n(1,\tau_2,\ldots,\tau_n) = \chi f_{n-1}(\tau_2,\ldots,\tau_n),$$

which can be rewritten as

$$\alpha WE\mathcal{A}V - \delta WD\mathcal{A}V = \chi W\mathcal{A}V \tag{2.35}$$

with the help of the abbreviation introduced in (2.31).

We conclude that Equation (2.23) is satisfied if Equations (2.34) and (2.35) hold. But Equations (2.34) and (2.35) are an immediate consequence of Matrix Ansatz Equation (2.21) (we just need to multiply by $\mathcal{A}V$ from the left).

Similarly we can reformulate the recursions (2.24) and (2.25) and derive their validity out of the generalized Matrix Ansatz equations. With the following abbreviations

$$\mathcal{B} = \prod_{i=1}^{j-1}(\tau_i D + (1-\tau_i)E), \tag{2.36}$$

$$\mathcal{C} = \prod_{i=j+2}^{n}(\tau_i D + (1-\tau_i)E), \tag{2.37}$$

Equation (2.24) yields

$$-qW\mathcal{B}ED\mathcal{C}V + pW\mathcal{B}ED\mathcal{C}V = \chi W\mathcal{B}E\mathcal{C}V + \chi W\mathcal{B}D\mathcal{C}V, \tag{2.38}$$

which certainly holds if (2.19) holds. On the other hand, abbreviating

$$\mathcal{D} = \prod_{i=2}^{n}(\tau_i D + (1-\tau_i)E),$$

Equation (2.25) yields

$$\beta\mathcal{D}V - \delta\mathcal{D}EV = \chi V. \tag{2.39}$$

This equation is implied by (2.25). Thus, we see that by assuming (2.19) – (2.21), Equations (2.23) – (2.25) hold, from which follows that $\frac{d}{dt}P_n = 0$. Setting

$$\chi = 1,$$

in (2.23) – (2.25), we obtain the Matrix Ansatz equations.

To finish the proof, we still need to show that (2.13) and (2.18) are indeed the same. Consider Equation (2.13) for a fixed state $S_k = (\tau_1, \dots, \tau_n)$:

$$P_n(S_1) * Q_{1,k} + \ldots + P_n(S_{2^n}) * Q_{2^n,k} = 0.$$

Recall that $Q_{i,k}$ is the transition rate from state $S_i$ to state $S_k$ For a lot of states the transition rate into state $S_k$ is equal to 0. Only three types of states can evolve directly into $S_k$ and therefore have non-zero transition rates:

- states which only differ in $\tau_1$ (the next move could be a particle entering respectively leaving at the left-hand side boundary, depending on $\tau_1$)
- states which only differ in $\tau_n$ (a particle could leave or enter at the right-hand side boundary)
- some states that differ in two neighboring sites $\tau_i, \tau_{i+1}$ — this corresponds to the fact that a particle could hop left or right. (As an example: the state $(\tau_1, \dots, \tau_{i-1}, 1, 0, \tau_{i+2}, \dots, \tau_n)$ can evolve into $(\tau_1, \dots, \tau_{i-1}, 0, 1, \tau_{i+2}, \dots, \tau_n)$).

We claim that each of these three types of states is contained in the Formula (2.18). Consider the first part of the formula.

$$\begin{aligned} \frac{d}{dt} P_n(\tau_1, \dots, \tau_n) = & \underbrace{\sum_{\sigma_1=0,1} (h_1)_{\sigma_1;\tau_1} P_n(\sigma_1, \tau_2, \dots, \tau_n)} \\ & + \ldots \\ & + \ldots \end{aligned} \tag{2.40}$$

The $P_n(\sigma_1, \tau_2, \dots, \tau_n)$ are exactly the states that differ from $P_n(\tau_1, \dots, \tau_n)$ by the first site (this is denoted by $\sigma_1 \in \{0, 1\}$). Recalling the definition of the matrix $h_1$ (see (2.14)) we see that the entry $(h_1)_{\sigma_1;\tau_1}$ is the transition rate from state $(\sigma_1, \tau_2, \dots, \tau_n)$ into $(\tau_1, \tau_2, \dots, \tau_n)$. So the term in (2.40) gives all $P_n$ which can evolve into $P_n(\tau)$ by a particle entering/leaving at left-hand side multiplied by the corresponding transition rate.

In the same vain, the third term in (2.18),

$$\begin{aligned} \frac{d}{dt} P_n(\tau_1, \dots, \tau_n) = & \quad \ldots \\ & + \ldots \\ & + \underbrace{\sum_{\sigma_n=0,1} (h_n)_{\sigma_n;\tau_n} P_n(\tau_1, \dots, \tau_{n-1}, \sigma_n),} \end{aligned}$$

contains all states that can evolve into $(\tau_1, \dots, \tau_n)$ by entering/leaving of a particle at right-hand side multiplied by the corresponding transition rates. The third type of states with non-zero transition rates are the states which differ from the fixed state $S_k$ only by two neighboring sites $\tau_i, \tau_{i+1}$ (as mentioned before). They are "encoded" in the remaining term of (2.18).

$$\frac{d}{dt}P_n(\tau_1,\ldots,\tau_n) = \ldots + \underbrace{\sum_{i=1}^{n-1}\sum_{\sigma_i,\sigma_{i+1}} (h)_{\sigma_i,\sigma_{i+1};\tau_i,\tau_{i+1}} P_n(\tau_1,\ldots,\sigma_i,\sigma_{i+1},\ldots,\tau_n)} + \ldots$$

We only need to consider the pairs $(\tau_i,\tau_{i+1}) = (0,1),(1,0)$ since in configurations $(\tau_i,\tau_{i+1}) = (1,1)$ or $(0,0)$ a jump of particles is suppressed (since no site is empty, respectively, there is no particle that could jump). This is reflected by the fact that $h$ has only 0 entries in rows and columns which are indexed by $(0,0)$ or $(1,1)$ (see (2.16) for definition and recall the non-standard labeling of the matrix entries). The remaining entries $(h)_{\sigma_i,\sigma_{i+1};\tau_i,\tau_{i+1}}$ of the matrix $h$ contain the transition rates corresponding to the remaining states — these are the states which can possible evolve into $S_k$ (including the state $S_k$ itself). Hence, we have seen that (2.13) and (2.18) are equal, and the theorem is proven. □

The Matrix Ansatz Theorem as stated above was first used to explicitly calculate the steady state probability or the *n-point function* (as defined in Chapter 3) in ([**9**], Section 5.-9.). A thorough inspection of the proof leads to the following two results which will turn out to be useful later: in one of the subsequent chapters we will need a somewhat more general version of the Matrix Ansatz Theorem which follows immediately out of the proof of Theorem 2.1 if we do not set $\chi = 1$:

Corollary 2.2 ([**7**], Corollary 5.2). *Let $\chi \in \mathbb{R}$, and suppose that $D$ and $E$ are matrices, $V$ a column vector, and $W$ a row vector such that the following conditions hold:*

$$pDE - qED = \chi(D+E) \tag{2.41}$$

$$(\beta D - \delta E)V = \chi V \tag{2.42}$$

$$W(\alpha E - \gamma D) = \chi W \tag{2.43}$$

*Then*

$$f_n(\tau_1,\ldots,\tau_n) = W\left(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E)\right)V, \tag{2.44}$$

*where we assume that $W(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))V$ has the same sign for all possible states $\tau \in \{0,1\}^n$ and that $f_n$ exists (i.e. that the matrix product is well-defined).*

### 2.2. Representations for the Matrix Ansatz

So far, we have seen that if some $D, E, V, W$ satisfy the Matrix Ansatz (2.3) – (2.5), then the unnormalized stationary distribution of the ASEP can be derived as a product of these matrices by (2.6).

But what do $D, E, V, W$ look like? We will (as the name already proposes) realize $D, E, V, W$ as matrices. Most of the time these shall be infinite-dimensional matrices. We could also interpret them as, e.g., linear operators between infinite-dimensional vector spaces (which we will actually do in the end of Chapter 6) or as any other abstract objects for which the equations of the Matrix Ansatz hold. But most of the time we will realize them as matrices.

The question arises of which form and size these matrices shall be. We will show that finite-dimensional matrices suffice if the following holds:

$$(p-q)(\alpha+\delta)(\beta+\gamma) = (\alpha+\delta+\beta+\gamma)(\alpha\beta-\gamma\delta). \tag{2.45}$$

In particular, one can then realize $D, E, V, W$ as one-dimensional matrices. If (2.45) does not hold (which is the common case), infinite-dimensional matrices are needed. To be precise, we show that then $D, E, V, W$ can not be finite-dimensional, and we will present some infinite-dimensional matrix representations for different choices of parameters later on.

### 2.2.1. Case 1: D and E commute.

Let us assume that $D$ and $E$ commute. Consider the first equation of the Matrix Ansatz, Equation (2.3),

$$(p-q)ED = D+E. \tag{2.46}$$

Multiplying by $W$ from the left and $V$ from the right yields

$$(p-q)WEDV = W(D+E)V = WDV + WEV. \tag{2.47}$$

We use the remaining two equations of the Matrix Ansatz to express the terms $WEDV$, $WDV$ and $WEV$.

Consider Equations (2.4) and (2.5). If we multiply (2.4) by $W$ from the left and (2.5) by $V$ from the right, we obtain

$$\text{I: } \beta WDV - \delta WEV = WV \tag{2.48}$$

$$\text{II: } \alpha WEV - \gamma WDV = WV. \tag{2.49}$$

For convenience, we set $\widehat{D} = WDV$ and $\widehat{E} = WEV$. Equations (2.48) and (2.49) then read

$$\text{I: } \beta\widehat{D} - \delta\widehat{E} = WV \tag{2.50}$$

$$\text{II: } \alpha\widehat{E} - \gamma\widehat{D} = WV. \tag{2.51}$$

We now solve for $\widehat{D}$ and $\widehat{E}$. We multiply the equations by $\alpha$ and $\delta$, respectively,

$$\text{I: } \alpha\beta\widehat{D} - \alpha\delta\widehat{E} = \alpha WV$$

$$\text{II: } \alpha\delta\widehat{E} - \gamma\delta\widehat{D} = \delta WV,$$

and by summing them we obtain

$$(\alpha\beta-\gamma\delta)\widehat{D} = (\alpha+\delta)WV,$$

from which follows that

$$\widehat{D} = \frac{(\alpha+\delta)}{(\alpha\beta-\gamma\delta)}WV. \tag{2.52}$$

Substituting $\widehat{D}$ into (2.51), we obtain

$$\begin{aligned}\alpha\widehat{E} &= WV + \gamma\widehat{D}\\ \alpha\widehat{E} &= \left(1 + \frac{\gamma(\alpha+\delta)}{(\alpha\beta-\gamma\delta)}\right)WV\\ \alpha\widehat{E} &= \frac{(\alpha\beta-\gamma\delta)+\gamma(\alpha+\delta)}{(\alpha\beta-\gamma\delta)}WV\\ \widehat{E} &= \frac{(\alpha\beta-\gamma\delta)+\gamma(\alpha+\delta)}{\alpha(\alpha\beta-\gamma\delta)}WV. \end{aligned} \tag{2.53}$$

Similarly, we multiply (2.4) by $WD$ from the left and (2.5) by $DV$ from the right to find

$$\text{I: } \beta WDDV - \delta WDEV = WDV$$
$$\text{II: } \alpha WEDV - \gamma WDDV = WDV,$$

which, since $DE = ED$ by assumption, is the same as

$$\text{I: } \beta WD^2V - \delta WEDV = \widehat{D} \tag{2.54}$$
$$\text{II: } \alpha WEDV - \gamma WD^2V = \widehat{D}. \tag{2.55}$$

By the same method as above we solve the equations, this time for $WEDV$.

$$\text{I: } \gamma\beta WD^2V - \gamma\delta WEDV = \gamma\widehat{D} \tag{2.56}$$
$$\text{II: } \alpha\beta WEDV - \gamma\beta WD^2V = \beta\widehat{D}. \tag{2.57}$$

Summing (2.56) and (2.57) yields

$$(\alpha\beta-\gamma\delta)WEDV = (\gamma+\beta)\widehat{D}.$$

We therefore find

$$WEDV = \frac{(\gamma+\beta)}{(\alpha\beta-\gamma\delta)}\widehat{D} = \frac{(\gamma+\beta)(\alpha+\delta)}{(\alpha\beta-\gamma\delta)^2}WV, \tag{2.58}$$

where we used (2.52).

We now plug $\widehat{D} = WDV$ , $\widehat{E} = WEV$, and $WDEV$ (Equations (2.52), (2.53), and (2.58)) into (2.47) which yields

$$\begin{aligned}(p-q)\frac{(\gamma+\beta)(\alpha+\delta)}{(\alpha\beta-\gamma\delta)^2}WV &= \frac{(\alpha+\delta)}{(\alpha\beta-\gamma\delta)}WV + \frac{\gamma(\alpha+\delta)+(\alpha\beta-\gamma\delta)}{\alpha(\alpha\beta-\gamma\delta)}WV\\ (p-q)\frac{(\gamma+\beta)(\alpha+\delta)}{(\alpha\beta-\gamma\delta)^2} &= \frac{\alpha(\alpha+\delta)+\gamma(\alpha+\delta)+(\alpha\beta-\gamma\delta)}{\alpha(\alpha\beta-\gamma\delta)}\\ (p-q)\frac{(\gamma+\beta)(\alpha+\delta)}{(\alpha\beta-\gamma\delta)} &= (\alpha+\delta+\gamma+\beta)\\ (p-q)(\gamma+\beta)(\alpha+\delta) &= (\alpha\beta-\gamma\delta)(\alpha+\delta+\gamma+\beta),\end{aligned}$$

as claimed in the beginning. Hence, if we suppose that $D$ and $E$ commute, Equality (2.45) has to be satisfied. In this case, we can choose $D$ and $E$ to be one-dimensional, namely

$$D = \frac{(\alpha + \delta)}{(\alpha\beta - \gamma\delta)} \tag{2.59}$$

$$E = \frac{\gamma(\alpha + \delta) + (\alpha\beta - \gamma\delta)}{\alpha(\alpha\beta - \gamma\delta)}. \tag{2.60}$$

To see this, just choose $V = W = 1$ (it is actually sufficient to demand $VW = 1$). Then Equations (2.52) and (2.53) give precisely $D, E$ as defined above. Clearly $D, E, V, W$ then satisfy the Matrix Ansatz since we constructed this representation from the Matrix Ansatz equations.

So, if (2.45) holds, then we can give a one-dimensional representation. If (2.45) does not hold, then $D$ and $E$ can not be commutative — we will examine the consequences for $D, E, V, W$ in the next subsection.

**2.2.2. Case 2: D and E do not commute.** What happens if $D$ and $E$ do not commute? In this case infinite-dimensional matrices are needed. To prove the claim, we consider the special case TASEP where the parameters $\alpha, \beta$ are chosen freely, $p = 1$ and $q = 0 = \gamma = \delta$. If for this special case finite-dimensional matrices do not work, then they clearly will not work for the general case where all parameters can be chosen freely (since, in particular, we can set the parameters as in the TASEP). For the TASEP the equations of the Matrix Ansatz (2.3) – (2.5) simplify to

$$\begin{aligned} DE &= D + E \\ (\beta D)V &= V \\ W(\alpha E) &= W \end{aligned}$$

which can be rewritten as

$$DE = D + E \tag{2.61}$$

$$DV = \frac{1}{\beta} V \tag{2.62}$$

$$WE = \frac{1}{\alpha} W \tag{2.63}$$

Remark. Note that then the vector $V$ is a (right) eigenvector to the matrix $D$ with eigenvalue $\frac{1}{\beta}$ and $W$ a (left) eigenvector to the matrix $E$ with eigenvalue $\frac{1}{\alpha}$.

Let us first assume that $D, E$ do not commute. We can then show that $D, E$ can not be finite as follows. Suppose $D, E$ were finite. From (2.61) we find

$$DE = D + E \tag{2.64}$$

$$\begin{aligned} DE - D &= E \\ D(E - 1) &= E \end{aligned}$$

$$D = E(E - 1)^{-1}, \tag{2.65}$$

where in (2.65) we assumed that $(E - 1)^{-1}$ is invertible. That this assumption holds is seen as follows: there is no non-zero vector $v$ such that $Ev = v$, because otherwise

$$Dv = DEv \overset{(2.64)}{=} (D + E)v = Dv + Ev \overset{Ev=v}{=} Dv + v \tag{2.66}$$

which implies that $v = 0$. So, no non-zero vector $v$ exists such that $Ev = v$, or equivalently, for all $v \neq 0$, $(E-1)v \neq 0$ from which follows (by theory of linear algebra) that $(E-1)$ is invertible. So, (2.65) holds. But $D = E(E-1)^{-1}$ does commute with $E$: first note that $E(E-1) = (E-1)E$:

$$E(E-1) = E^2 - E = (E-1)E.$$

Then, it follows that $(E-1)^{-1}E = E(E-1)^{-1}$:

$$\begin{aligned} E(E-1) &= (E-1)E \\ (E-1)^{-1}E &= E(E-1)^{-1}. \end{aligned}$$

Finally $DE = ED$ can be derived:

$$DE = E(E-1)^{-1}E = EE(E-1)^{-1} = ED. \tag{2.67}$$

That $D, E$ commute is a contradiction to the assumptions made in the beginning, therefore $D, E$ can not be of finite dimension.

In the general case where all parameters $p, q$ and $\alpha, \beta, \gamma, \delta$ can be chosen freely, it clearly follows that if there are non-commuting matrices $D, E$ that satisfy the Matrix Ansatz, then they have to be infinite-dimensional.

The infinite-dimensional matrices $D, E$ which we are going to consider will always be triangular matrices. We therefore avoid problems such as having to consider the convergence of infinite sums when forming the matrix product or having to exchange infinite sums when using associativity of matrix multiplication. The matrices $V, W$ will be realized as (infinite-dimensional) vectors most of the time — $V$ as a column vector and $W$ as a row vector. In the following chapters we will present some representations and give combinatorial interpretations of them. Not all of the results presented in the following chapters rely on the Matrix Ansatz, but a great part of them does — although some of them could also be derived by bypassing the Matrix Ansatz. It turns out to be a useful tool since it offers a simple way to calculate the stationary distribution.

## 2.3. A Stronger Version of the Matrix Ansatz Theorem

Let us return to the proof of Theorem 2.1 one more time. A thorough inspection of the proof actually leads to a another, stronger version of the Matrix Ansatz Theorem which we will need in the last chapter of this text. In this version we do not limit ourselves to matrices and have weaker assumptions than in the original version.

THEOREM 2.3. *Suppose that $D, E, V, W$ are operators, and $\chi \in \mathbb{R}$. Furthermore, let $\mathfrak{C}$ be the set of words of length less or equal $n-2$ in the language $\{D, E\}$. If*

$$pDECV - qEDCV = \chi(D+E)CV, \tag{2.68}$$

$$(\beta D - \delta E)V = \chi V, \tag{2.69}$$

$$W(\alpha E - \gamma D) = \chi W, \tag{2.70}$$

*hold for all $C \in \mathfrak{C}$, then*

$$f_n(\tau_1, \ldots, \tau_n) = W\left(\prod_{i=1}^{n} (\tau_i D + (1-\tau_i)E)\right)V, \tag{2.71}$$

*where we assume that* $W(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))V$ *has the same sign for all possible states* $\tau \in \{0,1\}^n$ *and that* $f_n$ *exists.*

PROOF. Consider the proof of Theorem 2.1. First, note that the Matrix Ansatz equations are algebraic equations and that we have never actually made use of the assumption that $D, E, V, W$ are matrices in the proof (we only mentioned that we will realize them as matrices most of the time). Hence, there is no obstacle to express $D, E, V, W$ in terms of any other operators.

Furthermore, consider Equation (2.38) again. We concluded that this equation holds if (2.19) is satisfied. But it is actually sufficient to assume that

$$pDECV - qEDCV = \chi(D+E)CV$$

for

$$C = \prod_{i=j+1}^{n} \tau_i D + (1-\tau_i)E$$

for all $j$ with $1 \le j \le n$. Hence, we can modify the Matrix Ansatz Equation (2.3) as claimed by the Theorem. □

CHAPTER 3

# TASEP and Lattice Path Interpretation

In this chapter we shall see some interesting combinatorial properties of the TASEP, or, more precisely, the **TASEP with maximal flow regime** — that is, the ASEP where particles enter the system at the left-hand side and travel to the right until leaving the system at the right-hand side, all at the same rate. This is equal to setting the parameters $p = \alpha = \beta$ and $q = \gamma = \delta = 0$. Since we can always assume that $p = 1$, the only non-zero parameters are $p = \alpha = \beta = 1$. Basically, the entire section will be dedicated to the above mentioned restriction of parameters, and we will discover some connections of the TASEP with maximal flow regime and various lattice paths. In the end we shall see a generalization for the case where we allow particles to hop to the left as well (Section 3.6) and we roughly outline results on the TASEP with general parameters (Section 3.7). We start by a short motivation.

In physics literature the so-called $n$-point correlation function is of great interest (see, e.g., [**8**] or ([**26**], p. 4988)). To avoid confusion with our notation we will simply call it the $k$-point correlation function. This function is defined as

$$P(\tau_{i_1} = 1, \tau_{i_2} = 1, \ldots, \tau_{i_k} = 1) \tag{3.1}$$

with $1 \leq i_1 \leq \ldots \leq i_k \leq n$ fixed. This is the probability of finding the PASEP in one of the states $\tau = (\tau_1, \ldots, \tau_n)$ with the determined sites $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_k}$ being occupied (other sites can be occupied as well, but do not have to).

Due to the Matrix Ansatz (Theorem 2.1), the $k$-point correlation function in (3.1) is given by

$$\frac{W(D+E)^{i_1-1}D(D+E)^{i_2-i_1-1}D\ldots(D+E)^{i_k-i_{k-1}-1}D(D+E)^{n-i_k}V}{Z_n}. \tag{3.2}$$

One can see this by considering the products in (2.6): since the sites $i_1, \ldots, i_k$ are occupied, they contribute a $D$ to the product at the corresponding position. The remaining elements of the product can either be $D$'s or $E$'s, each combination representing another state considered in the $k$-point correlation function. Summing up all these combinations yields (3.2).

While physicists are interested in the quantity or the form of the $k$-point correlation function (to see how particles at different positions influence each other, in other words, how they correlate) we use it for another purpose: by using a lattice path interpretation, we show that in the case of TASEP with $\alpha = \beta = p = 1$ and $q = \gamma = \delta = 0$, (3.2) leads to a set of lattice paths and that hence states can be represented in terms of paths. Afterwards, this is used to calculate the steady state probability of finding the system in a fixed state $\tau$ (see, e.g., Proposition 3.13 and Theorem 3.16). Through these consideration we shall also deduce a formula for finding exactly $l$ particles in the system in the long-time limit distribution (Proposition 3.17).

The basis for our considerations is a representation $D_0, E_0, V_0, W_0$ for which the Matrix Ansatz (Theorem 2.1) holds. Consider the following matrices:

$$D = \begin{pmatrix} \frac{1}{\beta} & \kappa & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}, \qquad E = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 & 0 & 0 & \cdots \\ \kappa & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix},$$

and

$$W = (1, 0, 0, 0, \ldots), \qquad V = (1, 0, 0, 0, \ldots)^T,$$

where we set $\kappa^2 = \frac{\alpha+\beta-1}{\alpha\beta}$. The fact that $\kappa^2$ might be negative is of no importance because in the equations $\kappa$ will only enter through $\kappa^2$ (see [**9**], p. 1500). In the case where $\alpha = \beta = 1$, which is relevant for us, the matrices take the following forms, which shall be our definitions:

$$D_0 := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}, \qquad E_0 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}, \tag{3.3}$$

and

$$W_0 := (1, 0, 0, 0, \ldots), \qquad V_0 := (1, 0, 0, 0, \ldots)^T, \tag{3.4}$$

We first show that the Matrix Ansatz for these matrices holds:

LEMMA 3.1. *For the matrices $D_0, E_0, V_0, W_0$ as defined in* (3.3) *and* (3.4)*, the Matrix Ansatz Equations* (2.3) – (2.5) *with $p = \alpha = \beta = 1$ and $q = 0 = \delta = \gamma$ hold.*

PROOF. First, note that by setting $\alpha = \beta = 1$ and $q = 0 = \delta = \gamma$ in Equations (2.3) – (2.5) the Matrix Ansatz equations read as follows:

$$D_0 E_0 = D_0 + E_0 \tag{3.5}$$

$$D_0 V_0 = V_0 \tag{3.6}$$

$$W_0 E_0 = W_0 \tag{3.7}$$

We will show that these equations are satisfied. We first consider Equation (3.5). By (3.3) it is immediately seen that the right-hand side is equal to

$$D_0 + E_0 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3.8}$$

For the left-hand side we consider the definition of the matrices $D_0$ and $E_0$ again:

$$(D_0)_{i,j} = \begin{cases} 1, & j = i, i+1 \\ 0, & \text{otherwise} \end{cases} \tag{3.9}$$

and

$$(E_0)_{i,j} = \begin{cases} 1, & j = i-1, i \\ 0, & \text{otherwise} \end{cases} \tag{3.10}$$

If we fix some $i$, the sum $(D_0E_0)_{i,j} = \sum_{k\geq 1} D_{i,k}E_{k,j}$ can be reduced to the terms that might differ from 0:

$$(D_0E_0)_{i,j} = \sum_{k\geq 1} D_{i,k}E_{k,j} = (D_0)_{i,i}(E_0)_{i,j} + (D_0)_{i,i+1}(E_0)_{i+1,j} \tag{3.11}$$

So, for the entries $(i,j)$ of $(D_0E_0)_{i,j}$ we see that

$$(D_0E_0)_{i,j} = (D_0)_{i,i}(E_0)_{i,j} + (D_0)_{i,i+1}(E_0)_{i+1,j} = \begin{cases} 1, & j = i-1 \\ 2, & j = i \\ 1, & j = i+1 \\ 0, & \text{otherwise} \end{cases} \tag{3.12}$$

which are precisely the entries $(D_0E_0)_{i,j}$ in (3.8). Therefore, Equation (3.5) is shown.

Since

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{3.13}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \end{pmatrix} \tag{3.14}$$

we see that (3.6) and (3.7) hold, too. We therefore have ensured that the matrices $D_0, E_0, V_0, W_0$ satisfy the Matrix Ansatz equations. □

An advantage of the choice of parameters $p = \alpha = \beta = 1$ and $q = \delta = \gamma = 0$ is that the first equation of the Matrix Ansatz, Equation (2.3), simplifies to $DE = D + E$. Hence, recalling (2.7), we can rewrite the partition function $Z_n$ as

$$Z_n = W(D+E)^nV = W(DE)^nV, \tag{3.15}$$

as well as we can rewrite the $k$-point correlation function (3.2)

$$\frac{W(DE)^{i_1-1}D(DE)^{i_2-i_1-1}D\ldots(DE)^{i_k-i_{k-1}-1}D(DE)^{n-i_k}V}{Z_n}. \tag{3.16}$$

Both will be useful for the following interpretation.

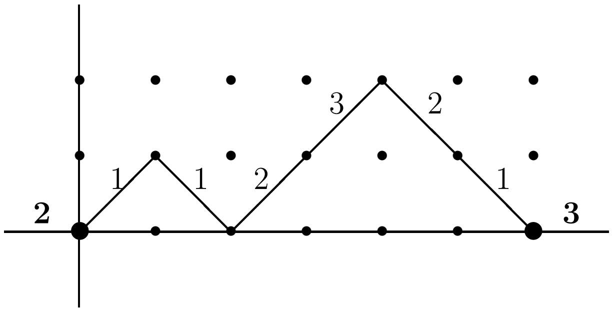



FIGURE 3.1. A lattice path $\omega$ of length 6 and of weight 72.

## 3.1. Lattice Paths and the Transfer Matrix Method

We now interpret the matrices introduced in the last section in terms of *transfer matrices*. In combinatorics, the *transfer matrix method* is a method used for counting walks in a directed graph and is essentially the same as the theory of finite Markov chains in probability theory. Here, we just present the ideas which will serve for our purpose [1]. This section mainly follows Section 2 in [**2**].

We consider paths with steps above the $x$-axis, more precisely, paths whose steps are between vertices which lie in $\Xi = \{(x, y)|x \in \mathbb{Z}, y \in \mathbb{Z}^+\}$, where $\mathbb{Z}$ is the set of integers and $\mathbb{Z}^+$ is the set of non-negative integers.

DEFINITION 3.2 ([**2**], Section 2, Definition 1). A **lattice path**, $\omega$, of length $n \geq 0$ is a sequence of vertices $(v_0, v_1, \ldots, v_n)$ where $v_i = (x_i, y_i) \in \Xi$ and, $v_i - v_{i-1} \in \mathbb{S}^\Omega$ where $\mathbb{S}^\Omega$ is the set of allowed steps, $i = 0, \ldots, n$. For a particular path, $\omega$, denote the corresponding sequence of steps by $\epsilon(\omega) = e_1 e_2 \cdots e_n$, $e_i = (v_{i-1}, v_i)$.

DEFINITION 3.3. Let $\omega$ be a lattice path of length $n$. We assign each step $e_i \in \epsilon(\omega)$ a weight $w_i$, the starting vertex $v_0$ the weight $w_0$ and the final vertex the weight $w_{n+1}$, where $w_i \in \mathbb{Z}^+$, $i = 0, \ldots, n+1$. Then, the **weight** $wt(\omega)$ **of a path** $\omega$ is defined as the product of these weights,

$$wt(\omega) = \prod_{i=0}^{n+1} w_i.$$

EXAMPLE. The lattice path $\omega$ shown in Figure 3.1 is of length 6. The weights are indicated above or below the steps. The bold numbers stand for the weights of the initial and final vertex, respectively. Hence, the weight of the path $\omega$ is given by

$$wt(\omega) = 2 \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdot 2 \cdot 1 \cdot 3 = 72.$$

As indicated before, we want to link matrices with lattice paths. The entries of these matrices shall represent the weight of steps of lattice paths. By matrix multiplication one can then calculate the weight of all paths between some starting and end vertices.

We label the rows of the matrix $D_0$ by odd integers $\mathbb{Z}_{odd} = \{1, 3, 5, \ldots\}$ and the columns by even integers starting with 0, $\mathbb{Z}_{even} = \{0, 2, 4, \ldots\}$. Then the entry $(D_0)_{k,l}$ gives the weight of a step from an odd height $k$ to an even height $l$. Analogously (but

[1]For more details on the method (and, eventually, further references) see, e.g., ([**22**], p. 241 – 262).

$$D_0 = \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 2 & 4 & 6 \end{array} \\ \left( \begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \end{array} \right) \end{array} \qquad E_0 = \begin{array}{c} \\ 0 \\ 2 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 5 & 7 \end{array} \\ \left( \begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \end{array} \right) \end{array}$$

FIGURE 3.2. Interpretation of $D_0$ and $E_0$ as lattice path matrices. The entries are the weights of steps of the heights indicated by the arrows.

the other way round) we label the rows of the matrix $E_0$ by even and the columns by odd integers. Therefore, $(E_0)_{l,m}$ represents the weight of a step from even height $l$ to odd height $m$ – see Figure 3.2.

Let $d_{[i,k]}$ be the entry of $D_0$ which is located in the row labeled by $i$ and column labeled by $j$. We will call $d_{[i,k]}$ the $[i, j]$ entry of $D_0$. Analogously we define $e_{[k,j]}$, the $[k, j]$ entry of $E_0$. (We write square brackets to remind ourselves that here it is not the standard integer labeling of rows and columns which is used). If we now form the product $D_0E_0 = C_0$, then all entries of this matrix are of the form

$$\sum_{k\in\mathbb{Z}_{even}} d_{[i,k]}e_{[k,j]}, \tag{3.17}$$

for some $i, j$. Actually, if we use the row labeling of $D_0$ and the column labeling of $E_0$, then (3.17) is the $[i, j]$ entry of the matrix $C_0$. So, how can we interpret this entry? Fix $k \in \mathbb{Z}_{even}$ and $i, j \in \mathbb{Z}_{odd}$. Then $d_{[i,k]}e_{[k,j]}$ is the product of the weights of a step from height $i$ to height $k$ and a step from height $k$ to height $j$. Hence, (3.17) sums up the weights of all paths from odd height $i$ to even height $k$ to odd height $j$ (where the weight of a path is defined as the product of the weights of its steps).

Multiplying $D_0E_0$ again by $D_0$ then leads to the weight of all paths starting at odd height $i$ to even height $k$ to odd height $j$ to even height $l$ in the $[i, l]$ entry of the resulting matrix (where, again, we label the column of the resulting matrix $D_0E_0D_0$ like the first matrix and the rows like the last matrix of the product).

This can be generalized for more than two steps, so the $[i, j]$ entry of any product $(D_0E_0)^n$ can be interpreted as the sum of the weights of all paths going from $i$ to $j$ with exactly $2n$ steps of odd height. Since we only use non-negative integers to label the rows and columns of the matrices, all steps are taken in the upper half of $\mathbb{Z}^2$. Similarly we label the entries of $W_0, V_0$ by odd integers; they will represent weights attached to the initial vertices ($W_0$) and the final vertices ($V_0$). Now consider the expression

$$Z_n = W_0(D_0E_0)^nV_0. \tag{3.18}$$

After having started a path at any odd height $k$, with weight $W_{[1,k]}$ attached to the beginning vertex, the matrices $D_0$ and $E_0$ contribute successively weights for each further step (odd to even heights and even to odd heights) ending after $2n$ steps at an odd height $m$ with weight $V_{[m,1]}$ attached to the final vertex. Therefore, we see that

$Z_n = W_0(D_0E_0)^nV_0$ is giving the weighted sum over all paths of length 2n that begin and end at odd height and stay above the $x$-axis all the time.

EXAMPLE. We illustrate the lattice path interpretation of matrices (or interpretation as transfer matrices) by an example. First, we shall consider lattice paths of length 2 without additional weights at the initial and final vertices. Set

$$D = \begin{array}{c c} & \begin{array}{cc} 0 & 2 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{pmatrix} \delta_1 & \delta_2 \\ 0 & \delta_3 \end{pmatrix} \end{array}, \qquad E = \begin{array}{c c} & \begin{array}{cc} 1 & 3 \end{array} \\ \begin{array}{c} 0 \\ 2 \end{array} & \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ 0 & \epsilon_3 \end{pmatrix} \end{array},$$

where $\delta_i, \epsilon_i > 0$ for $i = 1, 2, 3$. We now interpret these matrices as transfer matrices. Consider $D$ first: the transfer matrix $D$ admits steps from height 1 to height 0 (weighted by $\delta_1$), steps from height 1 to height 2 (weighted by $\delta_2$) and steps from height 3 to height 2 (weighted by $\delta_3$). Since the weight of a step from height 3 to height 0 is 0, we omit this step (paths containing steps of weight 0 have total weight 0 and are not of interest for us). The steps are shown in Figure 3.3(a). Analogously we interpret $E$ and obtain steps from height 0 to height 1 or from height 2 either to height 3 or height 1. The steps and their weights are shown in Figure 3.3(b). Now, We form a sequence of two steps by multiplication of $D$ and $E$. We find $DE$ to be equal to

$$\begin{array}{c c} & \begin{array}{cc} 1 & 3 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{pmatrix} \delta_1\epsilon_1 + \delta_2\epsilon_2 & \delta_2\epsilon_3 \\ \delta_3\epsilon_2 & \delta_3\epsilon_3 \end{pmatrix} \end{array}, \tag{3.19}$$

where we labeled the rows of the matrix in (3.19) like the rows of $D$ and the columns like the columns of $E$. The entries in (3.19) give the sum of the weights of the paths between to given vertices: the sum of weights of paths that start and end at height 1 is equal to $\delta_1\epsilon_1 + \delta_2\epsilon_2$, while the sum of weights of paths starting at height 1 and ending at height 3 is equal to $\delta_2\epsilon_3$. This is in correspondence with the paths formed by the solid and dashed steps in Figure 3.3(c).

Now, we can also attach weights to the initial and final vertices. We assign to the vertex $(0, 1)$ the weight 1 and the weight 0 to the vertex $(0, 3)$. The final vertices are weighted by 1. This is done by the following:

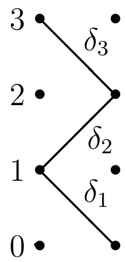
$$W = \begin{array}{c} \begin{array}{cc} 1 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}, \qquad V = \begin{array}{c} \begin{array}{cc} 1 & 3 \end{array} \\ \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}^T. \tag{3.20}$$

Therefore, paths starting at height 3 have a total weight of 0 and can be omitted. Only three different paths with non-zero weights remain. The sum of their weights is given by

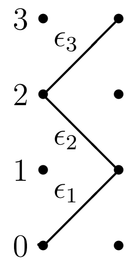
$$WDEV = \delta_1\epsilon_1 + \delta_2\epsilon_2 + \delta_2\epsilon_3. \tag{3.21}$$

Note that by setting $\delta_i = \epsilon_i = 1$ $(i = 1, 2, 3)$, $WDEV$ gives exactly the number of lattice paths that start at $(1, 0)$ and have length 2. This already suggests that by choosing the weights of the steps and vertices in a certain manner one can count certain types of lattice paths.

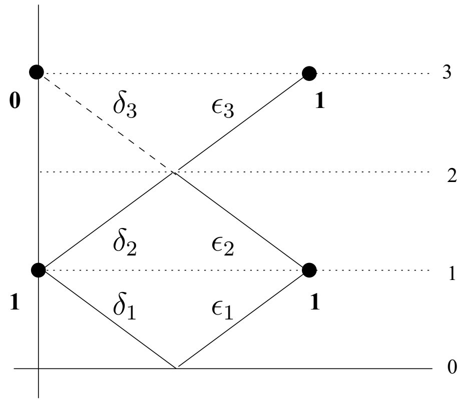
After this example we return to the matrices $D_0, E_0, V_0, W_0$ which were defined in (3.3) and (3.4) and examine the consequences of the lattice path interpretation.




(a) Steps permitted by $D$




(b) Steps permitted by $E$




(c) Lattice paths whose weights are summed by $WDEV$. The bold numbers indicate the weights of the initial and final vertices. The step starting in the vertex with weight 0 is dashed to indicate that paths containing this step do not contribute to the sum of weights of all paths.

FIGURE 3.3. An example for the lattice path interpretation by means of the transfer matrix method.

## 3.2. One Up Paths, Dyck Paths and the Partition Function of the TASEP

DEFINITION 3.4. The set of **One up paths** of length $2n$, $O^{2n}$, is the set of lattice paths which have step set $\mathbb{S}^O = \{(1,1),(1,-1)\}$, start at $v_0 = (0,1)$ and end at $v_{2n} = (2n,1)$.

An example of a One up path of length 10 is shown in Figure 3.4.

REMARK. Note that all One up paths have to be of even length, since they start and end at height 1 and their steps are of height 1.

LEMMA 3.5. *The partition function $Z_n$ for the matrix representation $D_0, W_0, V_0, W_0$ can be written as*

$$Z_n = \sum_{\omega \in O^{2n}} 1, \tag{3.22}$$

*where the sum ranges over all One up paths of length $2n$.*

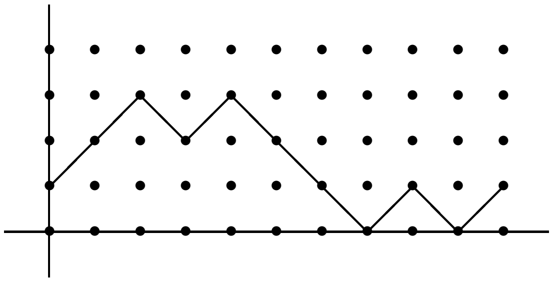


Figure 3.4. A One up path of length 10.

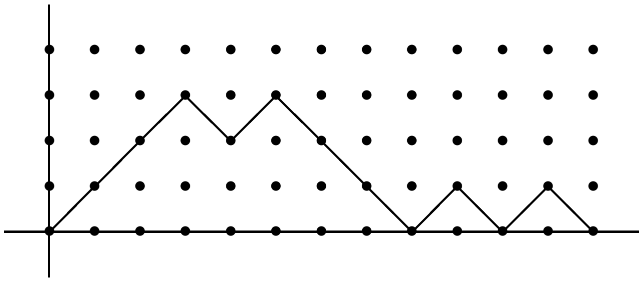


Figure 3.5. A Dyck path of length 12.

Proof. This is an immediate consequence of the interpretation of the matrices as transfer matrices. Note that due to odd labeling of the entries of $W_0 = (1, 0, 0, 0, \ldots)$, only paths starting at $v_0 = (0, 1)$ have a non-zero weight attached to their starting point, as well as only paths ending at $v_{2n} = (2n, 1)$ have a non-zero weight attached to their end point since $V_0 = (1, 0, 0, 0, \ldots)^T$. Consider the values of the entries of $D_0$ and $E_0$ together with their labeling:

$$D_0 = \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 7 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 2 & 4 & 6 & \cdots \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \qquad E_0 = \begin{array}{c} \\ 0 \\ 2 \\ 4 \\ 6 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 3 & 5 & 7 & \cdots \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \tag{3.23}$$

We see that $D_0$ and $E_0$ only assign non-zero weights to steps that are either up steps of height 1 or down steps of height 1. Also note that all weights of steps and vertices are equal to 1, therefore each path has weight 1. Putting this together we see that exactly One up paths are the paths whose weights are non-zero if we interpret the product for $Z_n$ in (3.18) in terms of transfer matrices. □

Definition 3.6. The set of **Dyck paths** of length $2n$, $P_{2n}^D$, is the set of lattice paths which have step set $\mathbb{S}^D = \{(1, 1), (1, -1)\}$, start at $v_0 = (0, 0)$ and end at $v_{2n} = (2n, 0)$.

An example of a Dyck path of length 12 is shown in Figure 3.5.

Proposition 3.7. *The number of Dyck paths of length $2n$ is given by the Catalan numbers*

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{3.24}$$

Proof. This is a well known fact. See, e.g., ([**23**], Corollary 6.2.3 (v)). □

As an immediate consequence of Lemma 3.5 we obtain for the TASEP with $\alpha = \beta = 1$ the following Corollary:

Corollary 3.8. *For the TASEP with $n$ sites, $p = 1$ and the boundary conditions $\alpha = \beta = 1$ the partition function $Z_n$ is given by*

$$Z_n = C_{n+1}, \tag{3.25}$$

*where $C_{n+1}$ is a Catalan number.*

Proof. This can be seen as follows: $Z_n$ can be expressed as the number of One up paths of length $2n$ (Lemma 3.5). One up paths are in bijection with Dyck paths. The bijection is not too hard to guess once we have noted that they only differ in the height of their starting and final vertices. First, add an up step to the starting point (this up step starts at height 0 and ends at height 1) and a down step to the final vertex. Then, move this path by one unit to the right (to ensure that this new path starts at $(0,0)$). This procedure turns every One up path of length $2n$ into a Dyck path of length $2n+2$. As an illustration compare Figure 3.4 and Figure 3.5. On the other hand, deleting the first and the last step of a Dyck path yields a One up path (after having moved the path to the left by one unit). We already know that the number of Dyck paths of length $2(n+1)$ is equal to $C_{n+1}$ (Corollary 3.7). Hence

$$\begin{aligned} Z_n &= \#\text{ number of One up paths of length } 2n \\ &= \#\text{ number of Dyck paths of length } 2(n+1) \\ &= C_{n+1}, \end{aligned}$$

and the corollary is proven. □

## 3.3. States of the TASEP in Terms of Lattice Paths

Let us now return to the $k$-point correlation function from (3.16). We will consider the numerator, the unnormalized $k$-point function

$$G(\tau_{i_1}, \ldots, \tau_{i_k}; n) := W(DE)^{i_1-1}D(DE)^{i_2-i_1-1}D\ldots(DE)^{i_k-i_{k-1}-1}D(DE)^{n-i_k}V. \tag{3.26}$$

We recall that we have obtained the function $G(\tau_{i_1}, \ldots, \tau_{i_k}; n)$ by exchanging the term $(DE)$ by $D$ in the corresponding site of $Z_n = W(DE)^n V$ (see the paragraph after (3.2)). So, if we want in the correlation function $G$ a particle at site $i_l$, we delete the $i_l$-th $(DE)$ in $Z_n$ and replace it by $D$, or, equivalently, we replace the $i_l$-th $E$ by the identity matrix $I$! Analogously, if we want the site $i_l$ to be empty, we replace in the partition function $Z_n$ the $i_l$-th $D$ by $I$. Since this is the crucial point in what is to follow we give a short example to illustrate it.

Example. We look for the probability $P(\tau_1 = 1, \tau_3 = 0)$ of a TASEP with 3 sites. The partition function is given by $Z_n = W(DE)^3V = W(DE)(DE)(DE)V$ where $D, E, W, V$ satisfy the Matrix Ansatz equations (e.g. $D = D_0$, $E = E_0$, $V = V_0$, $W = W_0$). The probability of finding the TASEP in state $(\bullet, \tau_2, \circ)$, where $\tau_2 \in \{\circ, \bullet\}$ is then given by

$$\frac{W(DI)(DE)(IE)V}{Z_n} = \frac{W(D)(DE)(E)V}{Z_n}.$$

Now, speaking in terms of One up paths (in particular referring to the representation $D_0, E_0, V_0, W_0$) recall that $Z_n$ is equal to the number of all One up paths of length $2n$ (see Lemma 3.5). We claim that replacing the $i_l$-th $E_0$ by the unit matrix $I$ in $Z_n$ leads to a One up path with the step $s_{2i_l}$ (starting at $x = 2i_l - 1$ and ending at $x = 2i_l$) being an up step. To see this, first note that the $i_l$-th $E_0$ is at the $(2i_l - 1)$-th position in the product $Z_n$ (not counting the vector $W_0$ that assigns weights to the starting vertices but does not represent a step) and then recall the labeling of the matrix $E_0$, the matrix that used to be at the $(2i_l - 1)$-th position:

$$E_0 = \begin{array}{c c} & \begin{array}{cccc} 1 & 3 & 5 & 7 \end{array} \\ \begin{array}{c} 0 \\ 2 \\ 4 \\ 6 \end{array} & \begin{pmatrix} \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \\ & & & \\ & & & \\ & & & \end{pmatrix} \end{array} \qquad I_0 = \begin{array}{c c} & \begin{array}{cccc} 1 & 3 & 5 & 7 \end{array} \\ \begin{array}{c} 0 \\ 2 \\ 4 \\ 6 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

The identity matrix has only non-zero entries in the diagonal, so this corresponds to an up step (with weight 1) in the lattice path interpretation — that is, the paths that have a *forced* up step, as we will call it, at the corresponding position do not have weight 0 and therefore are of importance for our considerations. So, a particle at site $i_l$ forces the step $s_{2i_l}$ in the corresponding One up paths to be an up step.

On the other hand, an empty site $i_l$ corresponds to replacing the $i_l$-th $D_0$ in $Z_n$ by the identity matrix. Recalling the labeling of the matrix at this position (see Figure 3.2) produces a *forced* down step $s_{2i_l-1}$ (starting at $x = 2i_l - 2$ and ending at $x = 2i_l - 1$) in the corresponding One up paths.

We summarize this. To represent certain states of the TASEP in terms of One up paths we need to put some restrictions upon certain steps: If site $i_l$ is supposed to be occupied, than the step $s_{2i_l}$ (ending at $x = 2i_l$) has to be an up step, if site $i_l$ is supposed to be empty, than the step $s_{2i_l-1}$ (ending at $x = 2i_l - 1$) has to be a down step. Consider a state $\tau = (\tau_1, \dots, \tau_n) \in \{0,1\}^n$ where entries $\tau_{l_1}, \dots, \tau_{l_v}$ are equal to 1 and $\tau_{j_1}, \dots, \tau_{j_w}$ are equal to 0. Clearly $v + w \leq n$. The remaining sites are not fixed. We define

$$O(\tau_{l_1}, \dots, \tau_{l_v}; \tau_{j_1}, \dots, \tau_{j_w}; 2n)$$

to be the set of One up paths of length $2n$, $o = (o_1, \dots, o_{2n})$, for which the following restrictions hold:

$$\begin{array}{ll} \text{steps } o_{2l_1}, \dots, o_{2l_v} & \text{are down steps,} \\ \text{steps } o_{2j_1-1}, \dots, o_{2j_w-1} & \text{are up steps.} \end{array}$$

By $O(2n)$ we denote the set of all One up paths of length $2n$ (without any restrictions).

LEMMA 3.9. *For the TASEP ($q = \gamma = \delta = 0$) with $n$ sites, $p = 1$, and the boundary conditions $\alpha = \beta = 1$, the probability of finding the system in a state $\tau$ with $v$ particles at sites $l_1, \dots, l_v$ and $w$ empty sites $j_1, \dots, j_w$ (with $v + w \leq n$) is given by*

$$P(\tau_{l_1} = \ldots = \tau_{l_v} = 1, \tau_{j_1} = \ldots = \tau_{j_w} = 0) = \frac{|O(\tau_{l_1}, \dots, \tau_{l_v}; \tau_{j_1}, \dots, \tau_{j_w}; 2n)|}{|O(2n)|}. \tag{3.27}$$

PROOF. This lemma follows from the lattice path interpretation and only summarizes the considerations made above. □

So we have seen how to "manipulate" One up paths (in terms of putting restrictions upon certain steps) to represent certain states of the TASEP. We will use the knowledge acquired so far to show to find simple formulas for calculating the steady state probabilities for certain states. Afterwards we formulate the Lemma 3.9 and its consequences in a more catchy way.

## 3.4. Formulas for the Stationary Distribution Through Pairs of Non-Intersecting Paths

We introduce a new type of paths, this time allowing steps to lie below the $x$-axis.

DEFINITION 3.10 ([**2**], Section 7, Definition 1). A **binomial path** is defined to be a path $\omega = (v_0, \dots, v_n)$ whose vertices lie in the plane, $v_i = (x_i, y_i) \in \mathbb{Z} \times \mathbb{Z}, i = 0, \dots, n$ and whose step set $\mathbb{S}^b$ is defined as $\mathbb{S}^b = \{(1, -1), (1, 1)\}$ (that is, the vertices are such that $v_i - v_{i-1} \in \mathbb{S}^b$).

Furthermore, we need the following two definitions as they will help us to express the steady state probabilities in a compact form.

DEFINITION 3.11 ([**2**], Section 7, Definition 3). Let $\omega_1$ and $\omega_2$ be two binomial paths both of length $n$. Let $\omega_1$ start at $(0, 0)$ and let $\omega_2$ start at $(0, -2)$. If the two paths have no vertices in common, then we call them **non-intersecting**.

For an example of a pair of non-intersecting paths see Figure 3.8(d).

DEFINITION 3.12 ([**2**], Section 7, Definition 2). Let $\tau = (\tau_1, \dots, \tau_n) \in \{\circ, \bullet\}^n$ a state of the TASEP with $n$ sites. A **state path** $\omega = (v_0, \dots, v_n)$ of type $\tau$ is a binomial path starting at $(0, 0)$ where

$$v_i - v_{i-1} = \begin{cases} (1, 1) & \text{if } \tau_i = \bullet \\ (1, -1) & \text{if } \tau_i = \circ. \end{cases} \tag{3.28}$$

So, a particle in the state $\tau$ is encoded by an up step in the state path, where an empty site contributes a down step.

EXAMPLE. Consider the TASEP with maximal flow regime (that is, particles only travel to the right, $p = \alpha = \beta = 1$) and 5 sites. Consider the state $(\circ, \bullet, \bullet, \circ, \circ)$. Then, the corresponding state path is shown in Figure 3.6.

REMARK. Note that until now there is no connection between a One up path corresponding to a state $\tau$ and the state path defined above. For the TASEP with $n$ sites, there usually is a number greater than 1 of One up paths linked to a state $\tau$, since only some steps, the forced steps, are determined by occupied or empty sites while there is a one-to-one correspondence between states and state paths of length $n$.

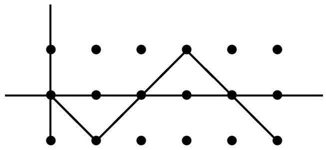


FIGURE 3.6. State path corresponding to the state $(\circ, \bullet, \bullet, \circ, \circ)$.

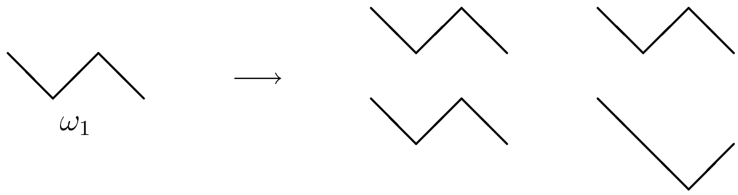



FIGURE 3.7. The state path for the state $(\circ, \bullet, \circ)$ and the two pairs of non-intersecting binomial paths.

PROPOSITION 3.13 ([**2**], Chapter 7, Proposition 5). *For the TASEP ($q = 0 = \gamma = \delta$) with $n$ sites and the boundary conditions $\alpha = \beta = 1$ the probability of finding the system in state $\tau$ with precisely $l$ particles is given by*

$$P_n(\tau) = \frac{|\{\omega_1, \omega_2\}|}{C_{n+1}} \tag{3.29}$$

*where $\{\omega_1, \omega_2\}$ is the set of all pairs of non-intersecting paths where $\omega_2$ is any binomial path that starts at $(0, -2)$ and ends at $(n, 2l - n - 2)$, and $\omega_1$ is the state path corresponding to the state $\tau$. $C_{n+1}$ is the $(n + 1)$-th Catalan number.*

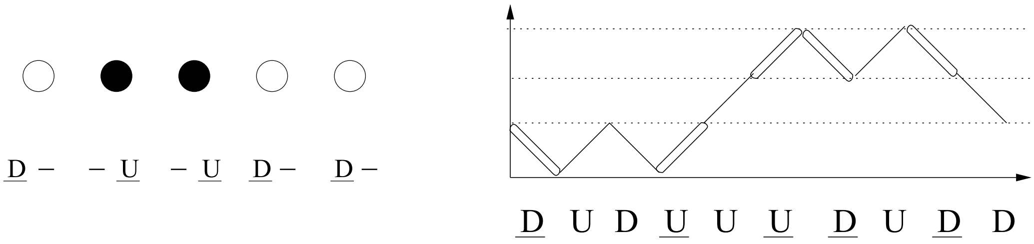
Before giving the proof we illustrate the proposition on an example:

EXAMPLE. Consider the TASEP with 3 sites and the state $(\circ, \bullet, \circ)$, that is, $\tau = (0, 1, 0)$. The corresponding state path $\omega_1$ is shown on the left-hand side in Figure 3.7. We need to count the number of pairs of non-intersecting paths $\omega_1, \omega_2$. The state $\tau$ contains 1 particle, hence $l = 1$. So, $\omega_2$ has to start at $(0, -2)$ and end at $(3, 2{\cdot}1-3-2) = (3, -3)$. There are two pairs of paths that satisfy these conditions; they are shown in Figure 3.7. Noting that $C_4 = 14$ we find

$$P_n((\circ, \bullet, \circ)) = P_n((0, 1, 0)) = \frac{2}{14} = \frac{1}{7}.$$

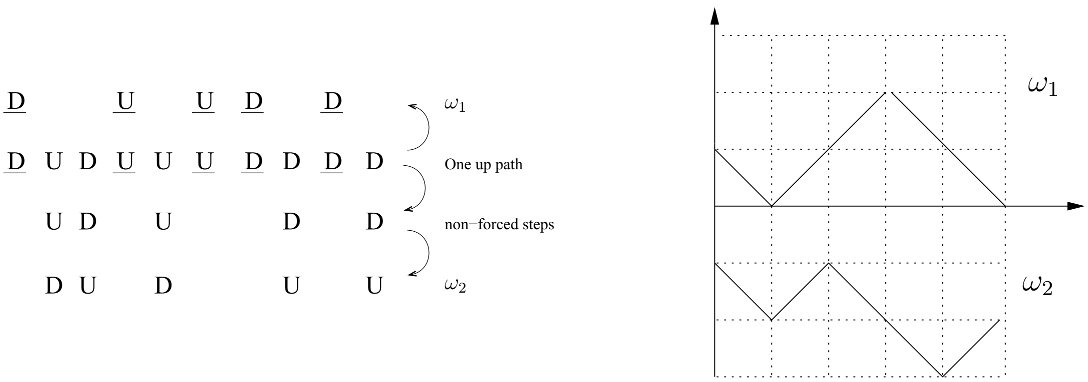
PROOF OF PROPOSITION 3.13. Let $\tau$ be a state of the TASEP with $n$ sites and the parameters as above. As seen in Lemma 3.9, representing certain states of the TASEP in terms of One up paths is equivalent to putting some restrictions upon certain steps: if site $i_l$ is occupied, then step $s_{2i_l}$ (ending at $x = 2i_l$) has to be an up step, if site $i_l$ is empty, then step $s_{2i_l-1}$ (ending at $x = 2i_l - 1$) has to be a down step. We introduce the following notation to encode this: we denote a One up path of length $2n$ as a word in $\{D, U\}^{2n}$:

$$\underbrace{- - - - - - - \cdots - - - - - - -}_{2n} \qquad - \in \{D, U\} \tag{3.30}$$




(a) A state and the forced steps in a corresponding One up path

(b) A possible choice for a One up path corresponding to the state (forced steps are emphasized)




(c) The One up path in the second line is separated into a pair of non-intersecting paths $\omega_1, \omega_2$

(d) The pair of non-intersecting binomial paths corresponding to the One up path from Figure (b)

FIGURE 3.8. An example for how to obtain a pair of non-intersecting binomial paths $\omega_1, \omega_2$ corresponding to a state of the ASEP.

Every dash, $-$, is either replaced by a $D$ (down steps) or a $U$ (up steps). Of course, these replacements have to be consistent with the One up path constraints, see Definition 3.4. So, due to the above mentioned "forced steps", we substitute "$-U$" at positions $2i_l - 1$ and $2i_l$ if site $i_l$ is occupied. This reflects the fact that step $2i_l - 1$ can be chosen freely while the step $2i_l$ has to be an up step. On the other hand, an empty site is represented by the pair "$D-$" at corresponding position:

$$\bullet \leftrightarrow -U \qquad \text{and} \qquad \circ \leftrightarrow D- \tag{3.31}$$

Having "translated" the state $\tau$ into a sequence of forced $D, U$'s and dashes, $n$ of the dashes remain to be filled with $U$'s and $D$'s to obtain all possible One up paths corresponding to the state $\tau$. Now we claim that from any of these sequences, we can construct a pair of non-intersecting paths $(\omega_1, \omega_2)$ and vice versa. The process is shown in Figure 3.8 and is described in the following.

In a One up path $o$ we separate the forced steps from the non-forced ones. The non-forced steps are inverted (each down step $D$ becomes an up step $U$ and vice versa) and form a path $\omega_2$ while the forced steps form the path $\omega_1$. This path $\omega_1$ is the state path from Definition 3.12. (This is seen easily as every particle in $\tau$ contributes a forced

$U$ in the One up path and at forming the path $\omega_1$ remains an up step. So, every particle causes an up step in the path $\omega_1$ and, conversely, every empty site causes a down step in path $\omega_1$. This is exactly the definition of the state path). Let us examine the paths $\omega_1, \omega_2$:

Clearly, the length of each path is $n$. By assumption, the state path $\omega_1$ starts at $(0,0)$ while we let the path $\omega_2$ start at $(0,-2)$. The ending height of state path $\omega_1$ results as the difference between the number of particles in $\tau$ (which produce an up step in the state path) and empty sites (which produce a down step in the state path), so $y = l - (n-l) = 2l - n$. What about the ending height of the path $\omega_2$? In the One up path $o$ every $U$ of $\omega_1$ used to be a $U$ as well. So, in the One up path there had to be a down step $D$ within the non-forced steps to balance the number of up and down steps (a One up path starts and ends at same height). Since the same argument holds for a step $D$ in $\omega_1$ we see that the number of $U$'s and $D$'s in $\omega_2$ is the same as the number of $U$'s and $D$'s in $\omega_1$. Hence, the ending height of $\omega_2$ is $y = 2l - n - 2$ (subtracting 2 is due to the fact that $\omega_2$ starts at height $-2$). It remains to show that these two paths $\omega_1, \omega_2$ are indeed non-intersecting.

To see this, we use a process which separates a One up path step by step into $\omega_1$ and $\omega_2$. For better understanding keep Figure 3.9 in mind. We draw the One up path $o$ together with the $x$-axis. This axis builds the bottom, or lower ground, of the path; it might be "touched" (meaning that there might lie some vertices on the axis) but there are no steps below the $x$-axis. We start reading the One up path from the left, distinguishing between forced up and forced down steps (denoted by $\mathbf{U}$ and $\mathbf{D}$) and non-forced up and non-forced down steps (denoted by $U$ and $D$). So, when reaching a non-forced up step $U$ we delete this step from the path. We close the gap by "gluing" the following up and down steps directly to the last vertex and call this new path $o_1$. After leaving the first step of the $x$-axis unchanged, we lower the rest of the $x$-axis by one level (see Figure 3.9). So, our bottom line (once the $x$-axis) now consists of two lines and all steps of the new path $o_1$ are above them. We note that a non-forced up step has caused a down shifting of the bottom line.

We turn again to the path $o_1$. We continue "reading" the path from the left-hand side. Let us assume that the next non-forced step we find in $o_1$ is a down step $D$. We proceed as follows: we delete the down step and again fill the gap by gluing the remaining tail to the last vertex. Let us denote this new path by $o_2$. We now lift the bottom line after a step: so the former $x$-axis now consists of one vertical step of length 1 at height 0, one vertical step of length 1 at height $-1$ and another vertical line at height 0 (see Figure 3.9). Again for the path $o_2$ this forms a bottom line which is never crossed. This time we note that a non-forced down step $D$ has caused the bottom line to rise by one unit.

Proceeding in the same way, we obtain a path $o_n$ which happens to be the state path $\omega_1$ ($o_n$ consists only of the forced steps, and they form the state path, as noted in the beginning) and a lower path which is staircase shaped, and which we denote by $s$. Due to the construction during the process, the path $\omega_1$ might touch this staircase path $s$, but never actually crosses it. Turning this staircase path into a binomial path (by replacing every corner $\urcorner$ by a down step $\diagdown$ and every corner $\lrcorner$ by an up step $\diagup$), we obtain a binomial path that lies below $\omega_1$ but might touch it at some point. By lowering this path by one unit we obtain a binomial path which we now call $\omega_2$. Note

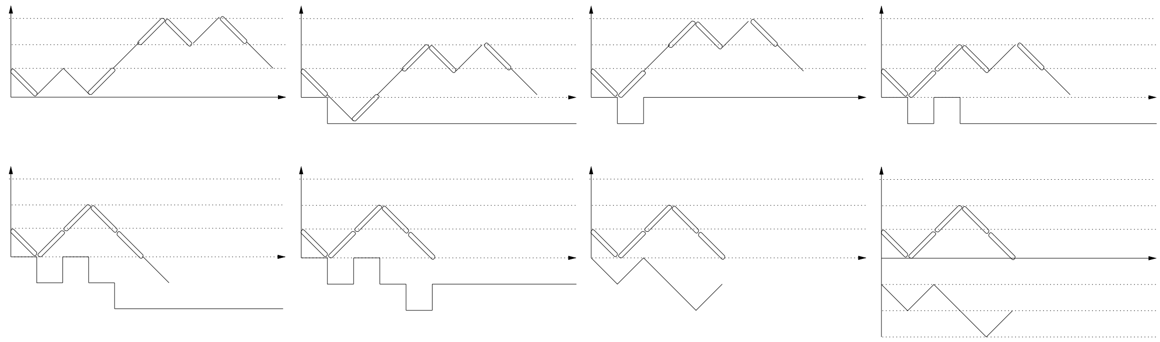


FIGURE 3.9. Turning a One up path into a pair of non-intersecting binomial paths.

that this path has no vertex in common with $\omega_1$. Hence the process has yielded two binomial paths $\omega_1, \omega_2$ that are non-intersecting (in the sense of Definition 3.11).

Finally, note that any given pair of non-intersecting paths $\omega_1, \omega_2$ starting at $(0,0)$ and $(0,-2)$, respectively, and ending at $(n, 2l-n)$ and $(n, 2l-n-2)$, respectively, can be "retranslated" into a One up path of length $2n$ which encodes information about the state $\tau$. So we have actually found a bijection between all One up paths representing a state $\tau$ and all paths $\omega_2$ starting at $(0,-2)$, ending at $(n, 2l-n-2)$ and being non-intersecting with the state path $\omega_1$. We therefore have proven the proposition. $\square$

For the proof of the next result we rotate the whole scenario by 45° counterclockwise and therefore need the following definition.

DEFINITION 3.14. A simple lattice path of width $n$ is a lattice path from $v_0 = (0, y_0)$ to $v_n = (n, y_n)$, where $n, y_0, y_n \geq 0$, with step set $\mathbb{S}^S = \{(1,0),(0,1)\}$, initial vertex $v_0$ and final vertex $v_n$.

For an example of a simple lattice path see Figure A.2 in the Appendix. Furthermore, we introduce the following restriction of the binomial coefficient:

DEFINITION 3.15. For $n, k \in \mathbb{Z}$ we define

$$\binom{n}{k}_+ := \begin{cases} \binom{n}{k} & n \geq k \geq 0, \\ 0 & n < k \text{ or } k < 0. \end{cases} \tag{3.32}$$

THEOREM 3.16. *Consider the TASEP ($q = 0 = \gamma = \delta$) with $n$ sites and the boundary conditions $\alpha = \beta = 1$. Fix a state $\tau = \{0,1\}^n$. Let $e$ be the number of empty sites and $\zeta_i$ ($i = 1, \ldots, e$) be the number of particles to the left of the $i$-th empty site. Then the probability $P_\tau$ of finding the system in the state $\tau$ in the long-time limit is given by*

$$P_\tau = \det_{1 \leq i,j \leq n} \left( \binom{\zeta_i + 1}{j - i + 1}_+ \right) / C_{n+1}. \tag{3.33}$$

PROOF. The theorem follows from Proposition 3.13 and Proposition A.4 in the Appendix. We know that we can calculate the probability of finding the TASEP in state $\tau$ with the help of non-intersecting paths $\omega_1, \omega_2$ due to Proposition 3.13. Let us rotate the whole scenario by 45° counterclockwise, hence the paths become lattice paths consisting of north and east steps, or, more precisely, simple lattice paths. Let the path that is obtained from of $\omega_1$ be denoted by $\overline{\omega_1}$. In general, the condition for the corresponding state path reads as follows:

$$v_i - v_{i-1} = \begin{cases} (1,0) & \text{if } \tau_i = \circ \\ (0,1) & \text{if } \tau_i = \bullet. \end{cases} \tag{3.34}$$

Let $\omega_2$, together with the entire scenario, also be rotated by 45° and lifted up by two units (that is, let the path start at $(0,0)$); we denote this path by $\overline{\omega_2}$. Then Proposition 3.13 can be reformulated as the task of finding all simple lattice paths $\overline{\omega_2}$ which lie on or above the $x$-axis, have $\overline{\omega_1}$ as their upper border and have the same final vertex as $\omega_1$. (We allow both paths to share steps, but the $s$-th vertex of $\overline{\omega_2}$ must not lie higher than the $s$-th vertex of the state path $\overline{\omega_1}$.)

That this reformulation is correct is not hard to see; it is indicated in Figure 3.10. Consider the state path $\omega_1$ and all possible paths $\omega_2$ such that a pair of non-intersecting paths is obtained. Consider the two extremes which form the upper and the lower boundary for the path $\omega_2$: for the lower boundary, denote it by $lb$, we choose as many down steps as possible followed by as many up steps as necessary to arrive at the fixed final vertex (with notation of Proposition 3.13 this is the vertex $(n, 2l-n-2)$). For the upper boundary, denote it by $ub$ we note that the step-sequence is the same as in the path $\omega_1$. This is true because for the upper boundary we want as many up steps as possible, but every time $\omega_1$ takes a down step, $\omega_2$ has to do the same, because otherwise the paths would already intersect. So, all other choices of step-sequences for $\omega_2$ lie in between these two paths. Turning the scenario by 45° we see that the reformulation holds.

The number of such simple lattice paths that run within some given borders is given through the determinant in Proposition A.4 in the Appendix; we only need to choose the borders $\mathbf{a}, \mathbf{b}$ correctly. Let $\tau$ be a state with $n$ sites. First, note that the state path ends at $(e, n-e)$, where $e$ is the number of empty sites in $\tau$. This is obvious since empty sites cause an east step in the state path, whereas particles cause an up step. Hence we choose the entries of $\mathbf{b}$ as follows:

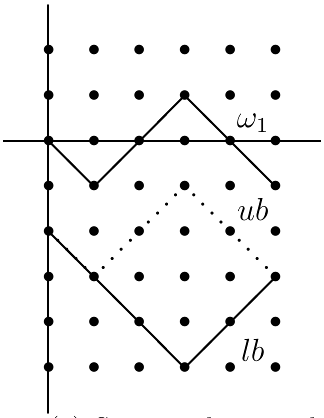
$$\mathbf{b} = (\underbrace{0, \ldots, 0}_{e})$$

As the height of the $i$-th step in the state path is determined by the number of particles within the first $i$ sites of $\tau$, we set

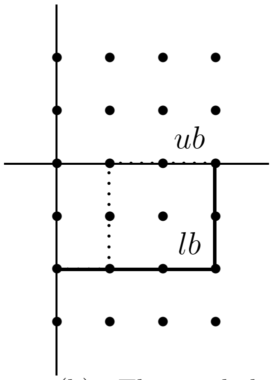
$$\mathbf{a} = (\zeta_1, \ldots, \zeta_e),$$

where $\zeta_i$ $(i = 1, \ldots, e)$ was the number of particles left to the $i$-th empty site. Now we can deduce that the number of paths that start at $(0,0)$, end at $(e, n-e)$ and stay below the state path all the time is given by (3.33). As seen, this number equals the number of non-intersecting paths in Proposition 3.13 and hence the theorem is proven. □

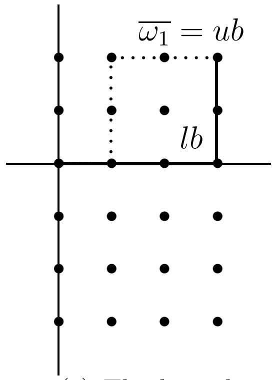
Proposition 3.17 ([**2**], Section 7, Proposition 6). *For the TASEP ($q = 0 = \gamma = \delta$) with $n$ sites and the boundary conditions $\alpha = \beta = 1$ the probability $P_{n,l}$ of finding $l$*




(a) State path $\omega_1$ and the lower and upper boundaries, $lb, ub$. For any path that lies within the borders a pair of non-intersecting paths is obtained.




(b) The whole scenario is rotated 45° counterclockwise and only the boundaries $lb, ub$ are examined




(c) The boundaries are lifted up by two. One sees that all possible pairs of non-intersecting paths can be expressed in terms of simple lattice path that stay within the borders

FIGURE 3.10. An example of the bijection between a pair of non-intersecting paths and a simple lattice path that stays within some borders. The state path to the state $(\circ, \bullet, \bullet, \circ, \circ)$, denoted by $\omega_1$, is fixed. $up$ and $bl$, the upper and lower boundaries, are such that any path that lies within them forms a pair of non-intersecting paths together with $\omega_1$. Through the bijection it follows that enumerating pairs on non-intersecting paths is equal to enumerating simple lattice paths which start at $(0,0)$ and stay below the path $\overline{\omega_1}$ before ending at the same final vertex.

*particles in the stationary state of the system is given by*

$$P_{n,l} = \sum_{\tau \in \{\bullet^l, \circ^{n-l}\}} P_n(\tau) = \frac{1}{C_{n+1}} \det \begin{vmatrix} \binom{n}{l} & \binom{n}{l-1} \\ \binom{n}{l+1} & \binom{n}{l} \end{vmatrix} \tag{3.35}$$

$$= \frac{1}{(n+1)C_{n+1}} \binom{n+1}{l} \binom{n+1}{n-l} \tag{3.36}$$

*where the sum is over all states $\tau$ with exactly $l$ particles and $n-l$ empty sites.*

PROOF. To show that the equality in (3.35) holds, we use Proposition 3.13: to any state $\tau$ with $l$ particles corresponds exactly one of the state paths ending at height $(n, n-2l)$. (This is seen easily: each state path starts at $(0,0)$ by definition. If in state $\tau$ there was no particle at all, then all steps would be up steps. So, the state path would end at $(n,n)$. But for any particle in $\tau$, we turn an up step into a down step, therefore lowering the end point by $y = 2$. Hence, each path starting at $(0,0)$ and ending at height $(n, n-2l)$ bijects to one state with exactly $l$ particles.) Recall that the steady state probability of any state $\tau$ is given by the number of paths which are non-intersecting with the state path of $\tau$ (see Proposition 3.13). So, the sum $\sum_{\tau \in \{\bullet^l, \circ^{n-l}\}} P_n(\tau)$ becomes a double sum over all state paths ending at $(n, n-2l)$ and all non-intersecting paths starting at $(0,2)$ and ending at $(n, n-2l+2)$. This number is given by the Lindstr"om-Gessel-Viennot determinant,

$$\det \begin{vmatrix} \binom{n}{l} & \binom{n}{l-1} \\ \binom{n}{l+1} & \binom{n}{l} \end{vmatrix}, \tag{3.37}$$

as shown in the Appendix, Lemma A.3. Bearing in mind the normalization factor (or partition function) $1/C_{n+1}$, the first equality follows.

To prove equality with (3.36), we need to show that the determinant (3.37) is equal to

$$\binom{n}{l}\binom{n}{l} - \binom{n}{l-1}\binom{n}{l+1} = \frac{1}{(n+1)}\binom{n+1}{l}\binom{n+1}{n-l} \tag{3.38}$$

We examine the left-hand side of (3.38):

$$\begin{aligned} &\binom{n}{l}\binom{n}{l} - \binom{n}{l-1}\binom{n}{l+1} = \\ &\frac{n!}{(l)!(n-l)!}\frac{n!}{(l)!(n-l)!} - \frac{n!}{(l-1)!(n-l+1)!}\frac{n!}{(l+1)!(n-l-1)!} = \\ &\frac{(n!)^2(l+1)(n-l+1) - (n!)^2(l)(n-l)}{(l+1)!(l)!(n-l+1)!(n-l)!} = \\ &\frac{(n!)^2(n+1)}{(l+1)!(l)!(n-l+1)!(n-l)!} \cdot \frac{n+1}{n+1} = \\ &\frac{1}{n+1}\binom{n+1}{l}\binom{n+1}{l+1}, \end{aligned} \tag{3.39}$$

which is equal to the right-hand side of (3.38) due to binomial coefficient identities. The numbers appearing in (3.39) are the so-called *Narayana numbers*, see, e.g., ([**23**], p. 237). ☐

Remark. Lemma 3.16 was already proven by Louis Shapiro and Doron Zeilberger ([**21**], Theorem 1.) more than ten years before Derrida et.al. even came up with the Matrix Ansatz in [**9**]. Although not mentioning it explicitly, they also considered non-intersecting lattice paths (in their paper they focused on sequences of 0's and 1's). Exchanging up steps with 1's and down steps with 0's in the step sequences of the non-intersecting paths $\omega_1, \omega_2$ in Lemma 3.13 yields exactly Theorem 1 in [**21**]. The proof of L. Shapiro and D. Zeilberger basically just uses the definition of the stationary distribution and considerations on how many ways there are to arrive at a certain state $\tau$, and how many ways there are to leave this state.

Remark. Note that we could use the definition of a state path (Definition 3.12) not only for the TASEP, but as well for the more general case of PASEP (where particles might also hop to the left). To a certain extant we will do so, although we will use more complex combinatorial objects to encode the states of the PASEP, since paths will turn out to be not sufficiently adopted to carry the information we would like to keep track of. But we will present another result that is still linked to paths – this time we use so called *bicolored Motzkin paths* – and extend the results established so far.

First, we establish a one-to-one correspondence between any state $\tau$ of the TASEP and bicolored Motzkin paths:

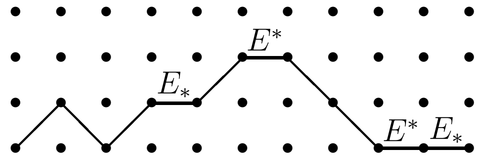



FIGURE 3.11. Bicolored Motzkin path of length 10. $E_*$ denotes a step $ES_*$ and $E^*$ denotes a step $ES^*$.

## 3.5. Motzkin Paths

A *bicolored Motzkin path* is a lattice path consisting of up, down, and east steps. While there is nothing special to the up and down steps, we admit two different varieties of side steps. One can distinguish them by coloring them differently. We simply give them different names.

DEFINITION 3.18. A **bicolored Motzkin path** of length $n$ is a lattice path that starts at $v_0 = (0, 0)$, ends at $v_n = (n, 0)$ and has step set

$$\mathbb{S}^M = \{(1, 1), (1, -1), (1, 0)\},$$

where east steps $v_i - v_{i-1} = (1, 0)$ are either of type $ES^*$ or $ES_*$. We denote up steps, $(1, 1)$, by $US$ and down steps, $(1, -1)$, by $DS$.

Furthermore, we denote by $M^n$ the set of all bicolored Motzkin paths of length $n$.

REMARK. Note that the definition of a lattice path does not require only steps above the $x$-axis but also admits steps *on* the $x$-axis.

REMARK. In figures, we sometimes shorten $ES^*$ to $E^*$ and $ES_*$ to $S_*$. See, e.g., Figure 3.11, which is an example of a bicolored Motzkin path.

The following observation concerning bicolored Motzkin paths shall turn out to be very useful for our purposes.

PROPOSITION 3.19. *Bicolored Motzkin paths of length $n$ are in bijection with One up paths of length $2n$.*

PROOF. Let $o = (o_1, \ldots, o_{2n})$ be a One up path of length $2n$ consisting of up and down steps, denoted by $U$ and $D$, respectively. Furthermore, we denote by $m = (m_1, \ldots, m_n)$ a bicolored Motzkin path with $m_i \in \{US, DS, ES^*, ES_*\}, 1 \leq i \leq n$, where $US$ and $DS$ represent an up or a down step, respectively, and $ES^*$ and $ES^*$ stand for the two types of east steps. We will define the bijection by considering two consecutive steps of the One up path $o$. If these two steps happen to be up steps, we change them to a single up step. Two consecutive down steps are turned into a single down step. If the pair consists of an up and a down step, we turn it into an east step of type $ES^*$, and into a colored east step of type $ES_*$ if the first step is a down step followed by an up step. See Figure 3.12 for an example. More precisely: we define

$$m_i = b((o_j, o_{j+1})) = \begin{cases} US & \text{if } (o_j, o_{j+1}) = (U, U) \\ DS & \text{if } (o_j, o_{j+1}) = (D, D) \\ ES^* & \text{if } (o_j, o_{j+1}) = (U, D) \\ ES_* & \text{if } (o_j, o_{j+1}) = (D, U) \end{cases} \tag{3.40}$$

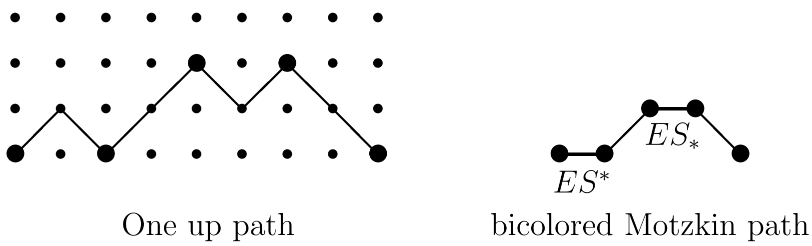



FIGURE 3.12. Example for the bijection between One up paths and Motzkin paths.

with $i, j$ as defined above, let the path $m = (m_1, \ldots, m_n)$ start in $(0, 0)$.

We claim that we what we obtain is a bicolored Motzkin path: first, it is clear, that $m$ is of length $n$ since $n$ pairs of steps are changed into $US, DS, ES^*$ or $ES_*$.

Next, we check that the path $m$ ends in $(n, 0)$.

To see this, we examine the number of up and down steps of the One up path. Let $r$ be the number of up steps, and consequently also be the number of down steps, in $o$ (that these two numbers are the same follows from the definition of a One up path since by definition it starts and ends at height 1). First, consider the pairs $(o_j, o_{j+1})$ (where $j$ was defined above) which form a "*hook*"; by this we mean the pairs $(U, D)$ and $(D, U)$. Through the mapping $b$ they become east steps, so the hooks do not affect the height (the $y$ coordinate) of the bicolored Motzkin path $m$. Let $h$ be the number of hooks in $o$, so $r - h$ up steps and $r - h$ down steps remain, which, if paired, are of the form $(D, D)$ or $(U, U)$. The number of pairs $(U, U)$ has to be equal to the number of pairs $(D, D)$ (namely $\frac{r-h}{2}$) and hence the number of up steps $US$ in $m$ is equal to the number of down steps $DS$.

Finally we have to ensure that $m$ does never lie below the $x$-axis. If it were below the $x$-axis, then, at some point, the number of down steps $DS$ would exceed the number of up steps $US$. This would imply that there were more down steps than up steps in the One up path at some point. But this can not occur since a One up path always stays above the $x$-axis.

On the other hand, each bicolored Motzkin path can be transformed into a One up path of length $2n$ by inverting (3.40):

$$((o_j, o_{j+1})) = b^{-1}(m_i) \begin{cases} (o_j, o_{j+1}) = (U, U) & \text{if } m_i = US \\ (o_j, o_{j+1}) = (D, D) & \text{if } m_i = DS \\ (o_j, o_{j+1}) = (U, D) & \text{if } m_i = ES^* \\ (o_j, o_{j+1}) = (D, U) & \text{if } m_i = ES_* \end{cases} \tag{3.41}$$

with $i, j$ as defined above. Let the resulting path start in $(0, 1)$. Then, one can check analogously as above that the resulting path

- is of length $2n$ (this is obvious)
- ends in $(2n, 1)$ (because if not then one can conclude that the bicolored Motzkin path does not end in $(n, 0)$)
- lies above the $x$-axis (because otherwise the bicolored Motzkin path would cross the $x$-axis at some point)

Therefore we have found a bijection between bicolored Motzkin paths and One up paths. □

Using bicolored Motzkin paths, we can phrase the observed results from Lemma 3.9 more succinctly:

As already seen, the probability of finding the TASEP with $n$ sites in state $\tau$ in the long-time limit can be represented by considerations of One Up paths, namely by

$$\frac{\#\text{ of One up paths of length } 2n \text{ with } n \text{ forced steps}}{\#\text{ of One up paths of length } 2n}, \tag{3.42}$$

where the $n$ forced steps depend on the state $\tau = (\tau_1, \ldots, \tau_n)$,

$$\tau_i = 0 \text{ if and only if step } o_{2i-1} \text{ (ending at } x = 2i-1) \text{ is a down step,}$$
$$\tau_i = 1 \text{ if and only if step } o_{2i} \quad \text{(ending at } x = 2i) \text{ is an up step,}$$

where $o = (o_1, \ldots, o_{2n})$ is a One up path. (Recall that this was a consequence of the lattice path interpretation). Now, using the bijection between One up paths and bicolored Motzkin paths (3.41), we see that for a pair of steps $(o_j, o_{j+1})$ there follows:

$$\tau_i = 0 \Leftrightarrow \text{step } o_{2i-1} \text{ is a down step} \Leftrightarrow (o_j, o_{j+1}) = (D, .),$$
$$\tau_i = 1 \Leftrightarrow \text{step } o_{2i} \quad \text{is an up step} \quad \Leftrightarrow (o_j, o_{j+1}) = (., U),$$

where the slot is either filled up with an up or down step. Hence, if $\tau_i = 0$, then the pair of steps $(o_j, o_{j+1})$ of a corresponding One up path can either be equal to $(D, D)$ or $(D, U)$. This implies that the bicolored Motzkin path has either a down step $DS$ or an east step of type $ES_*$ at position $\frac{j+1}{2}$. Conversely, if $\tau_i = 1$, the pair of steps $(o_j, o_{j+1})$ of a corresponding One up path has to be equal to $(D, U)$ or $(U, U)$. Hence, again using the bijection from (3.41), it follows that the step at position $\frac{j+1}{2}$ in the bicolored Motzkin path is either an up step $US$ or an east step of type $ES^*$. It follows that for a bicolored Motzkin path $m = (m_1, \ldots, m_n)$ corresponding to a state $\tau$ the following holds:

$$\tau_i = 0 \text{ if and only if } m_i \in \{DS, ES_*\} \tag{3.43}$$
$$\tau_i = 1 \text{ if and only if } m_i \in \{US, ES^*\} \tag{3.44}$$

We therefore can restate Lemma 3.9 as follows:

Lemma 3.20. *For the TASEP ($q = \gamma = \delta = 0$) with $n$ sites, $p = 1$, and the boundary conditions $\alpha = \beta = 1$, the probability of finding the system in a state $\tau$ with $v$ particles at sites $l_1, \ldots, l_v$ and $w$ empty sites $j_1, \ldots, j_w$ (with $v + w \leq n$) is given by*

$$P(\tau_{l_1} = \ldots = \tau_{l_v} = 1, \tau_{j_1} = \ldots = \tau_{j_w} = 0) = \frac{|M^n(l_1, \ldots, l_v; j_1, \ldots, j_w)|}{|M^n|}, \tag{3.45}$$

*where $M^n$ is the set of all bicolored Motzkinpaths of length $n$, and*

$$M^n(l_1, \ldots, l_v; j_1, \ldots, j_w)$$

*is the set of all bicolored Motzkin paths that either have an up step $US$ or an east step of type $ES^*$ at positions $l_1, \ldots, l_v$ and either a down step $DS$ or an east step of type $ES_*$ at positions $j_1, \ldots, j_w$ (while the remaining steps are arbitrary but in accordance with the Definition 3.18 of bicolored Motzkin paths).*

PROOF. The proof was already presented above. It follows from Lemma 3.9 and the bijection between One up paths and bicolored Motzkin paths given in (3.41), which leads to (3.43) - (3.44). □

Analogously to Definition 3.12, we use (3.43) – (3.44) to define a bicolored Motzkin path of type $\tau$:

DEFINITION 3.21. Fix a state $\tau = (\tau_1, \ldots, \tau_n)$. A bicolored Motzkin path is said to be **of type** $\tau$ if (3.43) – (3.44) hold, namely:

$$\tau_i = 0 \text{ if and only if } m_i \in \{DS, ES_*\}, \tag{3.46}$$

$$\tau_i = 1 \text{ if and only if } m_i \in \{US, ES^*\}. \tag{3.47}$$

We denote the set of bicolored Motzkin paths of type $\tau$ by $M_\tau$.

Using Lemma 3.20 and Definition 3.21, we can state the following corollary:

COROLLARY 3.22 ([**6**], Corollary 6.2.). *For the TASEP ($q = 0 = \gamma = \delta$) with $n$ sites and the boundary conditions $\alpha = \beta = 1$, the probability of finding the system in the state $\tau = (\tau_1, \ldots, \tau_n) \in \{0,1\}^n$ is given by*

$$\frac{|M_\tau|}{|M^n|} = \frac{\#\text{ of bicolored Motzkin paths of of type } \tau}{\#\text{ of bicolored Motzkin paths of length } n}. \tag{3.48}$$

PROOF. This is seen easily using Lemma 3.20. For a state $\tau = (\tau_1, \ldots, \tau_n)$, we obtain

$$P(\tau) = P(\tau_{l_1} = \ldots = \tau_{l_v} = 1, \tau_{j_1} = \ldots = \tau_{j_w} = 0) = \frac{|M^n(l_1, \ldots, l_v; j_1, \ldots, j_w)|}{|M^n|},$$

with $v+w = n$. Consider the numerator, and note that the steps $l_1, \ldots, l_v$ are either $US$ or $ES^*$, while steps $j_1 \ldots, j_w$ are either $DS$ or $ES_*$. Comparing this to Definition 3.21, we see that we obtain exactly the bicolored Motzkin paths of type $\tau$. The denominator does not change, and hence we arrive at (3.48). □

## 3.6. PASEP with Parameters $\alpha = \beta = 1$

We can even extend Corollary 3.22 to the case where particles might also hop to the left: for the PASEP with $\alpha = \beta = 1$ and $q \geq 0$. (Until now we have required $q$, the probability of a particle to hop left, to be equal to 0.) Recall that the weight of a lattice path was defined as the product of the weight of the steps and the weight of its starting and ending vertex. So far, we were only confronted with weights equal to 1. This will change now. We assign the steps of bicolored Motzkin paths different weights which depend on the value of $q$ and the height of the step. We will see that in the case of $q = 0$ the weights reduce to 1. Let $m = (m_1, \ldots, m_n)$ be a bicolored Motzkin path. Let the step $m_i$ end at the vertex $(x_i, y_i)$. Define $[y]$ to be $q$-analog of the number $y \in \mathbb{Z}_0$, namely $1 + q + \ldots + q^{y-1}$. We assign each step $m_i$ the weight $[y+1]$. Denote by $wt(m)$ the weight of a lattice path $m$ and let

$$M_\tau(q) := \sum_{m \in M_\tau} wt(m)$$

be the sum of the weights of all bicolored Motzkin paths of type $\tau$.

COROLLARY 3.23 ([**6**], Corollary 6.2.). *For the PASEP with $q \geq 0$ and boundary conditions $\alpha = \beta = 1$, $\gamma = \delta = 0$ the probability of finding the model in state $\tau = (\tau_1, \ldots, \tau_n) \in \{0,1\}^n$ is given by*

$$\frac{M_\tau(q)}{|M^n|} := \frac{\sum_{m \in M_\tau} wt(m)}{|M^n|} \tag{3.49}$$

We present the proof after the following preparations.

Corollary 3.23 is a generalization of Lemma 3.20, which is itself based upon the lattice path interpretation of the matrices $D_0, E_0, V_0, W_0$. Hence we can expect that these matrices show up again, somehow. Recall that $[y]$ is the $q$-analog of the number $y \in \mathbb{Z}_0$, namely $1 + q + \ldots + q^{y-1}$. We define

$$D_{\widehat{0}} := \begin{pmatrix} [1] & [2] & 0 & 0 & 0 & \cdots \\ 0 & [2] & [3] & 0 & 0 & \cdots \\ 0 & 0 & [3] & [4] & 0 & \cdots \\ 0 & 0 & 0 & [4] & [5] & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}, \qquad E_{\widehat{0}} := \begin{pmatrix} [1] & 0 & 0 & 0 & 0 & \cdots \\ [1] & [2] & 0 & 0 & 0 & \cdots \\ 0 & [2] & [3] & 0 & 0 & \cdots \\ 0 & 0 & [3] & [4] & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}, \tag{3.50}$$

while the matrices $W_{\widehat{0}}, V_{\widehat{0}}$ remain as before

$$W_{\widehat{0}} := W_0 = (1, 0, 0, 0, \ldots), \qquad V_{\widehat{0}} := V_0 = (1, 0, 0, 0, \ldots)^T. \tag{3.51}$$

Note that for $q = 0$ it follows that $D_{\widehat{0}} = D_0$ and $E_{\widehat{0}} = E_0$.

LEMMA 3.24. *For the matrices $D_{\widehat{0}}, E_{\widehat{0}}, V_{\widehat{0}}, W_{\widehat{0}}$ defined in* (3.50) – (3.51) *the Matrix Ansatz equations* (2.3) – (2.5) *hold.*

PROOF. We start with Equation (2.5). Due to the choice of parameters, the equation simplifies to

$$W_{\widehat{0}} E_{\widehat{0}} = W_{\widehat{0}}. \tag{3.52}$$

To see that this is indeed true, we just need to note that

$$W_0 = (1, 0, 0, 0, \ldots) \cdot \begin{pmatrix} [1] & 0 & 0 & 0 & 0 & \cdots \\ [1] & [2] & 0 & 0 & 0 & \cdots \\ 0 & [2] & [3] & 0 & 0 & \cdots \\ 0 & 0 & [3] & [4] & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} = ([1], 0, 0, \ldots). \tag{3.53}$$

Since $[1] = 1$, Equation (3.52) follows.

Similarly, we can see that Equation (2.4), which in our case reads

$$D_{\widehat{0}} V_{\widehat{0}} = V_{\widehat{0}}, \tag{3.54}$$

holds:

$$D_{\widehat{0}} := \begin{pmatrix} [1] & [2] & 0 & 0 & 0 & \cdots \\ 0 & [2] & [3] & 0 & 0 & \cdots \\ 0 & 0 & [3] & [4] & 0 & \cdots \\ 0 & 0 & 0 & [4] & [5] & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} [1] \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{3.55}$$

Finally, Equation (2.3) remains to be checked. Unfortunately, this equation does not simplify, so we have to show that

$$D_{\widehat{0}}E_{\widehat{0}} - qE_{\widehat{0}}D_{\widehat{0}} = D_{\widehat{0}} + E_{\widehat{0}}. \tag{3.56}$$

We start by examining the right-hand side of (3.56). We see that

$$D_{\widehat{0}} + E_{\widehat{0}} = \begin{pmatrix} 2[1] & [2] & 0 & 0 & 0 & \cdots \\ [1] & 2[2] & [3] & 0 & 0 & \cdots \\ 0 & [2] & 2[3] & [4] & 0 & \cdots \\ 0 & 0 & [3] & 2[4] & [5] & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \tag{3.57}$$

or, more precisely

$$D_{\widehat{0}} + E_{\widehat{0}} = \begin{cases} [i-1] & j = i-1, \\ 2[i] & j = i, \\ [i+1] & j = i+1, \\ 0 & \text{otherwise} \end{cases} \tag{3.58}$$

Now consider the left-hand side of (3.56). We start with the product $(D_{\widehat{0}}E_{\widehat{0}})_{i,j}$. If we fix some $i$, the sum $\sum_{k\geq 1}(D_{\widehat{0}})_{i,k}(E_{\widehat{0}})_{k,j} = (D_{\widehat{0}}E_{\widehat{0}})_{i,j}$ can be reduced to the terms that might differ from 0; these are, since by definition only $(D_{\widehat{0}})_{i,i}, (D_{\widehat{0}})_{i,i+1} \neq 0$ the following:

$$(D_{\widehat{0}}E_{\widehat{0}})_{i,j} = \sum_{k\geq 1}(D_{\widehat{0}})_{i,k}(E_{\widehat{0}})_{k,j} = (D_{\widehat{0}})_{i,i}(E_{\widehat{0}})_{i,j} + (D_{\widehat{0}})_{i,i+1}(D_{\widehat{0}})_{i+1,j} \tag{3.59}$$

So, for the entries $(i,j)$ of $(D_{\widehat{0}}E_{\widehat{0}})_{i,j}$, the following holds

$$(D_{\widehat{0}}E_{\widehat{0}})_{i,j} = (D_{\widehat{0}})_{i,i}(E_{\widehat{0}})_{i,j} + (D_{\widehat{0}})_{i,i+1}(E_{\widehat{0}})_{i+1,j} = \begin{cases} [i][i-1] & j = i-1, \\ [i][i] + [i+1][i] & j = i, \\ [i+1][i+1] & j = i+1, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.60}$$

On the other hand, the entries $(E_{\widehat{0}}D_{\widehat{0}})_{i,j}$ of the product $E_{\widehat{0}}D_{\widehat{0}}$ are given by:

$$(E_{\widehat{0}}D_{\widehat{0}})_{i,j} = (E_{\widehat{0}})_{i,i-1}(D_{\widehat{0}})_{i-1,j} + (E_{\widehat{0}})_{i,i}(D_{\widehat{0}})_{i,j} = \begin{cases} [i-1][i-1] & j = i-1, \\ [i-1][i] + [i][i] & j = i, \\ [i][i+1] & j = i+1, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.61}$$

Hence it follows from (3.60) and (3.61) that the left-hand side of (3.56) is equal to

$$(D_{\widehat{0}}E_{\widehat{0}})_{i,j} - q(E_{\widehat{0}}D_{\widehat{0}})_{i,j} = \begin{cases} [i][i-1] - q[i-1][i-1] & j = i-1 \\ [i][i] + [i+1][i] - q([i-1][i] + [i][i]) & j = i \\ [i+1][i+1] - q[i][i+1] & j = i+1 \\ 0 & \textit{otherwise} \end{cases} \tag{3.62}$$

Now, noting that

$$[i] - q[i-1] = (1 + q + \ldots + q^{i-1}) - q(1 + q + \ldots + q^{i-2}) = 1, \tag{3.63}$$

we find

$$[i][i-1] - q[i-1][i-1] = [i-1](\underbrace{[i] - q[i-1]}) = [i-1](\quad 1 \quad) \tag{3.64}$$

$$\begin{aligned}[i][i] + [i+1][i] - q([i-1][i] + [i][i]) &= [i]([i] + [i+1] - q[i-1] - q[i]) \\ &= [i](\underbrace{[i] - q[i-1]} + \underbrace{[i+1] - q[i]}) \\ &= [i](\quad 1 \quad + \quad 1 \quad) \\ &= [i](\quad 2 \quad)\end{aligned} \tag{3.65}$$

$$[i+1][i+1] - q[i][i+1] = [i+1](\underbrace{[i+1] - q[i]}) = [i+1](\quad 1 \quad) \tag{3.66}$$

where we have only been using (3.63) (and some rearranging). Substituting (3.64) – (3.66) into (3.62), we obtain precisely $D_{\widehat{0}} + E_{\widehat{0}}$ (as in (3.58)). Hence we have shown that Equation (3.56) holds. □

We are now ready for the proof of Corollary 3.23.

Proof of Corollary 3.23. We will show that the probability of finding the system in the state $\tau = (\tau_1, \ldots, \tau_n) \in \{0,1\}^n$ is given by

$$\frac{M_\tau}{M^n} = \frac{\text{sum of weights of bicolored Motzkin paths of of type } \tau}{\text{sum of weights of bicolored Motzkin paths of length } n}. \tag{3.67}$$

To this end, we use the matrices $D_{\widehat{0}}, E_{\widehat{0}}, V_{\widehat{0}}, W_{\widehat{0}}$ defined in (3.50) – (3.51), the Matrix Ansatz and a lattice path interpretation (similar to the one of One up paths). This time, let the rows and columns of $D_{\widehat{0}}, E_{\widehat{0}}$ be labeled by integers $\mathbb{Z}_0 = \{0, 1, 2 \ldots\}$ as shown in Figure 3.13. Analogously we label the entries of the vectors $V_{\widehat{0}}, W_{\widehat{0}}$ by $\mathbb{Z}_0 = \{0, 1, 2 \ldots\}$. Now fix a state $\tau = (\tau_1, \ldots, \tau_n) \in \{0,1\}^n$. Since for $D_{\widehat{0}}, E_{\widehat{0}}, V_{\widehat{0}}, W_{\widehat{0}}$ the Matrix Ansatz equations hold (see Lemma 3.24), and since all entries are positive (i.e., have the same sign), the steady-state probability of finding the system in state $\tau$ is given by

$$\frac{W_{\widehat{0}}(\prod_{i=1}^n (\tau_i D_{\widehat{0}} + (1-\tau_i)E_{\widehat{0}}))V_{\widehat{0}}}{W_{\widehat{0}}(D_{\widehat{0}} + E_{\widehat{0}})^n V_{\widehat{0}}}. \tag{3.68}$$

We claim that (3.68) and (3.67) are equal to each other. In order to see this, we examine the product

$$W_{\widehat{0}} \left( \prod_{i=1}^n (\tau_i D_{\widehat{0}} + (1-\tau_i)E_{\widehat{0}}) \right) V_{\widehat{0}} \tag{3.69}$$

in (3.68) for the state $\tau$. Using the lattice path interpretation described above (Section 3.1), we note the following: $W_{\widehat{0}}$ contributes the weight in the starting vertices, so only paths starting at height 0 can end up with non-zero weight (see (3.71)). Hence, the matrix $D_{\widehat{0}}$ offers two possibilities for steps which have non-zero weights assigned to it: either an up step of height 1 or an east step (use Figure 3.13 together with (3.70)). On the other hand, the matrix $E_{\widehat{0}}$ assigns non-zero weights only to down steps of height 1 as

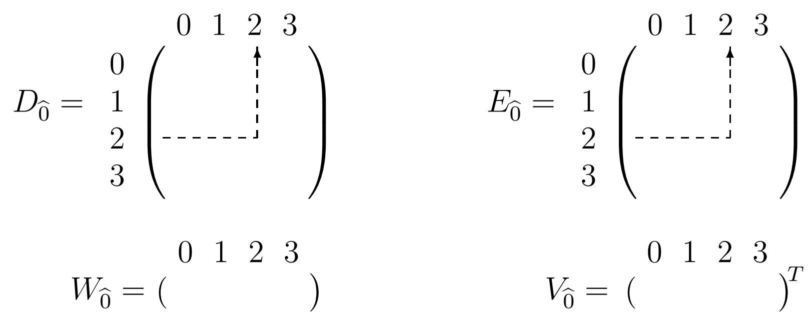



FIGURE 3.13. Scheme of the labeling of the matrices $D_{\widehat{0}}, E_{\widehat{0}}, V_{\widehat{0}}, W_{\widehat{0}}$.

well as to east steps (see Figure 3.13 together with (3.70)). So, all paths (with non-zero weights) might consist of down steps, up steps, and two different kinds of east steps. Note that, due to the entries of $V_{\widehat{0}}$, only paths ending at height 0 can have non-zero weights. Finally, consider the values of the entries of $D_{\widehat{0}}, E_{\widehat{0}}$ — we see that steps ending at height $h \geq 0$ have weight $[h+1]$ assigned to it. So, for fixed $\tau$, Equation (3.69) represents bicolored Motzkin paths. But even more: these bicolored Motzkin paths are of type $\tau$! This follows since $D_{\widehat{0}}$ enters in the product at position $i$ (respectively $i+1$ if one also takes the factor $W_{\widehat{0}}$ into account) if and only if $\tau_i = 1$. Since $D_{\widehat{0}}$ admits only an up or east step, we obtain (3.47) in Definition 3.21. On the other hand, if $\tau_i = 0$, then $E_{\widehat{0}}$ enters, and the resulting step in the path is either a down or an east step (as required in (3.46)).

$$D_{\widehat{0}} = \begin{pmatrix} [1] & [2] & 0 & 0 & 0 & \cdots \\ 0 & [2] & [3] & 0 & 0 & \cdots \\ 0 & 0 & [3] & [4] & 0 & \cdots \\ 0 & 0 & 0 & [4] & [5] & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \qquad E_{\widehat{0}} = \begin{pmatrix} [1] & 0 & 0 & 0 & 0 & \cdots \\ [1] & [2] & 0 & 0 & 0 & \cdots \\ 0 & [2] & [3] & 0 & 0 & \cdots \\ 0 & 0 & [3] & [4] & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \tag{3.70}$$

$$W_{\widehat{0}} = (1, 0, 0, 0, \ldots) \qquad V_{\widehat{0}} = (1, 0, 0, 0, \ldots)^T \tag{3.71}$$

So, we have seen that the numerators of (3.67) and (3.68) yield the same result. Because of this, we can also conclude that the denominators are equal: by calculating (or expanding) the term

$$W_{\widehat{0}}(D_{\widehat{0}} + E_{\widehat{0}})^n V_{\widehat{0}},$$

is equal to summing up

$$W_{\widehat{0}}(\mathrm{Word}(D_{\widehat{0}}, E_{\widehat{0}}))V_{\widehat{0}},$$

with $\mathrm{Word}(D_{\widehat{0}}, E_{\widehat{0}})$ being any word in $\{D_{\widehat{0}}, E_{\widehat{0}}\}^n$. This is the same as summing up all bicolored Motzkin paths of length $n$. This is exactly the denominator of (3.67) and hence we have shown that (3.67) and (3.68) are equal. $\square$

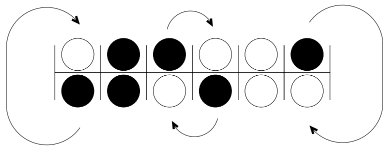


FIGURE 3.14. The Markov chain revealing a second row of particles traveling backwards for the TASEP.

### 3.7. Overview: Results for the TASEP with General Parameters $\alpha, \beta$

There was also some work done on the TASEP with general parameters $\alpha, \beta$. In [**2**], not only an interpretation of the partition function for TASEP with maximal flow regime in terms of lattice paths is given, but for the TASEP with general parameters interpretations in terms of various weighted lattice paths are presented; namely in terms of weighted One up paths, *Jump step paths* and *Cross paths* ([**2**], Lemma 1). The result is shown analogously to Lemma 3.5, by interpreting some matrix representations for the Matrix Ansatz as transfer matrices: the representations are those presented in ([**9**], p. 1499 – 1500) and one only has to check that they satisfy the Matrix Ansatz equations. Labeling them as we did for the TASEP with $p = \alpha = \beta = 1$ in Section 3.1, one can apply the transfer-matrix method and hence arrives at the weighted lattice paths named above.

Another interesting approach was presented by E. Duchi and G. Schaeffer [**11**]. In their work, they reveal a second row of sites (beneath the ASEP) in which the particles that have left the system travel backwards before eventually entering the ASEP again ([**11**], Subsection 1.4). The authors then define a Markov Chain on these two rows with the top row representing exactly the TASEP (see Figure 3.14) and the states being so-called *complete configurations*. These configurations turn out to be in bijection with bicolored Motzkin paths. The authors first obtain the results (the derivation of the steady-state probability) for the TASEP with $\alpha = \beta = 1$ ([**11**], Theorem 3.2.) before introducing a weight function on the complete configurations to obtain the steady-state probability in the case of arbitrary $\alpha, \beta$ ([**11**], Theorem 3.3).

CHAPTER 4

# PASEP and Permutation Tableaux

The following chapter is dedicated to the partially asymmetric simple exclusion process (PASEP). Recall that that for the **PASEP** only the parameters $\alpha, \beta, p, q$ are greater than 0; that is, particles can hop to both sides but only enter at the left-hand side and leave at the right-hand side. As before, we will always assume that $p = 1$. The combinatorial approach presented in this and the remaining chapters was provided by Sylvie Corteel and Lauren K. Williams [**5**–**7**]. They first considered *permutation tableaux*, a class of tableaux, introduced by E. Steingrímsson and Lauren K. Williams in [**24**]. These tableaux are a distinguished subset of so-called Ⅎ-diagrams of Alex Postnikov ([**19**], Section 6) and are in bijection with permutations ([**24**], Section 2). Through considerations of the combinatorics of permutation tableaux, S. Corteel and L. Williams succeeded in providing a solution to the Matrix Ansatz for the PASEP [**6**]. Furthermore, they established a connection between the stationary distribution of the PASEP and permutation tableaux in their work.

The representation for the Matrix Ansatz is substance of Lemma 4.4. The connection between the stationary distribution of the PASEP and permutation tableaux is established in Theorem 4.3. The formulation and proof of this theorem is the main task of this chapter.

## 4.1. Permutation Tableaux

We do some preparatory work before actually introducing *permutation tableaux*. First, we consider some diagrams, which we call *shapes*.

Let $k, l$ be non-negative integers such that their sum $n$ is greater than 0, $n = k + l > 0$.

Consider a rectangle consisting of $k \times l$ boxes with unit length sides. Let $p$ be a path within the rectangle, consisting of west and south steps of unit length, such that $p$ starts in the upper-rightmost corner and ends in the the lower-leftmost corner. See Figure 4.1. The path, interpreted as the right-hand outer border line, and the boxes in the rectangle which lie above the path form a diagram, as in the right-hand side of Figure 4.1. Such a diagram is called a **shape**. A shape is understood to consist of rows, and columns. The length of a row is given by the number of boxes in this row, and, equally, the length of a column is given by the number of boxes in a column. Note that we also allow rows or columns of length 0 (to which we will also refer as empty rows or empty columns). The shape in Figure 4.1 consists of 4 rows and of 5 columns. The **length** of a shape $\lambda$ is defined as the sum of the number of rows and the number of columns and denoted by $length(\lambda)$. The shape in Figure 4.1 is of length 9.

Now, every path $p$ that consists of $k$ west steps and $l$ south steps can be placed within a $k \times l$ rectangle as described above and yields a shape. (It is clear that the path $p$ stays within the rectangle and does not cross the borders.) Vice-versa, given a shape

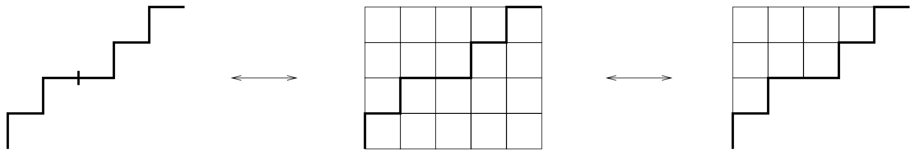


Figure 4.1. The path on the left-hand side is placed within the rectangle of $4 \times 5$ boxes in the middle of the figure. Considering only the path and the boxes that lie above this path, the shape on the right-hand side is obtained.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 | |
| 0 | 0 | 0 | 0 | 1 | |
| 0 | 1 | 1 | | | |
| 0 | 0 | | | | |

Figure 4.2. A permutation tableau of length 12.

$\lambda$ of length $n$, we can understand the right-hand border line as a path of $n$ south and west steps, starting in the upper-rightmost corner and ending in the lower-leftmost one. Hence, there is a one-to-one correspondence between paths consisting of $n$ south and west steps and shapes of length $n$ – see the left-hand side of Figure 4.1. For a shape $\lambda$ of length $n$ we denote the (to the right-hand border line) corresponding path by $p_\lambda$. We call $p_\lambda$ the **shape path** of $\lambda$. Denoting west steps by $W$ and south steps by $S$, we have $p_\lambda \in \{W, S\}^n$. Note that in the shape path a west step corresponds to the bottom of a column while a south step corresponds to the right end of a row. In general, we call a path $p_\lambda \in \{W, S\}^n$ a shape path of length $n$.

We can now define a *permutation tableau*:

Definition 4.1 ([**24**], Section 1). A **permutation tableau** $\mathcal{T}$ is a shape $\lambda$ whose shape path $p_\lambda$ starts with a south step and whose cells are filled with 0's and 1's such that the following conditions hold:

(1) Each column of the shape contains at least one 1.
(2) There is no 0 which has a 1 above it in the same column *and* a 1 to its left in the same row.

Such a filling will be called a *valid* filling of $\lambda$. We say that $\mathcal{T}$ is a permutation tableau of shape $\lambda$.

We denote the **set of permutation tableaux of shape $\boldsymbol{\lambda}$** by $\mathcal{T}_{shape[\lambda]}$ and the **set of permutation tableaux of length $\boldsymbol{n}$** by $\mathcal{T}^n$.

Note that condition (2) in aboves definition can be verified easily by the following steps: Read the columns of a permutation tableau from right to left. If in a column you find a 0 that lies beneath some 1, then the whole row left to this 0 must also contain only 0's.

Remark. Permutation tableaux inherit all concepts introduced for shapes so far, e.g., the length of a permutation tableau of shape $\lambda$ is defined as the length of its shape $\lambda$ and so on.

Now let us see how to link permutation tableaux to the PASEP. The idea is that every state $\tau$ of the PASEP corresponds to a shape $\lambda$. If we then consider the valid fillings of this shape, we obtain a set of permutation tableaux that correspond to the state $\tau$. Then we consider some statistics on the permutation tableaux and through them draw conclusions concerning the steady state probability of the state $\tau$. Let us make this more precise:

Let $\tau$ be a state of the PASEP with $n$ sides, that is $\tau \in \{0,1\}^n$. We associate to it a path $p$ of length $n+1$, $p = (p_1, \ldots, p_{n+1}) \in \{S, W\}^{n+1}$ by the following rules:

$$p_i = \begin{cases} S & i = 0 \\ S & \tau_i = 1 \\ W & \tau_i = 0 \end{cases} \tag{4.1}$$

where $i = 1, \ldots, n$. Hence, the path $p$ starts with a forced south step, while the remaining $n$ steps are encoding the state $\tau$ by defining the $i+1$ step to be a south step if and only if $\tau_i = 1$ (otherwise, $\tau_i = 0$). The path $p$ can now be interpreted as the shape path $p_\lambda$ of a shape $\lambda$ by understanding it as the right-hand border line of a shape – this was already shown above Definition 4.1. Thus, we have associated a shape $\lambda$ to the state $\tau$. We denote this shape associated to the state $\tau$ by $\lambda(\tau)$. Vice-versa, given a shape $\lambda$ we can consider the state associated to it by considering the shape path of $\lambda$ and inversing (4.1). (This simply yields the state $\tau$ for which $\lambda(\tau) = \tau$ holds.) We denote the state associated to $\lambda$ by $\tau(\lambda)$.

Finally, we consider the set of permutation tableaux of shape $\lambda$. To do so, we only need to add the valid fillings to the shape $\lambda$. Note that the forced first south step in (4.1) ensures that $\lambda$ yields a permutation tableaux (the shape path of a permutation tableaux has to start with a south step by definition). So, to a state $\tau$ we have associated the set $\mathcal{T}_{shape[\lambda]}$, the set of permutation tableaux of shape $\lambda = \lambda(\tau)$.

$$\text{state } \tau \;\rightarrow\; \text{(shape) path } p \;\rightarrow\; \text{shape } \lambda(\tau) \;\rightarrow\; \mathcal{T}_{shape[\lambda]} \tag{4.2}$$

We now define some statistics on permutation tableaux as in ([**7**], above Theorem 3.1) which we will use to “extract information” out of the tableaux:

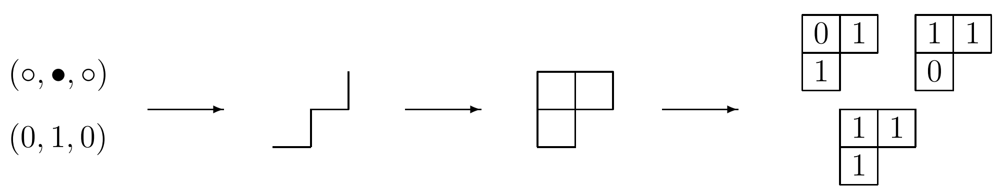



Figure 4.3. An example for he association of a state of the ASEP to the permutation tableaux of a certain shape.

Let $\mathcal{T}$ be a permutation tableaux of length $n$, with $m$ columns and $k$ rows. (Remember that a permutation tableau can only possess rows of length 0 but columns contain at least one cell). We define the *rank* $rk(\mathcal{T})$ as the number of 1's in the filling minus $m$. The subtraction of $m$ is due to the fact that a permutation tableau with $m$ columns has to contain at least $m$ 1's (see Definition 4.1). Therefore, the rank counts the number of extra 1's beyond the required ones. We define $f(\mathcal{T})$ to be the number of 1's in the first row of $\mathcal{T}$. If an entry in the first row is not 1, then we call the topmost 1 in the column *distinguished* and define $f'(\mathcal{T})$ to be the number of distinguished 1's. It is clear that $f(\mathcal{T}) + f'(\mathcal{T}) = m$. Concerning the 0's of a filling we call a 0 *restricted* if it has a 1 above it somewhere in the same column. The rightmost restricted 0 in a row is called a *distinguished* 0 (due to condition (2) of Definition 4.1 a distinguished 0 causes all entries to the left in the same row to be 0's). Now we define $u'(\mathcal{T})$ to be number of distinguished 0's in $\mathcal{T}$. A row is called *unrestricted* if it does not contain any restricted entry. Let $u(\mathcal{T})$ be the number of unrestricted rows of $\mathcal{T}$ minus 1. We subtract 1 since the first row is always unrestricted. Due to the fact that a row either contains a distinguished 0, which at the same time is a restricted 0, or is unrestricted, it is clear that $u(\mathcal{T}) + u'(\mathcal{T}) = k - 1$.

We summarize these definitions to provide a compact overview:

Definition 4.2. Let $\mathcal{T}$ be a permutation tableaux of length $n$, with $m$ columns and $k$ rows. We define

| | |
|---|---|
| **restricted** 0: | a 0 that lies below some 1; |
| **distinguished** 0: | the rightmost restricted 0 in its row; |
| **distinguished** 1: | the topmost 1 in a column if not located in the first row; |

and furthermore

| | |
|---|---|
| $rk(\mathcal{T})$: | as the number of 1's in the filling minus $m$ (also called the rank of the permutation tableau) |
| $f(\mathcal{T})$: | to be the numbers of 1's in the first row of $\mathcal{T}$, |
| $f'(\mathcal{T})$: | as the numbers of distinguished 1's, |
| $u(\mathcal{T})$: | to be the number of unrestricted rows minus 1, with a row being unrestricted if it does not contain a restricted 0, |
| $u'(\mathcal{T})$: | as the number of distinguished 0's. |

Example. Consider the permutation tableau $\mathcal{T}$ shown in Figure 4.2. First note that is of length 12 and consists of 6 columns and of 6 rows. It contains six restricted 0's of which three are also distinguished. Three is also the number of distinguished 1's that are found in the tableau. The rank $rk(\mathcal{T})$ is equal to $9 - 6 = 3$. There are three 1's in the first row, therefore $f(\mathcal{T}) = 3$, and, as already noted, $f'(\mathcal{T}) = 3$. We have that $f(\mathcal{T}) + f'(\mathcal{T})$ is equal to the number of columns. The tableau contains 3 unrestricted rows (rows that do not contain a restricted 0), and hence $u(\mathcal{T}) = 3 - 1 = 2$. We have already noted that $u'(\mathcal{T}) = 3$, and we have that $u'(\mathcal{T}) + u(\mathcal{T}) = 6 - 1 = 5$.

We can now state the first theorem that shows how, with the help of permutation tableaux, one can calculate the stationary distribution of the PASEP. (We recall that for the PASEP the probability of a particle entering at the left-hand side equals $\alpha$, of leaving at the right-hand side equals $\beta$, of hopping right equals 1 and of hopping to the left equals $q$, while for the remaining parameters we have $\gamma = \delta = 0$.) This observation was the starting point for a lot of other results (e.g. [**7**], [**5**]).

THEOREM 4.3 ([**6**], Theorem 3.1). *Consider the PASEP (with general parameters $\alpha, \beta, q$ but $\gamma = \delta = 0$) and let $\tau$ be a state of the process with $n$ sites. Let*

$$Z_n = \sum_{\mathcal{T} \in \mathcal{T}^{n+1}} q^{rk(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})}$$

*where the sum is over all permutation tableaux of length $n+1$. Then the steady state probability of the PASEP being in state $\tau$ is*

$$\frac{\sum_{\mathcal{T} \in \mathcal{T}_{shape[\lambda]}} q^{rk(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})}}{Z_n}, \tag{4.3}$$

*where the sum ranges over all permutation tableaux $\mathcal{T}$ of shape $\lambda$ with $\lambda = \lambda(\tau)$ being the shape associated to $\tau$.*

The proof is given in the next section; we first illustrate this theorem by the following example.

EXAMPLE ([**7**], Example 2.6.). Consider the PASEP with $n = 3$ sites and the state $(\circ, \bullet, \circ)$ which we denote by $\tau = (0, 1, 0)$. The path corresponding to $\tau$ is given by $(S, W, S, W)$ and therefore the associated shape consists of a row of length 2 and a row of length 1. In Figure 4.3 the shape and the three different permutation tableaux of shape $\lambda(\tau)$ are shown. So, the numerator in (4.3) is:

$$\alpha^{-2} + \alpha^{-1}\beta^{-1} + q\alpha^{-2}\beta^{-1}.$$

For the denominator $Z_n$ we find 24 different permutation tableaux of length $n + 1 = 4$ (with 8 different shapes). One can calculate

$$\begin{aligned} Z_3 = {} & \alpha^{-3} + 2\alpha^{-2} + 2\alpha^{-1} + \alpha^{-2}\beta^{-1} + 2\alpha^{-1}\beta^{-1} + 2\beta^{-1} + \alpha^{-1}\beta^{-2} + 2\beta^{-2} + \beta^{-3} \\ & + q(\alpha^{-2} + \alpha^{-2}\beta^{-1} + 4\alpha^{-1}\beta^{-1} + \alpha^{-1}\beta^{-2} + \beta^{-2}) + q^2(\alpha^{-2}\beta^{-1} + \alpha^{-1}\beta^{-2}). \end{aligned}$$

So, the steady state probability of finding the PASEP in the state (0,1,0) is given by

$$\frac{\alpha^{-2} + \alpha^{-1}\beta^{-1} + q\alpha^{-2}\beta^{-1}}{Z_3}.$$

## 4.2. Proof of Theorem 4.3

We will show that certain matrices $D_1, E_1, V_1, W_1$, for which the Matrix Ansatz equations hold, at the same time provide a generating function for permutation tableaux. More precisely, the term $\sum_{\mathcal{T} \in \mathcal{T}_{shape[\lambda]}} q^{rk(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})}$ can be expressed as

$$\sum_{\mathcal{T} \in \mathcal{T}_{shape[\lambda]}} q^{rk(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})} = W_1 \left( \prod_{i=1}^{n} (\tau_i D_1 + (1 - \tau_i) E_1) \right) V_1,$$

where $\tau = (\tau_1, \ldots, \tau_n) \in \{0, 1\}^n$ is the state of the PASEP which is associated to $\lambda$, $\tau = \tau(\lambda)$. We will also derive that $Z_n = W_1(D_1 + E_1)^n V_1$.

These relations were noted by S. Corteel and L. Williams and this section mainly follows their arguments provided in [**6**]. We define the following (infinite) matrices: set

$$D_1 = \begin{pmatrix} 0 & \beta^{-1} & 0 & 0 & \cdots \\ 0 & 0 & \beta^{-1} & 0 & \cdots \\ 0 & 0 & 0 & \beta^{-1} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{4.4}$$

and

$$E_1 = \begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 & \cdots \\ \alpha^{-1}\beta & 1+\alpha^{-1}q & 0 & 0 & \cdots \\ \alpha^{-1}\beta^2 & \beta(1+2\alpha^{-1}q) & 1+q+\alpha^{-1}q^2 & 0 & \cdots \\ \alpha^{-1}\beta^3 & \beta^2(1+3\alpha^{-1}q) & \beta(1+2q+3\alpha^{-1}q^2) & 1+q+q^2+\alpha^{-1}q^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{4.5}$$

with the $(i,j)$ entry of the lower triangular matrix $(E_1)_{i,j}$ being defined by

$$(E_1)_{i,j} = \begin{cases} \beta^{i-j}(\alpha^{-1}q^{j-1}\binom{i-1}{j-1} + \sum_{r=0}^{j-2}\binom{i-j+r}{r}q^r), & j \leq i \\ 0, & \text{otherwise} \end{cases}$$

Furthermore, let $W_1$ be the (row) vector

$$W_1 = (1, 0, 0, \ldots), \tag{4.6}$$

and $V_1$ be the (column) vector

$$V_1 = (1, 1, 1, \ldots)^T. \tag{4.7}$$

In order to prove Theorem 4.3 we first show in Lemma 4.4 that for the matrices $D_1, E_1, W_1, V_1$ the Matrix Ansatz equations (2.3) – (2.5) hold. Then, we check that the matrices have the desired combinatorial interpretation.

We start by proving the following lemma.

Lemma 4.4 ([**6**], Lemma 2.5.). *For the matrices $D_1, E_1, V_1, W_1$ defined in* (4.4) – (4.7) *the Matrix Ansatz Equations* (2.3) – (2.5) *(with $p = 1$) hold.*

Proof. We begin by showing that (2.3) holds, that is $D_1E_1 - qE_1D_1 = D_1 + E_1$. We first note that

$$(D_1E_1)_{i,j} = \beta^{-1}(E_1)_{i+1,j} = \beta^{i-j}\left(\alpha^{-1}q^{j-1}\binom{i}{j-1} + \sum_{r=0}^{j-2}\binom{i-j-r+1}{r}q^r\right),$$

if $j \leq i+1$ and it equals 0 otherwise. Then we note that

$$q(E_1D_1)_{i,j} = q\beta^{-1}(E_1)_{i,j-1} = \beta^{i-j}q\left(\alpha^{-1}q^{j-2}\binom{i-1}{j-2} + \sum_{r=0}^{j-3}\binom{i-j-r+1}{r}q^r\right)$$

if $1 \leq j-1 \leq i$ and it equals 0 otherwise. Putting this together, we obtain in the following four different cases for $((D_1E_1) - q(E_1D_1))_{i,j}$, depending on the values of $i, j$.

We find the term to be equal to

$$
\begin{aligned}
&((D_1E_1) - q(D_1E_1))_{i,j}\\
&\quad= \beta^{i-j}\left(\alpha^{-1}q^{j-1}\binom{i}{j-1} + \sum_{r=0}^{j-2}\binom{i-j+r+1}{r}q^r\right)\\
&\qquad\qquad - \beta^{i-j}q\left(\alpha^{-1}q^{j-2}\binom{i-1}{j-2} + \sum_{r=0}^{j-3}\binom{i-j-r+1}{r}q^r\right)\\
&\quad= \beta^{i-j}\alpha^{-1}q^{j-1}\binom{i-1}{j-1} + \beta^{i-j}\sum_{r=0}^{j-2}\binom{i-j+r+1}{r}q^r - \beta^{i-j}\sum_{s=1}^{j-2}\binom{i-j+s}{s-1}q^s\\
&\quad= \beta^{i-j}\alpha^{-1}q^{j-1}\binom{i-1}{j-1} + \beta^{i-j}\sum_{r=0}^{j-2}\binom{i-j+r}{r}q^r,
\end{aligned}
$$

which is the same as $(D_1+E_1)_{i,j} = (E_1)_{i,j}$ for $1 < j \le i$. Due to the above calculations, it is easy to see that $((D_1E_1)-q(E_1D_1))_{i,j}$ is equal to $\beta^{-1}$ for $j = i+1$, equal to $\alpha^{-1}\beta^{i-1}$ if $j = 1$, and it is equal to 0 when $j > i+1$, which are precisely the corresponding entries of the matrix $(D_1+E_1)_{i,j}$. Showing that the remaining Matrix Ansatz equations hold is a straightforward calculation: if we set $\gamma = \delta = 0$, then (2.4) and (2.5) read

$$D_1V_1 = \frac{1}{\beta}V_1, \tag{4.8}$$

$$W_1E_1 = \frac{1}{\alpha}W_1. \tag{4.9}$$

Now note that

$$D_1V_1 = \begin{pmatrix} 0 & \beta^{-1} & 0 & 0 & \cdots \\ 0 & 0 & \beta^{-1} & 0 & \cdots \\ 0 & 0 & 0 & \beta^{-1} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}\begin{pmatrix}1\\1\\1\\1\\\vdots\end{pmatrix} = \begin{pmatrix}\beta^{-1}\\\beta^{-1}\\\beta^{-1}\\\beta^{-1}\\\vdots\end{pmatrix} = \frac{1}{\beta}V_1. \tag{4.10}$$

We also note that multiplying $E_1$ by $W_1$ from the left gives a row vector whose entries are equal to the first row of $E_1$,

$$(\alpha^{-1},0,0,\ldots) = \frac{1}{\alpha}(1,0,0,\ldots),$$

which shows that $W_1E_1 = \frac{1}{\alpha}W_1$ also holds. □

So far we have shown that the matrices $D_1, E_1, V_1, W_1$ satisfy the Matrix Ansatz equations. Note, that all entries of the matrices $D_1, E_1, V_1, W_1$ are non-negative and hence the product $W_1(\prod_{i=1}^{n}(\tau_iD_1 + (1-\tau_i)E_1))V_1$ is always non-negative. So, all assumptions of the Matrix Ansatz Theorem (Theorem 2.1) hold. Hence the steady state probability for any state $\tau$ of the PASEP (with $\gamma = \delta = 0$) can be expressed as a product of these matrices. But our focus lies upon combinatorial approaches to the subject and hence we show how these matrices are linked to permutation tableaux. So, in the next step we shall see that the term $W_1(\prod_{i=1}^{n}(\tau_iD_1 + (1-\tau_i)E_1))V_1$ can be interpreted as a generating function for permutation tableaux:

THEOREM 4.5 ([**6**], Theorem 3.1.). *Let $\lambda$ be a shape of length $n+1$. Let $\tau(\lambda) = (\tau_1, \ldots, \tau_n) \in \{0,1\}^n$ be the state of the PASEP corresponding to the shape $\lambda$. Then, for $F_\lambda(q)$, the generating function for all permutation tableaux of shape $\lambda$ defined below, we have*

$$F_\lambda(q) := \sum_{\mathcal{T} \in \mathcal{T}_{shape[\lambda]}} q^{rk(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})} = W_1 \left( \prod_{i=1}^{n} (\tau_i D_1 + (1-\tau_i) E_1) \right) V_1.$$

*Moreover, for $F^{n+1}(q)$, the generating function for all permutation tableaux of length $n+1$ defined below, we have*

$$F^{n+1}(q) := \sum_{\mathcal{T} \in \mathcal{T}^{n+1}} q^{rk(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})} = W_1 (D_1 + E_1)^n V_1.$$

To prove this theorem we use the following *partition notation* (which is inspired by interpreting the shape $\lambda$ of a permutation tableau as the Young diagram of a partition). We see a shape as a collection of rows consisting of cells. So, we denote a shape $\lambda$ of a permutation tableau of length $n$ by $\lambda = (r_1, \ldots, r_k)$ with $k$ being the number of rows (therefore $k \geq 1$) and $r_i$ being the number of cells in the $i$-th row, where we start counting at the top row of the shape (therefore $r_i \in \mathbb{Z}$ and $r_k \geq r_{k-1} \ldots \geq 0$). It follows that $r_k + k = n$. As an example, the shape in Figure 4.2 is represented by $\lambda = (6, 5, 5, 3, 2, 0)$. This notation allows us to denote a shape in a short and intuitive way. (Note that this notation can only be used because we require the shape path of a permutation tableau to start with a south step. If we used shapes whose shape path starts with a west step, we would have at least one column of length 0 (caused by the west step) – this case could not be covered by the newly introduced notation).

PROOF. The theorem is proved by induction. We consider the matrix

$$M_\lambda := \prod_{i=1}^{n} (\tau_i D_1 + (1-\tau_i) E_1),$$

where $(\tau_1, \ldots, \tau_n) = \lambda(\tau)$, the shape associated to the state $\tau$. We shall see that the sum of the entries in the top row corresponds to the generating function $F_\lambda(q)$.

We denote by ${}_i\mathcal{T}_{shape[\lambda]}$ permutation tableaux $\mathcal{T}$ of shape $\lambda$ which have exactly $i$ unrestricted rows and define

$${}_iF_\lambda(q) := \sum_{\mathcal{T} \in {}_i\mathcal{T}_{shape[\lambda]}} q^{rk(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})} = \sum_{\mathcal{T} \in {}_i\mathcal{T}_{shape[\lambda]}} q^{rk(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-i+1}.$$

Recall that $u(\mathcal{T})$ was defined as the number of unrestricted rows minus 1 and that in every permutation tableau the top row is always unrestricted. It is then clear that $F_\lambda(q) = \sum_{i \geq 1} {}_iF_{shape[\lambda]}(q)$.

We will now show that the entry $M_\lambda[1,i]$ in position $(1,i)$ of $M_\lambda$ is ${}_iF_\lambda(q)$. To do this, we use induction on $n$, where $n$ is the number of sites in the PASEP (and $n+1$ is the length of the shape $\lambda$). Recall the notation for permutation tableaux introduced above.

$\mathbf{n = 1}$ : In the case of one site and length $l = 2$, there are two possible different shapes, namely $(0,0)$ and $(1)$ (partition notation!) as shown in Figure 4.4. If $\lambda = (0,0)$, then the corresponding state is $\tau(\lambda) = (1)$, and hence $M_\lambda = D_1$. So, the top row of

$M_\lambda$ is $(0, \beta^{-1}, 0, 0, \ldots)$. There is just one permutation tableau of shape $(0, 0)$ with no column but two rows. For this permutation tableau $\mathcal{T}$ we have $rk(\mathcal{T}) = 0$, $u(\mathcal{T}) = 1$ and $f(\mathcal{T}) = 0$ (since there are no 1's in the filling but two unrestricted rows). So the only ${}_iF_\lambda(q) \neq 0$ is ${}_2F_\lambda(q) = \beta^{-1}$. This corresponds to the top row of $D_1$. Similarly, if $\lambda = (1)$ then $M_\lambda = E_1$ since $\tau(\lambda) = (0)$. Again, there is just a single permutation tableau, this time consisting of one cell (that has to be filled with a 1). For this tableau the statistics are $rk(\mathcal{T}) = 0$ , $f(\mathcal{T}) = 1$ and $u(\mathcal{T}) = 0$ (since it has one unrestricted row). This corresponds to the top row of $E_1$ being $(\alpha^{-1}, 0, 0, \ldots)$.


(1) (0,0)

FIGURE 4.4. Permutation tableaux of length $l = 2$.

$\mathbf{n} \mapsto \mathbf{n+1}$ : Assuming that the claim is true for $n$ (and therefore for permutation tableaux of length less than or equal to $n + 1$), we can interpret the $i$-th entry in the top row of $M_\lambda$ as a generating function enumerating permutation tableaux of shape $\lambda$ with $i$ unrestricted rows, according to weight.

Now, consider a permutation tableau of length $n + 2$ of shape $\lambda'$. There are two different cases.

(1) The last row of the shape $\lambda'$ is an empty row. In this case the last step of the shape path is a south step (which forms the row of length 0).
(2) The last row of the shape $\lambda'$ is not an empty row. Then, the length of the leftmost column of $\lambda'$ is equal to the number of rows of the shape. This is the case if the last step of the shape path is a west step.

If we delete the last step of the shape path of $\lambda'$, we arrive at a shape $\lambda$ of length $n+1$. We will make use of this observation, but we go the other way around: if we consider a shape $\lambda$ of length $n + 1$, then we can add either a south step or a west step to the shape path to obtain a shape $\lambda'$ of length $n + 2$. Note that every shape of length $n + 2$ can be obtained through addition of a step to a shape of length $n + 1$. What we need to do is to check how one can obtain a valid filling of the shape $\lambda'$ out of a valid filling of the shape $\lambda$, and how this influences the generating function.

First, how does adding a south step to the shape path of $\lambda$ affect the generating function? Any permutation tableau $\mathcal{T}$ of shape $\lambda'$ can be obtained from a permutation tableau of shape $\lambda$ by adding an empty (and therefore unrestricted) row. So, in this case we have ${}_1F_{\lambda'}(q) = 0$ and ${}_iF_{\lambda'}(q) = \beta^{-1}\ {}_iF_\lambda(q)$ if $i > 1$. Comparing this to the matrix product $M_\lambda D_1$, we see that the top row $(0, \beta^{-1}M_\lambda[1, 1], \beta^{-1}M_\lambda[1, 2], \ldots)$ indeed reflects these identities.

The case of addition of a west step to the shape path of $\lambda = (\lambda_1, \ldots, \lambda_r)$ remains to be examined. In this case, an additional column is attached to the left-hand side of the tableau. Speaking in terms of partitions, a new partition $\lambda' = (\lambda_1 + 1, \ldots, \lambda_r + 1)$ is obtained. We will show that

$$ {}_aF_{\lambda'}(q) = \sum_{b \geq a} h_{a,b}(q)\ {}_bF_\lambda(q), $$

where

$$ h_{a,b}(q) := \beta^{b-a} \left( \alpha^{-1} q^{a-1} \binom{b-1}{a-1} + \sum_{j=0}^{a-2} q^j \binom{b-a+j}{j} \right). \tag{4.11} $$

To prove this, we will examine how addition of a column of length $r$ to the left-hand side to a permutation tableau in ${}_b\mathcal{T}_{shape[\lambda]}$ will affect the terms of the generating function ${}_bF_\lambda(q)$: fix a permutation tableau $\mathcal{T}$ of shape $\lambda$ with exactly $b$ unrestricted rows and consider the terms of the generating function ${}_bF_\lambda(q)$ containing $q^{rk(\mathcal{T})}$. If we now add a column to the left-hand side of the tableau $\mathcal{T}$, and consider the possibilities of adding a valid filling, we claim that the generating function ${}_aF_{\lambda'}(q)$ of the permutation tableaux $\mathcal{T}'$ of shape $\lambda'$ with exactly $a$ unrestricted rows can be obtained by multiplying the term containing $q^{rk(\mathcal{T})}$ of ${}_bF_\lambda(q)$ by $h_{a,b}(q)$. The claim can be proven by examining how an additional column can be filled with 0's and 1's such that a valid filling is obtained, and how this affects the statistics of the permutation tableau: the tableau $\mathcal{T}$ has exactly $b$ unrestricted rows; so in the newly added column we have to put a 0 in every already restricted row (since a row is restricted if it contains a 0 entry below some 1 and so all entries to the left of this 0 have to be 0's as well). Let us label the remaining cells yet to fill from top to bottom by $c_1, \ldots, c_b$. If we want the new tableau $\mathcal{T}$ to have $a$ unrestricted rows (where $a \leq b$), and if we suppose that the topmost 1 in the new column is in position $c_i$, then we will have $i-1$ entries that are already 0. Since we want our new permutation tableau to have $b-a$ additional restricted rows, we need to place another $b-a$ 0's within the $b-i$ cells below the topmost 1 in $c_i$; there are $\binom{b-i}{b-a}$ ways to do so. The remaining $(b-i)-(b-a) = a-i$ entries must be 1's. For these choices of filling the column, the weights change as follows: the rank $rk(\mathcal{T})$ rises by $a-i$, therefore the column contributes an extra weight of $q^{a-i}$ if $i \neq 1$, whereas in the case of $i = 1$ the extra weight contributed equals $\alpha^{-1}q^{a-i}$ for having also added an additional 1 to the first row. In both cases, we have added $b-a$ unrestricted rows and there is an extra weight of $\beta^{b-a}$. Summing over all possibilities of adding a column to the permutation tableau, we obtain

$$ \alpha^{-1} q^{a-1} \binom{b-1}{a-1} + \sum_{i=2}^{a} q^{a-i} \binom{b-i}{b-a}, \tag{4.12} $$

which is equal to

$$ \alpha^{-1} q^{a-1} \binom{b-1}{a-1} + \sum_{j=0}^{a-2} q^j \binom{b-a+j}{j}. \tag{4.13} $$

It therefore follows that ${}_aF_{\lambda'}(q) = \sum_{b \geq a} h_{a,b}(q)\ {}_bF_\lambda(q)$. As desired, this corresponds to the first row of the matrix product $M_\lambda E_1$, being equal to

$$ \left( \sum_{b\geq 1} h_{1,b}(q) M_\lambda[1,b], \sum_{b \geq 2} h_{2,b}(q) M_\lambda[1,b], \sum_{b\geq 3} h_{3,b}(q) M_\lambda[1,b], \ldots \right). $$

This shows that the entry $M_{\lambda'}[1,i]$ in position $(1,i)$ of $M_{\lambda'}$ is ${}_iF_{\lambda'}(q)$ as claimed in the beginning.

The first part of the theorem is now derived easily: fix a shape $\lambda$. Note that multiplying the product $M_\lambda = \prod_{i=1}^{n}(\tau_i D_1 + (1-\tau_i)E_1)$ by $W_1$ from the left and by $V_1$ from the right is equal to summing the top row of $M_\lambda$. As shown, the entries of the top row, $M_\lambda[1,i]$, are equal to ${}_iF_{\lambda'}(q)$. Now, recall that $\sum_{i\geq 1}\ {}_iF_{shape[\lambda]}(q) = F_\lambda$, and hence

$$\sum_{\mathcal{T}\in\mathcal{T}_{shape[\lambda]}} q^{rk(\mathcal{T})}\alpha^{-f(\mathcal{T})}\beta^{-u(\mathcal{T})} = F_\lambda = W_1\left(\prod_{i=1}^{n}(\tau_i D_1 + (1-\tau_i)E_1)\right)V_1.$$

The second part, stating that the generating function $F^{n+1}$ of all permutation tableaux with length $n+1$ is equal to $W_1(D_1+E_1)^n V_1$ follows from the fact that, for the generating function, summing over all permutation tableaux of length $n+1$ yields the same as summing up all the expressions $\prod_{i=1}^{n}(\tau_i D_1 + (1-\tau_i)E_1)$ over all words $\tau\in\{0,1\}^n$ which can also be written as $W_1(D_1+E_1)^n V_1$. $\square$

We now have seen a combinatorial interpretation of the matrices $D_1, E_1, V_1, W_1$ in terms of a generating function for permutation tableaux. Combining this last theorem and Lemma 4.4, Theorem 4.3 follows immediately:

Proof of Theorem 4.3. Due to Lemma 4.4, we know that for $D_1, E_1, V_1, W_1$ the Matrix Ansatz equations hold and that therefore the steady state probability of finding the PASEP in a state $\tau\in\{0,1\}^n$ is given by

$$\frac{W_1(\prod_{i=1}^{n}(\tau_i D_1 + (1-\tau_i)E_1))V_1}{W_1(\prod_{i=1}^{n}(D_1+E_1)^n)V_1}. \tag{4.14}$$

On the other hand, we have seen in Theorem 4.5 that numerator and denominator in (4.14) can be rewritten as

$$\frac{\sum_{\mathcal{T}_{shape[\lambda]}} q^{rk(\mathcal{T})}\alpha^{-f(\mathcal{T})}\beta^{-u(\mathcal{T})}}{\sum_{\mathcal{T}^{n+1}} q^{rk(\mathcal{T})}\alpha^{-f(\mathcal{T})}\beta^{-u(\mathcal{T})}}, \tag{4.15}$$

where $\lambda=\lambda(\tau)$, the shape corresponding to $\tau$. Since Theorem 2.1 claims that for any $D, E, V, W$ for which the Matrix Ansatz hold the term $W(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))V$ yields the same result (namely the steady state probability of finding the PASEP in state $\tau$), it follows that

$$\frac{W(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))V}{W(\prod_{i=1}^{n}(D+E)^n)V} \tag{4.16}$$

is equal to (4.14) and therefore as well to (4.15). This is precisely what the theorem claims. $\square$

Remark. As mentioned in the beginning, Theorem 4.3 is due to S. Corteel and L. Williams [**6**]. In a subsequent publication they were able to reproduce their result concerning the stationary distribution, this time bypassing the Matrix Ansatz. They constructed a Markov chain on permutation tableaux (and therefore on permutations) that *projects* to the PASEP, in a sense that a random walk on the constructed Markov chain is indistinguishable from a random walk on the PASEP. For more details on this approach we refer to [**5**].

CHAPTER 5

# PASEP and Alternative Tableaux

It was noted by several people (A. Burstein [**3**], S. Corteel and P. Nadeau [**4**]) that permutation tableaux are determined by the position of their distinguished 1's and 0's. (Recall that a 1 is called distinguished if it is the topmost one in its column but does not lie in the first row, whereas a 0 is called distinguished if it is the rightmost restricted 0 in its row.) This observation led to the definition of so called *alternative tableaux* which were first introduced by X. Viennot [**27**]. In the next section we will give the definition as well as a bijection between *alternative tableaux* and permutation tableaux. We will than restate Theorem 4.5, and we will give another result concerning the steady state distribution of the symmetric case of the simple exclusion process proved by S. Corteel and L. Williams [**7**].

The following definition was given by X. Viennot ([**27**]): an *alternative tableau* $\widehat{\mathcal{T}}$ is a shape $\widehat{\lambda}$ (where we allow rows and columns of length 0), in which cells are either empty, red or blue, such that the following conditions hold:

(1) there is no colored cell left of a blue cell;
(2) there is no colored cell above a red cell.

Another way to put this is the following, noted by Phillipe Nadeau, which is the definition we will use:

Definition 5.1. ([**17**], Definition 2) An **alternative tableau** $\widehat{\mathcal{T}}$ is a shape $\widehat{\lambda}$ (where we allow rows and columns of length 0), in which cells are either empty or contain a left arrow $\leftarrow$ or an up arrow $\uparrow$, such that the following conditions hold:

(1) there is no other arrow left of $\leftarrow$;
(2) there is no other arrow above a $\uparrow$.

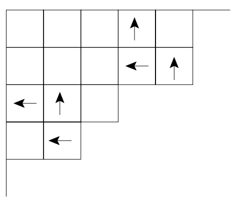


Figure 5.1. An alternative tableau of length 11.

We denote the **set of alternative tableaux of shape** $\widehat{\lambda}$ by $\widehat{\mathcal{T}}_{shape[\widehat{\lambda}]}$ and the **set of alternative tableaux of length** $n$ by $\widehat{\mathcal{T}}^n$.

Remark. Since an alternative tableau is based upon a shape (just like a permutation tableau), it inherits most of the concepts that we have introduced for permutation tableaux, i.e., length and shape path.

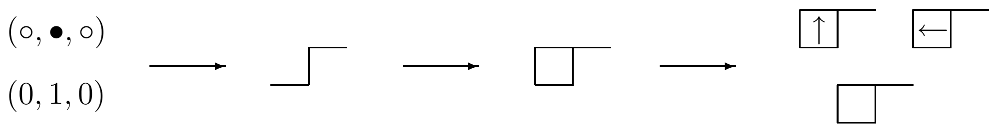



FIGURE 5.2. An example for the association of a state of the PASEP to the alternative tableaux of a certain shape.

Similarly to what we did in Section 4.1, we can associate alternative tableaux of a certain shape associate to states of the PASEP. Let $\tau = (\tau_1, \ldots, \tau_n) \in \{0,1\}^n$ be a state of the PASEP with $n$ sites. To the state $\tau$ we associate a path $p = (p_1, \ldots, p_n) \in \{W, S\}^n$ consisting of south steps (denoted by $S$) and west steps (denoted by $W$) by the rule:

$$p_i = \begin{cases} S & \tau_i = 1, \\ W & \tau_i = 0. \end{cases} \tag{5.1}$$

Analogue to the procedure in (4.2), we interpret $p$ as the shape path, and thus obtain a shape $\widehat{\lambda}(\tau) = \widehat{\lambda}$ associated to the state $\tau$. Lastly, we consider the set of alternative tableaux of shape $\widehat{\lambda}$, denoted by $\widehat{\mathcal{T}}_{\widehat{\lambda}}$. These are the alternative tableaux that we associate to the state $\tau$. The procedure is shown in (5.2) and Figure 5.2.

$$\text{state } \tau \;\rightarrow\; \text{(shape) path } p \;\rightarrow\; \text{shape } \widehat{\lambda}(\tau) \;\rightarrow\; \widehat{\mathcal{T}}_{shape[\widehat{\lambda}]} \tag{5.2}$$

Through considerations of statics on alternative tableaux, we will again be able to derive results concerning the steady state distribution of the PASEP. So, like permutation tableaux, alternative tableaux also encode information about the PASEP. We will even see that they encode it in a somehow more elegant way. But first we examine a bijection between permutation tableaux and alternative tableaux introduced in [**27**] and the correspondence of some statistics of permutation tableaux and alternative tableaux (noted in [**27**] and [**17**]).

## 5.1. Bijection Between Permutation Tableaux and Alternative Tableaux

First, as we will need this through the entire section, we introduce classes of entries of a filling. We therefore recall and extend Definition 4.2.

DEFINITION 5.2. We consider the entries of a filling of a permutation tableau. We recall that a restricted 0 is a 0 which lies below some 1 and we define:

**top-row** 0: a 0 which is located in the top row in the permutation tableau;
**distinguished** 0: is the rightmost restricted 0 in its row;
**limited** 0: a restricted 0 which is not a distinguished 0;
**additional** 0: all 0's which are neither restricted 0's nor top-row 0's;
**top-row** 1: a 1 which is located in the top row in the permutation tableau;
**distinguished** 1: a 1 which is the topmost one in its column, but not a top-row 1;
**additional** 1: all 1's that are neither top-row 1's nor distinguished 1's.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 2 | 1 | 0 | 1 | 1 | 0 | |
| 3 | 0 | 0 | 0 | 0 | 1 | |
| 4 | 0 | 1 | 1 | | | |
| 5 | 0 | 0 | | | | |
| 6 | | | | | | |

FIGURE 5.3. A permutation tableaux with rows and columns labeled.

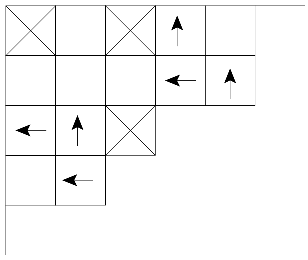


FIGURE 5.4. Example for free cells in an alternative tableau (marked with X).

One can easily verify that each entry of a permutation tableau belongs to exactly one of the classes. We call the classes **type of entry**.

EXAMPLE. Let us consider the permutation tableau in Figure 5.3. There are exactly three representatives of each class defined above. In the figure we have labeled the columns and rows by 1 through 6. By $(x, y)$ we denote the entry of the tableau that is found in the $x$-th row and the $y$-th column. Then, the representatives are to be found in the following positions:

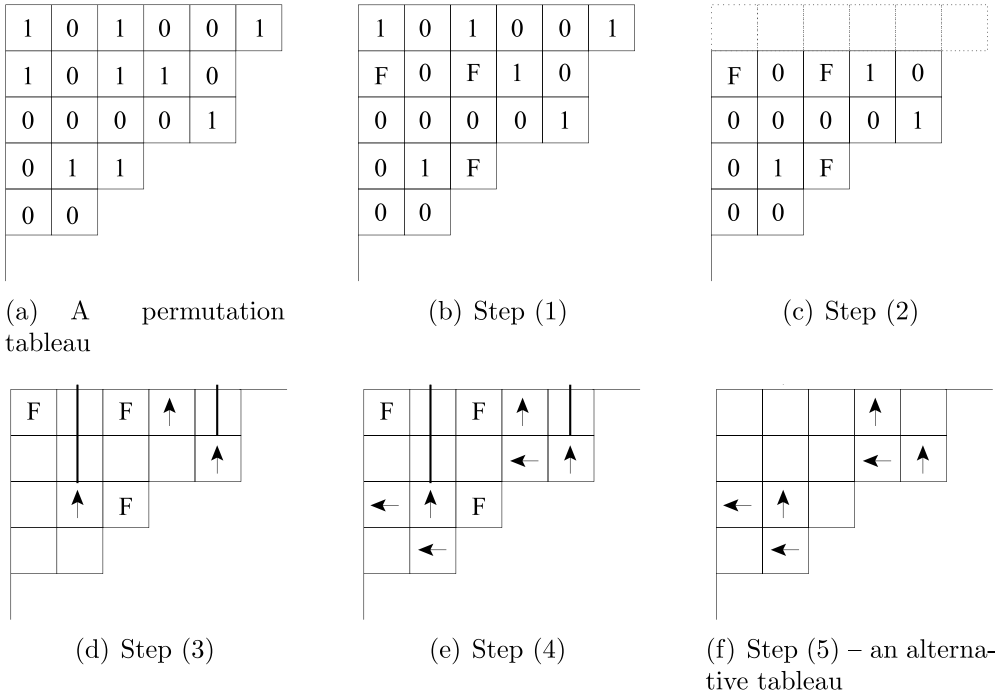
$$\begin{aligned} \text{top-row 0:} \quad & (1,2),(1,4),(1,5) \\ \text{distinguished 0:} \quad & (3,4),(4,1),(5,2) \\ \text{limited 0:} \quad & (3,3),(3,1),(5,1) \\ \text{additional 0:} \quad & (2,2),(2,5),(3,2) \\ \text{top-row 1:} \quad & (1,1),(1,3),(1,6) \\ \text{distinguished 1:} \quad & (2,4),(3,5),(4,2) \\ \text{additional 1:} \quad & (2,1),(2,3),(4,3) \end{aligned}$$

We also need an additional definition concerning the entries of alternative tableaux:

DEFINITION 5.3 ([**17**], Definition 2). In the filling of an alternative tableau we call an (empty) cell that does not have a $\uparrow$ below it and no $\leftarrow$ to its right a **free cell**. In other words, in an alternative tableau free cells are exactly the empty cells to which no arrow points to.

EXAMPLE. For an example of free cells see Figure 5.4. There, the three free cells in the tableau are marked with an X.

After these preparations we now aim to define a bijection between permutation tableaux and alternative tableaux. To this end, we define two maps $\varphi$ and $\psi$, both of



(a) A permutation tableau (b) Step (1) (c) Step (2)

(d) Step (3) (e) Step (4) (f) Step (5) – an alternative tableau

FIGURE 5.5. The steps of the algorithm $\varphi$.

them described through an algorithm. We then carefully check that these two algorithms actually have the desired properties, i.e., that $\varphi$ sends permutation tableaux to alternative tableaux, that $\psi$ does so vice-versa, and that the maps $\varphi$ and $\psi$ are inverse to each other. The maps $\phi, \psi$, as well as the outline of the proof of Theorem 5.8, were first presented in [**27**].

We start by defining $\varphi$, a map that sends a permutation tableau to an alternative tableau, through an algorithm. We will show then, that what we obtain by $\varphi$ is indeed an alternative tableau. The steps are illustrated in Figure 5.5.

(1) Every 1 which is not the topmost 1 in its column is marked by an $F$.
(2) Delete the first row and turn all the remaining 1's into a $\uparrow$.
(3) Delete all the remaining entries and mark all the cells above a $\uparrow$ (by extending the $\uparrow$ by a line).
(4) In every row that contains empty, not marked cells turn the rightmost of these cells into a $\leftarrow$.
(5) Delete the marks and the $F$'s.

Considering the single steps of the algorithm $\varphi$ (we identify the algorithm with the map $\varphi$), we note that it terminates with having produced a new shape (that differs from the original shape by lacking the first row) together with a filling. We still need to make sure that this filling satisfies the conditions of an alternative tableau, i.e., that no arrow is above a $\uparrow$ and that no arrow is left to a $\leftarrow$. We will do so after proving a lemma concerning the algorithm $\varphi$ and its actions upon the entries of a permutation tableau.

First, recall Definition 5.2. We will show that $\varphi$ treats entries of same type the same way. As every entry in a permutation tableau belongs to exactly one of these types, we can easily see how $\varphi$ acts on the entries. We examine the steps of the algorithm $\varphi$, bearing the definition of types of entries in mind.

(1′) In Step (1) all additional 1's are marked by an $F$.
(2′) In Step (2) all top-row 1's and top-row 0's are deleted. Furthermore, distinguished 1's are turned into ↑'s (these are the only 1-entries that are left at this time, since top-row 1's are already deleted and additional 1's are marked by an $F$).
(3′) In Step (3) all remaining entries are deleted. Hence, every cell that contains an additional, distinguished, or limited 0 is now emptied. Note that these are the only empty cells. Now all cells above a ↑ are marked. These cells correspond to the additional 0's, since ↑ used to be (distinguished) 1's, and neither distinguished nor limited 0's can lie above a 1.
(4′) In Step (4) the empty, non-marked cells that remain correspond to restricted 0's (i.e., limited or distinguished 0's). The rightmost empty cell in each row is, by definition, a distinguished 0 and turned into a ← by $\varphi$.
(5′) In Step (5) the marks and the $F$'s are deleted.

Through these considerations we can immediately derive the following lemma:

Lemma 5.4 ([**27**]). *The map $\varphi$ maps*

- *all distinguished* 1*'s to* ↑*'s but maps no other entries to* ↑*'s;*
- *all distinguished* 0*'s to* ←*'s but maps no other entries to* ←*'s;*

Proof. That the first claim holds is easy to see: in (2′) every distinguished 1 is turned into a ↑. Later on no ↑'s are introduced.

The second claim is shown similarly: in (4′) every distinguished 0 is turned into a ←. Again, this is the only step in which ← are introduced. □

We have not made sure yet that the application of $\varphi$ to a permutation tableau indeed yields an alternative tableau. To do so, we still have to check that there is no arrow above a ↑ or left to a ←. We use the lemma above.

Consider an up arrow ↑. By Lemma 5.4, ↑'s correspond to a distinguished 1's. Now, suppose that there is another ↑ above the fixed up arrow. This means that there are two distinguished 1's in the same column. This cannot be by definition of distinguished 1. On the other hand, suppose that there is a ← above the fixed up arrow. Then note that a ← corresponds to a distinguished 0, and a distinguished 0 lies below some 1. Hence, we would have a 1 above a distinguished 1 which cannot be by definition of a distinguished 1.

Now, fix a left arrow ←. ←'s used to be a distinguished 0's before the application of $\varphi$. Hence, if there is another ← to the left of the fixed one, then there are two distinguished 0's in a row which cannot occur in a permutation tableau. If there is a ↑ left to the fixed ←, then this corresponds to finding a (distinguished) 1 to the left of a distinguished 0. This cannot be, either.

Thus, we have shown the following:

Lemma 5.5 ([**27**]). *$\varphi$ is a map from the set of all permutation tableaux to the set of alternative tableaux.*

We note another property of $\varphi$ which will turn out to be important later:

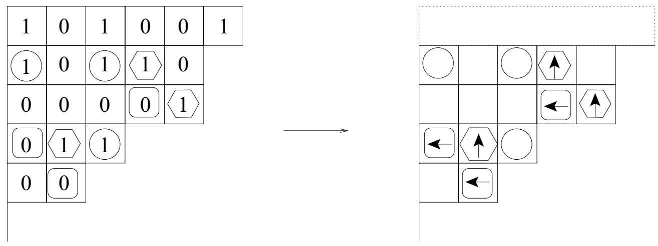


FIGURE 5.6. Correspondence between distinguished 0's, distinguished 1's and additional 1's in a permutation tableaux $\mathcal{T}$ and left arrows, up arrows and free cells in an alternative tableau.

LEMMA 5.6 ([**27**]). *The map $\varphi$ maps additional 1's to free cells but maps no other entries to free cells.*

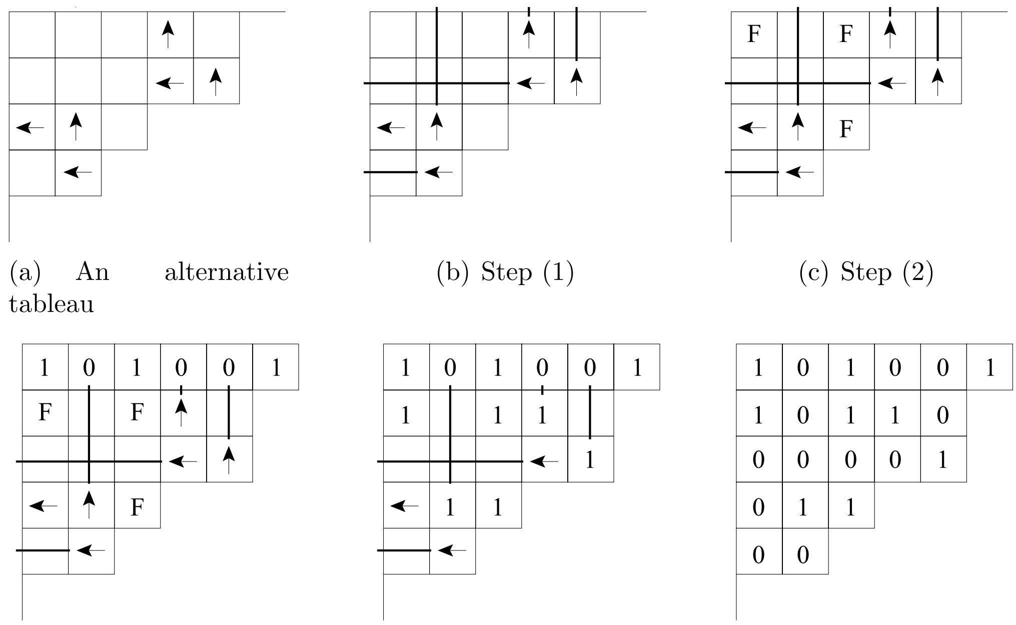
PROOF. To prove this claim, we have to go a little more into detail. Consider an additional 1. By $\varphi$ this cell is turned into an empty cell (first the cell is marked by an $F$, and then the $F$ is deleted). Now, suppose that there is a $\uparrow$ below this cell. We already know that a $\uparrow$ used to be a distinguished 1. But this would mean that there was a distinguished 1 below an additional 1 which cannot be by definition. On the other hand, suppose that a $\leftarrow$ points towards a cell that used to contain an additional 1. Since $\leftarrow$ used to be distinguished 0's (as seen before), we would have a 1 left to a distinguished 0. This cannot be by definition.

It is also true, that any other cell, which is not an additional 1, is either filled by an arrow or has some arrow pointing towards it: top row entries disappear through $\varphi$ so we do not need to consider them. Distinguished 1's and distinguished 0's correspond to $\uparrow$ and $\leftarrow$, respectively. So, it only remains to show that additional 0's are mapped to cells that at least have an arrow pointing towards it. But this is easy to prove, since by definition an additional 0 is not restricted and thus has to lie above a distinguished 1 (which is the topmost 1 in its column). Since distinguished 1's correspond to $\uparrow$, we have shown that there is an arrow pointing towards a cell that used to be filled with a 0. $\square$

Now, to go in the other direction, we define a map $\psi$ (again described through an algorithm) that maps any alternative tableau $\widehat{\mathcal{T}}$ to a permutation tableau $\mathcal{T}$. The steps are illustrated in Figure 5.7:

(1) Mark the cells left of $\leftarrow$ and above $\uparrow$ (by extending the arrows by a line, denoting the marked cells by ◫, ⊟, ⊞).
(2) Put an $F$ in the cells that are not marked nor contain an arrow.
(3) Add a new row whose length equals the number of columns in the tableau, and put a 0 in the cells to which a $\uparrow$ points to and fill the other cells with 1's.
(4) Turn all $\uparrow$'s and all $F$ into 1's.
(5) Turn all $\leftarrow$'s and the remaining marked cells ( ◫, ⊟, ⊞ ) into 0's.

It is clear that we obtain a shape with a filling of 0's and 1's: all cells are filled when the algorithm stops (there are non-empty cells containing arrows, empty cells that are marked in Step (2) or empty cells that are not marked (Step (5)), and all of them are somehow filled once during the algorithm). What we need to check is whether we always obtain a *valid* filling (see Definition 4.1). Seeing that the first condition holds




(a) An alternative tableau

(b) Step (1)

(c) Step (2)

(d) Step (3)

(e) Step (4)

(f) Step (5) – a permutation tableau

FIGURE 5.7. The steps of the algorithm $\psi$.

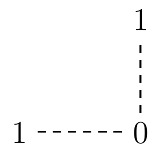



FIGURE 5.8. The forbidden pattern in a permutation tableau.

(i.e., that there is at least one entry being 1 in each column) is easy: either a column of the alternative tableau contains a $\uparrow$, then this arrow is turned into a 1 in Step (4), or the column does not contain a $\uparrow$, then a 1 is put into the corresponding entry of the additional row in Step (3). So, it remains to check that the second condition holds: if a 0 is below any 1 then there is no 1 to the left in the same row of this 0. We call this the forbidden pattern (see Figure 5.8).

So first, we check which cells of the alternative tableau are turned into 0's by the algorithm $\psi$. Then, we check whether this 0 will be below a 1, and if so, we check that there can only be 0's left to it. First, we recall the single steps of $\psi$ and note that by $\psi$ a 0 is introduced

(a) eventually in the added first row. Or in the rows below in one of the following cases:
(b) when replacing a $\leftarrow$,
(c) to the left of a $\leftarrow$ but not above a $\uparrow$

(d) above a $\uparrow$ and not to the left of a $\leftarrow$.
(e) to the left of a $\leftarrow$ *and* above a $\uparrow$

(The last three cases correspond to the marked cells ⊟, ◫, and ⊞ in the description of $\psi$). In case (a) the 0 cannot lie below any 1 (since it was injected in the first row). In the cases (b),(c),(e) the cell either contained a $\leftarrow$ or was marked in Step (1) by ⊟ or ⊞. So in all three of the cases the cells to the left are of type ⊟ or ⊞ and therefore become 0's in Step (4); the forbidden pattern from Figure 5.8 cannot occur. In the remaining case, case (d), all cells above $\uparrow$ are of type ◫ or ⊞ and hence become 0's in Step (4). Therefore there can not be a 1 above the considered cell and there is no need to check the entries to its left. This shows that through the map $\psi$ we obtained indeed a permutation tableau.

We summarize the considerations made above in the following lemma:

Lemma 5.7 ([**27**]). *$\psi$ is a map from the set of all alternative tableaux to the set of permutation tableaux.*

We can now state the main results of this section.

Theorem 5.8 ([**27**]). *The map $\varphi$ is a bijection between the set of all permutation tableaux and the set of all alternative tableaux, and its inverse is given by $\psi$.*

Proof. We have already proven that $\varphi$ maps any permutation tableau $\mathcal{T}$ to a unique alternative tableau, and that $\psi$ maps any alternative tableau to a unique permutation tableau. It remains to show that $\varphi^{-1} = \psi$. To do so, we will prove that for any fixed permutation tableau $\mathcal{T}$ it is true that $\mathcal{T} = \psi(\varphi(\mathcal{T}))$. First, we note that $\mathcal{T}$ and $\psi(\varphi(\mathcal{T}))$ have the same shape $\lambda$. (This is easy to see in terms of the corresponding shape paths: $\varphi$ erases the first south step of $\lambda$ by deleting the entire first row and $\psi$ adds a south step to the beginning of the shape path by adding a new row). It remains to show that the fillings of $\mathcal{T}$ and $\psi(\varphi(\mathcal{T}))$ are the same. We consider each cell separately and show how applications of $\varphi$ and $\psi$ changes their contents. We claim that every 0 in $\mathcal{T}$ is mapped to a 0 and every 1 to a 1 after having applied $\varphi$, followed by $\psi$. We start with the entries 1 and separate them into the following types, as defined in Definition 5.2.

- distinguished 1's: those are 1's that are topmost in their column but are not situated in the first row;
- additional 1's: those are 1's which lie below some other 1;
- top-row 1's: are located in the top-row.

We show that every type is mapped to a 1 again:

*distinguished* 1*'s*: $\varphi$ maps a distinguished 1 to a $\uparrow$ (Step (2)). Afterwards, $\psi$ maps every $\uparrow$ back to a 1 (Step (4)).

*additional* 1*'s*: $\varphi$ maps an additional 1 to a free cell, and $\psi$ maps free cells to 1's (Step (4)).

*top-row* 1*'s* The top row cells with a 1 are deleted by $\varphi$. But note that then no $\uparrow$ is in the corresponding column (Step (2)). When afterwards $\psi$ adds a new first row, then there is no arrow pointing to this cell and hence it is filled with a 1 (Step (3)).

It remains to check that $\psi(\varphi(\mathcal{T}))$ maps 0's to 0's: first, let us consider 0's in the top row. If a 0 is in the top row of the permutation tableau, then the cell is deleted after having applied $\varphi$ (because the whole row is deleted). But in the corresponding row there remains a $\uparrow$ (since the topmost 1 of every column, which obviously is not located in the top row, is turned into a $\uparrow$ in Step (2)). So, when applying $\psi$, a new row is added, and the cell in the column is filled with a 0 due to Step (3). If we consider a 0 that is not part of the top row, then it is either turned into a $\leftarrow$ or into an empty cell (which cannot be a free cell since only 1's are mapped into them, see Lemma 5.4). But since $\psi$ maps all cells of type $\leftarrow$, ◫, ⊟, ⊞ (which are exactly the cells that are not free cells or $\uparrow$) to 0's, we are done. □

Putting together some of the properties of the maps $\varphi$ and $\psi$, we make some observations.

Focusing on shapes of tableaux only, we easily note the following:

Corollary 5.9. *Let $\lambda$ be a shape and let $\widehat{\lambda}$ be the shape arising from $\lambda$ by removing the top row. Then $\varphi$ is a bijection between the set of permutation tableaux of shape $\lambda$ and the set of alternative tableaux of shape $\widehat{\lambda}$.*

So far we have seen that $\varphi$ is a bijection between permutation tableaux and alternative tableaux. If we consider an entry in a permutation tableau $\mathcal{T}$, we know that in the permutation tableau $\psi(\varphi(\mathcal{T}))$ we find the same entry in the same position. In particular, 1's are mapped to 1's again and 0's to 0's. But one could doubt that the types of entries (as defined in Definition 5.2) are preserved, e.g., that a distinguished 1 in $\mathcal{T}$ is again distinguished in the permutation tableau $\psi(\varphi(\mathcal{T}))$. But it is seen easily that types are preserved as well: note that the type of an entry depends only on its surrounding and its position in the tableau, e.g., a distinguished 0 is a 0 that lies below some 1 and is the rightmost 0 in its row. So, we can immediately formulate the following corollary which extends Lemma 5.4 and Lemma 5.6.

Corollary 5.10 ([**27**]). *Let $\mathcal{T}$ be a permutation tableau and let $\widehat{\mathcal{T}}$ be $\varphi(\mathcal{T})$, the corresponding alternative tableau. Then, there is a one-to-one correspondence between*

- *distinguished 0's in $\mathcal{T}$ and $\leftarrow$'s in $\widehat{\mathcal{T}}$,*
- *distinguished 1's in $\mathcal{T}$ and $\uparrow$'s in $\widehat{\mathcal{T}}$,*
- *additional 1's in $\mathcal{T}$ and free cells in $\widehat{\mathcal{T}}$.*

*In particular, the number of the corresponding entries is equal.*

Hence, the number of distinguished 0's in a permutation tableau $\mathcal{T}$ and the number of $\leftarrow$'s in the alternative tableau $\varphi(\mathcal{T}) = \widehat{\mathcal{T}}$ are equal, the number of distinguished 1's and $\uparrow$'s, and the number of additional 1's and free cells. This observation turns out to be important for the reformulation of Theorem 4.3 in terms of alternative tableaux.

## 5.2. Restatement in Terms of Alternative Tableaux

Our goal is to restate Theorem 4.3 in terms of alternative tableaux. We do so in Theorem 5.12 after having made some preparations.

Let $\widehat{\mathcal{T}}$ be an alternative tableau Extending the definitions in ([**17**], below Definition 2), we define a *free row* to be a row with no $\leftarrow$ and denote the number of free rows by $frow(\mathcal{T})$. Furthermore we define a *free column* to be a column without any $\uparrow$ and denote by $fcol(\mathcal{T})$ the number of free columns. Recall that a free cell is an empty cell without having any $\leftarrow$ up $\uparrow$ pointing towards it; by $fcell(\mathcal{T})$ we denote the number of free cells in an alternative tableaux. Finally, with $\tau \in \{0,1\}^n$ being a state of the PASEP, let $\widehat{\lambda}(\tau)$ be the shape $\lambda(\tau)$ with the top row removed.

When recalling Definition 4.2, we note an analogy to the statistics defined above. The parts which are important for us are made explicit.

LEMMA 5.11. *Let $\mathcal{T}$ be a permutation tableau of shape $\lambda$ and $\widehat{\mathcal{T}} = \varphi(\mathcal{T})$ the corresponding alternative tableau. Then, with the definitions from above and Definition 4.2, we have*

$$\begin{aligned} frow(\widehat{\mathcal{T}}) &= u(\mathcal{T}), \\ fcol(\widehat{\mathcal{T}}) &= f(\mathcal{T}), \\ fcell(\widehat{\mathcal{T}}) &= rk(\mathcal{T}). \end{aligned}$$

PROOF. Suppose that $\lambda$ is a shape with $k$ rows and $m$ columns, and set $\widehat{\mathcal{T}} = \varphi(\mathcal{T})$. By Corollary 5.9 we conclude that $\widehat{\mathcal{T}}$ is of shape $\widehat{\lambda}$, with $m$ columns and $k-1$ rows.

We show $frow(\widehat{\mathcal{T}}) = u(\mathcal{T})$. Recall that $u(\mathcal{T})$ is the number of unrestricted rows in $\mathcal{T}$ minus 1 and $u'(\mathcal{T})$ is the number of distinguished 0's. Also recall that $u(\mathcal{T})+u'(\mathcal{T}) = k-1$. Let $left(\widehat{\mathcal{T}})$ be the number of $\leftarrow$ in $\widehat{\mathcal{T}}$. Due to Corollary 5.10, we know that distinguished 0's correspond to $\leftarrow$, so

$$left(\widehat{\mathcal{T}}) = u'(\mathcal{T}).$$

Hence $k-1 = u(\mathcal{T}) + u'(\mathcal{T}) = u(\mathcal{T}) + left(\widehat{\mathcal{T}})$, from which follows that

$$u(\mathcal{T}) = k - 1 - left(\widehat{\mathcal{T}}). \tag{5.3}$$

In every row of an alternative tableau there can be at most one $\leftarrow$, so $left(\widehat{\mathcal{T}})$ is equal to the number of rows containing a $\leftarrow$. So, the right-hand side of (5.3) can also be read as the number of rows of $\widehat{\lambda}$ (which is equal to $k-1$, see above) minus the number of rows that contain a $\leftarrow$. This is exactly the definition of $frow(\widehat{\mathcal{T}})$, therefore $frow(\widehat{\mathcal{T}}) = u(\mathcal{T})$.

Now we show that $fcol(\widehat{\mathcal{T}}) = f(\mathcal{T})$. Recall that $f(\mathcal{T})$ is the number of 1's in the first row of the permutation tableau $\mathcal{T}$, and that $f'(\mathcal{T})$ is the number of distinguished 1's. Also recall that $f(\mathcal{T}) + f'(\mathcal{T}) = m$. If we denote the number of $\uparrow$'s in $\widehat{\mathcal{T}}$ by $up(\widehat{\mathcal{T}})$, we have

$$f'(\mathcal{T}) = up(\widehat{\mathcal{T}}),$$

again by Corollary 5.10. Hence, $m = f(\mathcal{T})+f'(\mathcal{T}) = f(\mathcal{T})+up(\widehat{\mathcal{T}})$, from which follows

$$f(\mathcal{T}) = m - up(\widehat{\mathcal{T}}). \tag{5.4}$$

In every column of an alternative tableau there can be at most one $\uparrow$, so $up(\widehat{\mathcal{T}})$ also equals the number of columns containing a $\uparrow$. Hence (5.4) can be read as follows: $f(\mathcal{T})$ equals the number of columns of $\widehat{\lambda}$ minus the number of columns containing an $\uparrow$. This is exactly the definition of $frow(\widehat{\mathcal{T}})$.

Finally, we show that $fcell(\widehat{\mathcal{T}}) = rk(\mathcal{T})$. Recall that $rk(\mathcal{T})$ is the number of 1's minus the number of columns of $\lambda$. If we can show that this is equal to the number of additional 1's in the filling (see Definition 5.2), then it follows from Corollary 5.10 that $rk(\mathcal{T}) = fcell(\widehat{\mathcal{T}})$. But the above claim is easy to verify: each column of $\mathcal{T}$ contains at least one 1. Consider the topmost one in each of the $m$ columns. Either it is located in the first row (then it is a top-row 1) or not (then it is a distinguished 1). All other 1's in the filling have to be additional 1's and their number is equal the number of 1's minus $m$, which was claimed. ☐

THEOREM 5.12 ([**17**], Corollary 5). *Consider the PASEP with $n$ sites, with $0 \leq q \leq p = 1$ and freely chosen boundary conditions $\alpha, \beta$ (and $\gamma = \delta = 0$). Let $\tau$ be a state of the PASEP. Set*

$$Z_n = \sum_{\widehat{\mathcal{T}} \in \widehat{\mathcal{T}}^n} q^{fcell(\widehat{\mathcal{T}})} \alpha^{-fcol(\widehat{\mathcal{T}})} \beta^{-frow(\widehat{\mathcal{T}})},$$

*with the sum ranging over all alternative tableaux of length $n$. Then the steady state probability of the PASEP being in state $\tau$ is*

$$\frac{\sum_{\widehat{\mathcal{T}} \in \widehat{\mathcal{T}}_{shape[\widehat{\lambda}]}} q^{fcell(\widehat{\mathcal{T}})} \alpha^{-fcol(\widehat{\mathcal{T}})} \beta^{-frow(\widehat{\mathcal{T}})}}{Z_n}, \tag{5.5}$$

*where the sum ranges over all alternative tableaux of shape $\widehat{\lambda} = \widehat{\lambda}(\tau)$, the shape associated to the state $\tau$.*

PROOF. The theorem follows by Theorem 4.3 and Lemma 5.11. We show that the numerator and denominator are equal to the ones in Theorem 4.3. First, we check the claim for the numerators

$$\sum_{\mathcal{T}_{shape(\lambda)}} q^{rk(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})} = \sum_{\mathcal{T}_{shape(\widehat{\lambda})}} q^{fcell(\widehat{\mathcal{T}})} \alpha^{-fcol(\widehat{\mathcal{T}})} \beta^{-frow(\widehat{\mathcal{T}})}. \tag{5.6}$$

The claim is proven as follows: fix $\mathcal{T}$, a permutation tableau of shape $\lambda$. There is an alternative tableau of shape $\widehat{\lambda}$ for which $\varphi(\mathcal{T}) = \widehat{\mathcal{T}}$ (Corollary 5.9). By Lemma 5.11 follows that the corresponding terms in the sums of (5.6) are the same. But Corollary 5.9 also states that every alternative tableau of shape $\widehat{\lambda}$ has a counterpart in the set of permutation tableaux of shape $\lambda$. Hence both sums yield the same result.

To see that the denominators in 4.3 and 5.5 are equal, apply the same argument as above, this time to all shapes $\lambda$ of permutation tableaux of length $n + 1$. ☐

Alternative tableaux seem to encode information about the PASEP in a somehow more natural way: The shapes that correspond to the states of a PASEP with $n$ sides are of length $n$, and also the analogous definitions of the so far essential statistics such as $fcell(\widehat{\mathcal{T}})$ and $fcol(\widehat{\mathcal{T}})$ seem more natural in case of alternative tableaux. But still, we consider as well permutation tableaux as alternative tableaux in the next chapter. On the one side, we want to cover as many starting points for further developments as possible. On the other side, the proofs presented in the last chapters relied on permutation tableaux and their structure, and the methods used so far can be applied to provide further results, as we shall see in the next chapter.

CHAPTER 6

# Symmetric ASEP and Bordered Tableaux

This chapter includes the most recent results in the literature of this text. As in the last chapter, the main results are due to S. Corteel and L. Williams. We will follow the arguments in [**7**].

The aim is to generalize Theorem 4.3, respectively Theorem 5.12, for the case where the parameters $\gamma, \delta$ are general, too. We will not succeed in doing so but we will get close, using the restriction of the symmetric displacement of the particles. This means that the probability of a particle hopping to the left is the same as hopping to the right ($p = q = 1$). To this end, we introduce the slightly more general *bordered permutation tableaux* and *bordered alternative tableaux*.

## 6.1. Bordered Tableaux and States of the Symmetric ASEP

The idea is to consider tableaux containing some additional information on their right-hand outer border line, namely labels $\alpha, \beta, \gamma$ and $\delta$. We want to derive results once in terms of permutation tableaux and once in terms of alternative tableaux. So, from a heuristic point of view, it can be expected that we need some adoptions on the definition of the border line, since the shapes of a permutation tableau and the corresponding alternative tableau differ by one row (compare to Corollary 5.9). Speaking in terms of shape paths, we note that if the shape path $p_{per}$ of a permutation tableau $\mathcal{T}_{per}$ is $p_{per} = (p_0, p_1, p_2, \dots, p_n)$, then the shape path $p_{alt}$ of the alternative tableau $\varphi(\mathcal{T}_{per}) = \mathcal{T}_{alt}$ is given by $p_{alt} = (p_1, p_2, \dots, p_n)$. To avoid having to distinguish between these two shape paths we give the following definition:

We define the **border path** $bp$ of an alternative tableau $\mathcal{T}_{alt}$ to be the shape path $p_{alt}$ of the tableau. For a permutation tableau we define the **border path** $bp$ to be the shape path $p_{per}$ omitting the first step $p_0$. Hence, with notation from above, we have that the border paths of $\mathcal{T}_{per}$ and $\varphi(\mathcal{T}_{per}) = \mathcal{T}_{alt}$ are the same, namely $bp = (p_1, p_2, \dots, p_n)$.

We assign to each step $p_i$ of the border path, respectively to the corresponding step in the border of the tableau $\mathcal{T}$, a weight $b_i \in \{\alpha, \beta, \gamma, \delta\}$ for $i = 1, \dots, n$. We call this sequence of weights a **border sequence**, denoted as $bord(\mathcal{T})$. Furthermore we define the $edge(\mathcal{T})$ as the product of the elements of the border sequence.

So, in other words, we take any alternative tableau and label the right-hand border line by $\alpha, \beta, \gamma$ or $\delta$. In the case of a permutation tableau, we also label the right-hand border line, but we ignore the topmost south step, which we do not consider to be part of the border. This is due to the fact that $p_0$ is a “forced” step of the shape path and does not contribute additional information about the state $\tau$ of the PASEP. To define bordered tableaux we will put a restriction on the labeling.

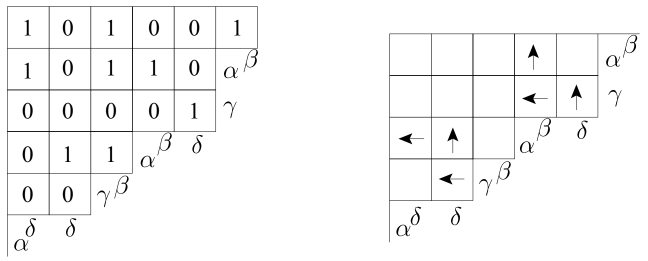



FIGURE 6.1. An example for a bordered permutation tableau and a bordered alternative tableau.

DEFINITION 6.1. A **bordered permutation tableau** is defined to be a permutation tableau $\mathcal{T}$ together with a labeling of the border path *bp* such that the following conditions hold:

- a vertical edge (respectively a south step) is either labeled by $\alpha$ or by $\gamma$;
- a horizontal edge (respectively a west step) is either labeled by $\beta$ or by $\delta$.

We call such a labeling a *valid labeling for the border* of a permutation tableaux. Denote by *bord* $\mathcal{T}^n$ the set of all bordered permutation tableaux of length $n$.

Analogously, we define bordered alternative tableaux:

DEFINITION 6.2. A *bordered alternative tableau* is defined to be a permutation tableau $\widehat{\mathcal{T}}$ together with a labeling of the border path *bp* such that the following conditions hold:

- a vertical edge (respectively a south step) is either labeled by $\alpha$ or by $\gamma$;
- a horizontal edge (respectively a west step) is either labeled by $\beta$ or by $\delta$.

We call such a labeling a *valid labeling for the border* of an alternative tableau. Denote by *bord* $\widehat{\mathcal{T}}^n$ the set of all bordered permutation tableaux of length $n$.

REMARK. Again, concepts introduced earlier, such as length, shape, and so on, are inherited by bordered permutation tableaux and bordered alternative tableaux.

Whereas in Chapter 4 and Chapter 5 we associated to a state $\tau \in \{0,1\}$ of the PASEP all permutation tableaux of a certain shape $\lambda(\tau)$, we will now generalize this and associate bordered tableaux of various shapes and bordered labellings to the state $\tau$.

DEFINITION 6.3. Let $\mathcal{T}$ be a bordered permutation tableau of length $n+1$ or an alternative tableau of length $n$. $\mathcal{T}$ is said to be of *type* $\tau$, $\tau = (\tau_1, \dots, \tau_n) \in \{0,1\}^n$, if for its border sequence $bord(\mathcal{T}) = (b_1, \dots, b_n)$ the following holds:

$$\tau_i = 1 \text{ if and only if } b_i = \alpha, \delta; \tag{6.1}$$

$$\tau_i = 0 \text{ if and only if } b_i = \beta, \gamma; \tag{6.2}$$

for $i = 1 \dots, n$. We denote the set of all bordered permutation tableaux of type $\tau$ by $\mathcal{T}_{type[\tau]}$ and the set of all bordered alternative tableaux of type $\tau$ by $\widehat{\mathcal{T}}_{type[\tau]}$.

EXAMPLE. We show how Definitions 6.1 and 6.3 are put together. Let $\tau$ be the state $(\circ, \bullet, \circ)$ of the TASEP with 3 sites (equivalently, $\tau = (0,1,0)$). We want to identify

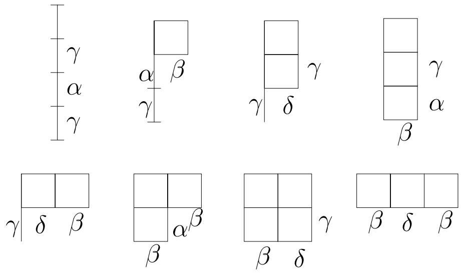



FIGURE 6.2. Labeled shapes of bordered permutation tableaux of type $(0, 1, 0)$.

the bordered permutation tableaux of type $\tau$. Note that the definition of type concerns shapes and their labeling, so for now, we focus on shapes only (and omit fillings of the tableau). So, due to Definition 6.3, the border path of the shapes has to satisfy the following:

$$b_1 = \beta, \gamma \qquad b_2 = \alpha, \delta \qquad b_3 = \beta, \gamma$$

We compare this to Definition 6.1 which says that steps of the border path must have the following appearance:

$$\big|\, \alpha, \gamma \qquad\qquad — \ \beta, \delta$$

Hence, the first step of the border path of a bordered permutation tableau of type $\tau$ can either be a west step labeled $\beta$ or a south step labeled $\gamma$. The second step of the bordered path might be a south step labeled $\alpha$ or a west step labeled $\delta$. And the third step is either a west step labeled $\beta$ or a south step labeled $\gamma$. So, there remain 8 different labeled shapes, as seen in Figure 6.2. Adding valid fillings yields all bordered permutation tableaux of type $\tau$.

Compare this to Figure 4.3 in Chapter 4. There, just one of the shapes was associated to the state $(\circ, \bullet, \circ)$ of the PASEP (with $\gamma = \delta = 0$ but without the restriction of $p = q = 1$). This shape is the one that remains in Figure 6.2, if the shapes with labels $\gamma$ or $\delta$ are omitted. We return to this observation after having established the connection between bordered shapes and the symmetric ASEP.

We now state the main result of this chapter. First, we do so in terms of permutation tableaux. Recall that $f'(\mathcal{T})$ and $u'(\mathcal{T})$ are the number of distinguished 1's and 0's, respectively. Furthermore recall the definition of *edge*, the product of the labels of a shape.

THEOREM 6.4. *Consider a state $\tau$ of the PASEP with $n$ sites, with general parameters $\alpha, \beta, \gamma, \delta$ and $q = 1$. Set*

$$Z_n = \sum_{\mathcal{T} \in bord\ \mathcal{T}^{n+1}} edge(\mathcal{T})(\alpha + \gamma)^{f'(\mathcal{T})}(\beta + \delta)^{u'(\mathcal{T})},$$

*where the sum is over all bordered permutation tableaux of length $n+1$. Then the steady state probability that the PASEP is at state $\tau$ is given by*

$$\frac{\sum_{\mathcal{T} \in \mathcal{T}_{type[\tau]}} edge(\mathcal{T})(\alpha + \gamma)^{f'(\mathcal{T})}(\beta + \delta)^{u'(\mathcal{T})}}{Z_n} \tag{6.3}$$

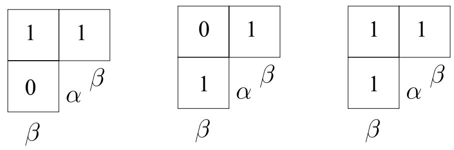



FIGURE 6.3. Some bordered permutation tableaux of type $(0, 1, 0)$.

*where the sum is over all bordered permutation tableaux $\mathcal{T}$ of type $\tau$.*

We can restate this theorem in terms of alternative tableaux. For an alternative tableau $\widehat{\mathcal{T}}$ we define $up(\widehat{\mathcal{T}})$ to be the numbers of $\uparrow$'s and $left(\widehat{\mathcal{T}})$ to be the number of $\leftarrow$'s in the tableau.

THEOREM 6.5. *Consider any state $\tau$ of the PASEP with $n$ sites, where the parameters $\alpha, \beta, \gamma, \delta$ are general and $q = 1$. Set*

$$Z_n = \sum_{\widehat{\mathcal{T}} \in bord\, \widehat{\mathcal{T}}^n} edge(\widehat{\mathcal{T}})(\alpha+\gamma)^{up(\widehat{\mathcal{T}})}(\beta+\delta)^{left(\widehat{\mathcal{T}})},$$

*where the sum is over all bordered alternative tableaux of length $n$. Then the steady state probability that the PASEP is at state $\tau$ is equal to*

$$\frac{\sum_{\widehat{\mathcal{T}} \in \widehat{\mathcal{T}}_{type[\tau]}} edge(\widehat{\mathcal{T}})(\alpha+\gamma)^{up(\widehat{\mathcal{T}})}(\beta+\delta)^{left(\widehat{\mathcal{T}})}}{Z_n} \tag{6.4}$$

*where the sum is over all bordered alternative tableaux $\widehat{\mathcal{T}}$ of type $\tau$.*

EXAMPLE. Again, consider the TASEP with 3 sites and the state $\tau = (\circ, \bullet, \circ)$ (which is equal to $\tau = (0, 1, 0)$). We indicate how to calculate (6.3). In the last example we have already seen that there are 8 different labeled shapes which yield to bordered permutation tableaux of type $\tau$ (see Figure 6.2). In one of the earlier examples we have noted that by adding valid fillings to these shapes we obtain 24 different permutation tableaux of type $\tau$. As an example, consider the shape $(2, 1)$ with border sequence $b = (\beta, \alpha, \beta)$ – this is the second shape in the second row in Figure 6.2. There are 3 ways to add a valid filling (see Figure 6.3). By doing so we obtain the following terms of the numerator in (6.3):

$$\alpha\beta^2(\alpha+\gamma) + \alpha\beta^2(\beta+\delta) + \alpha\beta^2.$$

Analogously, the remaining terms of $\sum_{\mathcal{T} \in \mathcal{T}_{type[\tau]}} edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})}$ can be found. For the denominator $Z_n$, one would have to do the same for all bordered permutation tableaux of length 4. So for each of the 24 permutation tableaux mentioned above there are $2^3 = 8$ different ways to label their border paths. Hence there are $24 \cdot 8 = 192$ differently labeled permutation tableaux.

If we compare Theorem 6.4 to Theorem 4.3, we note that we should be able to derive one out of the other. By setting $\gamma = \delta = 0$ in Theorem 6.4 we obtain Theorem 4.3 with $q = 1$: if $\gamma = \delta = 0$, then summing over all *bordered* permutation tableaux of length $n+1$ in (6.3) is like summing over all permutation tableaux of length $n+1$ : in the denominator $\sum_{\mathcal{T} \in \mathcal{T}^{n+1}} edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})}$ the term $edge(\mathcal{T})$ is different from 0 only when all south steps are labeled by $\alpha$ and all west steps are labeled by $\beta$. So, for each bordered permutation tableau of length $n+1$ just one representative of each shape contributes to the sum. The numerator in (6.3), $\sum_{\mathcal{T}\in\mathcal{T}_{type[\tau]}} edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})}$, becomes a sum over all permutation tableaux of shape $\lambda(\tau)$. To see this, note first that, again, the term $edge(\mathcal{T})$ becomes 0 if some weight on the border is chosen to be $\gamma$ or $\delta$. These tableaux do not contribute to the sum in the numerator. Therefore, we can implicitly omit these tableaux by modifying (6.1) and (6.2):

$$\tau_i = 1 \quad \text{if and only if} \quad b_i = \alpha,$$
$$\tau_i = 0 \quad \text{if and only if} \quad b_i = \beta.$$

Using the same arguments, we conclude for the weights of the border path in Definition 6.1 that:

$$\text{only south steps } S \text{ are weighted with } \alpha,$$
$$\text{only west steps } W \text{ are weighted with } \beta.$$

Putting this together we obtain for the shape path $p = (S, p_1, \dots, p_n)$:

$$\tau_i = 1 \quad \text{if and only if} \quad p_i = S,$$
$$\tau_i = 0 \quad \text{if and only if} \quad p_i = W,$$

which is exactly the bijection between a state $\tau$ and a shape $\lambda(\tau)$ given in Section 4.1. (Here we have used that the border path of a permutation tableaux does not contain the first step of the shape path, and we have used that a shape path of a permutation tableau always starts with a south step). Hence, the sum in the numerator can be seen as sum over permutation tableaux of type $\tau$. So, having seen how the range for the sums in Theorem 6.4 changes, we now let $\mathcal{T}$ be a permutation tableau with $k$ rows and $m$ columns. It follows then that

$$\begin{aligned}
edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})} &= edge(\mathcal{T})\alpha^{f'(\mathcal{T})}\beta^{u'(\mathcal{T})} \\
&= \alpha^{k-1}\beta^{m}\alpha^{f'(\mathcal{T})}\beta^{u'(\mathcal{T})} && (6.5)\\
&= \alpha^{k-1}\beta^{m}\alpha^{m-f(\mathcal{T})}\beta^{k-1-u(\mathcal{T})} && (6.6)\\
&= \alpha^{k-1+m-f(\mathcal{T})}\beta^{m+k-1-u(\mathcal{T})} && (6.7)\\
&= \alpha^{n-f(\mathcal{T})}\beta^{n-u(\mathcal{T})}. && (6.8)
\end{aligned}$$

In (6.5) we have used that the first south step in a permutation tableau is not part of the border path, in (6.6) we have used that $f(\mathcal{T})+f'(\mathcal{T}) = m$ and $u(\mathcal{T})+u'(\mathcal{T}) = k-1$, and in (6.8) we have used that $k+m = n+1$. So summarizing this, the fraction in (6.3) of Theorem 6.4 with $\gamma = \delta = 0$ and $q = 1$ can be read in the following way: the sum in the denominator is a sum over all permutation tableaux of length $n+1$, the sum in the numerator is a sum over all permutation tableaux of shape $\lambda(\tau)$, and, dividing the terms in both numerator and denominator by $\alpha^n\beta^n$ (see (6.8)), we arrive at (4.3) in Theorem 4.3.

## 6.2. Proof of Theorems 6.4 and 6.5

We will prove these theorems in a manner similar to the proof of Theorem 4.3. We define matrices $D_2, E_2, V_2, W_2$ for which the Matrix Ansatz equations hold and which, at the same time, have the desired combinatorial interpretation.

We define the following:

$$(D_2)_{i,j} = d_{i,j} := \begin{cases} \delta(\beta+\delta)^{i-j}\left(\binom{i-1}{j-1} + (\alpha+\gamma)\binom{i-1}{j-2}\right) & \text{for } j < i+1 \\ \alpha & j = i+1 \\ 0 & j > i+1 \end{cases} \tag{6.9}$$

$$(E_2)_{i,j} = e_{i,j} := \begin{cases} \beta(\beta+\delta)^{i-j}\left(\binom{i-1}{j-1} + (\alpha+\gamma)\binom{i-1}{j-2}\right) & \text{for } j < i+1 \\ \gamma & j = i+1 \\ 0 & j > i+1 \end{cases} \tag{6.10}$$

Furthermore, let $W_2$ be the following (row) vector

$$W_2 = (1, 0, 0, \ldots), \tag{6.11}$$

and $V_2$ be the (column) vector

$$V_2 = (1, 1, 1, \ldots)^T. \tag{6.12}$$

This time, we apply the "more general" version of the Matrix Ansatz Theorem, Theorem 2.2.

Lemma 6.6. *For the matrices $D_2, E_2, V_2, W_2$ defined as above the Matrix Ansatz equations* (2.41) – (2.43) *hold.*

Proof. We check that equations (2.41) - (2.43) hold for the matrices $D_2, E_2, V_2, W_2$ defined in (6.9) – (6.12) and for $\chi = \alpha\beta - \gamma\delta$. We first check Equation (2.43), that is

$$W_2(\alpha E_2 - \gamma D_2) = \chi W_2.$$

Due to the entries of $W_2$, it is enough to show that the first row of $\alpha E_2 - \gamma D_2$ is equal to $(\chi, 0, 0, \ldots)$. We note that the entries $(D_2)_{i,j}, (E_2)_{i,j}$ are very similar. If we write $X(i,j) = (\beta+\delta)^{i-j}\left(\binom{i-1}{j-1} + (\alpha+\gamma)\binom{i-1}{j-2}\right)$, then

$$(\alpha E_2 - \gamma D_2)_{i,j} = \begin{cases} \alpha\beta X(1,1) - \gamma\delta X(1,1) = \alpha\beta - \gamma\delta = \chi & i = 1, j = 1 \\ \alpha\gamma - \gamma\alpha = 0 & i = 1, j = 2 \\ 0 & i = 1, j \geq 3 \end{cases} \tag{6.13}$$

which gives the desired equality.

In the case of Equation (2.42), which in our setting reads

$$(\beta D_2 - \delta E_2)V_2 = \chi V_2,$$

we note that each row of the matrix $(\beta D_2 - \delta E_2)$ is added up by multiplication with $V_2$, so the sum of each row of $(\beta D_2 - \delta E_2)$ has to be equal to $\chi = \alpha\beta - \gamma\delta$ for Equation (2.43) to hold. This is the case since by definition of the matrices $D_2, E_2$, we find the row sum for each row $i$ to be equal to

$$\begin{aligned}
&\sum_{j\geq 1}(\beta D_2-\delta E_2)_{i,j}\\
&\quad=\sum_{j=1}^{i}(\beta D_2-\delta E_2)_{i,j} \quad + \sum_{j=i+1}(\beta D_2-\delta E_2)_{i,j} \; + \sum_{j\geq i+2}(\beta D_2-\delta E_2)_{i,j}\\
&\quad=\underbrace{\sum_{j=1}^{i}\beta(\delta X(i,j))-\delta(\beta X(i,j))}_{=0} + \quad (\beta\alpha-\delta\gamma) \quad + \underbrace{\sum_{j\geq i+2}(\beta D_2-\delta E_2)_{i,j}}_{=0}\\
&\quad= \; \chi.
\end{aligned}$$

To prove the remaining Equation (2.41),

$$D_2E_2-qE_2D_2=\chi(D_2+E_2),$$

we first find expressions for the terms $(D_2E_2)_{i,j}$ and $(E_2D_2)_{i,j}$. We claim that

$$(D_2E_2)_{i,j}=\begin{cases}0 & j>i+2\\ \alpha\gamma & j=i+2\\ \gamma d_{i,j-1}+\sum_{k=j}^{i}d_{i,k}e_{k,j}+\alpha e_{i+1,j} & j<i+2\end{cases} \tag{6.14}$$

and

$$(E_2D_2)_{i,j}=\begin{cases}0 & j>i+2\\ \alpha\gamma & j=i+2\\ \alpha e_{i,j-1}+\sum_{k=j}^{i}e_{i,k}d_{k,j}+\gamma d_{i+1,j} & j<i+2\end{cases} \tag{6.15}$$

To confirm these claims, we consider the matrix multiplication $(D_2E_2)_{i,j}=\sum_{k\geq 1}d_{i,k}e_{k,j}$. In the case of $j>i+2$, all products are equal to 0 because then at least one of the factors is equal to 0. To see that note that $e_{k,j}=0$ for $j>k+1$ or, equivalently, $k<j-1$, and $d_{i,k}=0$ for $k>i+1$. So only if $k\geq j-1$ and $k\leq i+1$ there are non-zero terms, but that can only happen if $j-1\leq i+1$ or, equivalently, $j\leq i+2$. In the case of $j=i+2$, it follows that exactly one summand is not 0, namely $d_{i,i+1}e_{i+1,i+2}=\alpha\gamma$. In the case of $j<i+2$, we can write the sum as follows:

$$\begin{aligned}
(D_2E_2)_{i,j}&=\sum_{k\geq 1}d_{i,k}e_{k,j}\\
&=\sum_{k=1}^{j-2}d_{i,k}\underbrace{e_{k,j}}_{=0}+d_{i,j-1}\underbrace{e_{j-1,j}}_{=\gamma}+\sum_{k=j}^{i}d_{i,k}e_{k,j}+\underbrace{d_{i,i+1}}_{=\alpha}e_{i+1,j}+\sum_{k\geq i+2}\underbrace{d_{i,k}}_{=0}e_{k,j}\\
&=\gamma d_{i,j-1}+\sum_{k=j}^{i}d_{i,k}e_{k,j}+\alpha e_{i+1,j}.
\end{aligned}$$

Interchanging the roles of $d_{i,j}$ and $e_{i,j}$, we obtain the analogous results for $(E_2D_2)_{i,j}$.

We now use (6.14) and (6.15) to calculate the term $(D_2E_2-E_2D_2)_{i,j}$, where we distinguish three cases: $j\leq i, j=i+1, j\geq i+2$. It immediately follows by (6.14) and (6.15) that

$$(D_2E_2 - E_2D_2)_{i,j} = 0, \qquad \text{for } j \geq i+2. \tag{6.16}$$

If $j = i+1$, then the term $\sum_{k=j}^{i} e_{i,k}d_{k,j}$ gives 0 in (6.14) and (6.15) (since the sum is empty), and we obtain

$$\begin{aligned}
(D_2E_2 - E_2D_2)_{i,i+1} &= (D_2E_2)_{i,i+1} - (E_2D_2)_{i,i+1} \\
&= (\gamma d_{i,i} + \alpha e_{i+1,i+1}) - (\alpha e_{i,i} + \gamma d_{i+1,i+1}) \\
&= (\gamma \frac{\delta}{\beta} e_{i,i} + \alpha e_{i+1,i+1}) - (\alpha e_{i,i} + \gamma \frac{\delta}{\beta} e_{i+1,i+1}) \\
&= (\alpha - \gamma\frac{\delta}{\beta})\,(e_{i+1,i+1} - e_{i,i}) \\
&= (\alpha - \gamma\frac{\delta}{\beta})\,\beta((1+(\alpha+\gamma)i) - (1+(\alpha+\gamma)(i-1))) \\
&= (\alpha\beta - \gamma\delta)\,(\alpha+\gamma) = \chi(\alpha+\gamma).
\end{aligned} \tag{6.17}$$

In the case of $j \leq i+1$, $(D_2E_2 - E_2D_2)_{i,j}$ reads

$$\begin{aligned}
&(D_2E_2 - E_2D_2)_{i,j} \\
&\qquad = (\gamma d_{i,j-1} + \sum_{k=j}^{i} d_{i,k}e_{k,j} + \alpha e_{i+1,j}) - (\alpha e_{i,j-1} + \sum_{k=j}^{i} e_{i,k}d_{k,j} + \gamma d_{i+1,j}).
\end{aligned} \tag{6.18}$$

We note that, since in the case of $j \leq k \leq i$ , we have $d_{i,k} = \frac{\delta}{\beta}e_{i,k}$ and $d_{k,j} = \frac{\delta}{\beta}e_{k,j}$, the sums cancel,

$$\sum_{k=j}^{i} d_{i,k}e_{k,j} - \sum_{k=j}^{i} e_{i,k}d_{k,j} = 0.$$

Thus (6.18) reduces to

$$\begin{aligned}
&(D_2E_2 - E_2D_2)_{i,j} \\
&\qquad = (\gamma d_{i,j-1} + \alpha e_{i+1,j}) - (\alpha e_{i,j-1} + \gamma d_{i+1,j}) \\
&\qquad = (\gamma\frac{\delta}{\beta}e_{i,j-1} + \alpha e_{i+1,j}) - (\alpha e_{i,j-1} + \gamma\frac{\delta}{\beta}e_{i+1,j}) \\
&\qquad = (\alpha - \gamma\frac{\delta}{\beta})(e_{i+1,j} - e_{i,j-1}) \\
&\qquad = (\alpha\beta - \gamma\delta)(\beta+\delta)^{i-j+1} \\
&\qquad\qquad \times \left( \binom{i}{j-1} + (\alpha+\gamma)\binom{i}{j-2} - \binom{i-1}{j-2} - (\alpha+\gamma)\binom{i-1}{j-3} \right) \\
&\qquad = \chi(\beta+\delta)^{i-j+1} \left( \binom{i-1}{j-1} + (\alpha+\gamma)\binom{i-1}{j-2} \right).
\end{aligned} \tag{6.19}$$

Putting equations (6.16), (6.17) and (6.19) together, we obtain

$$(D_2E_2 - E_2D_2)_{i,j} = \begin{cases} \chi(\beta+\delta)^{i-j+1}\left(\binom{i-1}{j-1} + (\alpha+\gamma)\binom{i-1}{j-2}\right) & j \le i \\ \chi(\alpha+\gamma) & j = i+1 \\ 0 & j \ge i+2. \end{cases} \tag{6.20}$$

On the other hand, we see that, due to the definitions of $D_2, E_2$, there are three different types of entries for the right hand side of the matrix Ansatz equation (2.41),

$$(D_2 + E_2)_{i,j} = \begin{cases} (\beta+\delta)^{i-j+1}\left(\binom{i-1}{j-1} + (\alpha+\gamma)\binom{i-1}{j-2}\right) & j \le i \\ (\alpha+\gamma) & j = i+1 \\ 0 & j \ge i+2 \end{cases} \tag{6.21}$$

which, multiplied by $\chi$, corresponds exactly to the expressions of $(D_2E_2 - E_2D_2)_{i,j}$ in (6.20). We therefore have shown that the Matrix Ansatz holds for the matrices $D_2, E_2, W_2, V_2$. □

Remark. Again, the matrices $D_2, E_2, V_2, W_2$ have only non-negative entries, and hence all their products have non-negative entries, as required by Theorem 2.1.

We now show that the matrices also have the desired combinatorial interpretation. So, analogously as in Chapter 4 we state the following theorem:

Theorem 6.7 ([**7**], Theorem 4.1.). *Let $(\tau_1, \ldots, \tau_n) \in \{0,1\}^n$ be a state of the PASEP with $n$ sites. Then, for the generating function for all bordered permutation tableaux of type $\tau$ as defined below, we have*

$$F_\tau := \sum_{\mathcal{T} \in \mathcal{T}_{type[\tau]}} edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})} = W_2\left(\prod_{i=1}^{n}(\tau_i D_2 + (1-\tau_i)E_2)\right)V_2.$$

*Moreover, for the generating function of all bordered permutation tableaux of length $n+1$ as defined below, the following holds:*

$$F^{n+1} := \sum_{\mathcal{T} \in bord\ \mathcal{T}^{n+1}} edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})} = W_2(D_2+E_2)^n V_2,$$

*where the sum ranges over all bordered permutation tableaux of length $n+1$.*

Proof. The theorem is proved by induction on $n$. We consider the matrix

$$M_\tau := \prod_{i=1}^{n}(\tau_i D_2 + (1-\tau_i)E_2),$$

where $\tau = (\tau_1, \ldots, \tau_n)$ is a state of the PASEP. W shall see that the sum of the entries in the top row corresponds to the generating function

$$F_\tau = \sum_{\mathcal{T} \in \mathcal{T}_{type[\tau]}} edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})}$$

.

We define ${}_jF_\tau$ to be the set of all permutation tableaux of type $\lambda$ which have exactly $j$ unrestricted rows, and we set ${}_jF_\tau := \sum_{\mathcal{T} \in {}_j\mathcal{T}_{type[\tau]}} edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})}$. It is clear then that $F_\tau = \sum_{j \ge 1} {}_jF_\tau$.

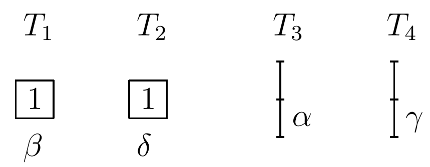



Figure 6.4. Bordered permutation tableaux of length 2.

We now show that the entry $M_\tau[1,j]$ in position $(1,j)$ of $M_\tau$ is ${}_jF_\tau$. For this, we use induction by $n$, where $n$ is the number of sites of the PASEP.

$\mathbf{n = 1}$ : If we consider the lattice path with just one side, then there are only two possibilities to choose a state $\tau$, namely $(\circ)$ or $(\bullet)$, respectively $\tau = (\tau_1) = (0)$ or $\tau = (\tau_1) = (1)$. We consider the permutation tableaux of those types. In general, there are two different shapes of permutation tableaux and four different bordered permutation tableaux that are linked to the states of the 1-side PASEP – see Figure 6.4.

Recalling the conditions on the types and borders of permutation tableaux

$$\tau_i = 1 \text{ if and only if } b_i = \alpha, \delta,$$
$$\tau_i = 0 \text{ if and only if } b_i = \beta, \gamma.$$

and considering Figure 6.4, we see that $\mathcal{T}_1$ and $\mathcal{T}_4$ are of type $\tau = (\tau_1) = (0)$. $T_1$ contains one unrestricted row, whereas $T_4$ contains two of them. Considering the statistics on the entries, we obtain for the generating function of permutation tableaux with 1 unrestricted row

$${}_1F_\tau = edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})} = \beta,$$

and for 2 unrestricted rows

$${}_2F_\tau = edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})} = \gamma,$$

since for both tableaux we have $f'(\mathcal{T}) = u'(\mathcal{T}) = 0$, and only the edges contribute to the term. Due to the fact that there are no other bordered permutation tableaux of type $\tau = (0)$, it follows that

$${}_jF_\tau = 0, \qquad \text{for } j \geq 3.$$

These observations correspond to the first row of $E_2$ being equal to $(\beta, \gamma, 0, 0, \ldots)$. Analogously, we see that the two bordered permutation tableaux in the middle of Figure 6.4, $T_2$ and $T_3$, are of type $\tau = (\tau_1) = (1)$, and we immediately obtain

$$\begin{aligned} {}_1F_\tau &= \delta \\ {}_2F_\tau &= \alpha \\ {}_jF_\tau &= 0, \qquad \text{for } j \geq 3, \end{aligned}$$

which, again, corresponds to the top row of $D_2$ , as desired. So, in the case of $n = 1$ it is true that $M[1,j] = {}_jF_\tau$.

$\mathbf{n \mapsto n+1}$ : Let $\widetilde{\tau}$ be $(\tau_1, \ldots, \tau_n, \tau_{n+1})$. We now want to show that the entry $M_{\widetilde{\tau}}[1,j]$ is equal to ${}_jF_{\widetilde{\tau}}$, the generating function of all permutation tableaux of type $\widetilde{\tau}$ with exactly $j$ unrestricted rows. To do so, we see how to obtain ${}_jF_{\widetilde{\tau}}$ out of ${}_jF_\tau$. We then compare this to $M_{\widetilde{\tau}}[1,j]$, which, by definition, is given by $M_\tau$ multiplied by $D_2$ or $E_2$.

It is clear, that every bordered permutation tableau of type $\widetilde{\tau}$ leads to a bordered permutation tableau of type $\tau$ by deleting the south-west most step of the shape (and therefore deleting the bottommost row or the leftmost column, depending on the deleted step). On the other hand, each bordered permutation tableau of type $\widetilde{\tau}$ can be obtained from a bordered permutation tableau of type $\tau$, namely by adding a step to the south-west corner (bearing in mind the restrictions that are valid for bordered permutation tableaux of a certain type) and by filling the newly created cells with 0's and 1's such that a valid filling results. We shall see how to obtain all bordered permutation tableaux of type $\widetilde{\tau}$ with exactly $j$ unrestricted rows out of the set of all bordered permutation tableaux of type $\tau$ with $i \geq j-1$ unrestricted rows (we will see that by adding a south step an unrestricted row is added, whereas by adding a west step we might reduce the number of rows due to the filling of the new column).

Note that the border path of a bordered permutation tableaux $\mathcal{T}$ of type $\widetilde{\tau}$ is of length $n+1$, and let the border path be denoted by $bp(\mathcal{T}) = (b_1, \dots, b_{n+1}) \in \{S, W\}$ (where $S$ denotes a south step and $W$ denotes a west step).

We obtain ${}_jF_{\widetilde{\tau}}$ in two steps,

$$ {}_jF_{\widetilde{\tau}} = {}_jF_{\widetilde{\tau}}(south) + {}_jF_{\widetilde{\tau}}(west), $$

where

$$ {}_jF_{\widetilde{\tau}}(south) := \sum_{\substack{\mathcal{T}\in {}_j\mathcal{T}_{type[\widetilde{\tau}]},\\ b_{n+1}=S}} edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})}, $$

with the sum ranging over all bordered permutation tableaux of type $\widetilde{\tau}$ with exactly $j$ unrestricted rows and with the last step of the corresponding border path being a south step, and analogously

$$ {}_jF_{\widetilde{\tau}}(west) := \sum_{\substack{\mathcal{T}\in {}_j\mathcal{T}_{type[\widetilde{\tau}]}\\ b_{n+1}=W}} edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})}, $$

with the sum ranging over all bordered permutation tableaux of type $\widetilde{\tau}$ with exactly $j$ unrestricted rows and with the last step of the corresponding border path being a west step.

We first consider $\widetilde{\tau} = (\tau_1, \dots, \tau_n, \tau_{n+1}) = (\tau_1, \dots, \tau_n, 1)$, the case were we add a occupied site to the state $\tau$ first. The case of adding an empty site is treated easily analogously.

Starting with ${}_jF_{\widetilde{\tau}}(south)$, we check which bordered permutation tableau of type $\widetilde{\tau}$ we can obtain out of a bordered permutation tableau of type $\tau$ by adding a south step. So, if we add a south step to a tableau, then this step has to be weighted with $\alpha$ due to Definitions 6.1 and 6.3. There are no cells to fill hence the statistics $f'(\mathcal{T})$ and $u'(\mathcal{T})$ do not change (since there is no additional distinguished 0 or 1). But we have added one unrestricted row and an additional $\alpha$ in the term $edge(\mathcal{T})$. We therefore find ${}_jF_{\widetilde{\tau}}(south) = \alpha\ {}_{j-1}F_\tau$.

Now we focus on ${}_jF_{\widetilde{\tau}}(west)$. We claim that

$$ {}_jF_{\widetilde{\tau}}(west) = \sum_{i\geq j} \delta(\beta+\delta)^{i-j} \left( \binom{i-1}{j-1} + (\alpha+\gamma)\binom{i-1}{j-2} \right) {}_iF_{\tau}. \tag{6.22} $$

To prove this, it is sufficient to see which bordered permutation tableaux with $j$ unrestricted rows can be obtained out of a given bordered permutation tableau with $i$ unrestricted rows by adding a west step. So, fix a bordered permutation tableau $\mathcal{T}$ with $i$ unrestricted rows of type $\tau$ and add a west step. Due to Definitions 6.1 and 6.3 (and the fact that we chose $\tau_{n+1} = 1$), this step is weighted with $\delta$. Since we are adding a whole column, this time, we have to check how many valid fillings are possible and how the number of unrestricted rows and the statistics $f'(\mathcal{T})$ and $u'(\mathcal{T})$ change due to these fillings. Since the tableau contains $i$ unrestricted rows, we can only choose freely how to fill $i$ newly created cells. The other cells lie in restricted rows and therefore have to be filled with a 0. Within these $i$ cells – let us label them by $c_1, \ldots, c_i$ from top to bottom – we consider the topmost 1 (any column has to contain at least one 1):

Case $(a)$: Let us assume that the topmost 1 is located in the top row ($c_1 = 1$). Recall that we want the resulting bordered permutation tableau to have $j$ unrestricted rows. Hence, we have to place exactly $i-j$ 0's below that topmost 1; there are $\binom{i-1}{i-j}$ ways to do so. The remaining cells have to be filled with 1's. So, in this case we obtain $\binom{i-1}{i-j}$ bordered permutation tableaux of type $\widetilde{\tau}$ with additional factors $\delta$ (weight on the border) and $(\beta+\delta)^{i-j}$ (since we have added $(i-j)$ distinguished 0's).

Case $(b)$: We now assume that the topmost 1 is not located in the top row, hence being a distinguished 1. Let the cell $c_s$, for some $s = 2, \ldots, n$, contain the topmost 1. There remain $\binom{i-s}{i-j}$ ways to place the desired $i-j$ (restricted) 0's in the cells below. Since the topmost 1 can be placed in any of the cells $c_2, \ldots, c_n$, we sum up to find that there are, all in all, $\sum_{s=2}^{i} \binom{i-s}{i-j}$ possibilities for the filling. The additional weights are $\delta$ (on the border), $(\beta+\delta)^{i-j}$ (due to $(i-j)$ additional distinguished 0's), and $(\alpha+\gamma)$ (due to the additional distinguished 1).

Putting the cases $(a)$ and $(b)$ together, we find

$$ {}_jF_{\widetilde{\tau}} = {}_jF_{\widetilde{\tau}}(south) + {}_jF_{\widetilde{\tau}}(west) \tag{6.23} $$

$$ = \alpha\ {}_{j-1}F_{\tau} + \sum_{i\geq j} \delta(\beta+\delta)^{i-j} \left( \binom{i-1}{i-j} + (\alpha+\gamma)\sum_{s=2}^{j}\binom{i-s}{i-j} \right) {}_iF_{\tau}. \tag{6.24} $$

Through the computations,

$$ \binom{i-1}{i-j} = \binom{i-1}{j-1}, $$

$$ \sum_{s=2}^{j}\binom{i-s}{i-j} = \sum_{s=0}^{j-2}\binom{i-2-s}{i-j} = \sum_{s=0}^{j-2}\binom{i-j+s}{i-j} = \binom{i-1}{i-j+1} = \binom{i-1}{j-2}, $$

we arrive at

$$
\begin{aligned}
{}_jF_{\tilde{\tau}} &= \alpha\ {}_{j-1}F_\tau + \sum_{i\geq j}\delta(\beta+\delta)^{i-j}\left(\binom{i-1}{i-j}+(\alpha+\gamma)\sum_{s=2}^{j}\binom{i-s}{i-j}\right)\ {}_iF_\tau && (6.25)\\
&= \alpha\ {}_{j-1}F_\tau + \sum_{i\geq j}\delta(\beta+\delta)^{i-j}\left(\binom{i-1}{j-1}+(\alpha+\gamma)\binom{i-1}{j-2}\right)\ {}_iF_\tau. && (6.26)
\end{aligned}
$$

This corresponds exactly to the entry $M_{\tilde{\tau}}[1,j]$:

$$
\begin{aligned}
M_{\tilde{\tau}}[1,j] &= (M_\tau D_2)[1,j] && (6.27)\\
&= \sum_{i\geq 1} M_{1,i}(D_2)_{i,j} && (6.28)\\
&= \sum_{i\geq 1}^{j-2} M_{1,i}\underbrace{(D_2)_{i,j-2}}_{=0} + M_{1,j-1}\underbrace{(D_2)_{j-1,j}}_{=\alpha} + \sum_{i\geq j} M_{1,i}(D_2)_{i,j} && (6.29)\\
&= \alpha M_{1,j-1} + \sum_{i\geq j}\delta(\beta+\delta)^{i-j}\left(\binom{i-1}{j-1}+(\alpha+\gamma)\binom{i-1}{j-2}\right) M_{1,i} && (6.30)\\
&= \alpha\ {}_{j-1}F_\tau + \sum_{i\geq j}\delta(\beta+\delta)^{i-j}\left(\binom{i-1}{j-1}+(\alpha+\gamma)\binom{i-1}{j-2}\right)\ {}_iF_\tau. && (6.31)
\end{aligned}
$$

In line (6.29) we have used the definition of the matrix $D_2$, and in line (6.31) we have used the induction hypothesis. We therefore have shown that the $(1,j)$ entry of the matrix $M_{\tilde{\tau}}$ is ${}_jF_{\tilde{\tau}}$.

To finish the proof, we note that multiplying $M_\tau$ by $W_2$ from the left and $V_2$ from the right is equal to summing up the top row of the matrix so that $W_2M_\tau V_2$ is indeed equal to $F_\tau = \sum_{\mathcal{T}_{type[\tau]}} edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})}$. As in the proof of Theorem 4.5, it follows that $W_2(D_2+E_2)^nV_2$ is equal to $\sum_{\mathcal{T}^{n+1}} edge(\mathcal{T})(\alpha+\gamma)^{f'(\mathcal{T})}(\beta+\delta)^{u'(\mathcal{T})}$. □

Theorem 6.4, our main goal, is now easy to prove:

Proof of Theorem 6.4. Using Lemma 6.6 and Theorem 6.7, we simply can follow the lines in the proof of Theorem 4.3. □

To prove the analogous result for alternative tableaux we need one more lemma:

Lemma 6.8. *The bijection $\varphi$ between permutation tableaux and alternative tableaux (see Section 5.1) can naturally be extended to a bijection $\overline{\varphi}$ between bordered permutation tableaux and bordered alternative tableaux. $\overline{\varphi}$ then preserves the type of a tableau.*

Proof. Considering the bijection $\varphi$ between permutation tableaux and alternative tableaux, we note that we can easily extend it to a bijection $\overline{\varphi}$ between bordered permutation tableaux and bordered alternative tableaux by mapping the weights of the border path of a bordered permutation tableau $\mathcal{T}$ to the weights of the border path of the corresponding alternative tableau $\varphi(\mathcal{T}) = \widehat{\mathcal{T}}$. (We simply transfer the weights on the right-hand outer border line, see Figure 6.1). It therefore is also clear that $\overline{\varphi}$ preserves the type $\tau$ of a tableau, since the type is determined by the weights of the border path, which are left unchanged by $\overline{\varphi}$. □

EXAMPLE. See Figure 6.1 for an example of a bordered permutation tableau and its corresponding bordered alternative tableau.

REMARK. Clearly other properties of $\varphi$ that concern the shape or filling of a permutation tableau also hold for $\overline{\varphi}$. In particular, Corollary 5.10 and Lemma 5.11 are also true for $\overline{\varphi}$.

Theorem 6.5 then follows immediately out of Theorem 6.4.

PROOF OF THEOREM 6.5. For the proof we only need to follow the lines in the proof of Theorem 5.12. Considering Theorem 6.4, the bijection $\overline{\varphi}$ maps each bordered permutation tableau of length $n+1$ to a bordered alternative tableau of length $n$ and each bordered permutation tableau of type $\tau$ to a bordered alternative tableau of type $\tau$ (see Lemma 6.8) where $f'(\mathcal{T}) = up(\widehat{\mathcal{T}})$ and $u'(\mathcal{T}) = left(\mathcal{T})$ (due to Corollary 5.11). Therefore Theorem 6.5 follows. □

## 6.3. Matrix Ansatz and Linear Operators – Alternative Proof of Theorem 6.4

In [**7**], another proof of Theorem 6.4 is given. It is actually equivalent to the proof stated above, as noted by the authors, but it avoids the explicit form of the matrices $D_2, E_2, V_2, W_2$. Instead, the matrices are interpreted as linear operators on the infinite-dimensional vector space whose basis is labeled by weighted permutation tableaux. We make this more precise in a moment. This subsection follows mainly the arguments in ([**7**], Section 5).

Consider the formula for $F_\tau$ in Theorem 6.7. There, every bordered permutation tableau contributes a certain weight, i.e., each distinguished 0 and distinguished 1 have a binomial associated to it: each distinguished 0 contributes the binomial $(\beta+\delta)$ while each distinguished 1 contributes the binomial $(\alpha+\gamma)$. Another interpretation is the following that each distinguished 0 is either labeled by $\beta$ or by $\delta$ and, depending on the labeling of the 0, contributes a monomial. So, instead of one bordered permutation tableau, we have a variety of *decorated bordered permutation tableaux*. For an example, see Figure 6.5, where we have two decorated bordered permutation tableaux instead of one bordered permutation tableau.

The above considerations motivate the following definition.

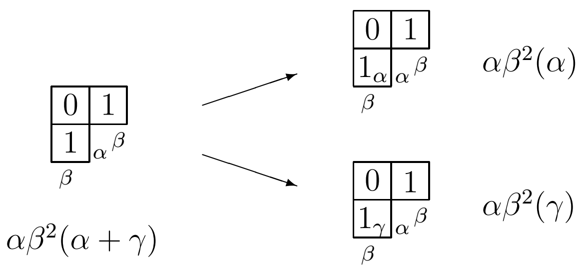



FIGURE 6.5. A bordered permutation tableau, the two corresponding decorated bordered permutation (dbp) tableaux, and their weights.

DEFINITION 6.9. A **dbp tableau** (decorated bordered permutation tableau) is a bordered permutation tableau where every distinguished 0 is either of type $0_\beta$ or of type $0_\delta$, and each distinguished 1 is either of type $1_\alpha$ or of type $1_\gamma$.

EXAMPLE. On the right-hand side of Figure 6.5 two dbp tableaux are shown.

REMARK. Due to the bijection between bordered permutation tableaux and bordered alternative tableaux, one could also define *decorated bordered alternative tableaux* by assigning weights to each arrow.

For a dbp tableau $\mathcal{T}$ we define a weight consisting of two compounds: we define $mon(\mathcal{T})$ to be the product of the labels of the distinguished entries $0_\beta, 0_\delta, 1_\alpha, 1_\gamma$. Thus, each distinguished entry contributes to the term $mon(\mathcal{T})$. For the second compound, recall $edge(\mathcal{T})$, the product of the labels on the border of $\mathcal{T}$ (or, equivalently, the product of the weights of the border path). We define the **weight** $wt(\mathcal{T})$ of a dpb tableau $\mathcal{T}$ by

$$wt(\mathcal{T}) = edge(\mathcal{T})mon(\mathcal{T}).$$

Now, let $\mathfrak{D}$ be the infinite-dimensional vector space with basis indexed by the set of all dbp tableaux. We define operators $D_\alpha, D_\delta, E_\beta, E_\gamma$ on this vector space $\mathfrak{D}$. They act by sending each basis vector $b_\mathcal{T}$ labeled by a dbp tableau $\mathcal{T}$ to a linear combination of basis vectors labeled by some other dbp tableaux $\mathcal{T}_1', \ldots, \mathcal{T}_l'$. The coefficients of the linear combination will turn out to be the weight of the dbp tableau labeling the vector. To simplify notation, we identify a basis vector $b_\mathcal{T}$ with the dbp tableau $\mathcal{T}$ labeling it. Hence, $\mathfrak{D}$ is the vector space formed by all dbp tableaux. Before defining the mentioned linear operators, we introduce the following sets: let $\mathcal{T}$ be a dbp tableau, and recall that the border path of a permutation tableau corresponds to the right-hand border line of the tableau (without the first south step). We define

- $\mathcal{U}_\alpha(\mathcal{T})$ to be the dbp tableaux which can be obtained from $\mathcal{T}$ by adding a new row of length 0 to each tableau and labeling the new edge by $\alpha$.
- $\mathcal{U}_\gamma(\mathcal{T})$ to be the dbp tableaux which can be obtained from $\mathcal{T}$ by adding a new row of length 0 to each tableau and labeling the new edge by $\gamma$.
- $\mathcal{U}_\delta(\mathcal{T})$ to be the set of dbp tableaux which can be obtained from $\mathcal{T}$ by adding a new column to the left of the tableau (with the same length as the left-hand border line in $\mathcal{T}$), labeling the new edge on the bottom by $\delta$, and adding all possible fillings of 0's and 1's to the new column such that a valid filling is obtained;
- $\mathcal{U}_\beta(\mathcal{T})$ to be the set of dbp tableaux which can be obtained from $\mathcal{T}$, by adding a new column to the left of the tableau (with the same length as the left-hand border line in $\mathcal{T}$), labeling the new edge on the bottom by $\beta$, and adding all possible fillings of 0's and 1's to the new column such that a valid filling is obtained.

Now we can define the operators $D_\alpha, D_\delta, E_\beta, E_\gamma$ on $\mathfrak{D}$ by their action on a basis vector of $\mathfrak{D}$, hence by their actions on a dbp tableau. Note that we apply the operators from the right-hand side. We define

$$(\mathcal{T})D_\alpha = \sum_{\mathcal{U}\in\mathcal{U}_\alpha(\mathcal{T})} \frac{wt(\mathcal{U})}{wt(\mathcal{T})}\mathcal{U}, \tag{6.32}$$

$$(\mathcal{T})E_\gamma = \sum_{\mathcal{U}\in\mathcal{U}_\gamma(\mathcal{T})} \frac{wt(\mathcal{U})}{wt(\mathcal{T})}\mathcal{U}, \tag{6.33}$$

$$(\mathcal{T})D_\delta = \sum_{\mathcal{U}\in\mathcal{U}_\delta(\mathcal{T})} \frac{wt(\mathcal{U})}{wt(\mathcal{T})}\mathcal{U}, \tag{6.34}$$

$$(\mathcal{T})E_\beta = \sum_{\mathcal{U}\in\mathcal{U}_\beta(\mathcal{T})} \frac{wt(\mathcal{U})}{wt(\mathcal{T})}\mathcal{U}. \tag{6.35}$$

Let us take a closer look at these operators. Note that in (6.32) and (6.33), $D_\alpha$ and $E_\beta$ yield only a single dbp tableau, namely the tableau $\mathcal{T}$ with an additional empty row labeled by $\alpha$ or $\beta$, respectively.

On the other hand, $D_\delta$ and $E_\gamma$ yield a sum of dbp tableaux that are obtained as follows: by adding an additional column to $\mathcal{T}$ (of the length of the left-hand border side of the shape $\mathcal{T}$) and filling this column with 0's and 1's such that a valid filling results. The new edge on the bottom is labeled by $\delta$ or $\gamma$, respectively. We can alternatively describe $D_\alpha, D_\delta, E_\beta, E_\gamma$ by their action on the shape of dbp tableau $\mathcal{T}$:

$D_\alpha$ acts by adding a south step to the south-west most corner of $\lambda$ with weight $\alpha$;
$E_\gamma$ acts by adding a south step to the south-west most corner of $\lambda$ with weight $\gamma$;
$D_\delta$ acts by adding a west step to the south-west most corner of $\lambda$ with weight $\delta$;
$E_\beta$ acts by adding a west step to the south-west most corner of $\lambda$ with weight $\beta$.

This is equivalent to adding an empty row $(D_\alpha, E_\gamma)$ or adding a new column $(D_\delta, E_\beta)$. Adding valid fillings to the shape (in accordance with the filling of $\mathcal{T}$), we obtain the dbp tableaux that are summed up in the definitions (6.32) – (6.35).

Furthermore, we define $W$ be the empty tableau consisting of a single, non-labeled south step, and set $mon(W) = edge(W) = 1$. Hence, the weight of $W$ is given by

$$wt(W) = 1^{\cdot}$$

Now, given a word $C = C_1 \dots C_n$ in $D_\alpha, D_\delta, E_\beta, E_\gamma$, we form $WC$; that is, we apply $C_1, \dots, C_n$ one after the other to the empty tableau. (Recall that we start the acting of the word $C$ from the left-hand side.) This yields a (finite) linear combination of basis vectors indexed by certain dbp tableaux. We denote by $\mathcal{C}$ the set of dbp tableaux that appear in the linear combination $WC$, and we say that these dbp tableaux are constructed by $C$. By the considerations above, we can conclude that these dbp tableaux all have the same shape and that this shape is constructed by the operators $D_\alpha, D_\delta, E_\beta, E_\gamma$ as described above. Adding a valid filling to this shapes yields all dbp tableaux that are summed up by $WC$.

Turning to the coefficients in the linear combination $WC$, we claim that they correspond to the weight of the dbp tableaux, hence can be written as

$$WC = \sum_{\mathcal{U}\in\,\mathcal{C}} edge(\mathcal{U}) mon(\mathcal{U})\, \mathcal{U}. \tag{6.36}$$

This is due to the term

$$\frac{wt(\mathcal{U})}{wt(\mathcal{T})}$$

in (6.32) – (6.35), since every time we apply an operator $D_\alpha, D_\delta, E_\beta, E_\gamma$ to $\mathcal{T}$, we divide by the weight of $\mathcal{T}$ which is the coefficient of $\mathcal{T}$. Only the term in the numerator, the weight $wt(\mathcal{U})$, remains.

Finally, we introduce one last operator in $\mathfrak{D}$. For a (finite) linear combination of basis vectors $\sum_{j=1}^{n} \epsilon_j d_j$, with $d_j \in \mathfrak{D}$ and $\epsilon_j \in \mathbb{R}$ we define

$$\sum_{j=1}^{n} \epsilon_j e_j = \sum_{j=1}^{n} \epsilon_j.$$

Assuming that $WC$ is a finite linear combination of basis vectors as above, we we have

$$WCV = \sum_{\mathcal{U} \in \mathcal{C}} wt(\mathcal{U}). \tag{6.37}$$

Now, we can give a second proof of Theorem 6.4. In this proof we will examine the structure of dbp tableaux and establish one-to-one correspondences between dbp tableaux of different shapes.

Second proof of Theorem 6.4. ([**7**], *Second proof of Theorem 4.1.*)

We define $D = D_\alpha + D_\delta$ and $E = E_\beta + E_\gamma$. $V, W$ are defined as above. This time, we apply Theorem 2.3, the stronger version of the Matrix Ansatz Theorem. Hence, we check that the operators $D, E, V, W$ satisfy Equations (2.68) – (2.70). We set $p = q = 1$ (since we are considering the symmetric case) and $\chi = \alpha\beta - \gamma\delta$. We start with (2.68), which in our setting reads

$$DECV - EDCV = (\alpha\beta - \gamma\delta)(D + E)CV. \tag{6.38}$$

We want to show that this equation holds for any (finite) word $C$ in $\{D, E\}$. Since $C$ is a product of the terms $D = D_\alpha + D_\delta$ and $E = E_\beta + E_\gamma$, it can also be expressed in terms of sums of words in $D_\alpha, D_\delta, E_\beta, E_\gamma$. As all operators are linear, it is sufficient to consider the case where $C$ is a word in $D_\alpha, D_\delta, E_\beta, E_\gamma$. Consider the terms of

$$\begin{aligned} DECV &= (D_\alpha + D_\delta)(E_\beta + E_\gamma)CV \\ &= D_\alpha E_\beta CV + D_\delta E_\beta CV + D_\alpha E_\gamma CV + D_\delta E_\gamma CV. \end{aligned} \tag{6.39}$$

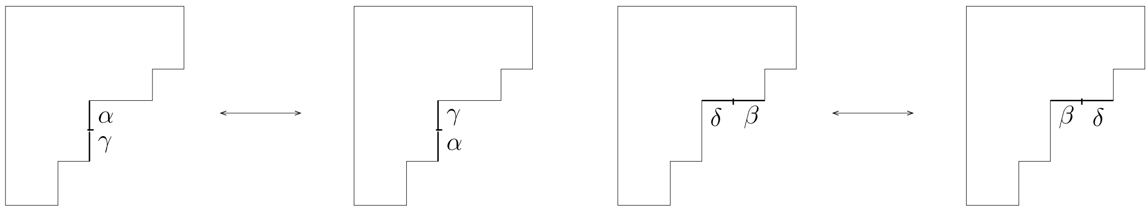
First, we claim that

$$D_\alpha E_\gamma CV = E_\gamma D_\alpha CV. \tag{6.40}$$

How to read this equation and how can we verify it? Equation (6.40) means that for any dbp tableau $\mathcal{T}$, we have

$$(\mathcal{T}) D_\alpha E_\gamma CV = (\mathcal{T}) E_\gamma D_\alpha CV. \tag{6.41}$$

We examine this equation: first, consider the left-hand side. $\mathcal{T} D_\alpha E_\gamma C$ is a linear combination consisting of all dbp tableaux whose labeled shape is constructed by $\mathcal{T} D_\alpha E_\gamma C$. (This is the labeled shape obtained from $\mathcal{T}$ by first adding a south step labeled $\alpha$ (due to $D_\alpha$), a south step labeled $\gamma$ (due to $E_\gamma$), and then adding the steps according to $C$). The coefficient in the linear combination of each tableau is given by the weight of the corresponding dbp tableaux (multiplied by $1/wt(\mathcal{T})$). Turn to to right-hand side of the Equation (6.41). $(\mathcal{T}) E_\gamma D_\alpha C$ also yields a linear combination of dbp tableaux. These




(a) $\mathcal{T}D_\alpha E_\gamma C \longleftrightarrow \mathcal{T}E_\gamma D_\alpha C$ (b) $\mathcal{T}D_\delta E_\beta \longleftrightarrow \mathcal{T}E_\beta D_\delta C$

FIGURE 6.6. An informal illustration of the correspondence of dbp tableaux.

tableaux are *not* the same tableaux that appear in $(\mathcal{T})E_\gamma D_\alpha C$; but they are all of same shape, only the labeling of the border is different, see Figure 6.6(a).

Nevertheless, Equation (6.41) only claims that *summation of the coefficients* of the linear combinations $(\mathcal{T})D_\alpha E_\gamma C$ and $(\mathcal{T})E_\gamma D_\alpha C$ yields the same result ($V$ sums up the coefficients of a linear combination).

To verify this claim, we compare the dbp tableaux appearing in each linear combination. The main idea is the following: if we find a one-to-one correspondence between tableaux of $(\mathcal{T})D_\alpha E_\gamma C$ and $(\mathcal{T})E_\gamma D_\alpha C$ with the same weights, then summation of the coefficients has to lead to the same quantity, as the coefficients are determined by the weights.

Such a correspondence is found relatively easily: since the shapes of the tableaux in $(\mathcal{T})D_\alpha E_\gamma C$ and $(\mathcal{T})E_\gamma D_\alpha C$ are the same, we can use the same valid fillings for both labeled shapes. So, to every dbp tableau $\mathcal{U}$ in $(\mathcal{T})D_\alpha E_\gamma C$ corresponds a dpt tableau $\mathcal{U}'$ in $(\mathcal{T})E_\gamma D_\alpha C$ that has the same filling and differs only in the order of the labels on the border. Now, since $\mathcal{U}$ and $\mathcal{U}'$ have the same filling, they also have the same distinguished entries, and hence $mon(\mathcal{U}) = mon(\mathcal{U}')$. Furthermore, $edge(\mathcal{U}) = edge(\mathcal{U}')$, since only the order of two labels $\alpha, \beta$ have changed, but the product of all labels gives the same quantity. Thus, we have $wt(\mathcal{U}) = wt(\mathcal{U}')$. So, for every dbp tableau $\mathcal{U}$ in $(\mathcal{T})D_\alpha E_\gamma C$ the corresponding $U'$ in $(\mathcal{T})E_\gamma D_\alpha C$ has the same weight, and hence the coefficients in the linear combinations are the same. Adding them up (through application of $V$) gives the same, and Equation (6.41) is shown.

In analogous manner we approach the following equation:

$$D_\delta E_\beta CV = E_\beta D_\delta CV. \tag{6.42}$$

Again, we have to check that

$$(\mathcal{T})D_\delta E_\beta CV = (\mathcal{T})E_\beta D_\delta CV,$$

for a dbp tableau $\mathcal{T}$ and a word $C \in \{D_\alpha, D_\delta, E_\beta, E_\gamma\}$. The term on the left-hand side, $(\mathcal{T})D_\delta E_\beta C$ adds two west steps with labels $\delta, \beta$ to $\mathcal{T}$ and then applies $C$. The term on the right-hand side, $(\mathcal{T})E_\beta D_\delta C$ also adds two west steps before applying $C$, but the label of the steps are in different order, see Figure 6.6(b). As before, we establish a correspondence between dbp tableaux of $(\mathcal{T})D_\delta E_\beta C$ and $(\mathcal{T})E_\beta D_\delta C$. The bordered shapes of the tableaux on both sides differ only in the order of the labels $\delta, \beta$. Hence, every filling of the labeled shape on the left-hand side in Figure 6.6(b) is also a valid filling for the labeled shape on the right-hand side in Figure 6.6(b). We associate the dbp tableaux of both sides to each other which have the same filling. Hence, *mon* gives the same (since the fillings are the same). *edge* yields the same quantity as well, since only the order of the labels $\delta, \beta$ is different, but not their values. So, the corresponding dbp tableaux have the same weight. As the coefficients in $(\mathcal{T})D_\delta E_\beta C$ and $(\mathcal{T})E_\beta D_\delta C$ are determined by the weight, they are equal, too. Therefore, summing them up by application of $V$ gives the same result. This is what Equation (6.42) claims.

The cases of other combinations of operators in (6.39) are a little more complicated. Take, for example, the following equation

$$D_\alpha E_\beta[1]CV = E_\beta D_\alpha CV. \tag{6.43}$$

Here, the [1] means that the corner which is created by adding a south step ($D_\alpha$) followed by a west step ($E_\beta$) is filled with an additional 1 (hence, it is not the topmost 1 in its column, see Definition 5.2). We will see afterwards why the distinction of the type of entry is important. We want to show that Equation (6.43) holds. With the same shorthand as above, it actually means

$$(\mathcal{T})D_\alpha E_\beta[1]CV = (\mathcal{T})E_\beta D_\alpha CV. \tag{6.44}$$

As before, we establish a correspondence of dbp tableau with same weights. Consider the second shape in Figure 6.7(a). A valid filling of this shape becomes a valid filling for the first shape if we add the entry [1]. To go into the other direction, note that in a valid filling of the first shape we can erase the box containing the additional [1] to obtain a valid filling for the lower shape. (Here it is important to note that after erasing the additional 1 another 1 remains in the column, as required for a valid filling). We still have to check that the weights of the corresponding dbp tableaux are equal: the quantity of *mon* is the same (since the distinguished entries are the same), and *edge* is the same (since only the order of the steps and therefore of labels have changed, but not the values). Thus, summing up the coefficients $(\mathcal{T})D_\alpha E_\beta[1]C$ or $(\mathcal{T})E_\beta D_\alpha C$ gives the same, and (6.44) is established.

With the same method (and shorthand) we show the following equalities.

$$D_\alpha E_\beta[1_\gamma]CV = \alpha\beta E_\gamma CV, \tag{6.45}$$

$$D_\alpha E_\beta[1_\alpha]CV = \alpha\beta D_\alpha CV, \tag{6.46}$$

$$D_\alpha E_\beta[0_\beta]CV = \alpha\beta E_\beta CV, \tag{6.47}$$

$$D_\alpha E_\beta[0_\delta]CV = \alpha\beta D_\delta CV. \tag{6.48}$$

For example, $[1_\gamma]$ means that the corner created by the south step ($D_\alpha$) and west step ($E_\beta$) contains a distinguished 1 labeled $\gamma$, whereas $[0_\beta]$ means that this corner contains a distinguished 0 labeled $\beta$, see Figure 6.7(b) and Figure 6.7(c).

First, we consider (6.45). To establish a correspondence of dbp tableaux constructed by $D_\alpha E_\beta[1_\gamma]C$ and $E_\gamma C$, see Figure 6.7(b) and consider a valid filling of the first labeled shape. If the corner created by $D_\alpha E_\beta$ contains a $1_\gamma$, then the whole column above must contain only 0's (by definition, since a distinguished 1 is the topmost 1 in its column). Cutting this column out, we obtain a valid filling for the labeled shape on the right-hand side in the figure. Conversely, we can introduce such a column of 0's and a 1 in a dbp tableau as on the right-hand side of Figure 6.7(b) without spoiling the validity of the filling: the crucial point here is that a column containing only 0's and a 1, which is the rightmost one in its row, does not influence any other entries in the filling. (If the 1 was located at a higher position in its column, it could lie left to a distinguished 0 which is forbidden). So, through addition or deletion of this row, we have a correspondence between the dbp tableaux in $D_\alpha E_\beta[1_\gamma]C$ and $E_\gamma CV$. We still have to compare the weights of the corresponding tableaux, to check whether the coefficients (determined by the weights) are the same. According to Figure 6.7(c), the left-hand shape differs from the right-hand shape by a distinguished 1 labeled $\gamma$ and by two steps labeled $\alpha, \beta$; hence there is an additional factor $\alpha\beta\gamma$ in the weight. On the other hand, the right-hand shape contains an additional step labeled $\gamma$. Multiplying by $\alpha\beta$, we adjust the weights of $E_\gamma C$, and so Equation (6.45) holds.

Equation (6.46) follows in analogous manner (only exchange $1_\alpha$ and $1_\gamma$ and one label).

Now, consider Equation (6.47). Here, the corner created by $D_\alpha, E_\beta$ is filled with a distinguished 0 labeled with $\beta$, see Figure 6.8(a). In the first labeled shape this corner is indicated. To establish a correspondence between $D_\alpha E_\beta[0_\beta]C$ and $E_\beta C$ consider the first shape in Figure 6.8(a). Since the entry $0_\beta$ is distinguished, all entries to its left have to be 0's (see Definition 5.2). If we cut this row out, we obtain a valid filling for the second shape. Conversely, if we add this row to the second shape again, the rightmost 0 becomes a distinguished 0 (since there is at least one 1 in the column above) and a valid filling for the first shape is obtained. We only need to check that the weights of the corresponding tableaux are the same. Compare the weight of a dbp tableau constructed by $D_\alpha E_\beta[0_\beta]C$ to the weight of the corresponding dbp tableau constructed by $E_\beta C$. We see that the weight of the first tableau differs by the factors $\alpha, \beta$ (contributed by the steps) and $\beta$ (the label of $0_\beta$) from the weight of the second tableau. On the other hand, an additional $\beta$ is contributed by the step $D_\delta$ to the weight of the second tableau. For this reason, we only need to multiply $(\mathcal{T})D_\delta C$ by $\alpha\beta$ to obtain equality of the weights, and so Equation (6.48) follows.

With the same notation we also have

$$D_\delta E_\gamma CV = E_\gamma D_\delta(1)CV \tag{6.49}$$
$$\gamma\delta E_\gamma CV = E_\gamma D_\delta(1_\gamma)CV \tag{6.50}$$
$$\gamma\delta D_\alpha CV = E_\gamma D_\delta(1_\alpha)CV \tag{6.51}$$
$$\gamma\delta E_\beta CV = E_\gamma D_\delta(0_\beta)CV \tag{6.52}$$
$$\gamma\delta D_\delta CV = E_\gamma D_\delta(0_\delta)CV. \tag{6.53}$$

This equations are similar to (6.43) and (6.45) – (6.48): if we exchange the left-hand sides and the right-hand sides, we obtain the same shapes as before, with the only difference that the labels of the border have to be changed.

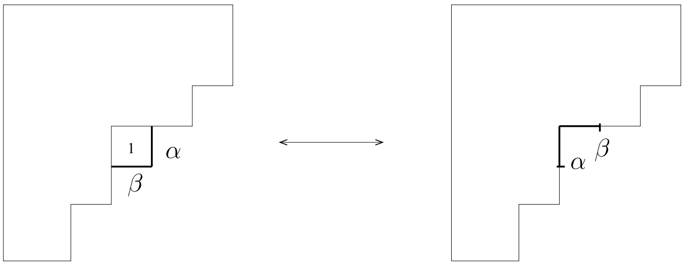
We now add up Equations (6.40), (6.42), (6.43), (6.45) – (6.48) and (6.49) – (6.53) step by step.

First note that (6.43), (6.45) – (6.48) yield to

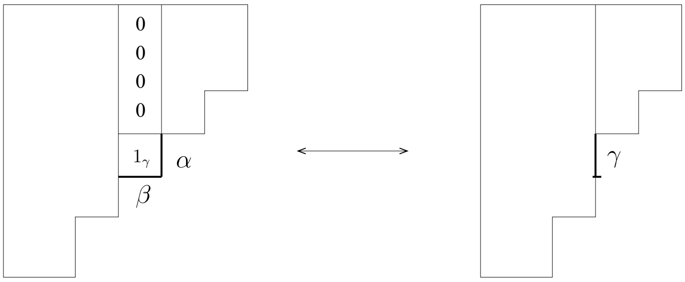
$$D_\alpha E_\beta CV = (E_\beta D_\alpha + \alpha\beta(D_\alpha + D_\delta + E_\beta + E_\gamma))CV. \tag{6.54}$$

Then, note that (6.49) – (6.53) add up to

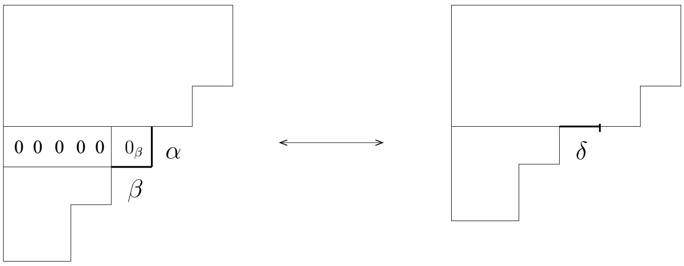
$$(D_\delta E_\gamma + \gamma\delta(D_\alpha + D_\delta + E_\beta + E_\gamma))CV = E_\gamma D_\delta CV. \tag{6.55}$$




(a) $\mathcal{T}D_\alpha E_\beta(1)C \leftrightarrow \alpha\beta\mathcal{T}E_\beta D_\alpha C$




(b) $\mathcal{T}D_\alpha E_\beta(1_\gamma)C \leftrightarrow \alpha\beta E_\gamma C$




(c) $\mathcal{T}D_\alpha E_\beta(0_\beta)C \leftrightarrow \alpha\beta E_\beta C$

FIGURE 6.7. An informal illustration of the correspondence of dbp tableaux.

Now (6.54) and (6.55) together with (6.40) and (6.42) give

$$\begin{aligned}(D_\alpha E_\gamma + D_\delta E_\beta + D_\alpha E_\beta + D_\delta E_\gamma + E_\gamma D_\delta + \gamma\delta(D_\alpha + D_\delta + E_\beta + E_\gamma))CV \\ = (E_\gamma D_\alpha + E_\beta D_\delta + E_\beta D_\alpha + \alpha\beta(D_\alpha + D_\delta + E_\beta + E_\gamma))CV. \quad (6.56)\end{aligned}$$

This can be rewritten as

$$\begin{aligned}((D_\alpha + D_\delta)(E_\beta + E_\gamma))CV + \gamma\delta(D_\alpha + D_\delta + E_\beta + E_\gamma)CV \\ = ((D_\alpha + D_\delta)(E_\beta + E_\gamma))CV + (\alpha\beta(D_\alpha + D_\delta + E_\beta + E_\gamma))CV. \quad (6.57)\end{aligned}$$

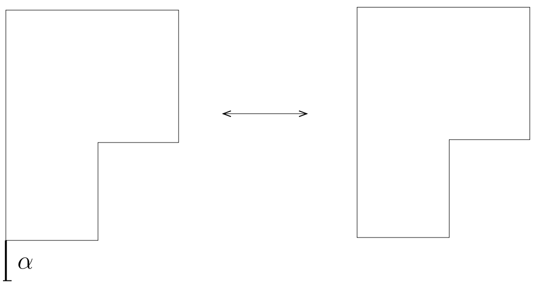
Recalling that $D = D_\alpha + D_\delta$ and $E = E_\beta + E_\gamma$, the equation above implies

$$DECV + \gamma\delta(D+E)CV = EDCV + \alpha\beta(D+E)CV, \quad (6.58)$$

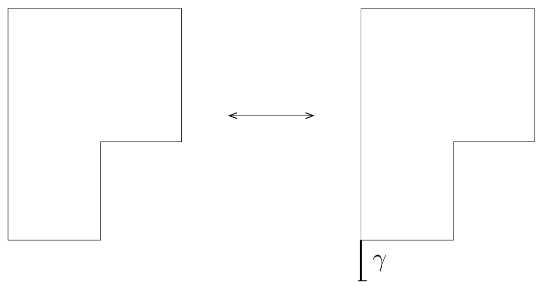
which is equivalent to the desired Equation (6.38).

In a similar manner, we show that the next Matrix Ansatz equation, Equation (2.69), holds. We need to show that

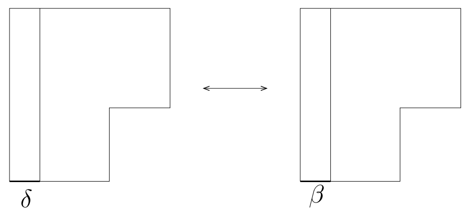
$$(\beta D - \delta E)V = (\alpha\beta - \gamma\delta)V. \quad (6.59)$$




(a) $\mathcal{T} D_\alpha \leftrightarrow \alpha \mathcal{T}$




(b) $\gamma \mathcal{T} \leftrightarrow \mathcal{T} E_\gamma$




(c) $\beta \mathcal{T} D_\delta \leftrightarrow \delta \mathcal{T} E_\beta$

FIGURE 6.8. An informal illustration of the correspondence of dbp tableaux.

We note that

$$D_\alpha V = \alpha V, \tag{6.60}$$
$$\beta D_\delta V = \delta E_\beta V, \tag{6.61}$$
$$\gamma V = E_\gamma V. \tag{6.62}$$

This equations are verified similarly to the ones before. We have to check that for a dbp tableau $\mathcal{T}$

$$(\mathcal{T}) D_\alpha V = \alpha (\mathcal{T}) V, \tag{6.63}$$
$$\beta (\mathcal{T}) D_\delta V = \delta (\mathcal{T}) E_\beta V, \tag{6.64}$$
$$\gamma (\mathcal{T}) V = (\mathcal{T}) E_\gamma V. \tag{6.65}$$

Again, this is done by establishing a correspondence of dbp tableau. They are illustrated in Figure 6.8. To establish a correspondence for the first equation, consider a dbp tableau constructed by $(\mathcal{T})D_\alpha$. Deleting the last south step labeled $\alpha$ (introduced by $D_\alpha$) gives $\mathcal{T}$. The lost of the label $\alpha$ is compensated by multiplication by $\alpha$ on the right-hand side in (6.63). This is illustrated in Figure 6.8(a). Changing the roles of $\alpha$ and $\gamma$, Equation (6.65) follows, too, see Figure 6.8(b). For Equation (6.64), see Figure 6.8(c). The bordered shapes only differ in their labels, hence the correspondence is established immediately. Only the weights have to be adjusted by multiplication of $\beta$ or $\delta$.

We rewrite Equations (6.60) – (6.62) to find

$$\begin{aligned} \beta D_\alpha V &= \alpha \beta V \\ \beta D_\delta V &= \delta E_\beta V \\ \gamma \delta V &= \delta E_\gamma V \end{aligned}$$

Summing up, we obtain

$$\beta(D_\alpha + D_\delta)V + \gamma\delta V = \alpha\beta V + \delta(E_\beta + E_\gamma)V. \tag{6.66}$$

This can be rewritten as

$$\begin{aligned} \beta DV + \gamma\delta V &= \alpha\beta V + \delta EV \\ \beta DV - \delta EV &= (\alpha\beta - \gamma\delta)V \end{aligned} \tag{6.67}$$

Equation (6.67) is exactly the desired Equation (6.59).

Finally, the third Matrix Ansatz equation, Equation (2.70) remains to be shown,

$$W(\alpha E - \gamma D) = (\alpha\beta - \gamma\delta)W. \tag{6.68}$$

Note that

$$\alpha W E_\gamma = \gamma W D_\alpha \tag{6.69}$$

$$\alpha W E_\beta = \alpha\beta W \tag{6.70}$$

$$\gamma\delta W = \gamma W D_\delta. \tag{6.71}$$

That is, for a word $C$ in $\{D_\alpha, E_\beta, E_\gamma, D_\delta\}$ we need to show that

$$\alpha W E_\gamma C = \gamma W D_\alpha C \tag{6.72}$$

$$\alpha W E_\beta C = \alpha\beta W C \tag{6.73}$$

$$\gamma\delta W C = \gamma W D_\delta C. \tag{6.74}$$

Equation (6.72) is evident, see Figure 6.9(a). $WE_\gamma$ and $WD_\alpha$ yield the same shape, only one of the labels $\alpha$ or $\gamma$ change. This is compensated by multiplication by $\alpha$ or $\gamma$, respectively. To check Equation (6.73), see Figure 6.9(b): the right-most column of every tableau constructed by $WE_\beta C$ contains a single cell filled with a 1 (there has to be at least one 1 in every column). Cutting this cell out yields another dbp tableau constructed by $WC$. The lost of the label $\beta$ is compensated by multiplication by $\beta$. Equation (6.74) follows analogously, only replacing the label of the west step, $\beta$, by $\delta$, see Figure 6.9(c). Again, through summation and rewriting we obtain

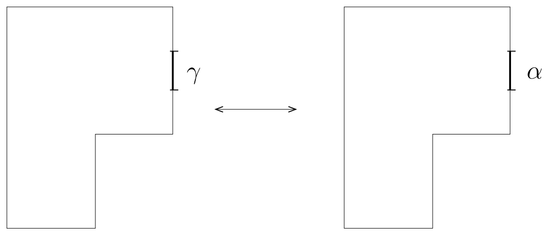
$$\begin{aligned} \alpha W E_\gamma + \alpha W E_\beta + \gamma\delta W &= \gamma W D_\alpha + \alpha\beta W + \gamma W D_\delta \\ W\alpha E + \gamma\delta W &= W\gamma D + \alpha\beta W \\ W(\alpha E - \gamma D) &= (\alpha\beta - \gamma\delta)W. \end{aligned}$$

This is exactly the desired equation.

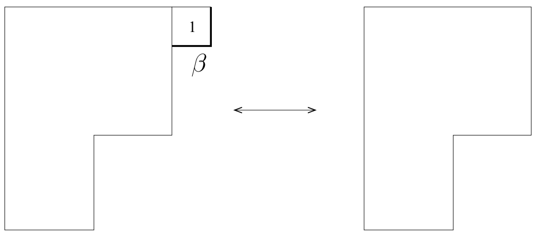
We have now seen that $D, E, V, W$ satisfy the Matrix Ansatz Theorem and that therefore the unnormalized probability $f_n$ of finding the TASEP in state $\tau \in \{0,1\}^n$ is given by

$$f_n = W\left(\prod_{i=1}^{n} (\tau_i D + (1-\tau_i)E)\right)V. \tag{6.75}$$

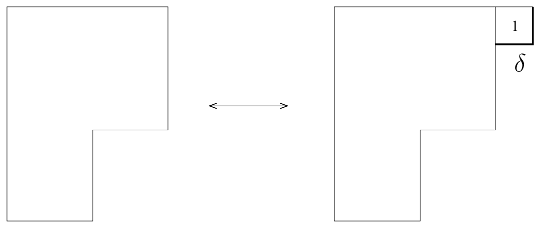
We claim that this quantity at the same time is the sum of the weights of all bordered permutation tableaux of type $\tau$. Once we have shown that this is true, the theorem is established. $D_\alpha, E_\beta, E_\gamma, D_\delta$ introduce south and west steps that are labeled by $\alpha, \beta, \gamma, \delta$. Recalling Definition 6.1, we see that all labels are such that a dbp tableau results. Furthermore, if $\tau_i = 1$, then $D = (D_\alpha + D_\delta)$ enters in the formula in (6.75). So, either a south step with label $\alpha$ is added to the tableau or a west step with label $\delta$. If $\tau_i = 0$




(a) $\alpha W E_\gamma C \leftrightarrow \gamma W D_\alpha C$




(b) $\alpha W E_\beta C \leftrightarrow \alpha\beta W C$




(c) $\gamma\delta W C \leftrightarrow \gamma W D_\delta C$

FIGURE 6.9. An informal illustration of the correspondence of dbp tableaux.

then $E = (E_\beta + E_\gamma)$ enters, either a south step with label $\gamma$ or a west step with label $\beta$ is added. Comparing this to Definition 6.3, we see that these are exactly the conditions for a dbp tableau to be of type $\tau = (\tau_1, \ldots, \tau_n)$. Since afterwards we consider the weights of the dbp tableaux, we can group the monomials that appear in the sum (due to the labeling of the distinguished 0's and 1's) and obtain the same result as summing the weights of bordered permutation tableaux (the grouping is seen in Figure 6.5 by reversing the arrows).

$\square$

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APPENDIX A

# Counting Lattice Paths and the Lindström-Gessel-Viennot-Determinant

In this appendix we prove two results concerning the number of certain lattice paths that are needed in Section 3.4. On the one hand, we shall see how to calculate the number of non-intersecting paths as defined in Definition 3.13 — this is done in Lemma A.3 — and, on the other hand, we shall see how to enumerate simple lattice paths that lie within some upper and lower boundaries — this is done in Lemma A.4. To this end, we first introduce the so-called *Lindström-Gessel-Viennot-Determinant* [2].

We start with some notation. Let $D$ be an acyclic graph (that is a directed graph containing no directed cycles). $D$ need not to be finite, but we assume that between two vertices there is only a finite number of paths.

Let $k$, a positive integer, be fixed and call a *$k$-vertex* a $k$-tuple of vertices in $D$. Choose two disjoint sets of vertices $\{u_1, \ldots, u_k\}, \{v_1, \ldots, v_k\}$ and let $\mathbf{u} = (u_1, \ldots, u_k)$ and $\mathbf{v} = (v_1, \ldots, v_k)$ be two $k$-vertices. Then, a $k$-tuple of paths $\omega = (\omega_1, \ldots, \omega_n)$, where $\omega_i$ is a path from $u_i$ to $v_i$, is called a $k$-path. This $k$-path is said to be *non-intersecting*, if all paths $A_i$ are vertex-disjoint (that is, they do not have a vertex in common). Let $S_k$ be the set of permutations of $\{1, \ldots, k\}$. For $\pi \in S_k$, we mean by $\pi(\mathbf{v})$ the $k$-vertex $(v_{\pi(1)}, \ldots, v_{\pi(k)})$.

We finally assign a weight to every edge of $D$. The weight of a path is defined as the product of the weights of its edges, and the weight of a $k$-path is defined as the product of the weights of its components. Let $P(u_i, v_i)$ be the set of all paths from $u_i$ to $v_i$. We define $W_P(u_i, v_i)$ to be the sum of the weights of the paths in $P(u_i, v_i)$. Analogously we define $P(\mathbf{u}, \mathbf{v})$ to be the set of all $k$-paths from $\mathbf{u}$ to $\mathbf{v}$. We define $W_P(\mathbf{u}, \mathbf{v})$ to be the sum of all weights of the paths in $P(\mathbf{u}, \mathbf{v})$. Finally, let $N(\mathbf{u}, \mathbf{v})$ be the subset of all disjoint $k$-paths in $P(\mathbf{u}, \mathbf{v})$, and let $W_N(\mathbf{u}, \mathbf{v})$ be the sum of their weights. Then, the Theorem of Lindström-Gessel-Viennot reads as follows:

THEOREM A.1 ([**15**], Lemma 1). *With the above notations, we have*

$$\sum_{\pi \in S_k} (\text{sgn } \pi) W_N(\mathbf{u}, \pi(\mathbf{v})) = \det_{1 \le i,j \le n} \left| W_P(u_i, v_j) \right| \tag{A.1}$$

[2]The result was originally obtained by B. Lindström in [**15**, Lemma 1]. By a curious coincidence the result was rediscovered in the 1980s in three different communities at about the same time. But since only I. Gessel and X. Viennot rediscovered it in its most general form (see, e.g. [**12**]), we refer to it as the Lindström-Gessel-Viennot-Determinant.

PROOF. First, we rewrite the right-hand side of (A.1):

$$\sum_{\pi\in S_k} (\text{sgn }\pi) W_N(\mathbf{u},\pi(\mathbf{v})) = \det_{1\le i,j\le n} \left|W_P(u_i,v_j)\right| = \sum_{\pi\in S_k} \text{sgn}(\pi)\; W_P(\mathbf{u},\pi(\mathbf{v})),$$

where we used the definition of the determinant of a matrix. We will prove that the equation derived above,

$$\sum_{\pi\in S_k} (\text{sgn }\pi) W_N(\mathbf{u},\pi(\mathbf{v})) = \sum_{\pi\in S_k} \text{sgn}(\pi)\; W_P(\mathbf{u},\pi(\mathbf{v})), \tag{A.2}$$

holds. To this end, we consider the set

$$\Upsilon = \bigcup_{\pi\in S_k} \left[P(\mathbf{u},\pi(\mathbf{v})) - N(\mathbf{u},\pi(\mathbf{v}))\right], \tag{A.3}$$

which is the set of $k$-paths from the $k$-vertex $(u_1,\dots,u_n)$ to $(v_{\pi(1)},\dots,v_{\pi(n)})$ which are non-intersecting. On this set $\Upsilon$ we construct a bijection $^* : A \mapsto A^*$ from the set to itself with the following properties:

(1) $A^{**} = A$ for $A \in \Upsilon$;
(2) the weight of $A$ equals the weight of $A^*$ for $A \in \Upsilon$;
(3) if $A = P(\mathbf{u},\pi(\mathbf{v}))$ and $A^* = P(\mathbf{u},\sigma(\mathbf{v}))$, then $\text{sgn }(\pi) = -\text{ sgn }(\sigma)$.

We then can group all elements in $\Upsilon$ in pairs $\{A, A^*\}$, and the terms corresponding to these pairs cancel in the sum on the right-hand side of (A.2). As noted before, these are exactly the terms corresponding to non-disjoint $k$-paths, and hence in (A.2) only the terms on the left-hand side remain in the sum of the right-hand side, which would prove the theorem.

So how to construct a bijection $^*$ with the desired properties ? Let

$$A = (A_1,\dots,A_k) \in \Upsilon$$

be a non-disjoint $k$-path. Set $1 \le i \le k$, such that $A_i$ is the path with the least index to intersect with another path. Furthermore, let $p$ be the point of the first intersection of $A_i$ with another path, and let $j$ be the least integer (greater $i$) such that $A_j$ also passes through $p$. We define

$$A^* = (A_1^*,\dots,A_i^*,\dots,A_j^*,\dots,A_k^*) \in \Upsilon$$

as follows: let $A_i^*$ be the path that follows $A_i$ up to $p$, and from $p$ follows the path $A_j$. On the other hand, let $A_j^*$ be the path that follows $A_j$ until $p$, and from there follows $A_i$ to the end. For $g \ne i,j$, set $A_g^* = A_g$. It easily follows that Properties (1)–(3) hold: first, note that by applying the mapping $^*$ again to $A^*$, the same components of the graph as before are chosen (namely the paths labeled by $i,j$), and thus the paths $A_i, A_j$ are recovered – this shows that (1) holds. Note that the weight of a $k$-path $A$ is equal to the product of all edges which form part of the components $A_1,\dots,A_k$ (this follows immediately from the definition). Since the components of $A^*$ contain the same edges as the ones of $A$ (only rearranged), we immediately see that the weights of $A$ and $A^*$ are the same, and thus Property (2) holds as well. Noting that by applying $^*$ we have interchanged exactly one pair of end vertices (the ones of $A_i$ and $A_j$), we see that for

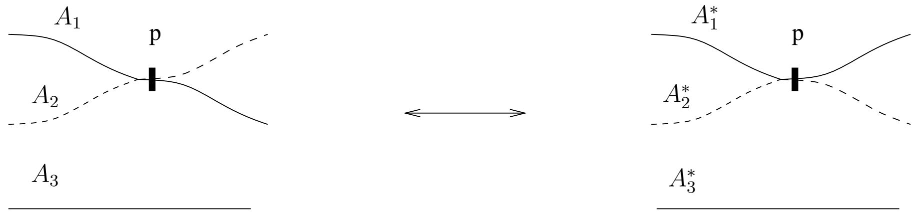



FIGURE A.1. The bijection $^* : A \mapsto A^*$ as defined in the proof of Theorem A.1.

any permutation $\pi$ the sign changes, and hence (3) follows. Thus, we are done and the theorem is established. □

## A.1. Enumerating Non-Intersecting Paths

Now, our first aim is to find a formula for counting non-intersecting paths of fixed length $n$. We will need the following lemma. Recall that binomial paths are lattice paths with step set $\mathbb{S}^b = \{(1,-1),(1,1)\}$.

LEMMA A.2. *The number of binomial paths of length $n$ that start at $(0,0)$ and end at $(n,y)$ is given by*

$$\big|P((0,0),(n,y))\big| = \begin{cases} \binom{n}{\frac{n+y}{2}} & \text{if } n+y \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{A.4}$$

PROOF. We check how many ways there are to construct binomial paths of length $n$ that end at height $y$. We have $n$ steps at disposal, up and down steps of height 1 (more precisely, diagonal steps of height 1). From this it follows that paths of odd length can only end at odd height and paths of even length can only end at even height – this explains why the number of binomial paths equals 0 in the case where $n+y$ is not even. Let us also assume $y \geq 0$; otherwise reflect the path along the $x$-axis. We count how many of the $n$ steps can, or actually have to be, chosen to be up steps. The number of up steps, denoted by *up*, has to exceed the number of down steps, denoted by *do*, by $y$,

$$up - y = do,$$

since we want our path to end at height $y$. The number of up and down steps has to add up to $n$, hence

$$up + do = n.$$

Putting this together, we find

$$up + do = n \tag{A.5}$$

$$up = n - do \tag{A.6}$$

$$up = n - (up - y) \tag{A.7}$$

$$2 \cdot up = n + y \tag{A.8}$$

$$up = \frac{n+y}{2}. \tag{A.9}$$

Therefore, there are $\binom{n}{\frac{n+y}{2}}$ ways to place the up steps within the path, and this (since the other steps are down steps, consequently) equals the total number of paths starting at $(0,0)$ and ending at $(n,y)$. $\square$

Now let $\omega_1, \omega_2$ be two non-intersecting paths, as in Definition 3.13, for which we assume the following: both paths are of fixed length $n \geq 1$. Furthermore, let $\omega_1$ be any binomial path that starts at $(0,0)$ and ends at $(n, 2n-l)$ and $\omega_2$ any binomial path that starts at $(0,2)$ and ends at $(n, n-2l+2)$, with $l \geq 0$. Then, we claim the following:

Lemma A.3 ([**2**], Chapter 7, Proposition 6). *The number of non-intersecting paths $\omega_1, \omega_2$ is given by*

$$\det \begin{vmatrix} \binom{n}{l} & \binom{n}{l-1} \\ \binom{n}{l+1} & \binom{n}{l} \end{vmatrix}.$$

Proof. We will make use of the Lindströ-Gessel-Viennot-Theorem. Consider the square grid $\mathbb{Z} \times \mathbb{Z}$ and turn it counter-clockwise in a $45°$ angle around the point $(0,0)$. We denote this new grid by $\Theta$. Let all edges in this grid be directed towards east and weight them by 1 – this shall be our graph $D$. Fix the integers $n \geq 1$ and $l \geq 0$. Furthermore, set $u_1 = (0,0), u_2 = (0,2)$ and $v_1 = (n, 2n-l)$, $v_2 = (n, 2n-l+2)$.

Now, we consider the left-hand side of (A.1),

$$\sum_{\pi \in S_k} (\text{sgn } \pi) W_N(\mathbf{u}, \pi(\mathbf{v})). \tag{A.10}$$

Note that we have chosen all weights to be equal 1, hence each path has total weight 1. So, in this case, $W_N(\mathbf{u}, \pi(\mathbf{v}))$, the sum of the weights of the paths, equals $N(\mathbf{u}, \pi(\mathbf{v}))$, the number of paths. Therefore, (A.10) equals

$$\sum_{\pi \in S_k} (\text{sgn } \pi) N(\mathbf{u}, \pi(\mathbf{v})). \tag{A.11}$$

Let us examine the terms $N(\mathbf{u}, \pi(\mathbf{v}))$ for $\pi \in S_k$. In our case, $k$, the number of paths, equals 2. Consider all permutations in $S_2$ and their signs:

**$\pi_1$:** $\pi_1(1) = 1, \pi_1(2) = 2$ and $\text{sgn}(\pi_1) = 1$;
**$\pi_2$:** $\pi_2(1) = 2, \pi_2(2) = 1$ and $\text{sgn}(\pi_2) = -1$.

So, the sum in (A.11) contains the following terms: $N(\mathbf{u}, \pi_1(\mathbf{v}))$ and $-N(\mathbf{u}, \pi_2(\mathbf{v}))$.

Let us first examine $N(\mathbf{u}, \pi_1(\mathbf{v})) = N(\mathbf{u}, \mathbf{v})$. This is the number of all disjoint $k$-paths from $\mathbf{u} = (u_1, u_2)$ to $\mathbf{v} = (v_1, v_2)$, namely all paths $\omega_1$ from $u_1$ to $v_1$ and $\omega_2$ from $u_2$ to $v_2$ that are non-intersecting. We claim that these paths $\omega_1$ and $\omega_2$ are non-intersecting in the sense of Definition 3.13. First, note that the paths are indeed binomial paths (the edges of $D$ correspond to the step set of a lattice path). Also note that starting and end points are $(0,0), (n, 2n-l)$ and $(0,2), (n, 2n-l+2)$, respectively. Furthermore, the paths are non-intersecting (in the sense that they do not share a vertex). Hence, $\omega_1, \omega_2$ are non-intersecting in the sense of Definition 3.13, and their number is given by $N(\mathbf{u}, \mathbf{v})$.

Now, $N(\mathbf{u}, \pi_2(\mathbf{v}))$ remains to be calculated. We note that $N(\mathbf{u}, \pi_2(\mathbf{v}))$ gives the number of non-intersecting $k$-paths going from $(u_1, u_2)$ to $(v_{\pi_2(1)}, v_{\pi_2(2)}) = (v_2, v_1)$. We claim that this number is equal to 0. There can not be any pair of disjoint paths from $u_1$ to $v_2$ and $u_2$ to $v_1$, respectively: assume that such a pair existed. These two paths would have to cross each other at some point (since the starting point of the first path, $u_1$, lies below the starting point of the second path, $u_2$, but the endpoint of the first path, $v_2$, lies above the end point of the second path, $v_1$). Both paths start at even height and have step heights equal to 1. Hence, at each time both paths are either at even or at odd height, and therefore have to share a vertex if they cross. So, the paths can not be non-intersecting, and it follows that $N(\mathbf{u}, \pi_2(\mathbf{v})) = 0$. This shows that the left-hand side of (A.1) is equal to $N(\mathbf{u}, \mathbf{v})$ which was shown to be the number of the non-intersecting paths $\omega_1, \omega_2$.

Let us turn to the right-hand side of equation (A.1). We need to consider the $2 \times 2$ matrix whose entries are $W_P(u_i, v_j)$ for $i, j \in \{1, 2\}$. Since we have chosen the weights of all steps to be equal to 1, $W_P u_i, v_j$ equals the number of paths in $D$ from $u_i$ to $v_j$; that is, the number of binomial paths from $u_i$ to $v_j$. Due to Lemma A.2, we know that the number of binomial paths from $(0,0)$ to $(n,k)$ is given by

$$\big|P((0,0),(n,y))\big| = \begin{cases} \binom{n}{\frac{n+y}{2}} & \text{if } n+y \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{A.12}$$

Hence, we conclude that

$$W_P(u_1, v_1) = |P((0,0),(n, n-2l))| = \binom{n}{\frac{2n-2l}{2}} = \binom{n}{n-l} = \binom{n}{l}, \tag{A.13}$$

$$W_P(u_1, v_2) = |P((0,0),(n, n-2l+2))| = \binom{n}{\frac{2n-2l+2}{2}} = \binom{n}{n-l+1} = \binom{n}{l+1}. \tag{A.14}$$

Through up-shifting of the starting vertex $u_2$ by two, we also obtain

$$W_P(u_2, v_1) = |P((0,2),(n, n-2l))| = P((0,0),(n, n-2l-2)) = \binom{n}{\frac{2n-2l-2}{2}} = \binom{n}{n-l-1} = \binom{n}{l-1}, \tag{A.15}$$

$$W_P(u_2, v_2) = |P((0,2),(n, n-2l+2))| = P((0,0),(n, n-2l)) = \binom{n}{\frac{2n-2l}{2}} = \binom{n}{n-l} = \binom{n}{l}. \tag{A.16}$$

Substituting (A.13) - (A.16) in the right-hand side of A.1, and recalling that the left-hand side of (A.1) is exactly the number of non-intersecting paths $\omega_1, \omega_2$, we see that Lemma A.3 holds. $\square$

## A.2. Enumerating Simple Lattice Paths

Our second aim is to derive a formula for counting simple lattice paths of fixed length $n$ that stay within some upper and lower boundaries (these boundaries are formed by simple lattice paths, as well). Recall that a simple lattice path consists of up steps and horizontal steps, and that its width is given by the $x$-coordinate of its final vertex. An example of a simple lattice path is shown in Figure A.2, the precise definition was given in Definition 3.14. Again, we use the Lindström-Gessel-Viennot-Determinant to obtain the desired result.

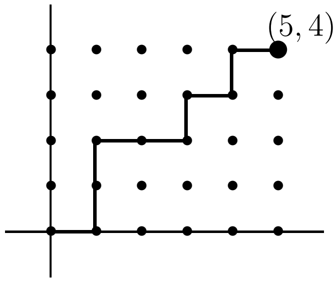



FIGURE A.2. A simple lattice path of width 5.

Recall Definition 3.32, the restriction of the binomial coefficient,

$$\binom{n}{k}_+ := \begin{cases} \binom{n}{k} & n \geq k \geq 0, \\ 0 & n < k \text{ or } k < 0, \end{cases} \tag{A.17}$$

for $n, k \in \mathbb{Z}$.

THEOREM A.4 ([**25**], p. 92). *Let* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$ *be integer sequences with* $a_1 \leq a_2 \leq \ldots \leq a_n$, $b_1 \leq b_2 \leq \ldots \leq b_n$, *and* $a_i \leq b_i$, $i = 1, 2, \ldots, n$. *The number of all simple lattice paths from* $(0, a_1)$ *to* $(n, b_n)$ *with the height of the* $i$*-th horizontal step being at least* $a_i$ *and not greater than* $b_i$ $(i = 1, 2, \ldots, n)$, *is given by*

$$\det_{1 \leq i,j \leq n} \left( \binom{b_i - a_j + 1}{j - i + 1}_+ \right). \tag{A.18}$$

PROOF. We want to count simple lattice paths whose steps lie in between fixed heights. To do so we establish a one-to-one correspondence between these lattice paths and non-intersecting $k$-paths whose number is given by the Lindström-Gessel-Viennot-Determinant in Theorem A.1.

More precisely: let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be integer sequences with $a_1 \leq a_2 \leq \ldots \leq a_n$, $b_1 \leq b_2 \leq \ldots \leq b_n$, and $a_i \leq b_i$, $i = 1, 2, \ldots, n$. For a simple lattice path of width $n$ denote by $h(i)$ the height of the $i$-th horizontal step $s_i$ (that is, a step of the form $(1, 0)$), and set $\mathbf{h} = (h(1) \ldots, h(n))$. We then want to count simple lattice paths of width $n$, starting at $(0, a_1)$ and ending at $(n, b_n)$ with $\mathbf{a} \leq \mathbf{h} \leq \mathbf{b}$ (this means that $a_i \leq h(i) \leq b_i$, for $i = 1, \ldots, n$).

Fix a simple lattice path $\rho$. First, note that it is sufficient to know the initial vertex, the end vertex, and the horizontal steps to reconstruct the path. Now, only focusing on horizontal steps, we consider the restriction $a_i \leq h(i) \leq b_i$. Having a horizontal step of height $b_i$ as upper boundary and of height $a_i$ as lower boundary for the $i$-th horizontal step $s_i$, we create a new path as seen in Figure A.3: we connect the starting vertex of the upper boundary by down steps with the step $s_i$. This step is then connected with the ending vertex of the lower boundary, again by down steps. Through this process each step and its boundaries are turned into a new lattice path (with step set consisting of down steps and a horizontal step to the right). Proceeding in the same manner for all horizontal steps of $\rho$, we obtain $n$ new lattice paths (see Figure A.4(b)). Note that they might be intersecting (namely if two vertical steps of $\rho$ are of same height). But through lifting each of the newly created paths up by one unit relatively to each other,

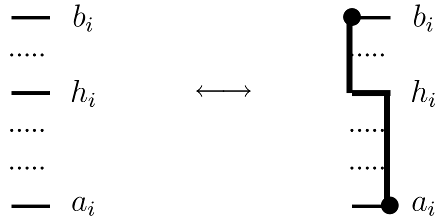
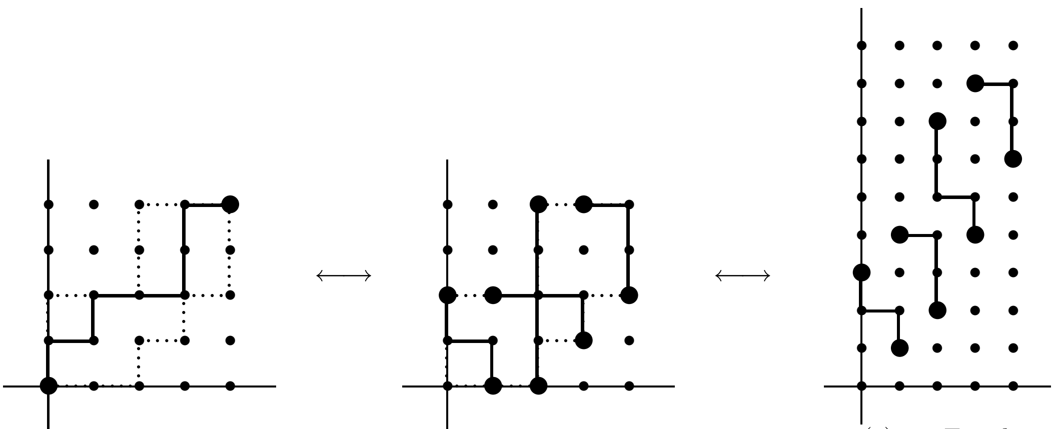



FIGURE A.3. A possible choice for the step height $h_i$ between the boundaries $a_i, b_i$ and a corresponding lattice path.



(a) A simple lattice path and its boundaries

(b) Family of lattice paths

(c) Family of disjoint lattice paths

FIGURE A.4. Bijection between bounded simple lattice paths and a family of disjoint lattice paths.

we ensure that they are non-intersecting. So, we have obtained a non-intersecting $k$-path $(P_1, \ldots, P_n)$, where the path $P_i$ runs from $(i-1, b_i + i)$ to $(i, a_i + i)$ and the step set consists of down steps and horizontal steps to the right.

Conversely, we can easily retrieve a simple lattice path from a non-intersecting $k$-path by reversing the process. Hence, we have found a bijection between the two counting problems. It only remains to evaluate the number of non-intersecting $k$-paths that run from

$$\mathbf{u} = ((0, b_1), (1, b_2 + 2), \ldots, (i-1, b_i + i), \ldots, (n-1, b_n + n)) \tag{A.19}$$

to

$$\mathbf{v} = ((1, a_1 + 1), (2, a_2 + 2), \ldots, (i, a_i + i), \ldots, (n, a_n + n)). \tag{A.20}$$

This number is given by the Lindström-Gessel-Viennot-Determinant (Theorem A.1); but we still have to specify the graph $D$: we choose $D$ to be the square grid $\mathbb{Z} \times \mathbb{Z}$ where all edges are directed towards south or towards east and are weighted by 1.

Now, consider equation (A.1),

$$\sum_{\pi \in S_k} (\text{sgn } \pi) W_N(\mathbf{u}, \pi(\mathbf{v})) = \left| W_P(u_i, v_j) \right|.$$

Since all weights are equal 1, each path has a total weight of 1. So, $W_N(\mathbf{u}, \pi(\mathbf{v}))$, the sum of the weight of all paths, is equal to $N(\mathbf{u}, \pi(\mathbf{v}))$. We can therefore rewrite (A.1) as

$$\sum_{\pi \in S_k} (\text{sgn } \pi) N(\mathbf{u}, \pi(\mathbf{v})) = \det \left| W_P(u_i, v_j) \right|. \tag{A.21}$$

We claim that on the left-hand side of (A.21) the only non-zero term is the term for which $\pi(\mathbf{v}) = \mathbf{v}$. For any other permutation there can not exist a non-intersecting $k$-path. To see this, note the following: consider a path $t$ that starts in $u_n = (n-1, b_n+n)$. Since only down steps and horizontal steps to the right are permitted in $D$, the path $t$ can only end in a vertex that lies to the right or below its initial vertex, namely in $v_n = (n, a_n + n)$ or $v_{n-1} = (n-1, a_{n-1} + n - 1)$. We show that the path has to end in the right-most of these vertices, $v_n$: suppose that this is not true. Then, $t$ runs from $u_n$ to $v_{n-1}$. But there has to be a path $\widetilde{t}$ that ends in $v_n$. For this path $\widetilde{t}$ an initial vertex $u_i$ (out of (A.19)) has to be chosen. If there is no vertex whose $y$-coordinate is at least the $y$-coordinate of $v_n$, we are already finished. Otherwise, denote the initial vertex of $\widetilde{t}$ by $(u_i^{(x)}, u_i^{(y)})$. Now, we claim that these two paths $t, \widetilde{t}$ share a vertex, see Figure A.5. First, recall that $t$ runs from $u_n = (n-1, b_n + n)$ to $v_{n-1} = (n-1, a_{n-1} + n - 1)$ and that $\widetilde{t}$ runs from $u_i = (u_i^{(x)}, u_i^{(y)})$ to $v_n = (n, a_n + n)$. It follows that $u_i^{(x)} < n - 1$, since $\widetilde{t}$ starts to the left of $t$. Hence, $u_i^{(y)} < b_n + n$, since $u_i^{(y)} = b_i + i$, and $b_1 \leq \cdots \leq b_n$. So, the initial vertex of $\widetilde{t}$ lies below and to the left of the initial vertex of $t$. On the other hand, the final vertex of $\widetilde{t}$ lies above and to the right of the final vertex of $t$. As we only allow down steps and horizontal steps to the right, the paths have to intersect at some point.

We have seen that for the right-most path of a non-intersecting $k$-path from $\mathbf{u}$ to $\mathbf{v}$ the final vertex can not be chosen freely. Now, we consider the next to the last initial vertex $u_{n-1}$ and omit the path $t$ which runs from $u_n$ to $v_n$. Applying the same argument as before, we see that out of the two remaining possible final vertices, $v_{n-1}$ and $v_{n-2}$, the right-most one has to be chosen. There is only one combination of initial and final vertices which yields a non-intersecting $k$-path going from $\mathbf{u}$ to $\mathbf{v}$, and its number is given by $N(\mathbf{u}, \mathbf{v})$. So, the only non-zero term in left-hand side of (A.21) is $N(\mathbf{u}, \mathbf{v})$. To calculate this quantity, we evaluate the terms $W_P(u_i, v_j)$. First, assume that $b_i \geq a_j$ and $j \geq i + 1$: to go from $u_i$ to $v_j$ we need $b_i + i - (a_j + j)$ down steps and $j - i + 1$ horizontal steps. Hence, the total number of steps is

$$b_i + i - a_j - j + j - i + 1 = b_i - a_j + 1,$$

and it follows that $W_P(u_i, v_j) = \binom{b_i - a_j + 1}{j - i + 1}$. Putting this together, Equation (A.1) reads

$$\begin{aligned} \#\text{ disjoint k-paths } &= N(\mathbf{u}, \mathbf{v}) = W_N(\mathbf{u}, \mathbf{v}) \\ &= \det_{1 \leq i,j \leq n} P(u_i, v_j) = \det_{1 \leq i,j \leq n} \left( \binom{b_i - a_j + 1}{j - i + 1}_+ \right). \end{aligned}$$

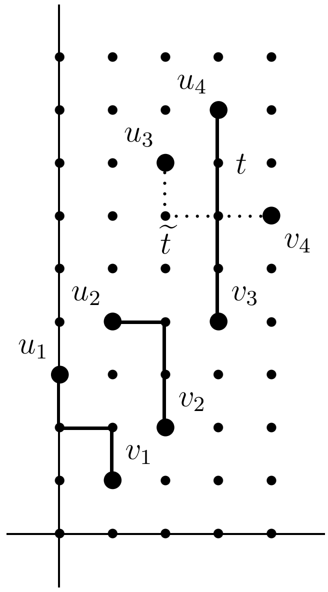



FIGURE A.5. Consider the path $t$ starting in $u_4$. Suppose that $t$ does not end in $v_4$. Then, $t$ has to have its final vertex in $v_3$, since only down steps and horizontal steps to the right are admitted. But some path $\widetilde{t}$ has to end in $v_4$. The only possible initial vertex for $\widetilde{t}$ is $u_3$, since $u_2$ and $u_1$ lie lower than $v_4$. $u_3$ lies below $u_4$ and $v_4$ lies above $v_3$, hence the paths $t, \widetilde{t}$ have to intersect each other.

In the case where $b_i \geq a_j$ or $j < i$, there is no path from $u_i$ to $v_j$ (since up steps are not permitted), neither in the case of $j < i$ (since left steps are not permitted); note that the definition of the restricted binomial coefficient reflects this and gives 0 for these cases. Hence, the theorem is established. $\square$

# Curriculum Vitae

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