MULTIDIMENSIONAL MATRIX INVERSIONS AND MULTIPLE BASIC HYPERGEOMETRIC SERIES

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Abstract. We compute the inverse of a specific infinite r-dimensional matrix, thus unifying multidimensional matrix inversions recently found by Milne, Lilly, and Bhatnagar. Our inversion is an r-dimensional extension of a matrix inversion previously found by Krattenthaler. We also compute the inverse of another infinite r-dimensional matrix. As applications of our matrix inversions, we derive new summation formulas for multidimensional basic hypergeometric series. We work in the setting of multiple basic hypergeometric series very-well-poised on the root systems A_r, C_r, and D_r. Our new summation formulas include D_r Jackson's ${}_8\phi_7$ summations, A_r and D_r quadratic, and D_r cubic summations. Further, we derive multivariable generalizations of Bailey's classical terminating balanced very-well-poised ${}_{10}\phi_9$ transformation. We obtain C_r and D_r ${}_{10}\phi_9$ transformations from an interchange of multisums, combined with A_r, C_r, and D_r extensions of Jackson's ${}_8\phi_7$ summation. Special cases of our ${}_{10}\phi_9$ transformations include multivariable generalizations of Watson's transformation of an ${}_8\phi_7$ into a multiple of a ${}_4\phi_3$. We also deduce multidimensional extensions of Sears' ${}_4\phi_3$ transformation. Furthermore, we derive summation formulas for a different kind of multidimensional basic hypergeometric series associated to root systems of classical type. We proceed by combining the classical one-dimensional summation formulas with certain determinant evaluations. Our theorems include A_r extensions of Ramanujan's bilateral ${}_1\psi_1$ sum, C_r extensions of Bailey's very-well-poised ${}_6\psi_6$ summation, and a C_r extension of Jackson's very-well-poised ${}_8\phi_7$ summation formula. We also derive multidimensional extensions, associated to the classical root systems of type A_r, B_r, C_r, and D_r, respectively, of Chu's bilateral transformation formula for basic hypergeometric series of Gasper-Karlsson-Minton type. Limiting cases of our various series identities include multidimensional generalizations of many of the most important summation and transformation theorems of the classical theory of basic hypergeometric series.

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CHAPTER I

Introduction

confugiamus ergo ad series
L. EULER

Hypergeometric series are so widespread, that referring to the hypergeometric function of Gauß, W. W. Sawyer [66, p. 63] claimed:

> There must be many universities today where 95 percent, if not 100 percent, of the functions studied by physics, engineering, and even mathematics students, are covered by this single symbol $F(a, b; c; x)$.

The analogous theory of q-hypergeometric functions, which arose initially in combinatorics and classical analysis, and contains hypergeometric functions as special cases, interacts similarly with number theory, statistics, physics, and representation theory of quantum Lie algebras.

The purpose of this dissertation is to study multidimensional extensions of some of the fundamental summation and transformation formulas of q-series. While there are several such theories being studied, the one we consider is motivated directly from some work in mathematical physics. Already, in 20 short years, there is evidence to believe that this is a natural generalization of the classical work, with the required depth and rich interaction with other parts of mathematics.

The study of hypergeometric series can be traced back many centuries. Perhaps the most fundamental result in this theory is the famous binomial theorem

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k,$$

for complex α, which was proved in 1665 by twenty-three year old Newton.

The study of basic hypergeometric series (also called q-hypergeometric series or

q-series) started essentially in 1748 when Euler [20] considered the infinite product

$$(q;q)_\infty^{-1} = \prod_{k=1}^{\infty}(1-q^k)^{-1}$$

as a generating function for $p(n)$, the number of partitions of a positive integer n into positive integers.

Euler [21] and Pfaff [62] studied many important properties of the hypergeometric series

$$ {}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};z\right] = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!\,(c)_k}z^k, \tag{I.1}$$

where the rising factorial $(a)_k$ is defined as

$$(a)_k = a(a+1)\cdots(a+k-1).$$

The hypergeometric series in (I.1) was extensively investigated by Gauß [27] in his doctoral dissertation of 1812, where he proved the following summation formula involving Gamma functions:

$$ {}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};1\right] = \frac{\Gamma(c-a-b)\,\Gamma(c)}{\Gamma(c-a)\,\Gamma(c-b)},$$

for integer $b \leq 0$ or $\Re c > \Re a + \Re b$. In 1846 Heine [36] converted the simple observation

$$\lim_{q\to 1}\frac{1-q^a}{1-q} = a$$

into a systematic study of q-hypergeometric series, analogous to the theory of ordinary hypergeometric series. The q-hypergeometric series are usually called basic hypergeometric series, where "basic" refers to the base q. From that time on basic hypergeometric series acquired independent interest, and the pace of progress in the last 150 years has been hectic.

Many important results on hypergeometric and basic hypergeometric series were derived by F. H. Jackson and by W. N. Bailey, and others. The short and brilliant career of Ramanujan, and his beautiful identities, has played its part in attracting many researchers to this area. In more recent years, G. E. Andrews has done much to popularize the area, by explaining the theory and pointing out the applications of q-series. In this dissertation, however, we often refer to Gasper and Rahman's textbook [25] which has become a standard reference to basic hypergeometric series.

An important feature of hypergeometric and basic hypergeometric series is that a symbolic algebra package can be used to discover as well as check new results. Moreover, algorithms are known (and implemented) for proving identities [76].

However, while we used such packages to discover and check our results, our identities cannot be proved using the known algorithms. This is the case because we are able to find an infinite family of results, only a few special cases of which can be proved using a computer. Our methods here are largely extensions of the classical techniques [25].

The multivariable series we study were introduced by the three mathematical physicists Biedenharn, Holman, and Louck [37] by comparing two ways of computing the matrix elements of multiplicity free Wigner and Racah coefficients for $SU(r)$. Milne [50] found q-analogues of these multiple series identities, and later Gustafson [31] defined multiple basic hypergeometric series associated to other Lie algebras of classical type. These types of multiple series (also called multiple series associated to root systems) are directly related to the corresponding Macdonald identities [51], [31]. Over the last fifteen years the theory of multiple basic hypergeometric series associated to root systems has been extensively developed by Gustafson, Milne, and by their co-workers. Many of the classical (one-dimensional) series identities have been extended to higher dimensions. It is interesting to observe that often many different extensions of the same classical identity exist. Further, it becomes apparent that a lot of facts of the classical theory of basic hypergeometric series are naturally motivated by the multivariable analysis. Thus the study of multiple q-series leads to a better understanding of the classical one-dimensional theory.

Recently, multiple basic hypergeometric series associated to root systems have appeared in combinatorial studies on plane partitions, tableaux, and cylindrical partitions. For example, Krattenthaler [44] has applied such multiple series to compute generating functions for plane partitions and tableaux. Other important work has been done by Gessel and Krattenthaler [28]. This makes it very likely that multiple basic hypergeometric series associated to the root systems will have many more applications in the future.

In this thesis, we derive various new summation and transformation formulas for multiple basic hypergeometric series associated to the root systems A_r, C_r and D_r. One of the main tools that we use for deriving our new summation theorems is matrix inversion. The idea is simple: Suppose, an infinite lower triangular matrix (f_{nk}), and two infinite sequences (a_k) and (b_k) are given, related by the equation

$$\sum_{0\le k\le n} f_{nk} a_k = b_n.$$

If the inverse of (f_{nk}), the infinite lower triangular matrix (g_{kl}), is known, then it is clear that this relation can be inverted, which is

$$\sum_{0\le l\le k} g_{kl} b_l = a_k.$$

Matrix inversions are very important in many fields of combinatorics and special functions. When dealing with combinatorial sums, application of matrix inversion may help to simplify problems, or yield new identities. Andrews [4] discovered that the Bailey transform [10], which is a very powerful tool in the theory of (basic) hypergeometric series, corresponds to the inversion of two infinite lower-triangular matrices. Gessel and Stanton [29] used a bibasic extension of that matrix inversion to derive a number of basic hypergeometric summations and transformations, and identities of Rogers-Ramanujan type. Even earlier, Carlitz [17] had found an even more general matrix inversion though without giving any applications.

Gasper and Rahman [24], [63], [26], [25, sec. 3.6] used Gasper's [24] bibasic matrix inversion together with an indefinite bibasic sum to derive numerous beautiful hypergeometric summation and transformation formulas.

The most general (1-dimensional) matrix inversion, however, which contained all the above inversions, was found by Krattenthaler [45] who applied his inversion to derive a number of hypergeometric summation formulas. The inverse matrices he gave are basically $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ ($\mathbb{Z}$ denotes the set of integers), where

$$f_{nk}=\frac{\prod\limits_{j=k}^{n-1}(a_j-c_k)}{\prod\limits_{j=k+1}^{n}(c_j-c_k)},$$

and

$$g_{kl}=\frac{(a_l-c_l)}{(a_k-c_k)}\frac{\prod\limits_{j=l+1}^{k}(a_j-c_k)}{\prod\limits_{j=l}^{k-1}(c_j-c_k)}.$$

In fact, the special case $a_j=aq^{-j}$, $c_k=q^k$ is equivalent to the matrix inversion of Andrews, and the case $a_j=ap^{-j}$, $c_k=q^k$ is equivalent to Gessel and Stanton's. Specializing $c_k=q^k$ we obtain Carlitz's matrix inversion, and $a_j=(bp^{-j}/a)+ap^j$, $c_k=q^{-k}+bq^k$ yields the inversion of Gasper.

Multidimensional matrix inversions were found by Milne, Lilly and Milne, and by Bhatnagar and Milne. The A_r (or equivalently $U(r+1)$) and C_r inversions (corresponding to the root systems A_r and C_r, respectively) of Milne and Lilly [55, Theorem 3.3], [57], [47], [48], which are higher-dimensional generalizations of Andrews' Bailey transform matrices, were used to derive A_r and C_r extensions [55], [58] of many of the classical hypergeometric summation and transformation formulas. Bhatnagar and Milne [11, Theorem 5.7], [13, Theorem 3.48] were even able to find an A_r extension of Gasper's bibasic hypergeometric matrix inversion. They used a special case of his matrix inversion, an A_r extension of Carlitz's inversion, to derive A_r identities of Abel-type. But none of these multidimensional matrix inversions contained Krattenthaler's inversion as a special case.

One of the main results of this thesis is a multidimensional extension of Krattenthaler's matrix inverse (see Theorem III.7). This multidimensional matrix inversion unifies all the matrix inversions mentioned so far as it contains them all as special cases. Besides, we present another interesting multidimensional matrix inversion (see Theorem III.29) which is of different type.

In order to prove our matrix inversions in Theorems III.7 and III.29 we utilize Krattenthaler's operator method [43] which we review in chapter II. We adapt a main theorem of [43] and add an appropriate multidimensional corollary (see Corollary II.18).

The main motivation for finding a multidimensional extension of Krattenthaler's matrix inverse came from prospective applications to basic hypergeometric series. These applications are the contents of chapter IV. We combine a special case of Theorem III.7 and a C_r ${}_8\phi_7$ summation theorem, found independently by Denis and Gustafson [19] and Milne and Lilly [58], to derive a D_r ${}_8\phi_7$ summation theorem, which is closely related to a D_r summation found by Bhatnagar [12]. We also derive A_r and D_r extensions of a quadratic hypergeometric summation formula of Gessel and Stanton [29]. Finally, we derive a D_r extension of a cubic summation formula of Gasper and Rahman [26]. The above work appears in [67].

Our multidimensional matrix inversions are very useful in the theory of basic hypergeometric series of type A_r, C_r, and D_r, respectively, and lead to the discovery of many more new identities. The D_r ${}_8\phi_7$ summation theorems, derived by matrix inversion in chapter IV, lead to new C_r and D_r extensions of Bailey's [7] very-well-poised balanced ${}_{10}\phi_9$ transformation. Previously, Milne and Newcomb [59] have found A_r extensions of Bailey's formula. In chapter V, we use A_r, C_r, and D_r extensions of Jackson's [40] very-well-poised balanced ${}_8\phi_7$ summation, stated in Appendix B, and an interchange of multisums, to generalize Bailey's ${}_{10}\phi_9$ transformation formula to multiple series. As special cases of our ${}_{10}\phi_9$ transformations, we find several multivariable generalizations of Watson's [73] transformation of an ${}_8\phi_7$ into a multiple of a ${}_4\phi_3$. Further, we deduce multidimensional extensions of Sears' [69] ${}_4\phi_3$ transformation. Our multiple Watson's and Sears' transformations are the contents of chapter VI. Chapter V and chapter VI is joint work with Bhatnagar [14].

Chapter VII can be read independently from the previous ones, but in fact, it is related to our previous work. Here we derive multiple generalizations of many of the classical basic hypergeometric summation formulas. The multiple series appearing in chapter VII were first considered by Gustafson and Krattenthaler [32], [33], who showed how to obtain multivariable summation and transformation formulas from determinant evaluations. The series we consider are different from those appearing elsewhere in this thesis, but they are also associated to root systems of classical type. Our theorems include new A_r extensions of Ramanujan's bilateral ${}_1\psi_1$ sum (cf. [35]), C_r extensions of Bailey's very-well-poised ${}_6\psi_6$ summation [9], and a C_r extension of Jackson's very-well-poised ${}_8\phi_7$ summation formula. We also derive multidimensional extensions, associated to the classical root systems of type A_r, B_r, C_r, and D_r, respectively, of Chu's [18] bilateral transformation formula for basic hypergeometric series of Gasper-Karlsson-Minton type [23]. Limiting cases of our various series identities include multidimensional generalizations of many of the most important summation theorems of the classical theory of basic hypergeometric series. This work will appear in a forthcoming paper [68].

A couple of determinant evaluations are necessary for proving the theorems of this thesis. Two determinant evaluations, which are elegant generalizations of the classical and "symplectic" Vandermonde determinants, turn out to be crucial for our computations in chapter III. Another determinant evaluation is needed in chapter VII. Since these lemmas may have value in a variety of other contexts as well we decided to give them separately in Appendix A.

This thesis has been written under the supervision of C. Krattenthaler. The author feels especially indebted to his supervisor who has patiently provided a lot of help and ideas. Further, it is a pleasure for me to thank G. Bhatnagar for contributing many valuable suggestions, during the period of our joint project on the multiple ${}_{10}\phi_9$ transformations and their consequences. Finally, I would like to thank my parents and friends for their support of any kind over the last years.

CHAPTER II

An operator method for proving matrix inversions

In this chapter we lay the ground work for proving our multidimensional matrix inversions in chapter III. We review Krattenthaler's operator method [43], adapt a main theorem of [43] and add an appropriate multidimensional corollary (see Corollary II.18).

Let $F = (f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ (as before, $\mathbb{Z}$ denotes the set of integers) be an infinite lower-triangular r-dimensional matrix; i.e. $f_{\mathbf{nk}} = 0$ unless $\mathbf{n} \geq \mathbf{k}$, by which we mean $n_i \geq k_i$ for all $i = 1, \ldots, r$. The matrix $G = (g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}}$ is said to be the *inverse matrix* of F if and only if

$$\sum_{\mathbf{n}\geq\mathbf{k}\geq\mathbf{l}} f_{\mathbf{nk}} g_{\mathbf{kl}} = \delta_{\mathbf{nl}}$$

for all $\mathbf{n}, \mathbf{l} \in \mathbb{Z}^r$, where $\delta_{\mathbf{nl}}$ is the usual Kronecker delta.

In [43] Krattenthaler gave a method for solving Lagrange inversion problems, which are closely connected with the problem of inverting lower-triangular matrices. We will use his operator method for proving our new theorems in chapter III. By a *formal Laurent series* we mean a series of the form $\sum_{\mathbf{n}\geq\mathbf{k}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}$, for some $\mathbf{k} \in \mathbb{Z}^r$, where $\mathbf{z}^{\mathbf{n}} = z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r}$. (Note that we actually consider *formal Laurent series truncated from below*, as in the usual definition the $\mathbf{n}$ in the sum runs over all elements of $\mathbb{Z}^r$.) Given the formal Laurent series $a(\mathbf{z})$ and $b(\mathbf{z})$ we introduce the bilinear form $\langle\, ,\, \rangle$ by

$$\langle a(\mathbf{z}), b(\mathbf{z})\rangle = \langle \mathbf{z}^{\mathbf{0}}\rangle (a(\mathbf{z}) \cdot b(\mathbf{z})), \tag{II.1}$$

where $\langle \mathbf{z}^{\mathbf{0}}\rangle c(\mathbf{z})$ denotes the coefficient of $\mathbf{z}^{\mathbf{0}}$ in $c(\mathbf{z})$. Since our Laurent series are truncated from below, (II.1) is well-defined. Given any linear operator L acting on formal Laurent series, L^* denotes the adjoint of L with respect to $\langle\, ,\, \rangle$; i.e. $\langle La(\mathbf{z}), b(\mathbf{z})\rangle = \langle a(\mathbf{z}), L^* b(\mathbf{z})\rangle$ for all formal Laurent series $a(\mathbf{z})$ and $b(\mathbf{z})$. We need the following special case of [43, Theorem 1].

Lemma II.2 (Krattenthaler). *Let $F = (f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ be an infinite lower-triangular r-dimensional matrix with $f_{\mathbf{kk}} \neq 0$ for all $\mathbf{k} \in \mathbb{Z}^r$. For $\mathbf{k} \in \mathbb{Z}^r$, define the formal Laurent series $f_{\mathbf{k}}(\mathbf{z})$ and $g_{\mathbf{k}}(\mathbf{z})$ by $f_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{n}\geq\mathbf{k}} f_{\mathbf{nk}}\mathbf{z}^{\mathbf{n}}$ and $g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\leq\mathbf{k}} g_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}$, where $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ is the uniquely determined inverse matrix of F. Suppose that for $\mathbf{k} \in \mathbb{Z}^r$ a system of equations of the form*

$$U_j f_{\mathbf{k}}(\mathbf{z}) = c_j(\mathbf{k}) V f_{\mathbf{k}}(\mathbf{z}), \qquad j = 1, \dots, r, \tag{II.3}$$

holds, where U_j, V are linear operators acting on formal Laurent series, V being bijective, and $(c_j(\mathbf{k}))_{\mathbf{k}\in\mathbb{Z}^r}$ are arbitrary sequences of constants. Moreover, we suppose that

$$\text{for all } \mathbf{m},\mathbf{n} \in \mathbb{Z}^r,\ \mathbf{m} \neq \mathbf{n}, \text{ there exists a } j \text{ with } 1 \leq j \leq r \text{ and } c_j(\mathbf{m}) \neq c_j(\mathbf{n}). \tag{II.4}$$

Then, if $h_{\mathbf{k}}(\mathbf{z})$ is a solution of the dual system

$$U_j^* h_{\mathbf{k}}(\mathbf{z}) = c_j(\mathbf{k}) V^* h_{\mathbf{k}}(\mathbf{z}), \qquad j = 1, \dots, r, \tag{II.5}$$

with $h_{\mathbf{k}}(\mathbf{z}) \not\equiv 0$ for all $\mathbf{k} \in \mathbb{Z}^r$, the series $g_{\mathbf{k}}(\mathbf{z})$ are given by

$$g_{\mathbf{k}}(\mathbf{z}) = \frac{1}{\langle f_{\mathbf{k}}(\mathbf{z}), V^* h_{\mathbf{k}}(\mathbf{z})\rangle} V^* h_{\mathbf{k}}(\mathbf{z}). \tag{II.6}$$

Proof. Suppose $(h_{\mathbf{k}}(\mathbf{z}))_{\mathbf{k}\in\mathbb{Z}^r}$ satisfies (II.5). By using (II.3) we have

$$\begin{aligned} c_j(\mathbf{k}) \langle f_{\mathbf{k}}(\mathbf{z}), V^* h_{\mathbf{n}}(\mathbf{z})\rangle =& c_j(\mathbf{k}) \langle V f_{\mathbf{k}}(\mathbf{z}), h_{\mathbf{n}}(\mathbf{z})\rangle = \langle U_j f_{\mathbf{k}}(\mathbf{z}), h_{\mathbf{n}}(\mathbf{z})\rangle \\ =& \langle f_{\mathbf{k}}(\mathbf{z}), U_j^* h_{\mathbf{n}}(\mathbf{z})\rangle = c_j(\mathbf{n}) \langle f_{\mathbf{k}}(\mathbf{z}), V^* h_{\mathbf{n}}(\mathbf{z})\rangle. \end{aligned}$$

From (II.4) this immediately implies $\langle f_{\mathbf{k}}(\mathbf{z}), V^* h_{\mathbf{n}}(\mathbf{z})\rangle = 0$ for $\mathbf{n} \neq \mathbf{k}$. For $\mathbf{n} = \mathbf{k}$ suppose $\langle f_{\mathbf{n}}(\mathbf{z}), V^* h_{\mathbf{n}}(\mathbf{z})\rangle = 0$ for some $\mathbf{n} \in \mathbb{Z}^r$. Together with the above calculation this means $\langle f_{\mathbf{k}}(\mathbf{z}), V^* h_{\mathbf{n}}(\mathbf{z})\rangle = 0$ for all $\mathbf{k} \in \mathbb{Z}^r$ and we conclude $V^* h_{\mathbf{n}}(\mathbf{z}) \equiv 0$. It is easy to show that with our assumptions V^* is bijective, and, therefore, $h_{\mathbf{n}}(\mathbf{z}) \equiv 0$ in contradiction to the condition $h_{\mathbf{k}}(\mathbf{z}) \not\equiv 0$ for all $\mathbf{k} \in \mathbb{Z}^r$.

Defining $g_{\mathbf{k}}(\mathbf{z}) = \langle f_{\mathbf{k}}(\mathbf{z}), V^* h_{\mathbf{k}}(\mathbf{z})\rangle^{-1} V^* h_{\mathbf{k}}(\mathbf{z})$, we evaluate

$$\langle f_{\mathbf{k}}(\mathbf{z}), g_{\mathbf{n}}(\mathbf{z})\rangle = \frac{\langle f_{\mathbf{k}}(\mathbf{z}), V^* h_{\mathbf{n}}(\mathbf{z})\rangle}{\langle f_{\mathbf{n}}(\mathbf{z}), V^* h_{\mathbf{n}}(\mathbf{z})\rangle} = \delta_{\mathbf{nk}}.$$

The uniqueness of the sequence $(g_{\mathbf{k}}(\mathbf{z}))_{\mathbf{k}\in\mathbb{Z}^r}$ is obvious. □

In our applications we will use a corollary of Lemma II.2 (see Corollary II.18). Let $\mathcal{S}_r$ be the symmetric group of order r. For possibly noncommuting operators V_{ij} let us define the *column determinant* by

$$\overrightarrow{\det_{1\leq i,j\leq r}} (V_{ij}) := \sum_{\sigma\in\mathcal{S}_r} \operatorname{sgn}(\sigma) V_{\sigma(r),r} V_{\sigma(r-1),r-1} \cdots V_{\sigma(1),1}. \tag{II.7}$$

An equivalent, recursive definition is by means of the expansion along the first column,

$$\overset{\rightarrow}{\det}_{1\le i,j\le r}(V_{ij}) = \sum_{k=1}^{r}(-1)^{k+1}\vec{V}^{(k,1)}V_{k1}, \tag{II.8}$$

where $\vec{V}^{(i,j)}$ denotes the column minor with the i-th row and j-th column being omitted.

PROPOSITION II.9. *Let $(V_{ij})_{i,j=1}^{r}$ be a matrix of linear operators acting on formal Laurent series. Suppose $V_{ij} = C_{ij} + A_{ij}$ for $i,j = 1,\dots,r$, where the operators C_{ij}, A_{ij} obey the following commutation rules*

$$C_{ij}C_{kl} = C_{kl}C_{ij}, \qquad i \neq k; \quad i,j,k,l = 1,\dots,r, \tag{II.10}$$

i.e. C_{ij} and C_{kl} commute when taken from different rows,

$$C_{ij}A_{kl} = A_{kl}C_{ij}, \qquad i \neq k; \quad i,j,k,l = 1,\dots,r, \tag{II.11}$$

i.e. C_{ij} and A_{kl} commute when taken from different rows, respectively the rules

$$A_{ij}A_{kl} = A_{kj}A_{il}, \qquad i,j,k,l = 1,\dots,r, \tag{II.12}$$

where the column indices j and l keep their order.

Then the column determinant $\overset{\rightarrow}{\det}_{1\le i,j\le r}(V_{ij})$, as defined in (II.7) *or* (II.8)*, can be reduced to a polynomial in the A_{ij}'s, $i,j = 1,\dots,r$, of degree ≤ 1,*

$$\overset{\rightarrow}{\det}_{1\le i,j\le r}(V_{ij}) = \overset{\rightarrow}{\det}_{1\le i,j\le r}(C_{ij}) + \sum_{i,j=1}^{r}(-1)^{i+j}\vec{C}^{(i,j)}A_{ij}, \tag{II.13}$$

where $\vec{C}^{(i,j)}$ again denotes the column minor with the i-th row and j-th column being omitted.

Moreover, for any $l = 1,\dots,r$ the expansion

$$\overset{\rightarrow}{\det}_{1\le i,j\le r}(V_{ij}) = \sum_{k=1}^{r}(-1)^{k+l}\vec{V}^{(k,l)}V_{kl} \tag{II.14}$$

holds, which means that the determinant can be expanded along any arbitrary column.

PROOF. Our column determinant can be expanded into

$$\overset{\rightarrow}{\det}_{1\le i,j\le r}(V_{ij}) = \sum_{\substack{\sigma\in\mathcal{S}_r\\ \mathbf{q}\in\{\gamma,\alpha\}^r}} \operatorname{sgn}(\sigma)X_{\sigma(r),r}(q_r)\cdots X_{\sigma(1),1}(q_1), \tag{II.15}$$

where $\mathbf{q} = (q_1, q_2, \dots, q_r)$ with q_i equal to either γ or α for $i = 1, \dots, r$, and where $X_{ij}(\gamma) = C_{ij}$ and $X_{ij}(\alpha) = A_{ij}$. Due to (II.10) and (II.11), we observe that in every term of (II.15) the $X_{ij}(\gamma)$ commute with all other variables. $X_{ij}(\alpha)$ and $X_{kl}(\alpha)$ do not commute unless $j = l$, but due to (II.12) we have $X_{ij}(\alpha)X_{kl}(\alpha) = X_{kj}(\alpha)X_{il}(\alpha)$. This important fact lets all terms in (II.15) cancel where α occurs more than once, since we can pair the terms

$$\operatorname{sgn}(\sigma)\dots X_{\sigma(i),i}(\alpha)\dots X_{\sigma(j),j}(\alpha)\dots$$

and

$$\operatorname{sgn}(\sigma\cdot(i,j))\dots X_{\sigma(j),i}(\alpha)\dots X_{\sigma(i),j}(\alpha)\dots,$$

having chosen the first two occurrences of α, for instance, and where all other factors are unchanged. After cancellation, we are left with terms with at most one occurrence of α and where all factors commute. This implies the first assertion of the proposition.

For proving the second assertion, we define for a given permutation $\tau \in \mathcal{S}_r$ a modified column determinant by

$$\overset{\rightarrow(\tau)}{\det_{1\le i,j\le r}}(V_{ij}) = \sum_{\sigma\in\mathcal{S}_r} \operatorname{sgn}(\sigma) V_{\sigma(\tau(r)),\tau(r)} \cdots V_{\sigma(\tau(1)),\tau(1)}, \tag{II.16}$$

i.e. the sequence of columns in expanding the determinant is determined by τ. Expanding $\overset{\rightarrow(\tau)}{\det}_{1\le i,j\le r}(V_{ij})$ as in (II.15) we see that the same cancellation argument applies. Thus we have $\overset{\rightarrow(\tau)}{\det}_{1\le i,j\le r}(V_{ij}) = \overset{\rightarrow}{\det}_{1\le i,j\le r}(V_{ij})$, which proves (II.14). □

Remark II.17. Note that in the determinant of Proposition II.9, we may not expand along rows because then the cancellation argument does not apply. (For a counterexample, consider the $r = 2$ case.)

Corollary II.18. *Let W_i, V_{ij} be linear operators acting on formal Laurent series, $c_j(\mathbf{k})$ arbitrary constants for* $\mathbf{k} \in \mathbb{Z}^r$ *and $i, j = 1, \dots, r$. Suppose $V_{ij} = C_{ij} + A_{ij}$, with the operators C_{ij}, A_{ij}, $i, j = 1, \dots, r$, satisfying the conditions* (II.10), (II.11), *and* (II.12) *of Proposition* II.9. *Suppose $W_i = W_i^{(c)} + W_i^{(a)}$, with the operators $W_i, W_i^{(c)}, W_i^{(a)}$, $i = 1, \dots, r$, satisfying*

$$C_{kl}W_i = W_iC_{kl}, \qquad i \ne k; \quad i,k,l = 1,\dots,r, \tag{II.19}$$

$$A_{kl}W_i^{(c)} = W_i^{(c)}A_{kl}, \qquad i \ne k; \quad i,k,l = 1,\dots,r, \tag{II.20}$$

$$A_{kl}W_i^{(a)} = A_{il}W_k^{(a)}, \qquad i,k,l = 1,\dots,r. \tag{II.21}$$

Moreover the $c_j(\mathbf{k})$ *are assumed to satisfy* (II.4), *and* $\overrightarrow{\det}_{1\le i,j\le r}(V_{ij})$ *is assumed to be invertible. With the notation of Lemma* II.2, *if*

$$\sum_{j=1}^{r} c_j(\mathbf{k}) V_{ij} f_{\mathbf{k}}(\mathbf{z}) = W_i f_{\mathbf{k}}(\mathbf{z}), \qquad i = 1, \dots, r, \tag{II.22}$$

then

$$g_{\mathbf{k}}(\mathbf{z}) = \frac{1}{\langle f_{\mathbf{k}}(\mathbf{z}), \overrightarrow{\det}(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z})\rangle} \overrightarrow{\det}(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}), \tag{II.23}$$

where $h_{\mathbf{k}}(\mathbf{z})$ *is a solution of*

$$\sum_{j=1}^{r} c_j(\mathbf{k}) V_{ij}^* h_{\mathbf{k}}(\mathbf{z}) = W_i^* h_{\mathbf{k}}(\mathbf{z}), \qquad i = 1, \dots, r, \tag{II.24}$$

with $h_{\mathbf{k}}(\mathbf{z}) \not\equiv 0$ *for all* $\mathbf{k} \in \mathbb{Z}^r$.

Proof. By Proposition II.9 it follows that the column determinant $\overrightarrow{\det}(V_{ij})$ may be expanded along any column. Hence, we can apply Cramer's rule to (II.22) to obtain

$$c_j(\mathbf{k}) \overrightarrow{\det}_{1\le i,l\le r}(V_{il}) f_{\mathbf{k}}(\mathbf{z}) = \sum_{i=1}^{r} (-1)^{i+j} \vec{V}^{(i,j)} W_i f_{\mathbf{k}}(\mathbf{z}),$$

for $j = 1, \dots, r$. The dual system reads

$$\begin{aligned} c_j(\mathbf{k}) \overrightarrow{\det}_{1\le i,l\le r}(V_{il}^*) h_{\mathbf{k}}(\mathbf{z}) &= \sum_{i=1}^{r} (-1)^{i+j} W_i^* \vec{V^*}^{(i,j)} h_{\mathbf{k}}(\mathbf{z}) \\ &= \sum_{i=1}^{r} (-1)^{i+j} \vec{V^*}^{(i,j)} W_i^* h_{\mathbf{k}}(\mathbf{z}), \end{aligned} \tag{II.25}$$

for $j = 1, \dots, r$, and is equivalent to (II.24). Notice that, because of (II.19), (II.20), and (II.21), we may apply Proposition II.9 in (II.25) and shift W_i^* to the right. Now apply Lemma II.2 with $V = \overrightarrow{\det}(V_{ij})$ and $U_j = \sum_{i=1}^{r} (-1)^{i+j} \vec{V}^{(i,j)} W_i$. $\square$

CHAPTER III

Multidimensional matrix inversions

In this chapter, we utilize the operator method of chapter II to derive two multidimensional matrix inversions, Theorem III.7 and Theorem III.29. We use special cases of these inversions to derive new multiple basic hypergeometric summation formulas in chapter IV. The matrix inversion in Theorem III.7 is an r-dimensional extension of a matrix inversion recently found by Krattenthaler [45] which contains a host of previously known one-dimensional matrix inversions. Our r-dimensional extension thereof generalizes multidimensional matrix inversions recently found by Milne, Lilly and Milne, and Bhatnagar and Milne, respectively, see Remark III.10. Our second matrix inversion in Theorem III.29 is of different type although it also includes the multidimensional matrix inversion of Lilly and Milne as a special case, see Remark III.32.

To motivate our multidimensional work in this chapter we recall Krattenthaler's matrix inverse with its proof [45] using the $r = 1$ case of Lemma II.2.

THEOREM III.1 (KRATTENTHALER'S MATRIX INVERSE). *Let $(a_i)_{i\in\mathbb{Z}}$, $(c_i)_{i\in\mathbb{Z}}$ be arbitrary sequences such that $c_i \neq c_j$ if $i \neq j$. Then $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are inverses of each other, where*

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j - c_k)}{\prod_{j=k+1}^{n}(c_j - c_k)}, \tag{III.2}$$

and

$$g_{kl} = \frac{(a_l - c_l)}{(a_k - c_k)} \frac{\prod_{j=l+1}^{k}(a_j - c_k)}{\prod_{j=l}^{k-1}(c_j - c_k)}. \tag{III.3}$$

Proof. From (III.2) we deduce for $n \geq k$ the recursion

$$(c_n - c_k) f_{nk} = (a_{n-1} - c_k) f_{n-1,k}. \tag{III.4}$$

We write

$$f_k(z) = \sum_{n \geq k} \frac{\prod_{j=k}^{n-1} (a_j - c_k)}{\prod_{j=k+1}^{n} (c_j - c_k)} z^n.$$

Moreover, we define linear operators $\mathcal{A}, \mathcal{C}$ by $\mathcal{A} z^n = a_n z^n$ and $\mathcal{C} z^n = c_n z^n$. Then we may write (III.4) in the form

$$(\mathcal{C} - z\mathcal{A}) f_k(z) = c_k (1 - z) f_k(z), \tag{III.5}$$

valid for all $k \in \mathbb{Z}$. Now we may apply the $r = 1$ case of Lemma II.2, since (III.5) is a system of type (II.3) with $U = \mathcal{C} - z\mathcal{A}$, $V = 1 - z$ and the functions c_k satisfy (II.4). The dual system (II.5) for the auxiliary formal Laurent series $h_k(z)$ in this case reads

$$(\mathcal{C}^* - \mathcal{A}^* z) h_k(z) = c_k (1 - z) h_k(z), \tag{III.6}$$

As is easily seen, we have $\mathcal{A}^* z^{-l} = a_l z^{-l}$ and $\mathcal{C}^* z^{-l} = c_l z^{-l}$. Thus, with $h_k(z) = \sum_{l \leq k} h_{kl} z^{-l}$, by comparing coefficients of z^{-l} in (III.6) we obtain

$$(c_l - c_k) h_{kl} = (a_l - c_k) h_{k,l+1}.$$

If we set $h_{kk} = 1$, we get

$$h_{kl} = \frac{\prod_{j=l}^{k-1} (a_j - c_k)}{\prod_{j=l}^{k-1} (c_j - c_k)}.$$

Taking into account (II.6), here we have $g_k(z) = (1 - z) h_k(z)$, where $g_k(z) = \sum_{l \leq k} g_{kl} z^{-l}$. (Since $f_{kk} = 1$, the pairing $\langle f_k(z), V^* h_k(z) \rangle$ is simply the coefficient of z^{-k} in $(1 - z) h_k(z)$ and equals 1.) So, comparing coefficients of z^{-l} leads to

$$g_{kl} = h_{kl} - h_{k,l+1} = \frac{(a_l - c_l)}{(a_k - c_k)} \frac{\prod_{j=l+1}^{k} (a_j - c_k)}{\prod_{j=l}^{k-1} (c_j - c_k)},$$

which is exactly (III.3). $\square$

Now we present one of the main results of this thesis, a multidimensional extension of Theorem III.1. For convenience, we introduce the notation $|\mathbf{n}| = n_1 + n_2 + \cdots + n_r$.

THEOREM III.7. *Let* $(a_t)_{t\in\mathbb{Z}}$, $(c_i(t_i))_{t_i\in\mathbb{Z}}$, $i=1,\dots,r$ *be arbitrary sequences,* b *arbitrary, such that none of the denominators in* (III.8) *or* (III.9) *vanish. Then* $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ *and* $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ *are inverses of each other, where*

$$f_{\mathbf{nk}}=\frac{\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}(a_t-b/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r\prod_{t_i=k_i+1}^{n_i}(c_i(t_i)-b/\prod_{j=1}^r c_j(k_j))}\frac{\prod_{i=1}^r\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}(a_t-c_i(k_i))}{\prod_{i,j=1}^r\prod_{t_i=k_i+1}^{n_i}(c_i(t_i)-c_j(k_j))} \tag{III.8}$$

and

$$\begin{aligned}g_{\mathbf{kl}}=\prod_{1\le i<j\le r}&\frac{(c_i(l_i)-c_j(l_j))}{(c_i(k_i)-c_j(k_j))}\\&\times\frac{(b-a_{|\mathbf{l}|}\prod_{j=1}^r c_j(l_j))}{(b-a_{|\mathbf{k}|}\prod_{j=1}^r c_j(k_j))}\prod_{i=1}^r\frac{(a_{|\mathbf{l}|}-c_i(l_i))}{(a_{|\mathbf{k}|}-c_i(k_i))}\\&\times\frac{\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}(a_t-b/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r\prod_{t_i=l_i}^{k_i-1}(c_i(t_i)-b/\prod_{j=1}^r c_j(k_j))}\frac{\prod_{i=1}^r\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}(a_t-c_i(k_i))}{\prod_{i,j=1}^r\prod_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_j(k_j))}.\end{aligned} \tag{III.9}$$

REMARK III.10. The special case $a_t=0$, $c_j(k_j)=x_j^{-1}q^{-k_j}$ is equivalent to the A_r Bailey transform of Milne and Lilly [55], [57], the specialization $a_t=0$, $c_j(k_j)=x_j^{-1}q^{-k_j}+x_jq^{k_j}$, $b=0$ is equivalent to their C_r Bailey transform [57], [47], [48]. The limiting case $a_t=b\,aq^{-t}$, $c_j(k_j)=x_j^{-1}q^{-k_j}$, then $b\to 0$, is equivalent to a second A_r Bailey transform of Milne [55, Theorem 8.26]. The specialization $c_j(k_j)=x_j^{-1}q^{-k_j}$ is equivalent to the A_r matrix inverse of Bhatnagar and Milne [11, Theorem 5.7], [13, Theorem 3.48]. Moreover, the $r=1$ case is a restatement of Krattenthaler's matrix inversion, Theorem III.1. Due to the fact that Theorem III.7 covers all known A_r matrix inversions (to the author's knowledge), we view Theorem III.7 as an A_r matrix inversion theorem (also see Remark III.32).

Another important special case of Theorem III.7 is a new multidimensional bibasic hypergeometric matrix inversion, stated separately as Corollary III.25, which we utilize in our applications in chapter IV.

PROOF OF THEOREM III.7. We will use the operator method of chapter II. From (III.8) we deduce for $\mathbf{n}\ge\mathbf{k}$ the recursion

$$\begin{aligned}(c_i(n_i)-b/\textstyle\prod_{j=1}^r c_j(k_j))&\prod_{s=1}^r(c_i(n_i)-c_s(k_s))f_{\mathbf{nk}}\\&=(a_{|\mathbf{n}|-1}-b/\textstyle\prod_{j=1}^r c_j(k_j))\prod_{s=1}^r(a_{|\mathbf{n}|-1}-c_s(k_s))f_{\mathbf{n}-\mathbf{e}_i,\mathbf{k}},\end{aligned} \tag{III.11}$$

for $i = 1, \dots, r$, where $\mathbf{e}_i$ denotes the vector of $\mathbb{Z}^r$ where all components are zero except the i-th, which is 1. We write

$$f_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{n} \geq \mathbf{k}} \frac{\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1} (a_t - b/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r \prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \frac{\prod_{i=1}^r \prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1} (a_t - c_i(k_i))}{\prod_{i,j=1}^r \prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - c_j(k_j))} \mathbf{z}^{\mathbf{n}}.$$

Moreover, we define linear operators $\mathcal{A}, \mathcal{C}_i$ by $\mathcal{A}\mathbf{z}^{\mathbf{n}} = a_{|\mathbf{n}|}\mathbf{z}^{\mathbf{n}}$ and $\mathcal{C}_i\mathbf{z}^{\mathbf{n}} = c_i(n_i)\mathbf{z}^{\mathbf{n}}$ for all $i = 1, \dots, r$. Then we may write (III.11) in the form

$$\begin{aligned}(\mathcal{C}_i - b/\textstyle\prod_{j=1}^r c_j(k_j)) \displaystyle\prod_{s=1}^r (\mathcal{C}_i - c_s(k_s)) f_{\mathbf{k}}(\mathbf{z}) \\ = z_i(\mathcal{A} - b/\textstyle\prod_{j=1}^r c_j(k_j)) \displaystyle\prod_{s=1}^r (\mathcal{A} - c_s(k_s)) f_{\mathbf{k}}(\mathbf{z}), \quad \text{(III.12)}\end{aligned}$$

valid for all $\mathbf{k} \in \mathbb{Z}^r$. We want to write our system of equations in a way such that Corollary II.18 is applicable. In order to achieve this, we expand the products on both sides of (III.12) in terms of the elementary symmetric functions (see [49, p.19])

$$e_j(c_1(k_1), c_2(k_2), \dots, c_r(k_r), b/\textstyle\prod_{s=1}^r c_s(k_s))$$

of order j, for which we write $e_j(\mathbf{c}(\mathbf{k}))$ for short. Our recurrence system then reads, using $e_{r+1}(\mathbf{c}(\mathbf{k})) = b$,

$$\begin{aligned}\sum_{j=1}^r e_j(\mathbf{c}(\mathbf{k}))[(-\mathcal{C}_i)^{r+1-j} - z_i(-\mathcal{A})^{r+1-j}] f_{\mathbf{k}}(\mathbf{z}) \\ = [z_i(-\mathcal{A})^{r+1} + bz_i - (-\mathcal{C}_i)^{r+1} - b] f_{\mathbf{k}}(\mathbf{z}), \quad i = 1, \dots, r. \quad \text{(III.13)}\end{aligned}$$

Now (III.13) is a system of type (II.22) with $V_{ij} = [(-\mathcal{C}_i)^{r+1-j} - z_i(-\mathcal{A})^{r+1-j}]$, $W_i = [z_i(-\mathcal{A})^{r+1} + bz_i - (-\mathcal{C}_i)^{r+1} - b]$, and $c_j(\mathbf{k}) = e_j(\mathbf{c}(\mathbf{k}))$. The operators $C_{ij} = (-\mathcal{C}_i)^{r+1-j}$, $A_{ij} = -z_i(-\mathcal{A})^{r+1-j}$, $W_i^{(c)} = [-(-\mathcal{C}_i)^{r+1} - b]$, $W_i^{(a)} = [z_i(-\mathcal{A})^{r+1} + bz_i]$ satisfy (II.10), (II.11), (II.12), (II.19), (II.20), and (II.21), the functions $c_j(\mathbf{k})$ satisfy (II.4). Hence we may apply Corollary II.18. The dual system (II.24) for the auxiliary formal Laurent series $h_{\mathbf{k}}(\mathbf{z})$ in this case reads

$$\begin{aligned}\sum_{j=1}^r e_j(\mathbf{c}(\mathbf{k}))[(-\mathcal{C}_i^*)^{r+1-j} - (-\mathcal{A}^*)^{r+1-j} z_i] h_{\mathbf{k}}(\mathbf{z}) \\ = [(-\mathcal{A}^*)^{r+1} z_i + bz_i - (-\mathcal{C}_i^*)^{r+1} - b] h_{\mathbf{k}}(\mathbf{z}), \quad i = 1, \dots, r.\end{aligned}$$

Equivalently, we have

$$(\mathcal{C}_i^* - b/\prod_{j=1}^r c_j(k_j)) \prod_{s=1}^r (\mathcal{C}_i^* - c_s(k_s)) h_{\mathbf{k}}(\mathbf{z}) \\ = (\mathcal{A}^* - b/\prod_{j=1}^r c_j(k_j)) \prod_{s=1}^r (\mathcal{A}^* - c_s(k_s)) z_i h_{\mathbf{k}}(\mathbf{z}), \quad \text{(III.14)}$$

for all $i = 1, \dots, r$ and $\mathbf{k} \in \mathbb{Z}^r$. As is easily seen, we have $\mathcal{A}^* \mathbf{z}^{-\mathbf{l}} = a_{|\mathbf{l}|} \mathbf{z}^{-\mathbf{l}}$ and $\mathcal{C}_i^* \mathbf{z}^{-\mathbf{l}} = c_i(l_i) \mathbf{z}^{-\mathbf{l}}$ for $i = 1, \dots, r$. Thus, with $h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}$, by comparing coefficients of $\mathbf{z}^{-\mathbf{l}}$ in (III.14) we obtain

$$(c_i(l_i) - b/\prod_{j=1}^r c_j(k_j)) \prod_{s=1}^r (c_i(l_i) - c_s(k_s)) h_{\mathbf{kl}} \\ = (a_{|\mathbf{l}|} - b/\prod_{j=1}^r c_j(k_j)) \prod_{s=1}^r (a_{|\mathbf{l}|} - c_s(k_s)) h_{\mathbf{k}, \mathbf{l}+\mathbf{e}_i}.$$

If we set $h_{\mathbf{kk}} = 1$, we get

$$h_{\mathbf{kl}} = \frac{\prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1} (a_t - b/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r \prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \frac{\prod_{i=1}^r \prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1} (a_t - c_i(k_i))}{\prod_{i,j=1}^r \prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - c_j(k_j))}.$$

Taking into account (II.23), we have to compute the action of

$$\overrightarrow{\det_{1\le i,j\le r}} (V_{ij}^*) = \overrightarrow{\det_{1\le i,j\le r}} [(-\mathcal{C}_i^*)^{r+1-j} - (-\mathcal{A}^*)^{r+1-j} z_i] \quad \text{(III.15)}$$

when applied to

$$h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} \frac{\prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1} (a_t - b/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r \prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \frac{\prod_{i=1}^r \prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1} (a_t - c_i(k_i))}{\prod_{i,j=1}^r \prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - c_j(k_j))} \mathbf{z}^{-\mathbf{l}}.$$

From Proposition II.9 it follows that the determinant (III.15) can be written as

$$\overrightarrow{\det_{1\le i,j\le r}} (V_{ij}^*) = \overrightarrow{\det_{1\le i,j\le r}} (C_{ij}^*) + \sum_{i,j=1}^r (-1)^{i+j} \overrightarrow{\det_{\substack{1\le m\le r, m\ne i \\ 1\le n\le r, n\ne j}}} (C_{mn}^*)\, A_{ij}^*,$$

or more explicitly,

$$\begin{aligned}\overrightarrow{\det_{1\le i,j\le r}}(V^*_{ij}) &= \overrightarrow{\det_{1\le i,j\le r}}[(-\mathcal{C}^*_i)^{r+1-j} - (-\mathcal{A}^*)^{r+1-j} z_i]\\ &= \overrightarrow{\det_{1\le i,j\le r}}[(-\mathcal{C}^*_i)^{r+1-j}]\\ &\quad + \sum_{i,j=1}^{r} (-1)^{i+j} \overrightarrow{\det_{\substack{1\le m\le r, m\ne i\\ 1\le n\le r, n\ne j}}}[(-\mathcal{C}^*_m)^{r+1-n}] \left(-(-\mathcal{A}^*)^{r+1-j} z_i\right). \qquad \text{(III.16)}\end{aligned}$$

Note that after expanding the column determinants by (II.16) all summands in (III.16) have pairwise commuting factors (when regarding $(-(-\mathcal{A}^*)^{r+1-j} z_i)$ as *one* factor). Since

$$z_i h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} \frac{(c_i(l_i) - b/\prod_{s=1}^r c_s(k_s))}{(a_{|\mathbf{l}|} - b/\prod_{s=1}^r c_s(k_s))} \prod_{s=1}^{r} \frac{(c_i(l_i) - c_s(k_s))}{(a_{|\mathbf{l}|} - c_s(k_s))} h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}},$$

we conclude that

$$\begin{aligned}&\overrightarrow{\det_{1\le i,j\le r}}(V^*_{ij}) h_{\mathbf{k}}(\mathbf{z})\\ &= \sum_{\mathbf{l}\le\mathbf{k}} \left(\overrightarrow{\det_{1\le i,j\le r}}(C^*_{ij}) + \sum_{i,j=1}^{r} (-1)^{i+j} \overrightarrow{\det_{\substack{1\le m\le r, m\ne i\\ 1\le n\le r, n\ne j}}}(C^*_{mn})\, A^*_{ij}\right) h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}\\ &= \sum_{\mathbf{l}\le\mathbf{k}} \Bigg(\overrightarrow{\det_{1\le i,j\le r}} \left[(-c_i(l_i))^{r+1-j}\right]\\ &\quad + \sum_{i,j=1}^{r} (-1)^{i+j} \overrightarrow{\det_{\substack{1\le m\le r, m\ne i\\ 1\le n\le r, n\ne j}}} \left[(-c_m(l_m))^{r+1-n}\right] \left(-(-a_{|\mathbf{l}|})^{r+1-j}\right)\\ &\quad \cdot \frac{(c_i(l_i) - b/\prod_{s=1}^r c_s(k_s))}{(a_{|\mathbf{l}|} - b/\prod_{s=1}^r c_s(k_s))} \prod_{s=1}^{r} \frac{(c_i(l_i) - c_s(k_s))}{(a_{|\mathbf{l}|} - c_s(k_s))}\Bigg) h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}. \qquad \text{(III.17)}\end{aligned}$$

We claim that

$$\overrightarrow{\det_{1\le i,j\le r}}(V^*_{ij}) h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} \det_{1\le i,j\le r}(v_{ij}) h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}, \qquad \text{(III.18)}$$

where

$$v_{ij} = (-c_i(l_i))^{r+1-j} - (-a_{|\mathbf{l}|})^{r+1-j} \frac{(c_i(l_i) - b/\prod_{s=1}^r c_s(k_s))}{(a_{|\mathbf{l}|} - b/\prod_{s=1}^r c_s(k_s))} \prod_{s=1}^{r} \frac{(c_i(l_i) - c_s(k_s))}{(a_{|\mathbf{l}|} - c_s(k_s))}.$$

The claim follows from the observation that $\det_{1\le i,j\le r}(v_{ij})$ is a determinant of commuting entries and so trivially satisfies the assumptions of Proposition II.9. Thus $\sum_{\mathbf{l}\le\mathbf{k}} \det_{1\le i,j\le r}(v_{ij}) h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}$ can also be transformed into the right side of (III.17).

For the computation of $\det_{1\le i,j\le r}(v_{ij})$ we utilize Lemma A.1 with $x_i = -c_i(l_i)$, $y_s = -c_s(k_s)$, $a = -a_{|\mathbf{l}|}$, and $c = (-1)^{r+1}b$, obtaining

$$\det_{1\le i,j\le r}(v_{ij}) = \frac{(a_{|\mathbf{l}|} - b/\prod_{j=1}^r c_j(l_j))}{(a_{|\mathbf{l}|} - b/\prod_{j=1}^r c_j(k_j))} \prod_{i=1}^r \frac{(a_{|\mathbf{l}|} - c_i(l_i))}{(a_{|\mathbf{l}|} - c_i(k_i))} \times \prod_{i=1}^r (-c_i(l_i)) \prod_{1\le i<j\le r} (c_j(l_j) - c_i(l_i)).$$

Plugging this determinant evaluation into (III.18) leads to

$$\overrightarrow{\det_{1\le i,j\le r}}(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} \Bigg(\prod_{1\le i<j\le r} (c_j(l_j) - c_i(l_i)) \prod_{i=1}^r (-c_i(l_i)) \times \frac{(a_{|\mathbf{l}|} - b/\prod_{j=1}^r c_j(l_j))}{(a_{|\mathbf{k}|} - b/\prod_{j=1}^r c_j(k_j))} \prod_{i=1}^r \frac{(a_{|\mathbf{l}|} - c_i(l_i))}{(a_{|\mathbf{k}|} - c_i(k_i))} \times \frac{\prod\limits_{t=|\mathbf{l}|+1}^{|\mathbf{k}|} (a_t - b/\prod_{j=1}^r c_j(k_j))}{\prod\limits_{i=1}^r \prod\limits_{t_i=l_i}^{k_i-1} (c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \frac{\prod\limits_{i=1}^r \prod\limits_{t=|\mathbf{l}|+1}^{|\mathbf{k}|} (a_t - c_i(k_i))}{\prod\limits_{i,j=1}^r \prod\limits_{t_i=l_i}^{k_i-1} (c_i(t_i) - c_j(k_j))} \mathbf{z}^{-\mathbf{l}} \Bigg). \quad \text{(III.19)}$$

Note that since $f_{\mathbf{kk}} = 1$, the pairing $\langle f_{\mathbf{k}}(\mathbf{z}), \overrightarrow{\det}(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z})\rangle$ is simply the coefficient of $\mathbf{z}^{-\mathbf{k}}$ in (III.19). Thus, equation (II.23) reads

$$g_{\mathbf{k}}(\mathbf{z}) = \prod_{1\le i<j\le r} (c_j(k_j) - c_i(k_i))^{-1} \prod_{i=1}^r (-c_i(k_i))^{-1} \overrightarrow{\det_{1\le i,j\le r}}(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}), \quad \text{(III.20)}$$

where $g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}$. So, extracting the coefficient of $\mathbf{z}^{-\mathbf{l}}$ in (III.20) we obtain exactly (III.9). □

For reference, we give two important basic hypergeometric specializations of Theorem III.7.

Corollary III.21 (Bhatnagar and Milne). *Let*

$$f_{\mathbf{nk}} = \frac{(ap^{|\mathbf{k}|} q^{-|\mathbf{k}|}/d; p)_{|\mathbf{n}|-|\mathbf{k}|}}{\prod\limits_{i=1}^r (dx_i q^{1+k_i+|\mathbf{k}|}; q)_{n_i-k_i}} \frac{\prod\limits_{i=1}^r (ax_i p^{|\mathbf{k}|} q^{k_i}; p)_{|\mathbf{n}|-|\mathbf{k}|}}{\prod\limits_{i,j=1}^r (q^{1+k_i-k_j} x_i/x_j; q)_{n_i-k_i}} \quad \text{(III.22)}$$

and

$$g_{\mathbf{kl}} = (-1)^{|\mathbf{k}|-|\mathbf{l}|}\, q^{\binom{|\mathbf{k}|-|\mathbf{l}|}{2}} \frac{(1-ap^{|\mathbf{l}|}q^{-|\mathbf{l}|}/d)}{(1-ap^{|\mathbf{k}|}q^{-|\mathbf{k}|})} \prod_{i=1}^{r} \frac{(1-ax_ip^{|\mathbf{l}|}q^{l_i})}{(1-ax_ip^{|\mathbf{k}|}q^{k_i})}$$
$$\times \frac{(ap^{1+|\mathbf{l}|}q^{-|\mathbf{k}|}/d;p)_{|\mathbf{k}|-|\mathbf{l}|}}{\prod_{i=1}^{r}(dx_iq^{l_i+|\mathbf{k}|};q)_{k_i-l_i}} \frac{\prod_{i=1}^{r}(ax_ip^{1+|\mathbf{l}|}q^{k_i};p)_{|\mathbf{k}|-|\mathbf{l}|}}{\prod_{i,j=1}^{r}(q^{1+l_i-l_j}x_i/x_j;q)_{k_i-l_i}}. \tag{III.23}$$

Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are infinite lower-triangular r-dimensional matrices being inverses of each other.

PROOF. In Theorem III.7 we set $a_t \mapsto aq^t$, $b \mapsto d/x_1\cdots x_n$, and $c_i(t_i) \mapsto q^{-t_i}/x_i$ for $i = 1,\dots,r$. After some elementary manipulations which include Lemma B.1 we obtain the inverse pair (III.22) and (III.23). □

REMARK III.24. The matrix inversion in Corollary III.21 is an A_r extension of Gasper's bibasic matrix inversion [24, (3.2) and (3.3)], [25, (3.6.19) and (3.6.20)], to which it reduces for $r = 1$. This inversion can easily be obtained from the inversion [13, Theorem 3.48] of Bhatnagar and Milne.

COROLLARY III.25. *Let*

$$f_{\mathbf{nk}} = \frac{\prod_{i=1}^{r}\left[(ap^{|\mathbf{k}|}q^{k_i}x_i;p)_{|\mathbf{n}|-|\mathbf{k}|}\,(ap^{|\mathbf{k}|}q^{-k_i}/x_i;p)_{|\mathbf{n}|-|\mathbf{k}|}\right]}{\prod_{i,j=1}^{r}\left[(q^{1+k_i-k_j}x_i/x_j;q)_{n_i-k_i}\,(q^{1+k_i+k_j}x_ix_j;q)_{n_i-k_i}\right]} \tag{III.26}$$

and

$$g_{\mathbf{kl}} = (-1)^{|\mathbf{k}|-|\mathbf{l}|}\, q^{\binom{|\mathbf{k}|-|\mathbf{l}|}{2}} \prod_{1\le i<j\le r} \frac{(1-x_ix_jq^{l_i+l_j})}{(1-x_ix_jq^{k_i+k_j})}$$
$$\times \prod_{i=1}^{r} \frac{(1-ap^{|\mathbf{l}|}q^{l_i}x_i)(1-ap^{|\mathbf{l}|}q^{-l_i}/x_i)}{(1-ap^{|\mathbf{k}|}q^{k_i}x_i)(1-ap^{|\mathbf{k}|}q^{-k_i}/x_i)}$$
$$\times \frac{\prod_{i=1}^{r}\left[(ap^{1+|\mathbf{l}|}q^{k_i}x_i;p)_{|\mathbf{k}|-|\mathbf{l}|}\,(ap^{1+|\mathbf{l}|}q^{-k_i}/x_i;p)_{|\mathbf{k}|-|\mathbf{l}|}\right]}{\prod_{i,j=1}^{r}\left[(q^{1+l_i-l_j}x_i/x_j;q)_{k_i-l_i}\,(q^{l_i+k_j}x_ix_j;q)_{k_i-l_i}\right]}. \tag{III.27}$$

Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are infinite lower-triangular r-dimensional matrices being inverses of each other.

PROOF. In Theorem III.7 we replace $b \mapsto 0$, $a_t \mapsto ap^t + p^{-t}/a$, and $c_i(t_i) \mapsto x_iq^{t_i} + q^{-t_i}/x_i$ for $i = 1,\dots,r$. After some elementary manipulations, including Lemma B.1, we obtain the inverse pair (III.26) and (III.27). □

Remark III.28. The inversion in Corollary III.25 is a D_r extension of Gasper's bibasic matrix inversion [24, (3.2) and (3.3)], [25, (3.6.19) and (3.6.20)], to which it reduces for $r=1$.

Next, we give another multidimensional matrix inversion which we can prove by the operator method of chapter II. The multidimensional matrix inversion in Theorem III.29 is of different type than the inversion of Theorem III.7, see Remark III.32.

Theorem III.29. *Let $(c_i(t_i))_{t_i\in\mathbb{Z}}$, $i=1,\dots,r$, be arbitrary sequences, b arbitrary, such that none of the denominators in* (III.30) *or* (III.31) *vanish. Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}}=\prod_{i=1}^r\frac{\prod\limits_{t_i=k_i}^{n_i-1}(1-bc_i(t_i)/\prod_{j=1}^r c_j(k_j))}{\prod\limits_{t_i=k_i+1}^{n_i}(c_i(t_i)-b/\prod_{j=1}^r c_j(k_j))}\prod_{i,j=1}^r\frac{\prod\limits_{t_i=k_i}^{n_i-1}(1-c_i(t_i)c_j(k_j))}{\prod\limits_{t_i=k_i+1}^{n_i}(c_i(t_i)-c_j(k_j))}\tag{III.30}$$

and

$$\begin{aligned}g_{\mathbf{kl}}=\prod_{1\le i<j\le r}&\left(\frac{(c_i(l_i)-c_j(l_j))}{(c_i(k_i)-c_j(k_j))}\frac{(1-c_i(l_i)c_j(l_j))}{(1-c_i(k_i)c_j(k_j))}\right)\\&\times\prod_{i=1}^r\frac{(1-c_i(l_i)^2)}{(1-c_i(k_i)^2)}\prod_{i=1}^r\frac{c_i(l_i)}{c_i(k_i)}\\&\times\prod_{i=1}^r\frac{\prod\limits_{t_i=l_i+1}^{k_i}(1-bc_i(t_i)/\prod_{j=1}^r c_j(k_j))}{\prod\limits_{t_i=l_i}^{k_i-1}(c_i(t_i)-b/\prod_{j=1}^r c_j(k_j))}\prod_{i,j=1}^r\frac{\prod\limits_{t_i=l_i+1}^{k_i}(1-c_i(t_i)c_j(k_j))}{\prod\limits_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_j(k_j))}.\end{aligned}\tag{III.31}$$

Remark III.32. The special case $c_j(k_j)=x_j^{-1}q^{-k_j}$ is a C_r generalization of Bressoud's matrix inversion formula [16], as pointed out in [48, second remark after Theorem 2.11]. Setting, in addition, $b=0$ yields a C_r Bailey transform which is equivalent to the one derived in [48]. Therefore, we view Theorem III.29 as a C_r matrix inversion theorem. See Corollary III.40 for a C_r Bressoud's matrix inverse.

Proof of Theorem III.29. Again, we will use the operator method of chapter II. From (III.30) we deduce for $\mathbf{n}\ge\mathbf{k}$ the recursion

$$\begin{aligned}(c_i(n_i)&-b/\textstyle\prod_{j=1}^r c_j(k_j))\displaystyle\prod_{s=1}^r(c_i(n_i)-c_s(k_s))f_{\mathbf{nk}}\\&=(1-bc_i(n_i-1)/\textstyle\prod_{j=1}^r c_j(k_j))\displaystyle\prod_{s=1}^r(1-c_i(n_i-1)c_s(k_s))f_{\mathbf{n}-\mathbf{e}_i,\mathbf{k}},\end{aligned}\tag{III.33}$$

for $i = 1, \ldots, r$. We write

$$f_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} \prod_{i=1}^{r} \frac{\prod_{t_i=k_i}^{n_i-1} (1 - bc_i(t_i)/\prod_{j=1}^{r} c_j(k_j))}{\prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - b/\prod_{j=1}^{r} c_j(k_j))} \prod_{i,j=1}^{r} \frac{\prod_{t_i=k_i}^{n_i-1} (1 - c_i(t_i)c_j(k_j))}{\prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - c_j(k_j))} \mathbf{z}^{\mathbf{n}}.$$

Moreover, we define linear operators $\mathcal{C}_i$ by $\mathcal{C}_i \mathbf{z}^{\mathbf{n}} = c_i(n_i)\mathbf{z}^{\mathbf{n}}$ for $i = 1, \ldots, r$. Then we may write (III.33) in the form

$$\begin{aligned} (\mathcal{C}_i - b/\textstyle\prod_{j=1}^{r} c_j(k_j)) \displaystyle\prod_{s=1}^{r} (\mathcal{C}_i - c_s(k_s)) f_{\mathbf{k}}(\mathbf{z}) \\ = z_i(I - \mathcal{C}_i b/\textstyle\prod_{j=1}^{r} c_j(k_j)) \displaystyle\prod_{s=1}^{r} (I - \mathcal{C}_i c_s(k_s)) f_{\mathbf{k}}(\mathbf{z}), \end{aligned} \quad \text{(III.34)}$$

I being the identity operator, valid for all $\mathbf{k} \in \mathbb{Z}^r$. We will write our system of equations in a way such that Corollary II.18 is applicable. Again, we expand the products on both sides of (III.34) in terms of the elementary symmetric functions

$$e_j(c_1(k_1), c_2(k_2), \ldots, c_r(k_r), b/\textstyle\prod_{s=1}^{r} c_s(k_s))$$

of order j, for which we write $e_j(\mathbf{c}(\mathbf{k}))$ for short. Our recurrence system then reads, again using $e_{r+1}(\mathbf{c}(\mathbf{k})) = b$,

$$\begin{aligned} \sum_{j=1}^{r} e_j(\mathbf{c}(\mathbf{k}))[(-\mathcal{C}_i)^{r+1-j} - (-1)^{r+1} z_i (-\mathcal{C}_i)^j] f_{\mathbf{k}}(\mathbf{z}) \\ = [(-1)^{r+1} z_i + b z_i \mathcal{C}_i^{r+1} - (-\mathcal{C}_i)^{r+1} - b] f_{\mathbf{k}}(\mathbf{z}), \quad i = 1, \ldots, r. \end{aligned} \quad \text{(III.35)}$$

Now (III.35) is a system of type (II.22) with $V_{ij} = [(-\mathcal{C}_i)^{r+1-j} - (-1)^{r+1} z_i (-\mathcal{C}_i)^j]$, $W_i = [(-1)^{r+1} z_i + b z_i \mathcal{C}_i^{r+1} - (-\mathcal{C}_i)^{r+1} - b]$ and $c_j(\mathbf{k}) = e_j(\mathbf{c}(\mathbf{k}))$. The operators $C_{ij} = V_{ij}$, $A_{ij} = 0$, $W_i^{(c)} = W_i$, $W_i^{(a)} = 0$ satisfy (II.10), (II.11), (II.12), (II.19), (II.20), and (II.21), the functions $c_j(\mathbf{k})$ satisfy (II.4). Hence we may apply Corollary II.18. The dual system (II.24) for the auxiliary formal Laurent series $h_{\mathbf{k}}(\mathbf{z})$ in this case reads

$$\begin{aligned} \sum_{j=1}^{r} e_j(\mathbf{c}(\mathbf{k}))[(-\mathcal{C}_i^*)^{r+1-j} - (-1)^{r+1} (-\mathcal{C}_i^*)^j z_i] h_{\mathbf{k}}(\mathbf{z}) \\ = [(-1)^{r+1} z_i + b(\mathcal{C}_i^*)^{r+1} z_i - (-\mathcal{C}_i^*)^{r+1} - b] h_{\mathbf{k}}(\mathbf{z}), \quad i = 1, \ldots, r. \end{aligned}$$

Equivalently, we have

$$\Big(\mathcal{C}_i^* - b/\textstyle\prod_{j=1}^r c_j(k_j)\Big)\displaystyle\prod_{s=1}^r (\mathcal{C}_i^* - c_s(k_s))h_{\mathbf{k}}(\mathbf{z})$$
$$= \Big(I - \mathcal{C}_i^* b/\textstyle\prod_{j=1}^r c_j(k_j)\Big)\displaystyle\prod_{s=1}^r (I - \mathcal{C}_i^* c_s(k_s)) z_i h_{\mathbf{k}}(\mathbf{z}), \quad \text{(III.36)}$$

for all $i = 1, \dots, r$ and $\mathbf{k} \in \mathbb{Z}^r$. Thus, with $h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}$, by comparing coefficients of $\mathbf{z}^{-\mathbf{l}}$ in (III.36) we obtain

$$\Big(c_i(l_i) - b/\textstyle\prod_{j=1}^r c_j(k_j)\Big)\displaystyle\prod_{s=1}^r (c_i(l_i) - c_s(k_s))h_{\mathbf{kl}}$$
$$= \Big(1 - bc_i(l_i)/\textstyle\prod_{j=1}^r c_j(k_j)\Big)\displaystyle\prod_{s=1}^r (1 - c_i(l_i)c_s(k_s))h_{\mathbf{k},\mathbf{l}+\mathbf{e}_i}.$$

If we set $h_{\mathbf{kk}} = 1$, we get

$$h_{\mathbf{kl}} = \prod_{i=1}^r \frac{\prod_{t_i=l_i}^{k_i-1}\big(1 - bc_i(t_i)/\prod_{j=1}^r c_j(k_j)\big)}{\prod_{t_i=l_i}^{k_i-1}\big(c_i(t_i) - b/\prod_{j=1}^r c_j(k_j)\big)} \prod_{i,j=1}^r \frac{\prod_{t_i=l_i}^{k_i-1}(1 - c_i(t_i)c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1}(c_i(t_i) - c_j(k_j))}.$$

Taking into account (II.23), we have to compute the action of

$$\overrightarrow{\det_{1\le i,j\le r}}(V_{ij}^*) = \overrightarrow{\det_{1\le i,j\le r}}\big[(-\mathcal{C}_i^*)^{r+1-j} - (-1)^{r+1}(-\mathcal{C}_i^*)^j z_i\big]$$

when applied to

$$h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} \prod_{i=1}^r \frac{\prod_{t_i=l_i}^{k_i-1}\big(1 - bc_i(t_i)/\prod_{j=1}^r c_j(k_j)\big)}{\prod_{t_i=l_i}^{k_i-1}\big(c_i(t_i) - b/\prod_{j=1}^r c_j(k_j)\big)} \prod_{i,j=1}^r \frac{\prod_{t_i=l_i}^{k_i-1}(1 - c_i(t_i)c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1}(c_i(t_i) - c_j(k_j))}\mathbf{z}^{-\mathbf{l}}.$$

Because $V_{i_1 j_1}^*$ and $V_{i_2 j_2}^*$ commute for $i_1 \neq i_2$ and all j_1, j_2, all the summands in $\overrightarrow{\det}_{1\le i,j\le r}(V_{ij}^*)$ have pairwise commuting factors. Since

$$z_i h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} \frac{\big(c_i(l_i) - b/\prod_{j=1}^r c_j(k_j)\big)}{\big(1 - bc_i(l_i)/\prod_{j=1}^r c_j(k_j)\big)} \prod_{j=1}^r \frac{(c_i(l_i) - c_j(k_j))}{(1 - c_i(l_i)c_j(k_j))} h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}},$$

we conclude that

$$\overrightarrow{\det_{1\le i,j\le r}}(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} \det_{1\le i,j\le r}(v_{ij}) h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}, \quad \text{(III.37)}$$

where

$$\begin{aligned} v_{ij} =&(-c_i(l_i))^{r+1-j} \\ &-(-1)^{r+1}(-c_i(l_i))^j \frac{(c_i(l_i)-b/\prod_{s=1}^r c_s(k_s))}{(1-bc_i(l_i)/\prod_{s=1}^r c_s(k_s))} \prod_{s=1}^r \frac{(c_i(l_i)-c_s(k_s))}{(1-c_i(l_i)c_s(k_s))}. \end{aligned}$$

For the computation of $\det_{1\le i,j\le r}(v_{ij})$ we utilize Lemma A.11 with $x_i = -c_i(l_i)$, $y_s = -c_s(k_s)$, and $c = (-1)^{r-1}b$, obtaining

$$\begin{aligned} \det_{1\le i,j\le r}(v_{ij}) = &\prod_{i=1}^r \frac{(1-c_i(k_i)b/\prod_{j=1}^r c_j(k_j))}{(1-c_i(l_i)b/\prod_{j=1}^r c_j(k_j))} \\ &\times \prod_{i=1}^r (1-c_i(l_i)^2) \prod_{i=1}^r (-c_i(l_i)) \prod_{i,j=1}^r (1-c_i(l_i)c_j(k_j))^{-1} \\ &\quad \times \prod_{1\le i<j\le r} [(c_j(l_j)-c_i(l_i))(1-c_i(l_i)c_j(l_j))(1-c_i(k_i)c_j(k_j))]. \end{aligned}$$

Plugging this determinant evaluation into (III.37) leads to

$$\begin{aligned} \overset{\rightarrow}{\det_{1\le i,j\le r}}(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} \Bigg(&\prod_{1\le i<j\le r} [(c_j(l_j)-c_i(l_i))(1-c_i(l_i)c_j(l_j))] \\ &\times \prod_{i=1}^r (1-c_i(l_i)^2) \prod_{i=1}^r (-c_i(l_i)) \prod_{i\le j} (1-c_i(k_i)c_j(k_j))^{-1} \\ \times \prod_{i=1}^r \frac{\prod_{t_i=l_i+1}^{k_i} (1-bc_i(t_i)/\prod_{j=1}^r c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1} (c_i(t_i)-b/\prod_{j=1}^r c_j(k_j))} &\prod_{i,j=1}^r \frac{\prod_{t_i=l_i+1}^{k_i} (1-c_i(t_i)c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1} (c_i(t_i)-c_j(k_j))} \mathbf{z}^{-\mathbf{l}} \Bigg). \end{aligned} \tag{III.38}$$

Again, the pairing $\langle f_{\mathbf{k}}(\mathbf{z}), \overset{\rightarrow}{\det}(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z})\rangle$ is simply the coefficient of $\mathbf{z}^{-\mathbf{k}}$ in (III.38). Thus, equation (II.23) reads

$$g_{\mathbf{k}}(\mathbf{z}) = \prod_{1\le i<j\le r} (c_j(k_j)-c_i(k_i))^{-1} \prod_{i=1}^r (-c_i(k_i))^{-1} \overset{\rightarrow}{\det_{1\le i,j\le r}}(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z}), \tag{III.39}$$

where $g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}$. So, extracting the coefficient of $\mathbf{z}^{-\mathbf{l}}$ in (III.39) we obtain exactly (III.31). $\square$

Finally, we state an important basic hypergeometric specialization of the matrix inversion in Theorem III.29.

COROLLARY III.40. *Let*

$$f_{\mathbf{nk}} = \prod_{i=1}^{r} \frac{(ax_i q^{k_i-|\mathbf{k}|};q)_{n_i-k_i}}{(x_i q^{1+k_i+|\mathbf{k}|}/a;q)_{n_i-k_i}} \prod_{i,j=1}^{r} \frac{(q^{k_i+k_j}x_ix_j;q)_{n_i-k_i}}{(q^{1+k_i-k_j}x_i/x_j;q)_{n_i-k_i}} \tag{III.41}$$

and

$$g_{\mathbf{kl}} = (-1)^{|\mathbf{k}|-|\mathbf{l}|} q^{\binom{|\mathbf{k}|-|\mathbf{l}|}{2}} \prod_{1\le i\le j\le r} \frac{(1-x_ix_jq^{l_i+l_j})}{(1-x_ix_jq^{k_i+k_j})} \\ \times \prod_{i=1}^{r} \frac{(ax_iq^{1+l_i-|\mathbf{k}|};q)_{k_i-l_i}}{(x_iq^{l_i+|\mathbf{k}|}/a;q)_{k_i-l_i}} \prod_{i,j=1}^{r} \frac{(q^{1+l_i+k_j}x_ix_j;q)_{k_i-l_i}}{(q^{1+l_i-l_j}x_i/x_j;q)_{k_i-l_i}}. \tag{III.42}$$

Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ *and* $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ *are infinite lower-triangular* r*-dimensional matrices being inverses of each other.*

PROOF. In Theorem III.29 we set $b \mapsto 1/ax_1\cdots x_r$, and $c_i(t_i) \mapsto q^{-t_i}/x_i$ for $i = 1,\dots,r$. After some elementary manipulations which include Lemma B.1 we obtain the inverse pair (III.41) and (III.42). $\square$

REMARK III.43. The inversion in Corollary III.40 is a C_r extension of Bressoud's matrix inversion [16], to which it reduces for $r = 1$. But Corollary III.40 is also equivalent to the $p = q$ case of Corollary III.25, with the roles of $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ being interchanged. Therefore, Corollary III.40 can also be viewed as a D_r basic hypergeometric matrix inversion. For an application of this inversion, see Theorem IV.22.

CHAPTER IV

Multiple basic hypergeometric summation theorems

In this chapter, we use our multidimensional matrix inversions of chapter III for deriving new multiple basic hypergeometric summation theorems. First we explain the standard technique we apply, namely of the so-called *inverse relations*, (IV.1) and (IV.2). Then we recall the standard definitions in basic hypergeometric series theory and note the conventions we use in stating our multidimensional series. Finally, we give a number of new summation theorems.

Probably, the most important application of matrix inversion is the derivation of hypergeometric series identities. There is a standard technique for deriving new summation formulas from known ones by using inverse matrices (cf. [4], [29], [64]). If $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are lower triangular matrices being inverses of each other, then of course the following is true:

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} f_{\mathbf{nk}} a_{\mathbf{k}} = b_{\mathbf{n}} \tag{IV.1}$$

if and only if

$$\sum_{\mathbf{0}\le\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}} b_{\mathbf{l}} = a_{\mathbf{k}}. \tag{IV.2}$$

Applications of our matrix inversions in Theorems III.7 and III.29 lead to many new identities for multidimensional (basic) hypergeometric series. As an illustration, we use special cases of our Theorem III.7 to derive A_r and D_r extensions of a terminating quadratic summation of Gessel and Stanton [29], D_r extensions of Jackson's ${}_8\phi_7$ summation [40], and a D_r extension of a cubic summation of Gasper and Rahman [26].

We now recall the standard definitions in basic hypergeometric series theory (cf. [25]). Let q be a complex number such that $|q| < 1$. Define

$$(a;q)_\infty := \prod_{j\geq 0}(1-aq^j),$$

and,

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty} \tag{IV.3}$$

$$= \prod_{j=0}^{k-1}(1-aq^j), \tag{IV.4}$$

where the equality (IV.4) holds when k is a non-negative integer. Classical (one-dimensional) basic hypergeometric series (or q-hypergeometric series) with s numerator parameters $a_1, \dots, a_s$ and t denominator parameters $b_1, \dots, b_t$ are defined by

Definition IV.5 (${}_s\phi_t$ basic hypergeometric series).

$${}_s\phi_t\left[\begin{matrix}a_1, \dots, a_s \\ b_1, \dots, b_t\end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1;q)_k\,(a_2;q)_k\,\cdots\,(a_s;q)_k}{(q;q)_k\,(b_1;q)_k\,\cdots\,(b_t;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+t-s} z^k, \tag{IV.6}$$

with $\binom{k}{2} = k(k-1)/2$, where $q \neq 0$ when $s > t+1$. The parameters $b_1, \dots, b_t$ are such that the denominator factors in the terms of the series (IV.6) are never zero.

Since $(q^{-n};q)_k = 0$, if $k = n+1, n+2, \dots$, an ${}_s\phi_t$ series terminates if one of its numerator parameters is of the form q^{-n} with $n = 0, 1, 2, \dots$, and $q \neq 0$. See [25] for the convergence criteria of these series when they do not terminate.

The theory of the (classical) basic hypergeometric series consists of several summation and transformation formulas involving ${}_s\phi_t$ series. Most of the fundamental summation and transformation formulas involve series where $s = t+1$. The classical summation theorems for terminating ${}_3\phi_2$, ${}_6\phi_5$, and ${}_8\phi_7$ series require that the parameters satisfy the additional condition of being either balanced and/or very-well-poised. An ${}_{s+1}\phi_s$ basic hypergeometric series is called *balanced* if $b_1\cdots b_s = a_1\cdots a_{s+1}q$ and $z = q$. An ${}_{s+1}\phi_s$ series is *well-poised* if $a_1 q = a_2 b_1 = \cdots = a_{s+1}b_s$. It is called *very-well-poised* if it is well-poised and if $a_2 = q\sqrt{a_1}$ and $a_3 = -q\sqrt{a_1}$. Note that in a very-well-poised series, the factor

$$\frac{1-a_1q^{2k}}{1-a_1} \tag{IV.7}$$

appears in the summand of the series. The parameter a_1 is usually referred to as the *special parameter* of the very-well-poised series.

One of the most powerful results in the theory and application of classical one-dimensional basic hypergeometric series is Jackson's [40] summation formula for a terminating ${}_8\phi_7$ series, which is both balanced and very-well-poised.

THEOREM IV.8 (JACKSON'S CLASSICAL ${}_8\phi_7$ SUMMATION). *Let a, b, c, and d be indeterminate, let n be a nonnegative integer and suppose that none of the denominators in* (IV.9) *vanish. Then*

$$
{}_8\phi_7\left[\begin{matrix} a,\ q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2q^{n+1}/bcd, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcdq^{-n}/a, aq^{n+1} \end{matrix}; q, q\right]
= \frac{(aq;q)_n\,(aq/bc;q)_n\,(aq/bd;q)_n\,(aq/cd;q)_n}{(aq/bcd;q)_n\,(aq/d;q)_n\,(aq/c;q)_n\,(aq/b;q)_n}. \tag{IV.9}
$$

Theorem IV.8 is equation (2.6.2) of [25], where we have chosen to do the replacement $e \to a^2q^{n+1}/bcd$ explicitly.

REMARK IV.10. Limiting cases of Theorem IV.8 include various important summation theorems such as the nonterminating and terminating very-well-poised ${}_6\phi_5$ summations, the balanced q-Pfaff-Saalschütz ${}_3\phi_2$ summation, the nonterminating q-Gauß ${}_2\phi_1$ summation, and the terminating q-Chu-Vandermonde ${}_2\phi_1$ summation theorem. For more details, consult [25].

Before proceeding to multiple basic hypergeometric summation theorems, we explain what we mean by A_r, C_r, or D_r basic hypergeometric series. We consider multiple series of the form

$$
\sum_{\substack{k_i \geq 0 \\ i=1,\dots,r}} S(\mathbf{k}), \tag{IV.11}
$$

where $\mathbf{k} = (k_1, \dots, k_r)$, which reduce to classical basic hypergeometric series when $r = 1$. We call such a multiple basic hypergeometric series *balanced* if it reduces to a balanced series when $r = 1$. *Well-poised* and *very-well-poised* series are defined similarly.

Further, such a multiple series is called a C_r basic hypergeometric series if the summand $S(\mathbf{k})$ contains the factor

$$
\prod_{1\leq i<j\leq r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{1\leq i<j\leq r}\left(\frac{1-q^{k_i+k_j}x_ix_j}{1-x_ix_j}\right)\prod_{i=1}^{r}\left(\frac{1-x_i^2q^{2k_i}}{1-x_i^2}\right). \tag{IV.12}
$$

Note that when $r = 1$, the first two factors disappear, and (IV.12) reduces to

$$
\frac{1-x_1^2q^{2k_1}}{1-x_1^2}. \tag{IV.13}
$$

The ratio (IV.13) is reminiscent of (IV.7). Indeed, in C_r series, x_1^2 acts like the special parameter of a very-well-poised series. In our statements of C_r theorems, we often set $x_i \mapsto \sqrt{a}x_i$ for $i = 1, \ldots, r$, and make similar changes to other parameters in $S(\mathbf{k})$. This is done in order to follow the classical notation in [25] as closely as possible. A typical example of a C_r basic hypergeometric series is the left hand side of (B.7).

D_r multiple basic series are closely related to C_r series. Instead of (IV.12), $S(\mathbf{k})$ only has the following factors:

$$\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i+k_j}x_ix_j}{1-x_ix_j}\right). \tag{IV.14}$$

A typical example is the left hand side of (IV.23), see Remark IV.25.

Finally, A_r basic hypergeometric series only have

$$\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \tag{IV.15}$$

as a factor of $S(\mathbf{k})$. A typical example is the left hand side of (IV.17). The other types of products occurring in the three types of series are similar. A reason for naming these series as A_r, C_r, or D_r series is that (IV.15), (IV.12), and (IV.14) are closely associated with the product side of the Weyl denominator formulas for the respective root systems, see Lemma VII.45 (cf. [72], [22], and [12]).

We terminate our multiple series in two ways: If the summand contains the factor

$$\prod_{i,j=1}^{r}(q^{-n_j}x_i/x_j;q)_{k_i}$$

in the numerator, then it vanishes if $k_i > n_i$ for $i = 1, \ldots, r$. In this case we have the natural bounds $0 \le k_i \le n_i$, for $i = 1, \ldots, r$, and we say that the series is summed over an r-rectangle. On the other hand, if the summand contains the factor

$$(q^{-N};q)_{k_1+\cdots+k_r}$$

in the numerator, then it vanishes if $k_1 + \cdots + k_r > N$. In this case we have the natural bounds $0 \le k_1 + \cdots + k_r \le N$ for all $k_i \ge 0$, and we say that the series is summed over an r-tetrahedron. Many identities involving series summed over an r-rectangle are equivalent to a corresponding identity, where the series involved are summed over an r-tetrahedron, see e.g. Theorem IV.16 and Theorem IV.20, or Theorem IV.22 and Theorem IV.26.

Finally, we mention some notation used to simplify our displays. We employ the notation $|\mathbf{k}|$ for $(k_1 + \cdots + k_r)$, where $\mathbf{k} = (k_1, \ldots, k_r)$. This notation is also applied to other vectors. We also find it useful to denote by $\mathbf{m}+\mathbf{k}$ the vector $(m_1+k_1,\ldots,m_r+k_r)$ formed by component-wise addition of the two vectors.

Now we can present our multiple basic hypergeometric summation theorems.

THEOREM IV.16 (AN A_r QUADRATIC SUM). *Let $x_1,\ldots,x_r$, a, b, and d be indeterminate, let $n_1,\ldots,n_r$ be nonnegative integers, let $r\geq 1$, and suppose that none of the denominators in* (IV.17) *vanish. Then*

$$\sum_{\substack{0\leq k_i\leq n_i\\ i=1,\ldots,r}} \Bigg(\prod_{i=1}^r \left(\frac{1-ax_iq^{2k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\leq i<j\leq r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{i,j=1}^r \frac{(q^{-2n_j}x_i/x_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}}$$
$$\times \prod_{i=1}^r \frac{(dx_i;q^2)_{k_i}\,(a^2x_iq^{1+2|\mathbf{n}|}/d;q^2)_{k_i}}{(ax_iq^2/b;q^2)_{k_i}\,(abx_iq;q^2)_{k_i}} \prod_{i=1}^r \frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+2n_i};q)_{|\mathbf{k}|}}$$
$$\times \frac{(b;q)_{|\mathbf{k}|}\,(q/b;q)_{|\mathbf{k}|}}{(aq/d;q)_{|\mathbf{k}|}\,(dq^{-2|\mathbf{n}|}/a;q)_{|\mathbf{k}|}}\, q^{-|\mathbf{k}|+2\sum_{i=1}^r ik_i}\Bigg)$$
$$= \frac{(aq^2/bd;q^2)_{|\mathbf{n}|}\,(abq/d;q^2)_{|\mathbf{n}|}}{(aq/d;q)_{2|\mathbf{n}|}} \prod_{i=1}^r \frac{(ax_iq;q)_{2n_i}}{(ax_iq^2/b;q^2)_{n_i}\,(abx_iq;q^2)_{n_i}}. \qquad \text{(IV.17)}$$

REMARK IV.18. This quadratic summation formula is an A_r extension of

$$\sum_{k=0}^n \frac{1-aq^{3k}}{1-a}\frac{(a;q)_k(b;q)_k(q/b;q)_k\ (d;q^2)_k(a^2q^{1+2n}/d;q^2)_k(q^{-2n};q^2)_k}{(q^2;q^2)_k(aq^2/b;q^2)_k(abq;q^2)_k\ (aq/d;q)_k(dq^{-2n}/a;q)_k(aq^{2n+1};q)_k}\,q^k$$
$$= \frac{(aq;q)_{2n}}{(aq/d;q)_{2n}}\frac{(abq/d;q^2)_n(aq^2/bd;q^2)_n}{(aq^2/b;q^2)_n(abq;q^2)_n}, \qquad \text{(IV.19)}$$

due to Gessel and Stanton [29, eq. (1.4), $q\to q^2$], to which it reduces for $r=1$. Many identities like (IV.19), involving bases of different powers of q, are known. Hypergeometric series with several bases were extensively studied by Gasper and Rahman [24], [63], [26], [25, sec. 3.8].

PROOF OF THEOREM IV.16. If we substitute $q\mapsto q^2$, $p\mapsto q$, and $d\mapsto a^2/d$ in Corollary III.21 we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}} = \prod_{i=1}^r \left(\frac{1-q^{1+2k_i+2|\mathbf{k}|}a^2x_i/d}{1-qa^2x_i/d}\right) \prod_{1\leq i<j\leq r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times \prod_{i,j=1}^r \frac{(q^{-2n_j}x_i/x_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}} \prod_{i=1}^r \frac{(ax_iq^{|\mathbf{n}|};q)_{2k_i}}{(a^2x_iq^{3+2n_i}/d;q^2)_{|\mathbf{k}|}} \cdot \frac{q^{2\sum_{i=1}^r ik_i}}{(aq^{2-|\mathbf{n}|}/d;q)_{2|\mathbf{k}|}}$$

and

$$g_{\mathbf{k}\mathbf{l}}=\prod_{i=1}^{r}\left(\frac{1-ax_iq^{2l_i+|\mathbf{l}|}}{1-ax_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{2l_i-2l_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i,j=1}^{r}\frac{(q^{-2k_j}x_i/x_j;q^2)_{l_i}}{(q^2x_i/x_j;q^2)_{l_i}}$$
$$\times\prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{l}|}}{(ax_iq^{1+2k_i};q)_{|\mathbf{l}|}}\prod_{i=1}^{r}\frac{(a^2x_iq^{1+2|\mathbf{k}|}/d;q^2)_{l_i}}{(a^2x_iq^3/d;q^2)_{l_i}}\prod_{i=1}^{r}\frac{(a^2x_iq/d;q^2)_{|\mathbf{k}|}}{(ax_iq;q)_{2k_i}}$$
$$\times\frac{(1-q^{1+|\mathbf{l}|}a/d)}{(1-qa/d)}\frac{(d/aq;q)_{|\mathbf{l}|}\,(qa/d;q)_{2|\mathbf{k}|}}{(q^{2-2|\mathbf{k}|}a/d;q)_{|\mathbf{l}|}}\,q^{-|\mathbf{l}|+2\sum_{i=1}^{r}il_i}.$$

Now (IV.1) holds for

$$a_{\mathbf{k}}=(baq/d;q^2)_{|\mathbf{k}|}\,(aq^2/bd;q^2)_{|\mathbf{k}|}\prod_{i=1}^{r}\frac{(a^2x_iq/d;q^2)_{|\mathbf{k}|}}{(ax_iq^2/b;q^2)_{k_i}\,(abx_iq;q)_{k_i}}$$

and

$$b_{\mathbf{n}}=\frac{(q^{2-|\mathbf{n}|}/b;q^2)_{|\mathbf{n}|}\,(bq^{1-|\mathbf{n}|};q^2)_{|\mathbf{n}|}}{(aq^{3-|\mathbf{n}|}/d;q^2)_{|\mathbf{n}|}\,(dq^{-|\mathbf{n}|}/a;q^2)_{|\mathbf{n}|}}\prod_{i=1}^{r}\frac{(a^2x_iq^3/d;q^2)_{n_i}\,(dx_i/a;q^2)_{n_i}}{(ax_iq^2/b;q^2)_{n_i}\,(abx_iq;q^2)_{n_i}}$$

by means of an A_r extension of Jackson's ${}_8\phi_7$-sum, Theorem B.3. This implies the inverse relation (IV.2) which is easily transformed into (IV.17). □

It is not hard to see from a polynomial identity argument that Theorem IV.16 implies the following summation theorem.

Theorem IV.20 (An A_r quadratic sum). *Let $x_1,\dots,x_r$, $c_1,\dots,c_r$, a, and d be indeterminate, let N be a nonnegative integer, let $r\ge 1$, and suppose that none of the denominators in* (IV.21) *vanish. Then*

$$\sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le N}}\Bigg(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{2k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i,j=1}^{r}\frac{(c_jx_i/x_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}}$$
$$\times\prod_{i=1}^{r}\frac{(dx_i;q^2)_{k_i}\,(a^2x_iq/d\prod_{j=1}^{r}c_j;q^2)_{k_i}}{(ax_iq^{2+N};q^2)_{k_i}\,(ax_iq^{1-N};q^2)_{k_i}}\prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq/c_i;q)_{|\mathbf{k}|}}$$
$$\times\frac{(q^{-N};q)_{|\mathbf{k}|}\,(q^{1+N};q)_{|\mathbf{k}|}}{(aq/d;q)_{|\mathbf{k}|}\,(d\prod_{j=1}^{r}c_j/a;q)_{|\mathbf{k}|}}\,q^{-|\mathbf{k}|+2\sum_{i=1}^{r}ik_i}\Bigg)$$
$$=\begin{cases}\dfrac{(dq/a;q^2)_M(aq^2/d\prod_{j=1}^{r}c_j;q^2)_M}{(aq^2/d;q^2)_M(dq\prod_{j=1}^{r}c_j/a;q^2)_M}\displaystyle\prod_{i=1}^{r}\frac{(ax_iq^2;q^2)_M(c_iq/ax_i;q^2)_M}{(q/ax_i;q^2)_M(ax_iq^2/c_i;q^2)_M} & (N=2M),\\[2ex] \dfrac{(d/a;q^2)_M(aq/d\prod_{j=1}^{r}c_j;q^2)_M}{(aq/d;q^2)_M(d\prod_{j=1}^{r}c_j/a;q^2)_M}\displaystyle\prod_{i=1}^{r}\frac{(ax_iq;q^2)_M(c_i/ax_i;q^2)_M}{(1/ax_i;q^2)_M(ax_iq/c_i;q^2)_M} & (N=2M-1).\end{cases}\tag{IV.21}$$

PROOF. First we write the right sides of (IV.21) as quotients of infinite products using (IV.3). Then by the $b = q^{-N}$ case of Theorem IV.16 it follows that the identity (IV.21) holds for $c_j = q^{-2n_j}$, $j = 1, \dots, r$. By clearing out denominators in (IV.21), we get a polynomial equation in c_1, which is true for q^{-2n_1}, $n_1 = 0, 1, \dots$. Thus we obtain an identity in c_1. By carrying out this process for $c_2, c_3, \dots, c_r$ also, we obtain Theorem IV.20. $\square$

In chapter III, we obtained by specialization an interesting bibasic hypergeometric matrix inversion, Corollary III.25. We use this inversion to derive D_r basic hypergeometric summation formulas.

THEOREM IV.22 (A D_r JACKSON'S SUM). *Let $x_1, \dots, x_r$, a, b, and c be indeterminate, let $n_1, \dots, n_r$ be nonnegative integers, let $r \geq 1$, and suppose that none of the denominators in* (IV.23) *vanish. Then*

$$\sum_{\substack{0 \leq k_i \leq n_i \\ i=1,\dots,r}} \left(\prod_{i=1}^{r} \left(\frac{1 - a x_i q^{k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \prod_{1 \leq i < j \leq r} (x_i x_j; q)^{-1}_{k_i + k_j} \right.$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i / x_j; q)_{k_i} \, (x_i x_j q^{n_j}; q)_{k_i}}{(q x_i / x_j; q)_{k_i}} \prod_{i=1}^{r} \frac{(a x_i; q)_{|\mathbf{k}|} \, (a q / x_i; q)_{|\mathbf{k}| - k_i}}{(a q^{1+n_i} x_i; q)_{|\mathbf{k}|} \, (a q^{1-n_i} / x_i; q)_{|\mathbf{k}|}}$$
$$\left. \times \frac{(b; q)_{|\mathbf{k}|} \, (c; q)_{|\mathbf{k}|} \, (a^2 q / bc; q)_{|\mathbf{k}|}}{\prod_{i=1}^{r} [(a x_i q / b; q)_{k_i} \, (a x_i q / c; q)_{k_i} \, (b c x_i / a; q)_{k_i}]} \, q^{\sum_{i=1}^{r} i \, k_i} \right)$$
$$= \prod_{i=1}^{r} \frac{(a x_i q; q)_{n_i} \, (a x_i q / bc; q)_{n_i} \, (b x_i / a; q)_{n_i} \, (c x_i / a; q)_{n_i}}{(x_i / a; q)_{n_i} \, (b c x_i / a; q)_{n_i} \, (a x_i q / b; q)_{n_i} \, (a x_i q / c; q)_{n_i}}. \tag{IV.23}$$

REMARK IV.24. For $r = 1$ Theorem IV.22 reduces to Jackson's balanced very-well-poised ${}_8\phi_7$ summation formula [40], [25, (II.22)], which is displayed in (IV.9). Theorem IV.22 is stated as Theorem B.12 in Appendix B again, where the notation is intended to follow Gasper-Rahman as closely as possible. In chapter V we use Theorem B.12 to derive multiple extensions of Bailey's [7] balanced very-well-poised ${}_{10}\phi_9$ transformation.

REMARK IV.25. We view the left hand side of (IV.23) as a D_r series since the summand contains the factor

$$\prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \prod_{1 \leq i < j \leq r} (x_i x_j; q)^{-1}_{k_i + k_j}$$
$$= \prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i + k_j} x_i x_j}{1 - x_i x_j} \right) \prod_{1 \leq i < j \leq r} (x_i x_j q; q)^{-1}_{k_i + k_j}.$$

As the series contains the factor (IV.14), at least in disguise, according to our conventions it belongs to type D_r. Also note that A_r series do not contain the factor $\prod_{1\le i<j\le r}(x_ix_j;q)_{k_i+k_j}^{-1}$ in the summand.

Proof of Theorem IV.22. By Corollary III.40 (or by setting $p=q$ in Corollary III.25, i.e. we consider a C_r/D_r extension of Bressoud's matrix inverse [16]) we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}} = \prod_{1\le i<j\le r} \frac{(1-q^{k_i-k_j}x_i/x_j)(1-x_ix_jq^{k_i+k_j})}{(1-x_i/x_j)(1-x_ix_j)} \prod_{i=1}^r \frac{(1-x_i^2q^{2k_i})}{(1-x_i^2)} \\ \times \prod_{i,j=1}^r \frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(x_ix_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(x_ix_jq^{1+n_j};q)_{k_i}} \prod_{i=1}^r \frac{(ax_iq^{|\mathbf{n}|};q)_{k_i}}{(x_iq^{1-|\mathbf{n}|}/a;q)_{k_i}} \cdot q^{\sum_{i=1}^r i\,k_i}$$

and

$$g_{\mathbf{kl}} = \prod_{1\le i<j\le r} \frac{(1-q^{l_i-l_j}x_i/x_j)(1-x_ix_jq^{l_i+l_j})}{(1-x_i/x_j)(1-x_ix_j)} \\ \times \prod_{i,j=1}^r \frac{(q^{-k_j}x_i/x_j;q)_{l_i}\,(x_ix_jq^{k_j};q)_{l_i}}{(qx_i/x_j;q)_{l_i}\,(x_ix_jq;q)_{l_i}} \prod_{i=1}^r \frac{(1-aq^{|\mathbf{l}|+l_i}x_i)(1-aq^{|\mathbf{l}|-l_i}/x_i)}{(1-ax_i)(1-a/x_i)} \\ \times \prod_{i=1}^r \frac{(x_i/a;q)_{k_i}}{(ax_iq;q)_{k_i}} \prod_{i=1}^r \frac{(ax_i;q)_{|\mathbf{l}|}\,(a/x_i;q)_{|\mathbf{l}|}}{(aq^{1+k_i}x_i;q)_{|\mathbf{l}|}\,(aq^{1-k_i}/x_i;q)_{|\mathbf{l}|}} \cdot q^{\sum_{i=1}^r i\,l_i}.$$

Now (IV.1) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^r \frac{(bx_i/a;q)_{k_i}\,(cx_i/a;q)_{k_i}\,(ax_iq/bc;q)_{k_i}}{(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(bcx_i/a;q)_{k_i}}$$

and

$$b_{\mathbf{n}} = \prod_{i=1}^r \frac{(x_i^2q;q)_{n_i}}{(bcx_i/a;q)_{n_i}\,(x_iq^{1-|\mathbf{n}|}/a;q)_{n_i}} \prod_{1\le i<j\le r} \frac{(x_ix_jq;q)_{n_i}}{(x_ix_jq^{1+n_j};q)_{n_i}} \\ \times \frac{(bcq^{-|\mathbf{n}|}/a^2;q)_{|\mathbf{n}|}\,(q^{1-|\mathbf{n}|}/b;q)_{|\mathbf{n}|}\,(c;q)_{|\mathbf{n}|}}{\prod_{i=1}^r [(ax_iq/b;q)_{n_i}\,(cq^{-n_i}/ax_i;q)_{n_i}]}$$

by Denis and Gustafson, Milne and Lilly's C_r ${}_8\phi_7$ summation B.6. This implies the inverse relation (IV.2) which is easily transformed into (IV.23). □

By using a polynomial argument we get

THEOREM IV.26 (A D_r JACKSON'S SUM). *Let $x_1,\ldots,x_r$, $c_1,\ldots,c_r$, a, and b be indeterminate, let N be a nonnegative integer, let $r\geq 1$, and suppose that none of the denominators in* (IV.27) *vanish. Then*

$$\sum_{\substack{k_1,\ldots,k_r\geq 0\\ 0\leq|\mathbf{k}|\leq N}}\Bigg(\prod_{i=1}^r\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\leq i<j\leq r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{1\leq i<j\leq r}(x_ix_j;q)^{-1}_{k_i+k_j}$$
$$\times\prod_{i,j=1}^r\frac{(c_jx_i/x_j;q)_{k_i}\,(x_ix_j/c_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{i=1}^r\frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-k_i}}{(ax_iq/c_i;q)_{|\mathbf{k}|}\,(ac_iq/x_i;q)_{|\mathbf{k}|}}$$
$$\times\frac{(b;q)_{|\mathbf{k}|}\,(a^2q^{1+N}/b;q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^r[(ax_iq/b;q)_{k_i}\,(bx_iq^{-N}/a;q)_{k_i}\,(ax_iq^{1+N};q)_{k_i}]}\,q^{\sum_{i=1}^r ik_i}\Bigg)$$
$$=\prod_{i=1}^r\frac{(ax_iq;q)_N\,(aq/x_i;q)_N\,(ax_iq/bc_i;q)_N\,(ac_iq/bx_i;q)_N}{(aq/bx_i;q)_N\,(ax_iq/b;q)_N\,(ac_iq/x_i;q)_N\,(ax_iq/c_i;q)_N}.\qquad\text{(IV.27)}$$

PROOF. First we write the right side of (IV.27) as quotient of infinite products using (IV.3). Then by the $c=q^{-N}$ case of Theorem IV.22 it follows that the identity (IV.27) holds for $c_j=q^{-n_j}$, $j=1,\ldots,r$. By clearing out denominators in (IV.27), we get a polynomial equation in c_1, which is true for q^{-n_1}, $n_1=0,1,\ldots$. Thus we obtain an identity in c_1. By carrying out this process for $c_2,c_3,\ldots,c_r$ also, we obtain Theorem IV.26. ☐

Limiting cases of Theorem IV.22 or Theorem IV.26 include various D_r summations. By reversing the multisum in Theorem IV.22 we obtain another D_r Jackson's sum, Theorem B.9, which was independently derived by G. Bhatnagar [12] using a different method. D_r extensions of many of the classical basic hypergeometric summation theorems are given in [12]. Further consequences of the new D_r ${}_8\phi_7$ summations, such as C_r and D_r extensions of Bailey's very-well-poised ${}_{10}\phi_9$ transformation formula [7], [25, (III.28)] are given in chapter V.

REMARK IV.28. We note that (IV.23) and (IV.27) could be written (with $a=1/x_{r+1}$ and $k_{r+1}:=-|\mathbf{k}|$) more compactly as

$$\sum_{k_1+\cdots+k_{r+1}=0}\Bigg(\prod_{1\leq i<j\leq r+1}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{1\leq i<j\leq r+1}(x_ix_j;q)^{-1}_{k_i+k_j}$$
$$\times\prod_{i=1}^{r+1}\prod_{j=1}^{r}\frac{(c_jx_i/x_j;q)_{k_i}\,(x_ix_j/c_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}$$
$$\times\prod_{i=1}^{r+1}[(x_iq/bx_{r+1};q)_{k_i}\,(x_iq/cx_{r+1};q)_{k_i}\,(bcx_ix_{r+1};q)_{k_i}]^{-1}$$

$$\times \prod_{i=1}^{r+1} x_i^{k_i} \cdot q^{\sum_{i=1}^{r+1}\left[\binom{k_i}{2}+i\,k_i\right]}\Bigg)$$

$$= \prod_{i=1}^{r} \frac{(x_iq/x_{r+1};q)_\infty (q/x_ix_{r+1};q)_\infty (x_iq/bc_ix_{r+1};q)_\infty (c_iq/bx_ix_{r+1};q)_\infty}{(q/bx_ix_{r+1};q)_\infty (x_iq/bx_{r+1};q)_\infty (c_iq/x_ix_{r+1};q)_\infty (x_iq/c_ix_{r+1};q)_\infty}$$
$$\times \prod_{i=1}^{r} \frac{(q/bcx_ix_{r+1};q)_\infty (x_iq/bcx_{r+1};q)_\infty (c_iq/cx_ix_{r+1};q)_\infty (x_iq/cc_ix_{r+1};q)_\infty}{(x_iq/cx_{r+1};q)_\infty (q/cx_ix_{r+1};q)_\infty (x_iq/bcc_ix_{r+1};q)_\infty (c_iq/bcx_ix_{r+1};q)_\infty}, \tag{IV.29}$$

provided the series terminates. However, we feel that the forms (IV.23) and (IV.27) are preferable since the dependence of summation indices in (IV.29) (which just hides what is really going on in the sum) is removed.

The remainder of this chapter is devoted to D_r quadratic and cubic summation theorems.

THEOREM IV.30 (A D_r QUADRATIC SUM). *Let $x_1,\dots,x_r$, a, and b be indeterminate, let $n_1,\dots,n_r$ be nonnegative integers, let $r\ge 1$, and suppose that none of the denominators in* (IV.31) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \left(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{2k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r}(x_ix_j;q^2)^{-1}_{k_i+k_j}\right.$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-2n_j}x_i/x_j;q^2)_{k_i}\,(x_ix_jq^{2n_j};q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}} \prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-2k_i}}{(ax_iq^{1+2n_i};q)_{|\mathbf{k}|}\,(aq^{1-2n_i}/x_i;q)_{|\mathbf{k}|}}$$
$$\left.\times \frac{(a^2q;q^2)_{|\mathbf{k}|}\,(b;q)_{|\mathbf{k}|}\,(q/b;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}\left[(abx_iq;q^2)_{k_i}\,(ax_iq^2/b;q^2)_{k_i}\right]}(-a)^{|\mathbf{k}|}\prod_{i=1}^{r}x_i^{-k_i}\cdot q^{2e_2(\mathbf{k})-\binom{|\mathbf{k}|+1}{2}+2\sum_{i=1}^{r}i\,k_i}\right)$$
$$= \prod_{i=1}^{r}\frac{(ax_iq;q)_{2n_i}\,(x_iq/ab;q^2)_{n_i}\,(bx_i/a;q^2)_{n_i}}{(x_i/a;q)_{2n_i}\,(abx_iq;q^2)_{n_i}\,(ax_iq^2/b;q^2)_{n_i}}, \tag{IV.31}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1,\dots,k_r\}$.

REMARK IV.32. This quadratic summation formula is a D_r extension of Gessel and Stanton's summation [29, eq. (1.4)], displayed in (IV.19), to which it reduces for $r=1$.

PROOF OF THEOREM IV.30. Doing the replacements $q\to q^2$, $p\to q$ in Corollary III.25 we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}} = \prod_{1\le i<j\le r}\frac{(1-q^{2k_i-2k_j}x_i/x_j)(1-x_ix_jq^{2k_i+2k_j})}{(1-x_i/x_j)(1-x_ix_j)}\prod_{i=1}^{r}\frac{(1-x_i^2q^{4k_i})}{(1-x_i^2)}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-2n_j}x_i/x_j;q^2)_{k_i}\,(x_ix_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}\,(x_ix_jq^{2+2n_j};q^2)_{k_i}} \prod_{i=1}^{r} \frac{(ax_iq^{|\mathbf{n}|};q)_{2k_i}}{(x_iq^{1-|\mathbf{n}|}/a;q)_{2k_i}} \cdot q^{2\sum_{i=1}^r ik_i}$$

and

$$g_{\mathbf{kl}} = \prod_{1\le i<j\le r} \frac{(1-q^{2l_i-2l_j}x_i/x_j)(1-x_ix_jq^{2l_i+2l_j})}{(1-x_i/x_j)(1-x_ix_j)}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-2k_j}x_i/x_j;q^2)_{l_i}\,(x_ix_jq^{2k_j};q^2)_{l_i}}{(q^2x_i/x_j;q^2)_{l_i}\,(x_ix_jq^2;q^2)_{l_i}} \prod_{i=1}^{r} \frac{(1-aq^{|\mathbf{l}|+2l_i}x_i)(1-aq^{|\mathbf{l}|-2l_i}/x_i)}{(1-ax_i)(1-a/x_i)}$$

$$\times \prod_{i=1}^{r} \frac{(x_i/a;q)_{2k_i}}{(ax_iq;q)_{2k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{l}|}\,(a/x_i;q)_{|\mathbf{l}|}}{(aq^{1+2k_i}x_i;q)_{|\mathbf{l}|}\,(aq^{1-2k_i}/x_i;q)_{|\mathbf{l}|}} \cdot q^{2\sum_{i=1}^r il_i}.$$

Now (IV.1) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^{r} \frac{(x_iq/ab;q^2)_{k_i}\,(bx_i/a;q^2)_{k_i}}{(abx_iq;q^2)_{k_i}\,(ax_iq^2/b;q^2)_{k_i}}$$

and

$$b_{\mathbf{n}} = \prod_{i=1}^{r} \frac{(x_i^2q^2;q^2)_{n_i}}{(abx_iq;q^2)_{n_i}\,(x_iq^{2-|\mathbf{n}|}/a;q^2)_{n_i}} \prod_{1\le i<j\le r} \frac{(x_ix_jq^2;q^2)_{n_i}}{(x_ix_jq^{2+2n_j};q^2)_{n_i}}$$

$$\times \frac{(bq^{1-|\mathbf{n}|};q^2)_{|\mathbf{n}|}\,(q^{2-|\mathbf{n}|}/b;q^2)_{|\mathbf{n}|}\,(a^2q;q^2)_{|\mathbf{n}|}}{\prod_{i=1}^{r}[(ax_iq^2/b;q^2)_{n_i}\,(aq^{1+|\mathbf{n}|-2n_i}/x_i;q^2)_{n_i}]}$$

by Denis and Gustafson, Milne and Lilly's C_r ${}_8\phi_7$ summation B.6. This implies the inverse relation (IV.2) which is easily transformed into (IV.31). $\square$

Remark IV.33. By reversing the multisum in (IV.31) we may obtain another, differently looking, extension of Gessel and Stanton's quadratic summation formula (IV.19).

Using the same technique as in the proofs of Theorems IV.20 and IV.26 we obtain from Theorem IV.30 the next two quadratic summation theorems (with minor substitutions of variables in Theorem IV.36).

Theorem IV.34 (A D_r quadratic sum). *Let $x_1,\dots,x_r$, $c_1,\dots,c_r$, and a be indeterminate, let N be a nonnegative integer, let $r\ge 1$, and suppose that none of the denominators in* (IV.35) *vanish. Then*

$$\sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le N}} \left(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{2k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r} \left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r} (x_ix_j;q^2)^{-1}_{k_i+k_j}\right.$$

$$\times \prod_{i,j=1}^{r} \frac{(c_j x_i/x_j;q^2)_{k_i}\,(x_i x_j/c_j;q^2)_{k_i}}{(q^2 x_i/x_j;q^2)_{k_i}} \prod_{i=1}^{r} \frac{(a x_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-2k_i}}{(a x_i q/c_i;q)_{|\mathbf{k}|}\,(a c_i q/x_i;q)_{|\mathbf{k}|}}$$

$$\left. \times \frac{(a^2 q;q^2)_{|\mathbf{k}|}\,(q^{1+N};q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}\left[(a x_i q^{2+N};q^2)_{k_i}\,(a x_i q^{1-N};q^2)_{k_i}\right]}\,(-a)^{|\mathbf{k}|} \prod_{i=1}^{r} x_i^{-k_i} \;\cdot\; q^{2e_2(\mathbf{k})-\binom{|\mathbf{k}|+1}{2}+2\sum_{i=1}^{r} i\,k_i}\right)$$

$$= \begin{cases} \displaystyle\prod_{i=1}^{r} \frac{(a x_i q^2;q^2)_M\,(aq^2/x_i;q^2)_M\,(c_i q/a x_i;q^2)_M\,(x_i q/a c_i;q^2)_M}{(q/a x_i;q^2)_M\,(x_i q/a;q^2)_M\,(a x_i q^2/c_i;q^2)_M\,(a c_i q^2/x_i;q^2)_M} & (N=2M), \\ \displaystyle\prod_{i=1}^{r} \frac{(a x_i q;q^2)_M\,(aq/x_i;q^2)_M\,(c_i/a x_i;q^2)_M\,(x_i/a c_i;q^2)_M}{(1/a x_i;q^2)_M\,(x_i/a;q^2)_M\,(a x_i q/c_i;q^2)_M\,(a c_i q/x_i;q^2)_M} & (N=2M-1), \end{cases} \tag{IV.35}$$

where $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\{k_1,\ldots,k_r\}$.

THEOREM IV.36 (A D_r QUADRATIC SUM). *Let* $x_1,\ldots,x_r$, $c_1,\ldots,c_r$, *and* b *be indeterminate, let* N *be a nonnegative integer, let* $r \geq 1$, *and suppose that none of the denominators in* (IV.37) *vanish. Then*

$$\sum_{\substack{k_1,\ldots,k_r\geq 0\\ 0\leq|\mathbf{k}|\leq N}} \left(\prod_{i=1}^{r}\left(\frac{1-x_i q^{2k_i+|\mathbf{k}|}}{1-x_i}\right) \prod_{1\leq i<j\leq r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\leq i<j\leq r} (x_i x_j q^{1+2N};q^2)^{-1}_{k_i+k_j}\right.$$

$$\times \prod_{i,j=1}^{r} \frac{(c_j x_i/x_j;q^2)_{k_i}\,(x_i x_j q^{1+2N}/c_j;q^2)_{k_i}}{(q^2 x_i/x_j;q^2)_{k_i}} \prod_{i=1}^{r} \frac{(x_i;q)_{|\mathbf{k}|}\,(q^{-2N}/x_i;q)_{|\mathbf{k}|-2k_i}}{(x_i q/c_i;q)_{|\mathbf{k}|}\,(c_i q^{-2N}/x_i;q)_{|\mathbf{k}|}}$$

$$\left. \times \frac{(b;q)_{|\mathbf{k}|}\,(q/b;q)_{|\mathbf{k}|}\,(q^{-2N};q^2)_{|\mathbf{k}|}}{\prod_{i=1}^{r}\left[(b x_i q;q^2)_{k_i}\,(x_i q^2/b;q^2)_{k_i}\right]} \prod_{i=1}^{r}(-x_i)^{-k_i} \;\cdot\; q^{2e_2(\mathbf{k})-\binom{|\mathbf{k}|}{2}-2N|\mathbf{k}|+2\sum_{i=1}^{r}(i-1)k_i}\right)$$

$$= \prod_{i=1}^{r} \frac{(x_i q;q)_{2N}\,(b x_i q/c_i;q^2)_N\,(x_i q^2/b c_i;q^2)_N}{(x_i q/c_i;q)_{2N}\,(x_i q^2/b;q^2)_N\,(b x_i q;q^2)_N}, \tag{IV.37}$$

where $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\{k_1,\ldots,k_r\}$.

Finally, we derive some cubic summations.

THEOREM IV.38 (A D_r CUBIC SUM). *Let* $x_1,\ldots,x_r$, *and* a *be indeterminate, let* $n_1,\ldots,n_r$ *be nonnegative integers, let* $r \geq 1$, *and suppose that none of the denominators in* (IV.39) *vanish. Then*

$$\sum_{\substack{0\leq k_i\leq n_i\\ i=1,\ldots,r}} \left(\prod_{i=1}^{r}\left(\frac{1-a x_i q^{3k_i+|\mathbf{k}|}}{1-a x_i}\right) \prod_{1\leq i<j\leq r}\left(\frac{1-q^{3k_i-3k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\leq i<j\leq r} (x_i x_j;q^3)^{-1}_{k_i+k_j}\right.$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-3n_j} x_i/x_j; q^3)_{k_i} \, (x_i x_j q^{3n_j}; q^3)_{k_i}}{(q^3 x_i/x_j; q^3)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i; q)_{|\mathbf{k}|} \, (aq/x_i; q)_{|\mathbf{k}|-3k_i}}{(ax_i q^{1+3n_i}; q)_{|\mathbf{k}|} \, (aq^{1-3n_i}/x_i; q)_{|\mathbf{k}|}}$$

$$\times \frac{(1/a^2; q)_{|\mathbf{k}|} \, (a^2 q; q)_{2|\mathbf{k}|}}{\prod_{i=1}^{r} (a^3 x_i q^3; q^3)_{k_i}} \, a^{2|\mathbf{k}|} \prod_{i=1}^{r} x_i^{-2k_i} \cdot q^{6e_2(\mathbf{k}) - \binom{2|\mathbf{k}|+1}{2} + 3\sum_{i=1}^{r} i\, k_i} \Bigg)$$

$$= \prod_{i=1}^{r} \frac{(ax_i q; q)_{3n_i} \, (x_i/a^3; q^3)_{n_i}}{(x_i/a; q)_{3n_i} \, (a^3 x_i q^3; q^3)_{n_i}}, \quad \text{(IV.39)}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1, \ldots, k_r\}$.

Remark IV.40. This cubic summation formula is a D_r extension of

$$\sum_{k=0}^{m} \frac{1 - aq^{4k}}{1 - a} \frac{(a;q)_k \, (b;q)_k \, (q/b;q)_{2k} \, (a^2 b q^{3n}; q^3)_k \, (q^{-3n}; q^3)_k}{(q^3;q^3)_k \, (aq^3/b; q^3)_k \, (ab;q)_{2k} \, (q^{1-3n}/ab; q)_k \, (aq^{3n+1}; q)_k} q^k$$

$$= \frac{(aq;q)_{3n}}{(ab;q)_{3n}} \frac{(ab^2;q^3)_n}{(aq^3/b;q^3)_n}, \quad \text{(IV.41)}$$

due to Gasper and Rahman [26, eq. (4.1), $c \to 1$], to which it reduces for $r = 1$.

Proof of Theorem IV.38. Doing the replacements $q \to q^3$, $p \to q$ in Corollary III.25 we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}} = \prod_{1 \le i < j \le r} \frac{(1 - q^{3k_i - 3k_j} x_i/x_j)(1 - x_i x_j q^{3k_i + 3k_j})}{(1 - x_i/x_j)(1 - x_i x_j)} \prod_{i=1}^{r} \frac{(1 - x_i^2 q^{6k_i})}{(1 - x_i^2)}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-3n_j} x_i/x_j; q^3)_{k_i} \, (x_i x_j; q^3)_{k_i}}{(q^3 x_i/x_j; q^3)_{k_i} \, (x_i x_j q^{3+3n_j}; q^3)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i q^{|\mathbf{n}|}; q)_{3k_i}}{(x_i q^{1-|\mathbf{n}|}/a; q)_{3k_i}} \cdot q^{3\sum_{i=1}^{r} i\, k_i}$$

and

$$g_{\mathbf{kl}} = \prod_{1 \le i < j \le r} \frac{(1 - q^{3l_i - 3l_j} x_i/x_j)(1 - x_i x_j q^{3l_i + 3l_j})}{(1 - x_i/x_j)(1 - x_i x_j)}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-3k_j} x_i/x_j; q^3)_{l_i} \, (x_i x_j q^{3k_j}; q^3)_{l_i}}{(q^3 x_i/x_j; q^3)_{l_i} \, (x_i x_j q^3; q^3)_{l_i}} \prod_{i=1}^{r} \frac{(1 - aq^{|\mathbf{l}|+3l_i} x_i)(1 - aq^{|\mathbf{l}|-3l_i}/x_i)}{(1 - ax_i)(1 - a/x_i)}$$

$$\times \prod_{i=1}^{r} \frac{(x_i/a; q)_{3k_i}}{(ax_i q; q)_{3k_i}} \prod_{i=1}^{r} \frac{(ax_i; q)_{|\mathbf{l}|} \, (a/x_i; q)_{|\mathbf{l}|}}{(aq^{1+3k_i} x_i; q)_{|\mathbf{l}|} \, (aq^{1-3k_i}/x_i; q)_{|\mathbf{l}|}} \cdot q^{3\sum_{i=1}^{r} i\, l_i}.$$

Now (IV.1) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^{r} \frac{(x_i/a^3; q^3)_{k_i}}{(a^3 x_i q^3; q^3)_{k_i}}$$

and

$$b_{\mathbf{n}} = \prod_{i=1}^{r} \frac{(x_i^2 q^3; q^3)_{n_i}}{(x_i q^{1-|\mathbf{n}|}/a; q^3)_{n_i}\,(x_i q^{3-|\mathbf{n}|}/a; q^3)_{n_i}} \prod_{1\le i<j\le r} \frac{(x_i x_j q^3; q^3)_{n_i}}{(x_i x_j q^{3+3n_j}; q^3)_{n_i}}$$
$$\times \frac{(q^{1-2|\mathbf{n}|}/a^2; q^3)_{|\mathbf{n}|}\,(a^2 q^{3-|\mathbf{n}|}; q^3)_{|\mathbf{n}|}\,(a^2 q^{1-|\mathbf{n}|}; q^3)_{|\mathbf{n}|}}{\prod_{i=1}^{r} [(a^3 x_i q^3; q^3)_{n_i}\,(a q^{1+|\mathbf{n}|-3n_i}/x_i; q^3)_{n_i}]}$$

by Denis and Gustafson, Milne and Lilly's C_r ${}_8\phi_7$ summation B.6. This implies the inverse relation (IV.2) which is easily transformed into (IV.39). $\square$

REMARK IV.42. By reversing the multisum in (IV.39) we may obtain another, differently looking, D_r extension of Gasper and Rahman's cubic summation formula (IV.41).

Using the same polynomial argument as above, we derive some more D_r cubic summation theorems from Theorem IV.38. For sake of brevity, we write them in compact form as follows.

THEOREM IV.43 (A D_r CUBIC SUM). *Let $x_1, \dots, x_r$, $c_1, \dots, c_r$, and a be indeterminate, let $r \ge 1$, and suppose that none of the denominators in* (IV.44) *vanish. Then,*

$$\sum_{\substack{k_i\ge 0\\ i=1,\dots,r}} \Bigg(\prod_{i=1}^{r} \left(\frac{1 - a x_i q^{3k_i+|\mathbf{k}|}}{1 - a x_i} \right) \prod_{1\le i<j\le r} \left(\frac{1 - q^{3k_i-3k_j} x_i/x_j}{1 - x_i/x_j} \right) \prod_{1\le i<j\le r} (x_i x_j; q^3)^{-1}_{k_i+k_j}$$
$$\times \prod_{i,j=1}^{r} \frac{(c_j x_i/x_j; q^3)_{k_i}\,(x_i x_j/c_j; q^3)_{k_i}}{(q^3 x_i/x_j; q^3)_{k_i}} \prod_{i=1}^{r} \frac{(a x_i; q)_{|\mathbf{k}|}\,(aq/x_i; q)_{|\mathbf{k}|-3k_i}}{(a x_i q/c_i; q)_{|\mathbf{k}|}\,(a c_i q/x_i; q)_{|\mathbf{k}|}}$$
$$\times \frac{(1/a^2; q)_{|\mathbf{k}|}\,(a^2 q; q)_{2|\mathbf{k}|}}{\prod_{i=1}^{r} (a^3 x_i q^3; q^3)_{k_i}}\, a^{2|\mathbf{k}|} \prod_{i=1}^{r} x_i^{-2k_i} \cdot q^{6e_2(\mathbf{k}) - \binom{2|\mathbf{k}|+1}{2} + 3\sum_{i=1}^{r} i k_i} \Bigg)$$
$$= \prod_{i=1}^{r} \frac{(a x_i q; q)_\infty\,(x_i/a c_i; q)_\infty\,(x_i/a^3; q^3)_\infty\,(a^3 x_i q^3/c_i; q^3)_\infty}{(a x_i q/c_i; q)_\infty\,(x_i/a; q)_\infty\,(x_i/a^3 c_i; q^3)_\infty\,(a^3 x_i q^3; q^3)_\infty}, \tag{IV.44}$$

provided the series terminates (and where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1, \dots, k_r\}$).

REMARK IV.45. The aforementioned D_r cubic summation theorems are the cases $a^2 = q^N$, with N being an arbitrary integer, of Theorem IV.43, where the right side of (IV.44) simplifies differently, depending on the sign of N and the residue class of N mod 6.

CHAPTER V

C_r and D_r very-well-poised ${}_{10}\phi_9$ transformations

Here we show how one of our main results which we obtained by matrix inversion, the D_r ${}_8\phi_7$ summation, Theorem IV.22, can be combined with other multidimensional results to obtain further, more complicated, identities. This chapter and the following one are part of a joint paper with G. Bhatnagar [14].

The Rogers–Ramanujan identities [1], [2], [3], [5], [6] are perhaps the most celebrated identities involving basic hypergeometric series. G. N. Watson [73] proved these identities by first finding a q-analogue of Whipple's [74] transformation formula, taking suitable limiting cases, and then invoking Jacobi's Triple Product Identity [41], [3]. Watson's proof of his transformation formula was analogous to Whipple's [75] second proof of his identity. He used a polynomial argument and Jackson's [39] q-Pfaff–Saalschütz theorem, a summation theorem for a terminating and balanced ${}_3\phi_2$ series. Bailey [7] observed that Whipple's ideas could be extended further, by using Jackson's [40] q-Dougall summation theorem, which is a summation theorem for a terminating and balanced ${}_8\phi_7$ sum (see Theorem IV.8), and which contains the q-Pfaff–Saalschütz summation as a special case. The result was Bailey's ${}_{10}\phi_9$ transformation formula, which converts a terminating, balanced, and very-well-poised ${}_{10}\phi_9$ series into a multiple of a series of the same kind.

In this chapter, we generalize Bailey's ${}_{10}\phi_9$ transformation formula to multiple series. As ingredients we use A_r, C_r, and D_r extensions of Jackson's [40] ${}_8\phi_7$ summation which are stated in Appendix B, and an interchange of multisums. As special cases of our ${}_{10}\phi_9$ transformations, we find several multivariable generalizations of Watson's [73] transformation of an ${}_8\phi_7$ into a multiple of a ${}_4\phi_3$ (see chapter VI). We also deduce multidimensional extensions of Sears' [69] ${}_4\phi_3$ transformation (which are also given in chapter VI).

Previously, Milne and Newcomb [59] have generalized Bailey's formula to $U(r+1)$ multiple series, or equivalently to A_r series. As special cases, they also recover many A_r generalizations of Watson's transformation found earlier by Milne [54],

[56]. We use similar techniques and find C_r and D_r generalizations of the ${}_{10}\phi_9$ transformation formula. Several different generalizations of Watson's transformation follow from any one multivariable extension of Bailey's formula. Thus we are able to unify all the previously known A_r, C_r, and D_r generalizations of Watson's transformations [12], [52], [54], [56], [58]. In addition, we find 14 new A_r and D_r generalizations of Watson's transformation.

The proofs of our results are generalizations of Bailey's [8], [10] second proof of his transformation formula. Following the distillation of Bailey's ideas by Gasper and Rahman [25], we use a suitable multivariable extension of the elementary interchange of summation argument:

$$\sum_{m=0}^{n}\sum_{k=0}^{m} A(m)\,C(k,m) = \sum_{k=0}^{n}\sum_{m=0}^{n-k} A(m+k)\,C(k,m+k). \tag{V.1}$$

When A and C are chosen so that the inner sums are both summable using Jackson's q-Dougall sum, we obtain the ${}_{10}\phi_9$ transformation formula. Essentially, Milne and Newcomb [59] use the same model in their derivation, except that they follow the organization of Bailey's ideas in Slater's [71] book, rather than the simpler exposition by Gasper and Rahman [25].

Milne and Newcomb [59] used Milne's [52] A_r generalization of Jackson's ${}_8\phi_7$ sum to sum up the inner sum on both sides of a suitable multivariable generalization of (V.1). However, this is not the only multivariable generalization of Jackson's sum. Denis and Gustafson [19] and Milne and Lilly [58] independently have found a C_r extension of Jackson's sum, and more recently, Bhatnagar [12] and the author [67] have found D_r extensions of Jackson's sum. (The author [68] has also found a new C_r extension of Jackson's sum, see Theorem VII.31 in this thesis, however it cannot be used to derive another multiple ${}_{10}\phi_9$ transformation, see Remark VII.33.) To derive our C_r and D_r ${}_{10}\phi_9$ transformations, we have used all possible combinations of these A_r, C_r and D_r summations (which are listed in Appendix B). As one strange consequence of combining C_r and D_r theorems, we find a generalization of Bailey's transformation, which converts a C_r ${}_{10}\phi_9$ series into a multiple of an A_r ${}_{10}\phi_9$ series. Indeed, in [14] and here, while we follow the convention set by previous authors [31], [58], [12] of labelling our formulas as A_r, C_r, or D_r theorems, we do not hesitate to mix the different types of series if necessary.

To motivate our multidimensional work in this chapter we state the classical (one-variable) ${}_{10}\phi_9$ transformation [25, (2.9.1)] and give a proof, basically following the proof displayed in [25, sec. 2.9]. We will extend this analysis in the sequel of this chapter to prove our multivariable ${}_{10}\phi_9$ transformations of type C_r and D_r, respectively.

THEOREM V.2 (BAILEY'S CLASSICAL ${}_{10}\phi_9$ TRANSFORMATION). *Let a, b, c, d, e, and f be indeterminate, let n be a nonnegative integer, and suppose that none of the denominators in* (V.3) *vanish. Then*

$$
\begin{aligned}
&{}_{10}\phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f, \lambda a q^{n+1}/ef, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-n}/\lambda, aq^{n+1} \end{matrix}; q, q\right] \\
&\qquad = \frac{(aq;q)_n\,(aq/ef;q)_n\,(\lambda q/e;q)_n\,(\lambda q/f;q)_n}{(\lambda q/ef;q)_n\,(\lambda q;q)_n\,(aq/f;q)_n\,(aq/e;q)_n} \\
&\qquad \times {}_{10}\phi_9\left[\begin{matrix} \lambda, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda a q^{n+1}/ef, q^{-n} \\ \sqrt{\lambda}, -\sqrt{\lambda}, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-n}/a, \lambda q^{n+1} \end{matrix}; q, q\right], \qquad (V.3)
\end{aligned}
$$

where $\lambda = qa^2/bcd$.

PROOF OF THEOREM V.2. To derive (V.3), we start by writing the sum on the left hand side of (V.1) in the form

$$\sum_{0\le m\le n} A(m)\,B(m), \qquad (V.4)$$

where

$$
\begin{aligned}
A(m) = \frac{(1-aq^{2m})}{(1-a)}\,&\frac{(a;q)_m\,(e;q)_m\,(f;q)_m}{(q;q)_m\,(aq/e;q)_m\,(aq/f;q)_m} \\
&\times \frac{(a\lambda q^{1+n}/ef;q)_m\,(q^{-n};q)_m\,(a/\lambda;q)_m}{(efq^{-n}/\lambda;q)_m\,(aq^{1+n};q)_m\,(\lambda q;q)_m}\,q^m
\end{aligned}
$$

and

$$B(m) = \frac{(b;q)_m\,(c;q)_m\,(d;q)_m\,(\lambda q;q)_m}{(aq/b;q)_m\,(aq/c;q)_m\,(aq/d;q)_m\,(a/\lambda;q)_m}.$$

By Theorem IV.8 we may substitute

$$
\begin{aligned}
\sum_{0\le k\le m} \frac{(1-\lambda q^{2k})}{(1-\lambda)}\,&\frac{(\lambda;q)_k\,(\lambda b/a;q)_k\,(\lambda c/a;q)_k}{(q;q)_k\,(aq/b;q)_k\,(aq/c;q)_k} \\
&\times \frac{(\lambda d/a;q)_k\,(aq^m;q)_k\,(q^{-m};q)_k}{(aq/d;q)_k\,(\lambda q^{1-m}/a;q)_k\,(\lambda q^{1+m};q)_k}\,q^k \qquad (V.5)
\end{aligned}
$$

for $B(m)$ in (V.4). Write $C(k,m)$ for the summand in (V.5). Thus, the left side of (V.3) equals

$$\sum_{0\le m\le n}\ \sum_{0\le k\le m} A(m)\,C(k,m).$$

Now we interchange the summations

$$\sum_{0\le k\le n}\ \sum_{k\le m\le n} A(m)\,C(k,m),$$

and after shifting m to $m+k$ in the inner sum while changing the range of summation we obtain

$$\sum_{0\le k\le n}\ \sum_{0\le m\le n-k} A(m+k)\, C(k,m+k). \tag{V.6}$$

Explicitly, after rearranging terms, (V.6) is

$$\begin{aligned}\sum_{0\le k\le n} &\frac{(1-\lambda q^{2k})}{(1-\lambda)}\frac{(\lambda;q)_k\,(\lambda b/a;q)_k\,(\lambda c/a;q)_k\,(\lambda d/a;q)_k\,(aq^k;q)_k\,(q^{-k};q)_k}{(q;q)_k\,(aq/b;q)_k\,(aq/c;q)_k\,(aq/d;q)_k\,(\lambda q^{1-k}/a;q)_k\,(\lambda q^{1+k};q)_k}\, q^k\\
&\times\frac{(1-aq^{2k})}{(1-a)}\frac{(a;q)_k\,(e;q)_k\,(f;q)_k\,(a/\lambda;q)_k\,(a\lambda q^{1+n}/ef;q)_k\,(q^{-n};q)_k}{(q;q)_k\,(aq/e;q)_k\,(aq/f;q)_k\,(\lambda q;q)_k\,(efq^{-n}/\lambda;q)_k\,(aq^{1+n};q)_k}\, q^k\\
&\qquad\times\sum_{0\le m\le n-k}\frac{(1-aq^{2k+2m})}{(1-aq^{2k})}\frac{(aq^{2k};q)_m\,(eq^k;q)_m\,(fq^k;q)_m}{(q;q)_m\,(aq^{1+k}/e;q)_m\,(aq^{1+k}/f;q)_m}\\
&\qquad\qquad\times\frac{(a/\lambda;q)_m\,(a\lambda q^{1+k+n}/ef;q)_m\,(q^{k-n};q)_m}{(\lambda q^{1+2k};q)_m\,(efq^{k-n}/\lambda;q)_m\,(aq^{1+k+n};q)_m}\, q^m.\end{aligned}$$

Summing the inner sum by means of the $a\mapsto aq^k$, $b\mapsto eq^k$, $c\mapsto fq^k$, $d\mapsto a/\lambda$, and $n\mapsto n-k$ case of Theorem IV.8 and simplifying terms by some elementary manipulations (cf. [25, Appendix I]), we obtain the right side of (V.3). $\square$

REMARK V.7. Note that the choice of λ is motivated by our desire to sum the inner sum of (V.6) using (IV.9). In our choice of $B(m)$, we have to choose one of our products in the numerator as corresponding to the product $(aq;q)_n$ in the product side of (IV.9). So we start by choosing λ to correspond to a. Next we try to keep the other three parameters b, c, and d, as general as possible. Now forcing $B(m)$ to fit the product side of (IV.9), leads us to $\lambda = qa^2/bcd$.

Notice also that the product side of (IV.9) satisfies the well-poised condition. That is,

$$(aq)(aq/bcd) = (aq/bc)(aq/d) = (aq/bd)(aq/c) = (aq/cd)(aq/b) = q^2a^2/bcd.$$

So if any of these products survive when we use (IV.9) on the right hand side of (V.6), the "special parameter" is forced to be qa^2/bcd. This is another justification for the choice of λ.

The choice of $A(m)$ (and of course also of $B(m)$) is motivated by our desire to sum the inner sum by some summation theorem which is as general as possible (i.e. Jackson's sum).

We proceed to higher-dimensional ${}_{10}\phi_9$ transformations. First, we present a C_r generalization of Theorem V.2. We use a straightforward multiple series extension of (V.1), and two multiple series generalizations of Theorem IV.8. These are: The D_r extension of Jackson's sum derived in chapter IV and a C_r summation theorem found independently by Denis and Gustafson [19] and Milne and Lilly [58]. Both results are listed (for completeness and better reference) in the appendix, see Theorem B.12 and Theorem B.6.

Consider the substitutions $a \mapsto qa^2/bcd$, $b \mapsto aq/cd$, $c \mapsto aq/bd$, $d \mapsto aq/bc$ in Theorem V.2. With these substitutions, λ becomes a, and the ${}_{10}\phi_9$ on the right hand side of (V.3) gets transformed into the one on the left, and vice-versa. Once again, we obtain Bailey's formula (V.3). This is quite a striking symmetry of Bailey's transformation. This symmetry holds for quite all the multiple series extensions of (V.3) in [59] and in this chapter. There is only one exception: Remarkably enough, it does not hold in the following generalization of Theorem V.2:

THEOREM V.8 (A C_r ${}_{10}\phi_9$ TRANSFORMATION). *Let $x_1, \dots, x_r$, a, b, c, d, e, and f be indeterminate, let $n_1, \dots, n_r$ be nonnegative integers with $r \geq 1$, and suppose that none of the denominators in* (V.9) *vanish. Then*

$$\sum_{\substack{0 \le k_i \le n_i \\ i=1,2,\dots,r}} \Bigg(\prod_{i=1}^r \left(\frac{1 - a x_i^2 q^{2k_i}}{1 - a x_i^2} \right) \prod_{1 \le i < j \le r} \left(\frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \, \frac{1 - a x_i x_j q^{k_i + k_j}}{1 - a x_i x_j} \right)$$
$$\times \prod_{i,j=1}^r \frac{(q^{-n_j} x_i/x_j; q)_{k_i} \, (a x_i x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i} \, (a x_i x_j q^{1+n_j}; q)_{k_i}}$$
$$\times \prod_{i=1}^r \frac{(b x_i; q)_{k_i} \, (c x_i; q)_{k_i} \, (d x_i; q)_{k_i}}{(a x_i q/b; q)_{k_i} \, (a x_i q/c; q)_{k_i} \, (a x_i q/d; q)_{k_i}}$$
$$\times \prod_{i=1}^r \frac{(e x_i; q)_{k_i} \, (f x_i; q)_{k_i} \, (a \lambda x_i q^{1+|\mathbf{n}|}/ef; q)_{k_i}}{(a x_i q/e; q)_{k_i} \, (a x_i q/f; q)_{k_i} \, (ef x_i q^{-|\mathbf{n}|}/\lambda; q)_{k_i}} \, q^{\sum_{i=1}^r i k_i} \Bigg)$$
$$= \prod_{1 \le i < j \le r} (a x_i x_j q; q)^{-1}_{n_i + n_j} \prod_{i,j=1}^r (a x_i x_j q; q)_{n_i}$$
$$\times \frac{(\lambda q/e; q)_{|\mathbf{n}|} \, (\lambda q/f; q)_{|\mathbf{n}|} \, (aq/ef; q)_{|\mathbf{n}|}}{\prod_{i=1}^r [(\lambda x_i q; q)_{n_i} \, (a x_i q/e; q)_{n_i} \, (a x_i q/f; q)_{n_i} \, (\lambda q^{1+|\mathbf{n}|-n_i}/ef x_i; q)_{n_i}]}$$
$$\times \sum_{\substack{0 \le k_i \le n_i \\ i=1,2,\dots,r}} \Bigg(\prod_{i=1}^r \left(\frac{1 - \lambda x_i q^{k_i + |\mathbf{k}|}}{1 - \lambda x_i} \right) \prod_{1 \le i < j \le r} \left(\frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \right)$$
$$\times \prod_{i,j=1}^r \frac{(q^{-n_j} x_i/x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i}} \prod_{i=1}^r \frac{(\lambda x_i; q)_{|\mathbf{k}|}}{(\lambda x_i q^{1+n_i}; q)_{|\mathbf{k}|}}$$
$$\times \prod_{i=1}^r \frac{(e x_i; q)_{k_i} \, (f x_i; q)_{k_i} \, (a \lambda x_i q^{1+|\mathbf{n}|}/ef; q)_{k_i}}{(a x_i q/b; q)_{k_i} \, (a x_i q/c; q)_{k_i} \, (a x_i q/d; q)_{k_i}}$$

$$\times \frac{(\lambda b/a;q)_{|\mathbf{k}|}\,(\lambda c/a;q)_{|\mathbf{k}|}\,(\lambda d/a;q)_{|\mathbf{k}|}}{(\lambda q/e;q)_{|\mathbf{k}|}\,(\lambda q/f;q)_{|\mathbf{k}|}\,(efq^{-|\mathbf{n}|}/a;q)_{|\mathbf{k}|}}\; q^{\sum_{i=1}^r i k_i}\Bigg), \tag{V.9}$$

where $\lambda = qa^2/bcd$.

REMARK V.10. We call Theorem V.8 a C_r theorem, since a limiting case yields Milne and Lilly's [58] C_r generalization of Watson's transformation. However, note that (V.9) transforms a terminating, balanced, and very-well-poised C_r ${}_{10}\phi_9$ sum into a multiple of a terminating A_r series, which is also balanced and very-well-poised. In view of our earlier remarks, it may also be written so that it transforms an A_r series into a C_r series.

PROOF. The left side of (V.9) can be written in the form

$$\sum_{\substack{0\le m_i\le n_i\\ i=1,2,\ldots,r}} A(\mathbf{m})\,B(\mathbf{m}), \tag{V.11}$$

where

$$A(\mathbf{m}) = \prod_{i=1}^r \left(\frac{1-ax_i^2q^{2m_i}}{1-ax_i^2}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{m_i-m_j}x_i/x_j}{1-x_i/x_j}\;\frac{1-ax_ix_jq^{m_i+m_j}}{1-ax_ix_j}\right)$$
$$\times \prod_{i=1}^r \frac{(ex_i;q)_{m_i}\,(fx_i;q)_{m_i}\,(a\lambda x_iq^{1+|\mathbf{n}|}/ef;q)_{m_i}\,(ax_i/\lambda;q)_{m_i}}{(ax_iq/e;q)_{m_i}\,(ax_iq/f;q)_{m_i}\,(efx_iq^{-|\mathbf{n}|}/\lambda;q)_{m_i}\,(\lambda x_iq;q)_{m_i}}$$
$$\times \prod_{i,j=1}^r \frac{(q^{-n_j}x_i/x_j;q)_{m_i}\,(ax_ix_j;q)_{m_i}}{(qx_i/x_j;q)_{m_i}\,(ax_ix_jq^{1+n_j};q)_{m_i}}\; q^{\sum_{i=1}^r i m_i}$$

and

$$B(\mathbf{m}) = \prod_{i=1}^r \frac{(bx_i;q)_{m_i}\,(cx_i;q)_{m_i}\,(dx_i;q)_{m_i}\,(\lambda x_iq;q)_{m_i}}{(ax_iq/b;q)_{m_i}\,(ax_iq/c;q)_{m_i}\,(ax_iq/d;q)_{m_i}\,(ax_i/\lambda;q)_{m_i}}.$$

By Theorem B.12 we may substitute

$$\sum_{\substack{0\le k_i\le m_i\\ i=1,2,\ldots,r}} \left(\prod_{i=1}^r\left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\right.$$
$$\times \prod_{1\le i<j\le r}(ax_ix_j;q)^{-1}_{k_i+k_j}\prod_{i,j=1}^r (ax_ix_jq^{m_j};q)_{k_i}$$
$$\times \prod_{i,j=1}^r \frac{(q^{-m_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^r \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda q/ax_i;q)_{|\mathbf{k}|-k_i}}{(\lambda x_iq^{1+m_i};q)_{|\mathbf{k}|}\,(\lambda q^{1-m_i}/ax_i;q)_{|\mathbf{k}|}}$$

$$\times \frac{(\lambda b/a;q)_{|\mathbf{k}|}\,(\lambda c/a;q)_{|\mathbf{k}|}\,(\lambda d/a;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}[(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}]}\, q^{\sum_{i=1}^{r} i\,k_i}\Bigg) \tag{V.12}$$

for $B(\mathbf{m})$ in (V.11). Write $C(\mathbf{k},\mathbf{m})$ for the summand in (V.12). Thus, the left side of (V.9) equals

$$\sum_{\substack{0\le m_i\le n_i\\ i=1,\dots,r}}\ \sum_{\substack{0\le k_i\le m_i\\ i=1,\dots,r}} A(\mathbf{m})\,C(\mathbf{k},\mathbf{m}).$$

Now we interchange the summations

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\ \sum_{\substack{k_i\le m_i\le n_i\\ i=1,\dots,r}} A(\mathbf{m})\,C(\mathbf{k},\mathbf{m}),$$

and after shifting each m_i to m_i+k_i in the inner sum while changing the range of summation we obtain

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\ \sum_{\substack{0\le m_i\le n_i-k_i\\ i=1,\dots,r}} A(\mathbf{m}+\mathbf{k})\,C(\mathbf{k},\mathbf{m}+\mathbf{k}). \tag{V.13}$$

Explicitly, after rearranging terms, (V.13) is

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\dots,r}} \Bigg(\prod_{i=1}^{r}\left(\frac{1-\lambda x_i q^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$\times \prod_{1\le i<j\le r}(ax_ix_j;q)^{-1}_{k_i+k_j} \prod_{i,j=1}^{r}(ax_ix_jq^{k_j};q)_{k_i}$$

$$\times \prod_{i,j=1}^{r}\frac{(q^{-k_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r}\frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda q/ax_i;q)_{|\mathbf{k}|-k_i}}{(\lambda x_iq^{1+k_i};q)_{|\mathbf{k}|}\,(\lambda q^{1-k_i}/ax_i;q)_{|\mathbf{k}|}}$$

$$\times \frac{(\lambda b/a;q)_{|\mathbf{k}|}\,(\lambda c/a;q)_{|\mathbf{k}|}\,(\lambda d/a;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}[(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}]}\, q^{\sum_{i=1}^{r} i\,k_i}$$

$$\times \prod_{i=1}^{r}\left(\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\ \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right)$$

$$\times \prod_{i=1}^{r}\frac{(ex_i;q)_{k_i}\,(fx_i;q)_{k_i}\,(a\lambda x_iq^{1+|\mathbf{n}|}/ef;q)_{k_i}\,(ax_i/\lambda;q)_{k_i}}{(ax_iq/e;q)_{k_i}\,(ax_iq/f;q)_{k_i}\,(efx_iq^{-|\mathbf{n}|}/\lambda;q)_{k_i}\,(\lambda x_iq;q)_{k_i}}$$

$$\times \prod_{i,j=1}^{r}\frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(ax_ix_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq^{1+n_j};q)_{k_i}}\, q^{\sum_{i=1}^{r} i\,k_i}$$

$$\times \sum_{\substack{0\le m_i\le n_i-k_i\\ i=1,2,\dots,r}} \prod_{1\le i<j\le r}\left(\frac{1-q^{m_i-m_j+k_i-k_j}x_i/x_j}{1-q^{k_i-k_j}x_i/x_j}\,\frac{1-ax_ix_jq^{k_i+k_j+m_i+m_j}}{1-ax_ix_jq^{k_i+k_j}}\right)$$
$$\times\prod_{i=1}^r\left(\frac{1-ax_i^2q^{2k_i+2m_i}}{1-ax_i^2q^{2k_i}}\right)\prod_{i=1}^r\frac{(ex_iq^{k_i};q)_{m_i}\,(fx_iq^{k_i};q)_{m_i}}{(ax_iq^{1+k_i}/e;q)_{m_i}\,(ax_iq^{1+k_i}/f;q)_{m_i}}$$
$$\times\prod_{i=1}^r\frac{(a\lambda x_iq^{1+k_i+|\mathbf{n}|}/ef;q)_{m_i}\,(ax_iq^{k_i-|\mathbf{k}|}/\lambda;q)_{m_i}}{(efx_iq^{k_i-|\mathbf{n}|}/\lambda;q)_{m_i}\,(\lambda x_iq^{1+k_i+|\mathbf{k}|};q)_{m_i}}$$
$$\left.\times\prod_{i,j=1}^r\frac{(q^{k_i-n_j}x_i/x_j;q)_{m_i}\,(ax_ix_jq^{k_i+k_j};q)_{m_i}}{(q^{1+k_i-k_j}x_i/x_j;q)_{m_i}\,(ax_ix_jq^{1+k_i+n_j};q)_{m_i}}\,q^{\sum_{i=1}^r im_i}\right).$$

Summing the inner sum by means of the

$$b\mapsto e,\qquad c\mapsto f,\qquad d\mapsto aq^{-|\mathbf{k}|}/\lambda,$$
$$x_i\mapsto x_iq^{k_i},\quad n_i\mapsto n_i-k_i,\quad \text{for } i=1,2,\dots,r$$

case of Theorem B.6 and simplifying terms by the $l_1=l_2=\cdots=l_r=0$ case of Lemma B.1 and some elementary manipulations (cf. [25, Appendix I]), we obtain the right side of (V.9). □

REMARK V.14. An alternative way to derive Theorem V.8 is to use Denis and Gustafson's, respectively Milne and Lilly's, C_r ${}_8\phi_7$ summation, Theorem B.6, in the inner sum, and then employ Bhatnagar's D_r ${}_8\phi_7$ summation, Theorem B.9, after exchanging sums. Then we would arrive at identity V.9 too, but with the right hand side on the left and vice-versa.

We will discuss some special cases of our C_r ${}_{10}\phi_9$ transformation in chapter VI.

Some more multiple ${}_{10}\phi_9$ transformations can be derived by the above method. If we combine both D_r ${}_8\phi_7$ summations, Theorem B.9 and Theorem B.12 we obtain the following D_r ${}_{10}\phi_9$ transformation formula.

THEOREM V.15 (A D_r ${}_{10}\phi_9$ TRANSFORMATION). *Let $x_1,\dots,x_r$, a, b, c, d, e, and f be indeterminate, let $n_1,\dots,n_r$ be nonnegative integers with $r\ge 1$, and suppose that none of the denominators in* (V.16) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\dots,r}}\left(\prod_{i=1}^r\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\right.$$
$$\times\prod_{1\le i<j\le r}\frac{(ax_ix_jq/d;q)_{k_i+k_j}}{(\lambda ax_ix_jq/ef;q)_{k_i+k_j}}\prod_{i=1}^r\frac{(ef/\lambda x_i;q)_{|\mathbf{k}|-k_i}}{(d/x_i;q)_{|\mathbf{k}|-k_i}}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i/x_j;q)_{k_i}\,(\lambda a x_i x_j q^{1+n_j}/ef;q)_{k_i}}{(q x_i/x_j;q)_{k_i}\,(a x_i x_j q/d;q)_{k_i}}$$
$$\times \prod_{i=1}^{r} \frac{(a x_i;q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|}}{(a x_i q^{1+n_i};q)_{|\mathbf{k}|}\,(ef q^{-n_i}/\lambda x_i;q)_{|\mathbf{k}|}}$$
$$\times \prod_{i=1}^{r} \frac{(b x_i;q)_{k_i}\,(c x_i;q)_{k_i}}{(a x_i q/e;q)_{k_i}\,(a x_i q/f;q)_{k_i}}$$
$$\left.\times \frac{(e;q)_{|\mathbf{k}|}\,(f;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}}\; q^{\sum_{i=1}^{r} i k_i}\right)$$
$$= \prod_{i=1}^{r} \frac{(a x_i q;q)_{n_i}\,(a x_i q/ef;q)_{n_i}\,(\lambda x_i q/e;q)_{n_i}\,(\lambda x_i q/f;q)_{n_i}}{(\lambda x_i q/ef;q)_{n_i}\,(\lambda x_i q;q)_{n_i}\,(a x_i q/f;q)_{n_i}\,(a x_i q/e;q)_{n_i}}$$
$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}} \left(\prod_{i=1}^{r} \left(\frac{1-\lambda x_i q^{k_i+|\mathbf{k}|}}{1-\lambda x_i} \right) \prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j} x_i/x_j}{1-x_i/x_j} \right) \right.$$
$$\times \prod_{1\le i<j\le r} \frac{(a x_i x_j q/d;q)_{k_i+k_j}}{(\lambda a x_i x_j q/ef;q)_{k_i+k_j}} \prod_{i=1}^{r} \frac{(ef/a x_i;q)_{|\mathbf{k}|-k_i}}{(\lambda d/a x_i;q)_{|\mathbf{k}|-k_i}}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i/x_j;q)_{k_i}\,(\lambda a x_i x_j q^{1+n_j}/ef;q)_{k_i}}{(q x_i/x_j;q)_{k_i}\,(a x_i x_j q/d;q)_{k_i}}$$
$$\times \prod_{i=1}^{r} \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda d/a x_i;q)_{|\mathbf{k}|}}{(\lambda x_i q^{1+n_i};q)_{|\mathbf{k}|}\,(ef q^{-n_i}/a x_i;q)_{|\mathbf{k}|}}$$
$$\times \prod_{i=1}^{r} \frac{(\lambda b x_i/a;q)_{k_i}\,(\lambda c x_i/a;q)_{k_i}}{(\lambda x_i q/e;q)_{k_i}\,(\lambda x_i q/f;q)_{k_i}}$$
$$\left.\times \frac{(e;q)_{|\mathbf{k}|}\,(f;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}}\; q^{\sum_{i=1}^{r} i k_i}\right), \qquad \text{(V.16)}$$

where $\lambda = qa^2/bcd$.

Proof. The left side of (V.16) can be written in the form (V.11) where

$$A(\mathbf{m}) = \prod_{i=1}^{r} \left(\frac{1-a x_i q^{m_i+|\mathbf{m}|}}{1-a x_i} \right) \prod_{1\le i<j\le r} \left(\frac{1-q^{m_i-m_j} x_i/x_j}{1-x_i/x_j} \right)$$
$$\times \prod_{1\le i<j\le r} (\lambda a x_i x_j q/ef;q)^{-1}_{m_i+m_j} \prod_{i,j=1}^{r} (\lambda a x_i x_j q^{1+n_j}/ef;q)_{m_i}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i/x_j;q)_{m_i}}{(q x_i/x_j;q)_{m_i}} \prod_{i=1}^{r} \frac{(a x_i;q)_{|\mathbf{m}|}\,(ef/\lambda x_i;q)_{|\mathbf{m}|-m_i}}{(a x_i q^{1+n_i};q)_{|\mathbf{m}|}\,(ef q^{-n_i}/\lambda x_i;q)_{|\mathbf{m}|}}$$

$$\times \frac{(e;q)_{|\mathbf{m}|}\,(f;q)_{|\mathbf{m}|}\,(a/\lambda;q)_{|\mathbf{m}|}}{\prod_{i=1}^{r}\left[(ax_iq/e;q)_{m_i}\,(ax_iq/f;q)_{m_i}\,(\lambda x_iq;q)_{m_i}\right]}\, q^{\sum_{i=1}^{r} i\,m_i}$$

and

$$B(\mathbf{m}) = \prod_{1\le i<j\le r}(ax_ix_jq/d;q)_{m_i+m_j}\prod_{i,j=1}^{r}(ax_ix_jq/d;q)_{m_i}^{-1}$$
$$\times \frac{\prod_{i=1}^{r}\left[(bx_i;q)_{m_i}\,(cx_i;q)_{m_i}\,(dq^{|\mathbf{m}|-m_i}/x_i;q)_{m_i}\,(\lambda x_iq;q)_{m_i}\right]}{(aq/b;q)_{|\mathbf{m}|}\,(aq/c;q)_{|\mathbf{m}|}\,(a/\lambda;q)_{|\mathbf{m}|}}.$$

By Theorem B.9 we may substitute

$$\sum_{\substack{0\le k_i\le m_i\\ i=1,2,\ldots,r}}\Bigg(\prod_{i=1}^{r}\left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times\prod_{1\le i<j\le r}(ax_ix_jq/d;q)_{k_i+k_j}\prod_{i,j=1}^{r}(ax_ix_jq/d;q)_{k_i}^{-1}$$
$$\times\prod_{i,j=1}^{r}\frac{(q^{-m_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{i=1}^{r}\frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|}}{(\lambda x_iq^{1+m_i};q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|-k_i}}$$
$$\times\frac{\prod_{i=1}^{r}\left[(\lambda bx_i/a;q)_{k_i}\,(\lambda cx_i/a;q)_{k_i}\,(ax_iq^{|\mathbf{m}|};q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(\lambda q^{1-|\mathbf{m}|}/a;q)_{|\mathbf{k}|}}\,q^{\sum_{i=1}^{r} i\,k_i}\Bigg) \tag{V.17}$$

for $B(\mathbf{m})$ in (V.11). Write $C(\mathbf{k},\mathbf{m})$ for the summand in (V.17). As in the proof of Theorem V.8, the left side of (V.16) equals

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\ldots,r}}\ \sum_{\substack{0\le m_i\le n_i-k_i\\ i=1,\ldots,r}} A(\mathbf{m}+\mathbf{k})\,C(\mathbf{k},\mathbf{m}+\mathbf{k}). \tag{V.18}$$

Explicitly, after rearranging terms, (V.18) is

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}}\Bigg(\prod_{i=1}^{r}\left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times\prod_{1\le i<j\le r}(ax_ix_jq/d;q)_{k_i+k_j}\prod_{i,j=1}^{r}(ax_ix_jq/d;q)_{k_i}^{-1}$$
$$\times\prod_{i,j=1}^{r}\frac{(q^{-k_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{i=1}^{r}\frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|}}{(\lambda x_iq^{1+k_i};q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|-k_i}}$$

$$\times \frac{\prod_{i=1}^{r}\left[(\lambda b x_i/a;q)_{k_i}\,(\lambda c x_i/a;q)_{k_i}\,(a x_i q^{|\mathbf{k}|};q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(\lambda q^{1-|\mathbf{k}|}/a;q)_{|\mathbf{k}|}}\; q^{\sum_{i=1}^{r} i k_i}$$
$$\times \prod_{i=1}^{r}\left(\frac{1-a x_i q^{k_i+|\mathbf{k}|}}{1-a x_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times \prod_{1\le i<j\le r}(\lambda a x_i x_j q/ef;q)^{-1}_{k_i+k_j} \prod_{i,j=1}^{r}(\lambda a x_i x_j q^{1+n_j}/ef;q)_{k_i}$$
$$\times \prod_{i,j=1}^{r}\frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(q x_i/x_j;q)_{k_i}} \prod_{i=1}^{r}\frac{(a x_i;q)_{|\mathbf{k}|}\,(ef/\lambda x_i;q)_{|\mathbf{k}|-k_i}}{(a x_i q^{1+n_i};q)_{|\mathbf{k}|}\,(ef q^{-n_i}/\lambda x_i;q)_{|\mathbf{k}|}}$$
$$\times \frac{(e;q)_{|\mathbf{k}|}\,(f;q)_{|\mathbf{k}|}\,(a/\lambda;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}\left[(a x_i q/e;q)_{k_i}\,(a x_i q/f;q)_{k_i}\,(\lambda x_i q;q)_{k_i}\right]}\; q^{\sum_{i=1}^{r} i k_i}$$
$$\times \sum_{\substack{0\le m_i\le n_i-k_i\\ i=1,2,\dots,r}} \prod_{i=1}^{r}\left(\frac{1-a x_i q^{k_i+|\mathbf{k}|+m_i+|\mathbf{m}|}}{1-a x_i q^{k_i+|\mathbf{k}|}}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{m_i-m_j+k_i-k_j}x_i/x_j}{1-q^{k_i-k_j}x_i/x_j}\right)$$
$$\times \prod_{1\le i<j\le r}(\lambda a x_i x_j q^{1+k_i+k_j}/ef;q)^{-1}_{m_i+m_j} \prod_{i,j=1}^{r}(\lambda a x_i x_j q^{1+k_i+n_j}/ef;q)_{m_i}$$
$$\times \prod_{i,j=1}^{r}\frac{(q^{k_i-n_j}x_i/x_j;q)_{m_i}}{(q^{1+k_i-k_j}x_i/x_j;q)_{m_i}} \prod_{i=1}^{r}\frac{(a x_i q^{k_i+|\mathbf{k}|};q)_{|\mathbf{m}|}\,(ef q^{|\mathbf{k}|-k_i}/\lambda x_i;q)_{|\mathbf{m}|-m_i}}{(a x_i q^{1+|\mathbf{k}|+n_i};q)_{|\mathbf{m}|}\,(ef q^{|\mathbf{k}|-n_i}/\lambda x_i;q)_{|\mathbf{m}|}}$$
$$\left.\times \frac{(e q^{|\mathbf{k}|};q)_{|\mathbf{m}|}\,(f q^{|\mathbf{k}|};q)_{|\mathbf{m}|}\,(a/\lambda;q)_{|\mathbf{m}|}}{\prod_{i=1}^{r}\left[(a x_i q^{1+k_i}/e;q)_{m_i}\,(a x_i q^{1+k_i}/f;q)_{m_i}\,(\lambda x_i q^{1+k_i+|\mathbf{k}|};q)_{m_i}\right]}\; q^{\sum_{i=1}^{r} i m_i}\right).$$

Summing the inner sum by means of the

$$a\mapsto a q^{|\mathbf{k}|},\quad b\mapsto e q^{|\mathbf{k}|},\quad c\mapsto f q^{|\mathbf{k}|},\quad d\mapsto a/\lambda,$$
$$x_i\mapsto x_i q^{k_i},\quad n_i\mapsto n_i-k_i,\quad \text{for } i=1,2,\dots,r$$

case of Theorem B.12 and simplifying terms by the $l_1=l_2=\cdots=l_r=0$ case of Lemma B.1 and some elementary manipulations, we obtain the right side of (V.16). □

REMARK V.19. Some special limiting cases of Theorem V.15 give multiple Watson transformations. The various possibilities of specializing for Watson's will be analyzed in chapter VI. This remark holds for all our multiple ${}_{10}\phi_9$ transformations.

By using a polynomial argument we easily obtain

THEOREM V.20 (A D_r ${}_{10}\phi_9$ TRANSFORMATION). *Let $x_1,\ldots,x_r$, a, b, c, d, e, and $f_1\ldots,f_r$ be indeterminate, let N be a nonnegative integer, let $r\geq 1$, and suppose that none of the denominators in* (V.21) *vanish. Then*

$$
\begin{aligned}
&\sum_{\substack{k_1,k_2,\ldots,k_r\geq 0\\ 0\leq|\mathbf{k}|\leq N}}\Bigg(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\leq i<j\leq r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\\
&\qquad\times\prod_{1\leq i<j\leq r}\frac{(ax_ix_jq/d;q)_{k_i+k_j}}{(\lambda ax_ix_jq^{1+N}/e;q)_{k_i+k_j}}\prod_{i=1}^{r}\frac{(eq^{-N}/\lambda x_i;q)_{|\mathbf{k}|-k_i}}{(d/x_i;q)_{|\mathbf{k}|-k_i}}\\
&\qquad\times\prod_{i,j=1}^{r}\frac{(f_jx_i/x_j;q)_{k_i}\,(\lambda ax_ix_jq^{1+N}/ef_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/d;q)_{k_i}}\\
&\qquad\times\prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|}}{(ax_iq/f_i;q)_{|\mathbf{k}|}\,(ef_iq^{-N}/\lambda x_i;q)_{|\mathbf{k}|}}\\
&\qquad\times\prod_{i=1}^{r}\frac{(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}}{(ax_iq/e;q)_{k_i}\,(ax_iq^{1+N};q)_{k_i}}\\
&\qquad\times\frac{(e;q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}}\,q^{\sum_{i=1}^{r}i\,k_i}\Bigg)\\
&=\prod_{i=1}^{r}\frac{(ax_iq;q)_N\,(ax_iq/ef_i;q)_N\,(\lambda x_iq/e;q)_N\,(\lambda x_iq/f_i;q)_N}{(\lambda x_iq/ef_i;q)_N\,(\lambda x_iq;q)_N\,(ax_iq/f_i;q)_N\,(ax_iq/e;q)_N}\\
&\times\sum_{\substack{k_1,k_2,\ldots,k_r\geq 0\\ 0\leq|\mathbf{k}|\leq N}}\Bigg(\prod_{i=1}^{r}\left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right)\prod_{1\leq i<j\leq r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\\
&\qquad\times\prod_{1\leq i<j\leq r}\frac{(ax_ix_jq/d;q)_{k_i+k_j}}{(\lambda ax_ix_jq^{1+N}/e;q)_{k_i+k_j}}\prod_{i=1}^{r}\frac{(eq^{-N}/ax_i;q)_{|\mathbf{k}|-k_i}}{(\lambda d/ax_i;q)_{|\mathbf{k}|-k_i}}\\
&\qquad\times\prod_{i,j=1}^{r}\frac{(f_jx_i/x_j;q)_{k_i}\,(\lambda ax_ix_jq^{1+N}/ef_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/d;q)_{k_i}}\\
&\qquad\times\prod_{i=1}^{r}\frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|}}{(\lambda x_iq/f_i;q)_{|\mathbf{k}|}\,(ef_iq^{-N}/ax_i;q)_{|\mathbf{k}|}}\\
&\qquad\times\prod_{i=1}^{r}\frac{(\lambda bx_i/a;q)_{k_i}\,(\lambda cx_i/a;q)_{k_i}}{(\lambda x_iq/e;q)_{k_i}\,(\lambda x_iq^{1+N};q)_{k_i}}\\
&\qquad\qquad\times\frac{(e;q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}}\,q^{\sum_{i=1}^{r}i\,k_i}\Bigg),\qquad\text{(V.21)}
\end{aligned}
$$

where $\lambda = qa^2/bcd$.

PROOF. First we write the product in front of the sum of the right side of (V.21) as quotient of infinite products using (IV.3). Then by the $f = q^{-N}$ case of Theorem V.15 it follows that the identity (V.21) holds for $f_j = q^{-n_j}$, $j = 1, \ldots, r$. By clearing out denominators in (V.21), we get a polynomial equation in f_1, which is true for q^{-n_1}, $n_1 = 0, 1, \ldots$. Thus we obtain an identity in f_1. By carrying out this process for $f_2, f_3, \ldots, f_r$ also, we obtain Theorem V.20. $\square$

Instead of using Theorem B.12 in the derivation of Theorem V.15, we may also use Milne's A_r ${}_8\phi_7$ summation, Theorem B.3, to get another ${}_{10}\phi_9$ transformation.

THEOREM V.22 (A D_r ${}_{10}\phi_9$ TRANSFORMATION). *Let $x_1, \ldots, x_r$, a, b, c, d, e, and f be indeterminate, let $n_1, \ldots, n_r$ be nonnegative integers with $r \geq 1$, and suppose that none of the denominators in* (V.23) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}} \left(\prod_{i=1}^{r} \left(\frac{1 - a x_i q^{k_i+|\mathbf{k}|}}{1 - a x_i} \right) \prod_{1\le i<j\le r} \left(\frac{1 - q^{k_i-k_j} x_i/x_j}{1 - x_i/x_j} \right) \right.$$
$$\times \prod_{1\le i<j\le r} (a x_i x_j q/d; q)_{k_i+k_j} \prod_{i,j=1}^{r} (a x_i x_j q/d; q)_{k_i}^{-1}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i/x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i}} \prod_{i=1}^{r} \frac{(a x_i; q)_{|\mathbf{k}|}\, (d/x_i; q)_{|\mathbf{k}|}}{(a x_i q^{1+n_i}; q)_{|\mathbf{k}|}\, (d/x_i; q)_{|\mathbf{k}|-k_i}}$$
$$\times \frac{\prod_{i=1}^{r} \left[(b x_i; q)_{k_i} (c x_i; q)_{k_i} (e x_i; q)_{k_i} (\lambda a x_i q^{1+|\mathbf{n}|}/ef; q)_{k_i} \right]}{(aq/b; q)_{|\mathbf{k}|} (aq/c; q)_{|\mathbf{k}|} (aq/e; q)_{|\mathbf{k}|} (ef q^{-|\mathbf{n}|}/\lambda; q)_{|\mathbf{k}|}}$$
$$\left. \times \frac{(f; q)_{|\mathbf{k}|}}{\prod_{i=1}^{r} (a x_i q/f; q)_{k_i}}\, q^{\sum_{i=1}^{r} i k_i} \right)$$
$$= \frac{(aq/ef; q)_{|\mathbf{n}|} (\lambda q/e; q)_{|\mathbf{n}|}}{(\lambda q/ef; q)_{|\mathbf{n}|} (aq/e; q)_{|\mathbf{n}|}} \prod_{i=1}^{r} \frac{(a x_i q; q)_{n_i} (\lambda x_i q/f; q)_{n_i}}{(\lambda x_i q; q)_{n_i} (a x_i q/f; q)_{n_i}}$$
$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}} \left(\prod_{i=1}^{r} \left(\frac{1 - \lambda x_i q^{k_i+|\mathbf{k}|}}{1 - \lambda x_i} \right) \prod_{1\le i<j\le r} \left(\frac{1 - q^{k_i-k_j} x_i/x_j}{1 - x_i/x_j} \right) \right.$$
$$\times \prod_{1\le i<j\le r} (a x_i x_j q/d; q)_{k_i+k_j} \prod_{i,j=1}^{r} (a x_i x_j q/d; q)_{k_i}^{-1}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i/x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i}} \prod_{i=1}^{r} \frac{(\lambda x_i; q)_{|\mathbf{k}|}\, (\lambda d/a x_i; q)_{|\mathbf{k}|}}{(\lambda x_i q^{1+n_i}; q)_{|\mathbf{k}|}\, (\lambda d/a x_i; q)_{|\mathbf{k}|-k_i}}$$

$$\times \frac{\prod_{i=1}^{r} \left[(\lambda b x_i/a;q)_{k_i}\,(\lambda c x_i/a;q)_{k_i}\,(e x_i;q)_{k_i}\,(\lambda a x_i q^{1+|\mathbf{n}|}/ef;q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(\lambda q/e;q)_{|\mathbf{k}|}\,(efq^{-|\mathbf{n}|}/a;q)_{|\mathbf{k}|}} \times \frac{(f;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}(\lambda x_i q/f;q)_{k_i}}\, q^{\sum_{i=1}^{r} i\,k_i} \Bigg), \quad \text{(V.23)}$$

where $\lambda = qa^2/bcd$.

PROOF. The left side of (V.23) can be written in the form (V.11) where

$$A(\mathbf{m}) = \prod_{i=1}^{r}\left(\frac{1-ax_iq^{m_i+|\mathbf{m}|}}{1-ax_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{m_i-m_j}x_i/x_j}{1-x_i/x_j}\right) \times \prod_{i,j=1}^{r}\frac{(q^{-n_j}x_i/x_j;q)_{m_i}}{(qx_i/x_j;q)_{m_i}} \prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{m}|}}{(ax_iq^{1+n_i};q)_{|\mathbf{m}|}} \times \prod_{i=1}^{r}\frac{(ex_i;q)_{m_i}\,(\lambda a x_i q^{1+|\mathbf{n}|}/ef;q)_{m_i}}{(ax_iq/f;q)_{m_i}\,(\lambda x_i q;q)_{m_i}} \times \frac{(f;q)_{|\mathbf{m}|}\,(a/\lambda;q)_{|\mathbf{m}|}}{(aq/e;q)_{|\mathbf{m}|}\,(efq^{-|\mathbf{n}|}/\lambda;q)_{|\mathbf{m}|}}\, q^{\sum_{i=1}^{r} i\,m_i}$$

and

$$B(\mathbf{m}) = \prod_{1\le i<j\le r}(ax_ix_jq/d;q)_{m_i+m_j} \prod_{i,j=1}^{r}(ax_ix_jq/d;q)_{m_i}^{-1} \times \frac{\prod_{i=1}^{r}\left[(bx_i;q)_{m_i}\,(cx_i;q)_{m_i}\,(dq^{|\mathbf{m}|-m_i}/x_i;q)_{m_i}\,(\lambda x_i q;q)_{m_i}\right]}{(aq/b;q)_{|\mathbf{m}|}\,(aq/c;q)_{|\mathbf{m}|}\,(a/\lambda;q)_{|\mathbf{m}|}}.$$

By Theorem B.9 we may substitute (V.17) for $B(\mathbf{m})$ in (V.11). Write $C(\mathbf{k},\mathbf{m})$ for the summand in (V.17). As in the proofs of Theorems V.8 and V.15, the left side of (V.23) equals

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \sum_{\substack{0\le m_i\le n_i-k_i\\ i=1,\dots,r}} A(\mathbf{m}+\mathbf{k})\; C(\mathbf{k},\mathbf{m}+\mathbf{k}).$$

Explicitly, after rearranging terms, this is

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\dots,r}} \Bigg(\prod_{i=1}^{r}\left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$\times \prod_{1\le i<j\le r} (ax_ix_jq/d;q)_{k_i+k_j} \prod_{i,j=1}^{r} (ax_ix_jq/d;q)_{k_i}^{-1}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-k_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|}}{(\lambda x_iq^{1+k_i};q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|-k_i}}$$
$$\times \frac{\prod_{i=1}^{r}\Big[(\lambda bx_i/a;q)_{k_i}\,(\lambda cx_i/a;q)_{k_i}\,(ax_iq^{|\mathbf{k}|};q)_{k_i}\Big]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(\lambda q^{1-|\mathbf{k}|}/a;q)_{|\mathbf{k}|}}\; q^{\sum_{i=1}^{r} ik_i}$$
$$\times \prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+n_i};q)_{|\mathbf{k}|}}$$
$$\times \prod_{i=1}^{r} \frac{(ex_i;q)_{k_i}\,(\lambda ax_iq^{1+|\mathbf{n}|}/ef;q)_{k_i}}{(ax_iq/f;q)_{k_i}\,(\lambda x_iq;q)_{k_i}}$$
$$\times \frac{(f;q)_{|\mathbf{k}|}\,(a/\lambda;q)_{|\mathbf{k}|}}{(aq/e;q)_{|\mathbf{k}|}\,(efq^{-|\mathbf{n}|}/\lambda;q)_{|\mathbf{k}|}}\; q^{\sum_{i=1}^{r} ik_i}$$
$$\times \sum_{\substack{0\le m_i\le n_i-k_i\\ i=1,2,\ldots,r}} \prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|+m_i+|\mathbf{m}|}}{1-ax_iq^{k_i+|\mathbf{k}|}}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{m_i-m_j+k_i-k_j}x_i/x_j}{1-q^{k_i-k_j}x_i/x_j}\right.$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{k_i-n_j}x_i/x_j;q)_{m_i}}{(q^{1+k_i-k_j}x_i/x_j;q)_{m_i}} \prod_{i=1}^{r} \frac{(ax_iq^{k_i+|\mathbf{k}|};q)_{|\mathbf{m}|}}{(ax_iq^{1+|\mathbf{k}|+n_i};q)_{|\mathbf{m}|}}$$
$$\times \prod_{i=1}^{r} \frac{(ex_iq^{k_i};q)_{m_i}\,(\lambda ax_iq^{1+k_i+|\mathbf{n}|}/ef;q)_{m_i}}{(ax_iq^{1+k_i}/f;q)_{m_i}\,(\lambda x_iq^{1+k_i+|\mathbf{k}|};q)_{m_i}}$$
$$\left.\times \frac{(fq^{|\mathbf{k}|};q)_{|\mathbf{m}|}\,(a/\lambda;q)_{|\mathbf{m}|}}{(aq^{1+|\mathbf{k}|}/e;q)_{|\mathbf{m}|}\,(efq^{|\mathbf{k}|-|\mathbf{n}|}/\lambda;q)_{|\mathbf{m}|}}\; q^{\sum_{i=1}^{r} im_i}\right).$$

Summing the inner sum by means of the

$$a\mapsto aq^{|\mathbf{k}|},\quad b\mapsto fq^{|\mathbf{k}|},\quad c\mapsto a/\lambda,\quad d\mapsto e,$$
$$x_i\mapsto x_iq^{k_i},\quad n_i\mapsto n_i-k_i,\quad \text{for } i=1,2,\ldots,r$$

case of Theorem B.3 and simplifying terms by the $l_1=l_2=\cdots=l_r=0$ case of Lemma B.1 and some elementary manipulations, we obtain the right side of (V.23). ☐

By using a polynomial argument we obtain

THEOREM V.24 (A D_r ${}_{10}\phi_9$ TRANSFORMATION). *Let $x_1, \ldots, x_r$, a, b, c, d, e, and $f_1 \ldots, f_r$ be indeterminate, let N be a nonnegative integer, let $r \geq 1$, and suppose that none of the denominators in* (V.25) *vanish. Then*

$$
\begin{aligned}
&\sum_{\substack{k_1,k_2,\ldots,k_r\geq 0\\ 0\leq|\mathbf{k}|\leq N}} \Bigg(\prod_{i=1}^{r} \left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\leq i<j\leq r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \\
&\qquad\qquad \times \prod_{1\leq i<j\leq r} (ax_ix_jq/d;q)_{k_i+k_j} \prod_{i,j=1}^{r} (ax_ix_jq/d;q)_{k_i}^{-1} \\
&\qquad\qquad \times \prod_{i,j=1}^{r} \frac{(f_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|}}{(ax_iq/f_i;q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|-k_i}} \\
&\qquad\quad \times \frac{\prod_{i=1}^{r}\Big[(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}\,(ex_i;q)_{k_i}\,(\lambda ax_iq^{1+N}/ef_1\cdots f_r;q)_{k_i}\Big]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(aq/e;q)_{|\mathbf{k}|}\,(ef_1\cdots f_rq^{-N}/\lambda;q)_{|\mathbf{k}|}} \\
&\qquad\qquad\qquad\qquad \times \frac{(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}(ax_iq^{1+N};q)_{k_i}}\, q^{\sum_{i=1}^{r} i\,k_i} \Bigg) \\
&\quad = \frac{(aq/ef_1\cdots f_r;q)_N\,(\lambda q/e;q)_N}{(\lambda q/ef_1\cdots f_r;q)_N\,(aq/e;q)_N} \prod_{i=1}^{r} \frac{(ax_iq;q)_N\,(\lambda x_iq/f_i;q)_N}{(\lambda x_iq;q)_N\,(ax_iq/f_i;q)_N} \\
&\quad \times \sum_{\substack{k_1,k_2,\ldots,k_r\geq 0\\ 0\leq|\mathbf{k}|\leq N}} \Bigg(\prod_{i=1}^{r} \left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right) \prod_{1\leq i<j\leq r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \\
&\qquad\qquad \times \prod_{1\leq i<j\leq r} (ax_ix_jq/d;q)_{k_i+k_j} \prod_{i,j=1}^{r} (ax_ix_jq/d;q)_{k_i}^{-1} \\
&\qquad\qquad \times \prod_{i,j=1}^{r} \frac{(f_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|}}{(\lambda x_iq/f_i;q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|-k_i}} \\
&\qquad \times \frac{\prod_{i=1}^{r}\Big[(\lambda bx_i/a;q)_{k_i}\,(\lambda cx_i/a;q)_{k_i}\,(ex_i;q)_{k_i}\,(\lambda ax_iq^{1+N}/ef_1\cdots f_r;q)_{k_i}\Big]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(\lambda q/e;q)_{|\mathbf{k}|}\,(ef_1\cdots f_rq^{-N}/a;q)_{|\mathbf{k}|}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \frac{(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}(\lambda x_iq^{1+N};q)_{k_i}}\, q^{\sum_{i=1}^{r} i\,k_i} \Bigg), \qquad \text{(V.25)}
\end{aligned}
$$

where $\lambda = qa^2/bcd$.

PROOF. First we write the product in front of the sum of the right side of

(V.25) as quotient of infinite products using (IV.3). Then by the $f = q^{-N}$ case of Theorem V.22 it follows that the identity (V.25) holds for $f_j = q^{-n_j}$, $j = 1, \dots, r$. By clearing out denominators in (V.25), we get a polynomial equation in f_1, which is true for q^{-n_1}, $n_1 = 0, 1, \dots$. Thus we obtain an identity in f_1. By carrying out this process for $f_2, f_3, \dots, f_r$ also, we obtain Theorem V.24. $\square$

By reversing sums in Theorem V.22 we get

THEOREM V.26 (A D_r ${}_{10}\phi_9$ TRANSFORMATION). *Let $x_1, \dots, x_r$, a, b, c, d, e, and f be indeterminate, let $n_1, \dots, n_r$ be nonnegative integers with $r \geq 1$, and suppose that none of the denominators in* (V.27) *vanish. Then*

$$\sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\dots,r}} \Bigg(\prod_{i=1}^{r} \left(\frac{1 - a x_i q^{k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right)$$
$$\times \prod_{1 \leq i < j \leq r} (\lambda a x_i x_j q / ef; q)^{-1}_{k_i + k_j} \prod_{i,j=1}^{r} (\lambda a x_i x_j q^{1+n_j} / ef; q)_{k_i}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i / x_j; q)_{k_i}}{(q x_i / x_j; q)_{k_i}} \prod_{i=1}^{r} \frac{(a x_i; q)_{|\mathbf{k}|} (ef / \lambda x_i; q)_{|\mathbf{k}| - k_i}}{(a x_i q^{1+n_i}; q)_{|\mathbf{k}|} (ef q^{-n_i} / \lambda x_i; q)_{|\mathbf{k}|}}$$
$$\times \frac{(c; q)_{|\mathbf{k}|} (d; q)_{|\mathbf{k}|} (e; q)_{|\mathbf{k}|} (f; q)_{|\mathbf{k}|}}{\prod_{i=1}^{r} [(a x_i q / c; q)_{k_i} (a x_i q / d; q)_{k_i} (a x_i q / e; q)_{k_i} (a x_i q / f; q)_{k_i}]}$$
$$\times \frac{\prod_{i=1}^{r} (b x_i; q)_{k_i}}{(aq/b; q)_{|\mathbf{k}|}} q^{\sum_{i=1}^{r} i k_i} \Bigg)$$
$$= \prod_{i=1}^{r} \frac{(a x_i q; q)_{n_i} (a x_i q / ef; q)_{n_i} (\lambda x_i q / e; q)_{n_i} (\lambda x_i q / f; q)_{n_i}}{(\lambda x_i q / ef; q)_{n_i} (\lambda x_i q; q)_{n_i} (a x_i q / f; q)_{n_i} (a x_i q / e; q)_{n_i}}$$
$$\times \sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\dots,r}} \Bigg(\prod_{i=1}^{r} \left(\frac{1 - \lambda x_i q^{k_i + |\mathbf{k}|}}{1 - \lambda x_i} \right) \prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right)$$
$$\times \prod_{1 \leq i < j \leq r} (\lambda a x_i x_j q / ef; q)^{-1}_{k_i + k_j} \prod_{i,j=1}^{r} (\lambda a x_i x_j q^{1+n_j} / ef; q)_{k_i}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i / x_j; q)_{k_i}}{(q x_i / x_j; q)_{k_i}} \prod_{i=1}^{r} \frac{(\lambda x_i; q)_{|\mathbf{k}|} (ef / a x_i; q)_{|\mathbf{k}| - k_i}}{(\lambda x_i q^{1+n_i}; q)_{|\mathbf{k}|} (ef q^{-n_i} / a x_i; q)_{|\mathbf{k}|}}$$
$$\times \frac{(\lambda c / a; q)_{|\mathbf{k}|} (\lambda d / a; q)_{|\mathbf{k}|} (e; q)_{|\mathbf{k}|} (f; q)_{|\mathbf{k}|}}{\prod_{i=1}^{r} [(a x_i q / c; q)_{k_i} (a x_i q / d; q)_{k_i} (\lambda x_i q / e; q)_{k_i} (\lambda x_i q / f; q)_{k_i}]}$$

$$\times \frac{\prod_{i=1}^{r} (\lambda b x_i/a;q)_{k_i}}{(aq/b;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^r i\,k_i}\Bigg), \qquad \text{(V.27)}$$

where $\lambda = qa^2/bcd$.

PROOF. First, replace k_i by $n_i - k_i$, for $i = 1, 2, \ldots, r$, on both sides of Theorem V.22 and simplify terms by the $l_1 = l_2 = \cdots = l_r = 0$ case of Lemma B.1 and some elementary manipulations (cf. [25, Appendix I]). Finally, relabel

$$\begin{aligned} &a \mapsto q^{-|\mathbf{n}|}/a, \quad && b \mapsto c/a, && c \mapsto d/a, && d \mapsto \lambda q^{1-|\mathbf{n}|}/ef, \\ &e \mapsto e/a, && f \mapsto bq^{-|\mathbf{n}|}/a, && x_i \mapsto q^{-n_i}/x_i, && \text{for } i = 1, 2, \ldots, r \end{aligned}$$

in the resulting identity to obtain (V.27). □

By using a polynomial argument we obtain

THEOREM V.28 (A D_r ${}_{10}\phi_9$ TRANSFORMATION). *Let $x_1, \ldots, x_r$, a, b, c, d, e, and $f_1, \ldots, f_r$ be indeterminate, let N be a nonnegative integer, let $r \geq 1$, and suppose that none of the denominators in* (V.29) *vanish. Then*

$$\sum_{\substack{k_1,k_2,\ldots,k_r\geq 0\\ 0\leq|\mathbf{k}|\leq N}} \Bigg(\prod_{i=1}^{r} \left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\leq i<j\leq r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$\times \prod_{1\leq i<j\leq r} (\lambda a x_i x_j q^{1+N}/e;q)^{-1}_{k_i+k_j} \prod_{i,j=1}^{r} (\lambda a x_i x_j q^{1+N}/ef_j;q)_{k_i}$$

$$\times \prod_{i,j=1}^{r} \frac{(f_j x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(eq^{-N}/\lambda x_i;q)_{|\mathbf{k}|-k_i}}{(ax_iq/f_i;q)_{|\mathbf{k}|}\,(ef_iq^{-N}/\lambda x_i;q)_{|\mathbf{k}|}}$$

$$\times \frac{(c;q)_{|\mathbf{k}|}\,(d;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}\left[(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\,(ax_iq/e;q)_{k_i}\,(ax_iq^{1+N};q)_{k_i}\right]}$$

$$\times \frac{\prod_{i=1}^{r}(bx_i;q)_{k_i}}{(aq/b;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^r i\,k_i}\Bigg)$$

$$= \prod_{i=1}^{r} \frac{(ax_iq;q)_N\,(ax_iq/ef_i;q)_N\,(\lambda x_iq/e;q)_N\,(\lambda x_iq/f_i;q)_N}{(\lambda x_iq/ef_i;q)_N\,(\lambda x_iq;q)_N\,(ax_iq/f_i;q)_N\,(ax_iq/e;q)_N}$$

$$\times \sum_{\substack{k_1,k_2,\ldots,k_r\geq 0\\ 0\leq|\mathbf{k}|\leq N}} \Bigg(\prod_{i=1}^{r} \left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right) \prod_{1\leq i<j\leq r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$\times \prod_{1\le i<j\le r} (\lambda a x_i x_j q^{1+N}/e;q)^{-1}_{k_i+k_j} \prod_{i,j=1}^{r} (\lambda a x_i x_j q^{1+N}/ef_j;q)_{k_i}$$
$$\times \prod_{i,j=1}^{r} \frac{(f_j x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(eq^{-N}/ax_i;q)_{|\mathbf{k}|-k_i}}{(\lambda x_i q/f_i;q)_{|\mathbf{k}|}\,(ef_i q^{-N}/ax_i;q)_{|\mathbf{k}|}}$$
$$\times \frac{(\lambda c/a;q)_{|\mathbf{k}|}\,(\lambda d/a;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^{r} [(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\,(\lambda x_i q/e;q)_{k_i}\,(\lambda x_i q^{1+N};q)_{k_i}]}$$
$$\left. \times \frac{\prod_{i=1}^{r} (\lambda b x_i/a;q)_{k_i}}{(aq/b;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^{r} i\,k_i} \right), \qquad \text{(V.29)}$$

where $\lambda = qa^2/bcd$.

PROOF. First we write the product in front of the sum of the right side of (V.29) as quotient of infinite products using (IV.3). Then by the $f = q^{-N}$ case of Theorem V.26 it follows that the identity (V.29) holds for $f_j = q^{-n_j}$, $j = 1, \ldots, r$. By clearing out denominators in (V.29), we get a polynomial equation in f_1, which is true for q^{-n_1}, $n_1 = 0, 1, \ldots$. Thus we obtain an identity in f_1. By carrying out this process for $f_2, f_3, \ldots, f_r$ also, we obtain Theorem V.28. $\square$

CHAPTER VI

Watson's and Sears' transformations

In this chapter, we work out the A_r, C_r, and D_r extensions of Watson's transformation and some of the multiple Sears' transformations which follow from our results in chapter V. As special cases of our generalizations of Bailey's ${}_{10}\phi_9$ transformation, we obtain A_r, C_r, and D_r generalizations of Watson's transformation found by Milne [52], [54], [56], Milne and Lilly [58], and Bhatnagar [12], respectively. See also [46], [59], [61] for some of Milne's A_r generalizations of Watson's theorem. In addition to recovering all such transformations known previously, we also obtain two new A_r, and several D_r generalizations of Watson's theorem.

We commence this chapter by stating Watson's classical q-Whipple transformation and our new multiple extensions thereof. Subsequently, we list some of Milne's A_r Watson's transformations, which follow immediately from our multiple ${}_{10}\phi_9$ transformations. We thank Professor S. C. Milne for showing us his notes for [56]. Finally, after stating a D_r Watson's transformation recently found by Bhatnagar [12], we conclude this chapter with some multidimensional Sears' transformations.

Watson's formula follows from Bailey's ${}_{10}\phi_9$ transformation in one of many ways. For instance, if we take the limit as $d \to \infty$ in Theorem V.2, and replace f by d in the resulting identity, we obtain [25, equation (2.5.1)]:

Theorem VI.1 (Watson's classical q-Whipple transformation). *Let a, b, c, d, and e be indeterminate, let n be a nonnegative integer, and suppose that none of the denominators in* (VI.2) *vanish. Then*

$$
{}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix}; q, \frac{a^2q^{2+n}}{bcde}\right]
= \frac{(aq;q)_n\,(aq/de;q)_n}{(aq/d;q)_n\,(aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix} aq/bc, d, e, q^{-n} \\ aq/b, aq/c, deq^{-n}/a \end{matrix}; q, q\right]. \quad \text{(VI.2)}
$$

This is not the only the only way to obtain Theorem VI.1 from (V.3). We may also take the limit $b \to \infty$, or $c \to \infty$. Further, we may first restate Theorem V.2 by replacing e or f by $\lambda a q^{n+1}/ef$ in (V.3), and then take the limits as b, c, or $d \to \infty$, to obtain Watson's transformation.

All the above limiting cases of Bailey's formula lead to the same identity, up to a trivial relabelling of parameters. However, in our multivariable generalizations of Bailey's formula, some of its symmetry is broken, and we obtain many different generalizations of Watson's transformation. For example, while the b, c, or $d \to \infty$ case of Theorem V.8 yields the same identity, letting $b \to \infty$ and $d \to \infty$ in Theorem V.15 yield different generalizations of Theorem VI.1.

Finally, as noted in chapter V we may set $a \mapsto qa^2/bcd$, $b \mapsto aq/cd$, $c \mapsto aq/bd$, $d \mapsto aq/bc$ in Theorem V.2. With these substitutions, λ becomes a, and the ${}_{10}\phi_9$ on the right hand side of (V.3) gets transformed into the one on the left, and vice-versa. Once again, we obtain Bailey's ${}_{10}\phi_9$ formula. Now, we may take limits as above and obtain Watson's transformation. This observation does not lead to any further generalizations of (VI.2) from our D_r ${}_{10}\phi_9$ transformations in chapter V, but it does apply to our C_r ${}_{10}\phi_9$ transformation, Theorem V.8.

If we perform the above substitutions and let $d \to \infty$ in Theorem V.8, we obtain one of Milne's [56] A_r Watson's transformations, see [59, Theorem 5.1].

Instead, if we simply let $d \to \infty$ in Theorem V.8, we obtain Milne and Lilly's [58, Theorem 6.6] C_r generalization of Theorem VI.1.

Next we consider the multivariable generalizations of Watson's transformations following from Theorem V.15. First we have an A_r extension of Theorem VI.1.

Theorem VI.3 (An A_r Watson's transformation). *Let $x_1, \ldots, x_r$, a, b, c, d, and e be indeterminate, let $n_1, \ldots, n_r$ be nonnegative integers for with $r \geq 1$, and suppose that none of the denominators in* (VI.4) *vanish. Then*

$$\sum_{\substack{0 \le k_i \le n_i \\ i=1,2,\ldots,r}} \left(\prod_{i=1}^{r} \left(\frac{1 - a x_i q^{k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \le i < j \le r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \right.$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i / x_j; q)_{k_i}}{(q x_i / x_j; q)_{k_i}} \prod_{i=1}^{r} \frac{(a x_i; q)_{|\mathbf{k}|}}{(a x_i q^{1+n_i}; q)_{|\mathbf{k}|}}$$
$$\times \prod_{i=1}^{r} \frac{(b x_i; q)_{k_i} (c x_i; q)_{k_i}}{(a x_i q/d; q)_{k_i} (a x_i q/e; q)_{k_i}}$$
$$\left. \times \frac{(d; q)_{|\mathbf{k}|} (e; q)_{|\mathbf{k}|}}{(aq/b; q)_{|\mathbf{k}|} (aq/c; q)_{|\mathbf{k}|}} \left(\frac{a^2 q^{1+|\mathbf{n}|}}{bcde} \right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r} i k_i} \right)$$

$$= \prod_{i=1}^{r} \left[\frac{(ax_i q; q)_{n_i} \, (ax_i q/de; q)_{n_i}}{(ax_i q/d; q)_{n_i} \, (ax_i q/e; q)_{n_i}} \right]$$

$$\times \sum_{\substack{0 \le k_i \le n_i \\ i=1,2,\ldots,r}} \left(\prod_{1 \le i < j \le r} \left(\frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \right) \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i/x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i}} \right.$$

$$\times \prod_{i=1}^{r} \left[\frac{(de/ax_i; q)_{|\mathbf{k}|-k_i} \, (aq/bcx_i; q)_{|\mathbf{k}|}}{(aq/bcx_i; q)_{|\mathbf{k}|-k_i} \, (deq^{-n_i}/ax_i; q)_{|\mathbf{k}|}} \right]$$

$$\left. \times \frac{(d; q)_{|\mathbf{k}|} \, (e; q)_{|\mathbf{k}|}}{(aq/b; q)_{|\mathbf{k}|} \, (aq/c; q)_{|\mathbf{k}|}} \, q^{\sum_{i=1}^{r} i k_i} \right). \tag{VI.4}$$

PROOF. We first replace λ by qa^2/bcd in Theorem V.15, and let $d \to \infty$. Finally, relabel $f \mapsto d$ in the resulting identity to obtain (VI.4). $\square$

REMARK VI.5. Theorem VI.3 concerns series summed over an r-rectangle. By using a polynomial argument similar to the proof of Theorem V.20, we may find another A_r extension of Theorem VI.1, where the series involved are summed over an r-tetrahedron. This may also be accomplished by taking the limit as $d \to \infty$ in (V.21).

REMARK VI.6. Theorem VI.3 is different from any of the generalizations of Theorem VI.1 which appears in [56]. The series on the left hand side of (VI.4) is the same as two such transformations appearing in [59, Theorems A10 and 5.1]. By comparing the other two sides we obtain generalizations of the Sears ${}_4\phi_3$ transformations [69], [25, equations (III.15) and (III.16)], see [59] for an example of similar calculations.

The rest of our extensions of Theorem VI.1 concern D_r series. After taking limits, we sometimes use

$$\binom{|\mathbf{k}|}{2} - e_2(\mathbf{k}) = \binom{k_1}{2} + \binom{k_2}{2} + \cdots + \binom{k_r}{2}$$

to simplify the powers of q.

THEOREM VI.7 (A D_r WATSON'S TRANSFORMATION). *Let $x_1, \ldots, x_r$, a, b, c, d, and e be indeterminate, let $n_1, \ldots, n_r$ be nonnegative integers with $r \ge 1$, and suppose that none of the denominators in* (VI.8) *vanish. Then*

$$\sum_{\substack{0 \le k_i \le n_i \\ i=1,2,\ldots,r}} \left(\prod_{i=1}^{r} \left(\frac{1 - ax_i q^{k_i + |\mathbf{k}|}}{1 - ax_i} \right) \prod_{1 \le i < j \le r} \left(\frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \right) \right.$$

$$\times \prod_{1\le i<j\le r} (ax_ix_jq/c;q)_{k_i+k_j} \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/c;q)_{k_i}}$$

$$\times \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(c/x_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+n_i};q)_{|\mathbf{k}|}\,(c/x_i;q)_{|\mathbf{k}|-k_i}} \; \frac{\prod_{i=1}^{r}(bx_i;q)_{k_i}}{(aq/b;q)_{|\mathbf{k}|}}$$

$$\times \frac{(d;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}[(ax_iq/d;q)_{k_i}\,(ax_iq/e;q)_{k_i}]} \left(\frac{a^2q^{1+|\mathbf{n}|}}{bcde}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r} i\,k_i} q^{-2e_2(\mathbf{k})} \prod_{i=1}^{r} x_i^{2k_i}\Bigg)$$

$$= \prod_{i=1}^{r}\left[\frac{(ax_iq;q)_{n_i}\,(ax_iq/de;q)_{n_i}}{(ax_iq/d;q)_{n_i}\,(ax_iq/e;q)_{n_i}}\right]$$

$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}} \Bigg(\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r}(ax_ix_jq/c;q)_{k_i+k_j}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/c;q)_{k_i}} \prod_{i=1}^{r}\frac{(de/ax_i;q)_{|\mathbf{k}|-k_i}}{(deq^{-n_i}/ax_i;q)_{|\mathbf{k}|}}$$

$$\times \prod_{i=1}^{r}(ax_iq/bc;q)_{k_i} \cdot \frac{(d;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}} q^{\sum_{i=1}^{r} i\,k_i}\Bigg), \tag{VI.8}$$

where $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\mathbf{k}$.

PROOF. We first replace λ by qa^2/bcd in Theorem V.15, and let $c\to\infty$. Finally, relabel $d\mapsto c$, and $f\mapsto d$ in the resulting identity to obtain (VI.8). □

REMARK VI.9. Theorem VI.7 concerns series summed over an r-rectangle. By using a polynomial argument similar to the proof of Theorem V.20, we may find another transformation formula between series summed over an r-tetrahedron. This may also be accomplished by taking the limit as $c\to\infty$ in (V.21).

Next, we consider the special cases of the equivalent formulation of Theorem V.15, obtained by replacing e by $\lambda aq/ef$ in (V.16).

THEOREM VI.10 (A D_r WATSON'S TRANSFORMATION). *Let* $x_1,\ldots,x_r$, a, b, c, d, *and* e *be indeterminate, let* $n_1,\ldots,n_r$ *be nonnegative integers with* $r\ge 1$, *and suppose that none of the denominators in* (VI.11) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}} \Bigg(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$\times \prod_{1\le i<j\le r} \frac{(ax_ix_jq/c;q)_{k_i+k_j}}{(ex_ix_j;q)_{k_i+k_j}} \prod_{i=1}^{r} \frac{(aq/ex_i;q)_{|\mathbf{k}|-k_i}}{(c/x_i;q)_{|\mathbf{k}|-k_i}}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{n_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/c;q)_{k_i}}$$
$$\times \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(c/x_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+n_i};q)_{|\mathbf{k}|}\,(aq^{1-n_i}/ex_i;q)_{|\mathbf{k}|}}$$
$$\times \prod_{i=1}^{r}\left[\frac{(bx_i;q)_{k_i}}{(ax_iq/d;q)_{k_i}}\right]\cdot\frac{(d;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}}\left(\frac{qa^2}{bcde}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r} i\,k_i}\Bigg)$$
$$= \prod_{i=1}^{r}\left[\frac{(ax_iq;q)_{n_i}\,(dex_i/a;q)_{n_i}}{(ex_i/a;q)_{n_i}\,(ax_iq/d;q)_{n_i}}\right]\left(\frac{1}{d}\right)^{|\mathbf{n}|}$$
$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}} \left(\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r}\frac{(ax_ix_jq/c;q)_{k_i+k_j}}{(ex_ix_j;q)_{k_i+k_j}}\right.$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{n_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/c;q)_{k_i}}$$
$$\left.\times \prod_{i=1}^{r}\left[\frac{(ax_iq/bc;q)_{k_i}}{(dex_i/a;q)_{k_i}}\right]\cdot\frac{(d;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^{r} i\,k_i}\right). \quad \text{(VI.11)}$$

Proof. We first replace e by $\lambda aq/ef$ in (V.16) to obtain an equivalent formulation of Theorem V.15. Next, we replace λ by qa^2/bcd, and let $c\to\infty$. Finally, relabel $f\mapsto d$ and $d\mapsto c$ in the resulting identity to obtain (VI.11). $\square$

Remark VI.12. Theorem VI.10 concerns series summed over an r-rectangle. By using a polynomial argument similar to the proof of Theorem V.20, we may find another transformation formula between series summed over an r-tetrahedron. This may also be accomplished by first replacing e by $\lambda aq/ef_1\cdots f_r$ in (V.21), and then taking the limit as $c\to\infty$ in (V.21).

Theorem VI.13 (A D_r Watson's transformation). *Let $x_1,\ldots,x_r$, a, b, c, d, and e be indeterminate, let be nonnegative integers with $r\ge 1$, and suppose that none of the denominators in* (VI.14) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}} \left(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\right.$$

$$\times \prod_{1\le i<j\le r} \frac{1}{(ex_ix_j;q)_{k_i+k_j}} \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{n_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}}$$

$$\times \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/ex_i;q)_{|\mathbf{k}|-k_i}}{(ax_iq^{1+n_i};q)_{|\mathbf{k}|}\,(aq^{1-n_i}/ex_i;q)_{|\mathbf{k}|}} \frac{\prod_{i=1}^{r}\left[(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}}$$

$$\times \frac{(d;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}(ax_iq/d;q)_{k_i}} \left(\frac{qa^2}{bcde}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r} i\,k_i} q^{2e_2(\mathbf{k})} \prod_{i=1}^{r} x_i^{-2k_i}\Bigg)$$

$$= \prod_{i=1}^{r}\left[\frac{(ax_iq;q)_{n_i}\,(dex_i/a;q)_{n_i}}{(ex_i/a;q)_{n_i}\,(ax_iq/d;q)_{n_i}}\right] \left(\frac{1}{d}\right)^{|\mathbf{n}|}$$

$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}} \Bigg(\prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r} \frac{1}{(ex_ix_j;q)_{k_i+k_j}}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{n_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(aq/bcx_i;q)_{|\mathbf{k}|}}{(aq/bcx_i;q)_{|\mathbf{k}|-k_i}}$$

$$\times \prod_{i=1}^{r}\left[\frac{1}{(dex_i/a;q)_{k_i}}\right] \cdot \frac{(d;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}} q^{\sum_{i=1}^{r} i\,k_i}\Bigg), \qquad \text{(VI.14)}$$

where $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\mathbf{k}$.

Proof. We first replace e by $\lambda aq/ef$ in (V.16) to obtain an equivalent formulation of Theorem V.15. Next, we replace λ by qa^2/bcd, and let $d\to\infty$. Finally, relabel $f\mapsto d$ in the resulting identity to obtain (VI.14). □

Remark VI.15. Theorem VI.13 concerns series summed over an r-rectangle. By using a polynomial argument similar to the proof of Theorem V.20, we may find another transformation formula between series summed over an r-tetrahedron. This may also be accomplished by first replacing e by $\lambda aq/ef_1\cdots f_r$ in (V.21), and then taking the limit as $d\to\infty$ in the resulting identity.

Next, we obtain the Watson's transformations which follow from Theorem V.22. If we take the limit as $d\to\infty$ in Theorem V.22, we obtain one of Milne's transformations, Theorem VI.25, see [61, Theorem 4.5].

Another limiting case yields:

Theorem VI.16 (A D_r Watson's transformation). *Let* $x_1,\ldots,x_r$, a, b, c, d, *and* e *be indeterminate, let* $n_1,\ldots,n_r$ *be nonnegative integers with* $r\ge 1$, *and suppose that none of the denominators in* (VI.17) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\dots,r}} \left(\prod_{i=1}^r \left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \right.$$
$$\times \prod_{1\le i<j\le r} (ax_ix_jq/c;q)_{k_i+k_j} \prod_{i,j=1}^r \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/c;q)_{k_i}}$$
$$\times \prod_{i=1}^r \frac{(ax_i;q)_{|\mathbf{k}|}\,(c/x_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+n_i};q)_{|\mathbf{k}|}\,(c/x_i;q)_{|\mathbf{k}|-k_i}} \frac{\prod_{i=1}^r [(bx_i;q)_{k_i}\,(ex_i;q)_{k_i}]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/e;q)_{|\mathbf{k}|}}$$
$$\left. \times \frac{(d;q)_{|\mathbf{k}|}}{\prod_{i=1}^r (ax_iq/d)_{k_i}} \left(\frac{a^2q^{1+|\mathbf{n}|}}{bcde}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^r i\,k_i} q^{-e_2(\mathbf{k})} \prod_{i=1}^r x_i^{k_i} \right)$$
$$= \prod_{i=1}^r \left[\frac{(ax_iq;q)_{n_i}}{(ax_iq/d;q)_{n_i}}\right] \frac{(aq/de;q)_{|\mathbf{n}|}}{(aq/e;q)_{|\mathbf{n}|}}$$
$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,2,\dots,r}} \left(\prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r} (ax_ix_jq/c;q)_{k_i+k_j} \right.$$
$$\times \prod_{i,j=1}^r \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/c;q)_{k_i}} \prod_{i=1}^r [(ax_iq/bc;q)_{k_i}\,(ex_i;q)_{k_i}]$$
$$\left. \times \frac{(d;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(deq^{-|\mathbf{n}|}/a;q)_{|\mathbf{k}|}} q^{\sum_{i=1}^r i\,k_i} \right), \qquad \text{(VI.17)}$$

where $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\mathbf{k}$.

Proof. We first replace λ by qa^2/bcd in Theorem V.22, and let $c\to\infty$. Finally, relabel $d\mapsto c$, and $f\mapsto d$ in the resulting identity to obtain (VI.17). □

Remark VI.18. Theorem VI.16 concerns series summed over an r-rectangle. By using a polynomial argument similar to the proof of Theorem V.24, we may find another transformation formula between series summed over an r-tetrahedron. This may also be accomplished by taking the limit as $c\to\infty$ in (V.25).

Next, we consider the special cases of the equivalent formulation of Theorem V.22, obtained by replacing f by $\lambda aq^{1+|\mathbf{n}|}/ef$ in (V.23). If we take the limit as $c\to\infty$ in the resulting identity, we obtain a D_r generalization of Theorem VI.1 found by Bhatnagar [12], see Theorem VI.35.

Instead, if we take the limit as $d\to\infty$, we obtain an A_r transformation theorem found by Milne [56], see Theorem VI.28. Further, if we take the limit as $d\to$

∞ in Theorem V.26, we obtain an A_r generalization of Theorem VI.1 found by Milne [56], see Theorem VI.30. Leininger and Milne [46] have found some elegant applications of this case of Theorem V.26. The $b \to \infty$ case of Theorem V.26 was also found by Milne [56], see Theorem VI.33.

Finally, we consider the equivalent formulation of Theorem V.26, obtained by replacing e by $\lambda aq/ef$ in (V.27).

THEOREM VI.19 (A D_r WATSON'S TRANSFORMATION). *Let $x_1, \ldots, x_r$, a, b, c, d, and e be indeterminate, let $n_1, \ldots, n_r$ be nonnegative integers with $r \geq 1$, and suppose that none of the denominators in* (VI.20) *vanish. Then*

$$\sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\ldots,r}} \Bigg(\prod_{i=1}^{r} \left(\frac{1 - a x_i q^{k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right)$$
$$\times \prod_{1 \leq i < j \leq r} \frac{1}{(e x_i x_j; q)_{k_i + k_j}} \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i / x_j; q)_{k_i} \, (e x_i x_j q^{n_j}; q)_{k_i}}{(q x_i / x_j; q)_{k_i}}$$
$$\times \prod_{i=1}^{r} \frac{(a x_i; q)_{|\mathbf{k}|} \, (aq/e x_i; q)_{|\mathbf{k}| - k_i}}{(a x_i q^{1+n_i}; q)_{|\mathbf{k}|} \, (a q^{1-n_i}/e x_i; q)_{|\mathbf{k}|}}$$
$$\times \frac{(b; q)_{|\mathbf{k}|} \, (c; q)_{|\mathbf{k}|} \, (d; q)_{|\mathbf{k}|}}{\prod_{i=1}^{r} \left[(a x_i q/b; q)_{k_i} \, (a x_i q/c; q)_{k_i} (a x_i q/d; q)_{k_i} \right]} \left(\frac{q a^2}{b c d e} \right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r} i \, k_i} \Bigg)$$
$$= \prod_{i=1}^{r} \left[\frac{(a x_i q; q)_{n_i} \, (d e x_i / a; q)_{n_i}}{(e x_i / a; q)_{n_i} \, (a x_i q/d; q)_{n_i}} \right] \left(\frac{1}{d} \right)^{|\mathbf{n}|}$$
$$\times \sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\ldots,r}} \Bigg(\prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \prod_{1 \leq i < j \leq r} \frac{1}{(e x_i x_j; q)_{k_i + k_j}}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i / x_j; q)_{k_i} \, (e x_i x_j q^{n_j}; q)_{k_i}}{(q x_i / x_j; q)_{k_i}}$$
$$\times \prod_{i=1}^{r} \left[\frac{(a x_i q/bc; q)_{k_i}}{(a x_i q/b; q)_{k_i} \, (a x_i q/c; q)_{k_i} \, (d e x_i / a; q)_{k_i}} \right] \cdot (d; q)_{|\mathbf{k}|} \, q^{\sum_{i=1}^{r} i \, k_i} \Bigg). \quad \text{(VI.20)}$$

PROOF. We first replace e by $\lambda aq/ef$ in (V.27) to obtain an equivalent formulation of Theorem V.26. Next, we replace λ by qa^2/bcd, and let $b \to \infty$. Finally, relabel $d \mapsto b$, $f \mapsto d$ in the resulting identity to obtain (VI.20). $\square$

REMARK VI.21. Theorem VI.19 concerns series summed over an r-rectangle. By using a polynomial argument similar to the proof of Theorem V.28, we may find another transformation formula between series summed over an r-tetrahedron.

This may also be accomplished by first replacing e by $\lambda aq/ef_1\cdots f_r$ in (V.29), and then taking the limit as $b\to\infty$ in the resulting identity.

THEOREM VI.22 (A D_r WATSON'S TRANSFORMATION). *Let $x_1,\ldots,x_r$, a, b, c, d, and e be indeterminate, let $n_1,\ldots,n_r$ be nonnegative integers with $r\geq 1$, and suppose that none of the denominators in* (VI.23) *vanish. Then*

$$\begin{aligned}
&\sum_{\substack{0\leq k_i\leq n_i\\ i=1,2,\ldots,r}} \Bigg(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\leq i<j\leq r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\\
&\qquad\times \prod_{1\leq i<j\leq r}\frac{1}{(ex_ix_j;q)_{k_i+k_j}} \prod_{i,j=1}^{r}\frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{n_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\\
&\qquad\times \prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/ex_i;q)_{|\mathbf{k}|-k_i}}{(ax_iq^{1+n_i};q)_{|\mathbf{k}|}\,(aq^{1-n_i}/ex_i;q)_{|\mathbf{k}|}}\,\frac{\prod\limits_{i=1}^{r}(bx_i;q)_{k_i}}{(aq/b;q)_{|\mathbf{k}|}}\\
&\quad\times \frac{(c;q)_{|\mathbf{k}|}\,(d;q)_{|\mathbf{k}|}}{\prod\limits_{i=1}^{r}\left[(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\right]}\cdot\left(\frac{qa^2}{bcde}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r} i\,k_i}\,q^{e_2(\mathbf{k})}\prod_{i=1}^{r}x_i^{-k_i}\Bigg)\\
&\qquad= \prod_{i=1}^{r}\left[\frac{(ax_iq;q)_{n_i}\,(dex_i/a;q)_{n_i}}{(ex_i/a;q)_{n_i}\,(ax_iq/d;q)_{n_i}}\right]\left(\frac{1}{d}\right)^{|\mathbf{n}|}\\
&\qquad\times \sum_{\substack{0\leq k_i\leq n_i\\ i=1,2,\ldots,r}}\Bigg(\prod_{1\leq i<j\leq r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\leq i<j\leq r}\frac{1}{(ex_ix_j;q)_{k_i+k_j}}\\
&\qquad\qquad\times \prod_{i,j=1}^{r}\frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{n_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\\
&\quad\times \prod_{i=1}^{r}\left[\frac{1}{(ax_iq/c;q)_{k_i}\,(dex_i/a;q)_{k_i}}\right]\frac{(aq/bc;q)_{|\mathbf{k}|}\,(d;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}}\,q^{\sum_{i=1}^{r} i\,k_i}\Bigg), \qquad \text{(VI.23)}
\end{aligned}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\mathbf{k}$.

PROOF. We first replace e by $\lambda aq/ef$ in (V.27) to obtain an equivalent formulation of Theorem V.26. Next, we replace λ by qa^2/bcd, and let $d\to\infty$. Finally, relabel $f\mapsto d$ in the resulting identity to obtain (VI.23). $\square$

REMARK VI.24. Theorem VI.22 concerns series summed over an r-rectangle. By using a polynomial argument similar to the proof of Theorem V.28, we may find another transformation formula between series summed over an r-tetrahedron.

This may also be accomplished by first replacing e by $\lambda aq/ef_1 \cdots f_r$ in (V.29), and then taking the limit as $d \to \infty$ in the resulting identity.

Before proceeding to some of Milne's Watson's transformations, we indicate the effect of reversing the sum on the various generalizations of Theorem VI.1 presented above. Theorem VI.3 and Theorem VI.10 are invariant if we reverse the series on both sides of the transformation, and relabel parameters. If we do the same for Theorem VI.7, we obtain Theorem VI.13, and vice-versa. Similarly, Theorem VI.16 and Theorem VI.22 are transformed into each other by reversing series. Finally, Theorem VI.19 and Bhatnagar's [12] D_r Watson's transformation, Theorem VI.35, are similarly seen to be equivalent.

THEOREM VI.25 ((MILNE) AN A_r WATSON'S TRANSFORMATION). *Let $x_1, \ldots, x_r$, a, b, c, d, and e be indeterminate, let $n_1, \ldots, n_r$ be nonnegative integers with $r \geq 1$, and suppose that none of the denominators in* (VI.26) *vanish. Then*

$$\begin{aligned}
\sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\ldots,r}} \Bigg(\prod_{i=1}^{r} \left(\frac{1 - a x_i q^{k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \\
\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i/x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i}} \prod_{i=1}^{r} \frac{(a x_i; q)_{|\mathbf{k}|}}{(a x_i q^{1+n_i}; q)_{|\mathbf{k}|}} \\
\times \frac{\prod_{i=1}^{r} \left[(b x_i; q)_{k_i} (c x_i; q)_{k_i} (e x_i; q)_{k_i} \right]}{(aq/b; q)_{|\mathbf{k}|} (aq/c; q)_{|\mathbf{k}|} (aq/e; q)_{|\mathbf{k}|}} \frac{(d; q)_{|\mathbf{k}|}}{\prod_{i=1}^{r} (a x_i q/d; q)_{k_i}} \\
\times \left(\frac{a^2 q^{1+|\mathbf{n}|}}{bcde} \right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r} i k_i} q^{e_2(\mathbf{k})} \prod_{i=1}^{r} x_i^{-k_i} \Bigg) \\
= \prod_{i=1}^{r} \left[\frac{(a x_i q; q)_{n_i}}{(a x_i q/d; q)_{n_i}} \right] \frac{(aq/de; q)_{|\mathbf{n}|}}{(aq/e; q)_{|\mathbf{n}|}} \\
\times \sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\ldots,r}} \Bigg(\prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i/x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i}} \\
\times \prod_{i=1}^{r} \left[\frac{(aq/bc x_i; q)_{|\mathbf{k}|} (e x_i; q)_{k_i}}{(aq/bc x_i; q)_{|\mathbf{k}| - k_i}} \right] \\
\times \frac{(d; q)_{|\mathbf{k}|}}{(aq/b; q)_{|\mathbf{k}|} (aq/c; q)_{|\mathbf{k}|} (de q^{-|\mathbf{n}|}/a; q)_{|\mathbf{k}|}} q^{\sum_{i=1}^{r} i k_i} \Bigg), \qquad \text{(VI.26)}
\end{aligned}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\mathbf{k}$.

Proof. We first replace λ by qa^2/bcd in Theorem V.22, and let $d \to \infty$. Finally, relabel $f \mapsto d$ in the resulting identity to obtain (VI.26). $\square$

Remark VI.27. The transformation formula in Theorem VI.25 is one of Milne's A_r Watson's transformations, see [61, Theorem 4.5].

The following three A_r transformation formulas have initially been derived by Milne, see [56], but they also follow immediately from our results.

Theorem VI.28 ((Milne) An A_r Watson's transformation). *Let $x_1, \ldots, x_r$, a, b, c, d, and e be indeterminate, let $n_1, \ldots, n_r$ be nonnegative integers with $r \geq 1$, and suppose that none of the denominators in* (VI.29) *vanish. Then*

$$\sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\ldots,r}} \Bigg(\prod_{i=1}^{r} \left(\frac{1 - a x_i q^{k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right)$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i / x_j; q)_{k_i}}{(q x_i / x_j; q)_{k_i}} \prod_{i=1}^{r} \frac{(a x_i; q)_{|\mathbf{k}|}}{(a x_i q^{1+n_i}; q)_{|\mathbf{k}|}}$$
$$\times \frac{\prod_{i=1}^{r} \left[(b x_i; q)_{k_i} (c x_i; q)_{k_i} (d x_i; q)_{k_i} (e x_i; q)_{k_i} \right]}{(aq/b; q)_{|\mathbf{k}|} (aq/c; q)_{|\mathbf{k}|} (aq/d; q)_{|\mathbf{k}|} (aq/e; q)_{|\mathbf{k}|}}$$
$$\times \left(\frac{a^2 q^{1+|\mathbf{n}|}}{bcde} \right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r} i k_i} q^{2 e_2(\mathbf{k})} \prod_{i=1}^{r} x_i^{-2k_i} \Bigg)$$
$$= \frac{\prod_{i=1}^{r} \left[(a x_i q; q)_{n_i} (a q^{1+|\mathbf{n}| - n_i} / d e x_i; q)_{n_i} \right]}{(aq/d; q)_{|\mathbf{n}|} (aq/e; q)_{|\mathbf{n}|}}$$
$$\times \sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\ldots,r}} \Bigg(\prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i / x_j; q)_{k_i}}{(q x_i / x_j; q)_{k_i}}$$
$$\times \prod_{i=1}^{r} \left[\frac{(aq/bc x_i; q)_{|\mathbf{k}|}}{(aq/bc x_i; q)_{|\mathbf{k}| - k_i} (d e x_i q^{-|\mathbf{n}|}/a; q)_{k_i}} \right]$$
$$\times \frac{\prod_{i=1}^{r} \left[(d x_i; q)_{k_i} (e x_i; q)_{k_i} \right]}{(aq/b; q)_{|\mathbf{k}|} (aq/c; q)_{|\mathbf{k}|}} q^{\sum_{i=1}^{r} i k_i} \Bigg), \qquad \text{(VI.29)}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\mathbf{k}$.

Proof. We first replace f by $\lambda a q^{1+|\mathbf{n}|}/ef$ in (V.23) to obtain an equivalent formulation of Theorem V.22. Next, we replace λ by qa^2/bcd, and let $d \to \infty$. Finally, relabel $f \mapsto d$ in the resulting identity to obtain (VI.29). $\square$

Theorem VI.30 ((Milne) An A_r Watson's transformation). *Let $x_1, \ldots, x_r$, a, b, c, d, and e be indeterminate, let $n_1, \ldots, n_r$ be nonnegative integers with $r \geq 1$, and suppose that none of the denominators in* (VI.31) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}} \Bigg(\prod_{i=1}^{r} \left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i} \right) \prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j} \right)$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+n_i};q)_{|\mathbf{k}|}}$$
$$\times \frac{\prod_{i=1}^{r}(bx_i;q)_{k_i}}{(aq/b;q)_{|\mathbf{k}|}} \frac{(c;q)_{|\mathbf{k}|}\,(d;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}[(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\,(ax_iq/e;q)_{k_i}]}$$
$$\times \left(\frac{a^2q^{1+|\mathbf{n}|}}{bcde} \right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r} i\,k_i}\, q^{-e_2(\mathbf{k})} \prod_{i=1}^{r} x_i^{k_i} \Bigg)$$
$$= \prod_{i=1}^{r} \left[\frac{(ax_iq;q)_{n_i}\,(ax_iq/de;q)_{n_i}}{(ax_iq/d;q)_{n_i}\,(ax_iq/e;q)_{n_i}} \right]$$
$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}} \Bigg(\prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j} \right) \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}$$
$$\times \prod_{i=1}^{r} \left[\frac{(de/ax_i;q)_{|\mathbf{k}|-k_i}}{(deq^{-n_i}/ax_i;q)_{|\mathbf{k}|}\,(ax_iq/c;q)_{k_i}} \right]$$
$$\times \frac{(aq/bc;q)_{|\mathbf{k}|}\,(d;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}} q^{\sum_{i=1}^{r} i\,k_i} \Bigg), \quad \text{(VI.31)}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\mathbf{k}$.

Proof. We first replace λ by qa^2/bcd in Theorem V.26, and let $d \to \infty$. Finally, relabel $f \mapsto d$ in the resulting identity to obtain (VI.31). $\square$

Remark VI.32. Leininger and Milne [46] have found some elegant applications of Theorem VI.30.

Theorem VI.33 ((Milne) An A_r Watson's transformation). *Let $x_1, \ldots, x_r$, a, b, c, d, and e be indeterminate, let $n_1, \ldots, n_r$ be nonnegative integers* *with $r \geq 1$, and suppose that none of the denominators in* (VI.34) *vanish. Then*

$$\sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\dots,r}} \left(\prod_{i=1}^r \left(\frac{1 - a x_i q^{k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \right.$$
$$\times \prod_{i,j=1}^r \frac{(q^{-n_j} x_i / x_j; q)_{k_i}}{(q x_i / x_j; q)_{k_i}} \prod_{i=1}^r \frac{(a x_i; q)_{|\mathbf{k}|}}{(a x_i q^{1+n_i}; q)_{|\mathbf{k}|}}$$
$$\times \frac{(b; q)_{|\mathbf{k}|} (c; q)_{|\mathbf{k}|} (d; q)_{|\mathbf{k}|} (e; q)_{|\mathbf{k}|}}{\prod_{i=1}^r [(a x_i q/b; q)_{k_i} (a x_i q/c; q)_{k_i} (a x_i q/d; q)_{k_i} (a x_i q/e; q)_{k_i}]}$$
$$\left. \times \left(\frac{a^2 q^{1+|\mathbf{n}|}}{bcde} \right)^{|\mathbf{k}|} q^{\sum_{i=1}^r i k_i} q^{-2 e_2(\mathbf{k})} \prod_{i=1}^r x_i^{2k_i} \right)$$
$$= \prod_{i=1}^r \left[\frac{(a x_i q; q)_{n_i} (a x_i q/de; q)_{n_i}}{(a x_i q/d; q)_{n_i} (a x_i q/e; q)_{n_i}} \right]$$
$$\times \sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\dots,r}} \left(\prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \prod_{i,j=1}^r \frac{(q^{-n_j} x_i / x_j; q)_{k_i}}{(q x_i / x_j; q)_{k_i}} \right.$$
$$\times \prod_{i=1}^r \left[\frac{(de/a x_i; q)_{|\mathbf{k}| - k_i} (a x_i q/bc; q)_{k_i}}{(de q^{-n_i} / a x_i; q)_{|\mathbf{k}|}} \right]$$
$$\left. \times \frac{(d; q)_{|\mathbf{k}|} (e; q)_{|\mathbf{k}|}}{\prod_{i=1}^r [(a x_i q/b; q)_{k_i} (a x_i q/c; q)_{k_i}]} q^{\sum_{i=1}^r i k_i} \right), \qquad \text{(VI.34)}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\mathbf{k}$.

PROOF. We first replace λ by qa^2/bcd in Theorem V.26, and let $b \to \infty$. Finally, relabel $d \mapsto b$ and $f \mapsto d$ in the resulting identity to obtain (VI.34). $\square$

Finally, we state a D_r transformation formula which was recently found by Bhatnagar [12].

THEOREM VI.35 ((BHATNAGAR) A D_r WATSON'S TRANSFORMATION). *Let $x_1, \dots, x_r$, a, b, c, d, and e be indeterminate, let $n_1, \dots, n_r$ be nonnegative integers with $r \geq 1$, and suppose that none of the denominators in* (VI.36) *vanish. Then*

$$\sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\dots,r}} \left(\prod_{i=1}^r \left(\frac{1 - a x_i q^{k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \right.$$

$$\times \prod_{1\le i<j\le r} (ax_ix_jq/c;q)_{k_i+k_j} \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/c;q)_{k_i}}$$

$$\times \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(c/x_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+n_i};q)_{|\mathbf{k}|}\,(c/x_i;q)_{|\mathbf{k}|-k_i}}$$

$$\times \frac{\prod_{i=1}^{r}[(bx_i;q)_{k_i}\,(dx_i;q)_{k_i}\,(ex_i;q)_{k_i}]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/d;q)_{|\mathbf{k}|}\,(aq/e;q)_{|\mathbf{k}|}} \left(\frac{a^2q^{1+|\mathbf{n}|}}{bcde}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r} i k_i}\Bigg)$$

$$= \frac{\prod_{i=1}^{r}\left[(ax_iq;q)_{n_i}\,(aq^{1+|\mathbf{n}|-n_i}/dex_i;q)_{n_i}\right]}{(aq/d;q)_{|\mathbf{n}|}\,(aq/e;q)_{|\mathbf{n}|}}$$

$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}} \Bigg(\prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r} (ax_ix_jq/c;q)_{k_i+k_j}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/c;q)_{k_i}}$$

$$\times \prod_{i=1}^{r}\left[\frac{(ax_iq/bc;q)_{k_i}\,(dx_i;q)_{k_i}\,(ex_i;q)_{k_i}}{(dex_iq^{-|\mathbf{n}|}/a;q)_{k_i}}\right]\cdot\frac{1}{(aq/b;q)_{|\mathbf{k}|}}\,q^{\sum_{i=1}^{r} i k_i}\Bigg). \qquad \text{(VI.36)}$$

Proof. We first replace f by $\lambda aq^{1+|\mathbf{n}|}/ef$ in (V.23) to obtain an equivalent formulation of Theorem V.22. Next, we replace λ by qa^2/bcd, and let $c\to\infty$. Finally, relabel $f\mapsto d$ and $d\mapsto c$ in the resulting identity to obtain (VI.36). $\square$

In the following, we indicate how multivariable Sears' [69] transformations can be worked out. Many of these follow from our Watson's transformations of this chapter, respectively from our Bailey's ${}_{10}\phi_9$ transformations of chapter V. Special cases of our generalizations of Watson's ${}_8\phi_7$ transformation include A_r, C_r, and D_r generalizations of Sears' transformation found by Milne and Newcomb [59], [61], Milne and Lilly [58], and Bhatnagar [12], respectively. In addition to recovering all such transformations known previously, we can also use our results to obtain several new A_r and D_r generalizations of Sears' theorem.

Sears' formula follows from Watson's ${}_8\phi_7$ transformation in one of many ways. Observe that the ${}_8\phi_7$ on the left side of (VI.2) is termwise unchanged if we interchange c with d, b with e. Equate the resulting sums corresponding to the right side of (VI.2), and then relabel the parameters $a\mapsto bcq^{-n}/d$, $d\mapsto e/a$, and $e\mapsto bcq^{1-n}/de$. After some simplification we obtain [25, equation (2.10.4), explicitly]:

THEOREM VI.37 (SEARS' CLASSICAL ${}_4\phi_3$ TRANSFORMATION). *Let a, b, c, d, and e be indeterminate, let n be a nonnegative integer, and suppose that none of the denominators in* (VI.38) *vanish. Then*

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, abcq^{1-n}/de \end{matrix}; q, q\right] = \frac{(e/a;q)_n\,(de/bc;q)_n}{(e;q)_n\,(de/abc;q)_n}\, {}_4\phi_3\left[\begin{matrix} q^{-n}, a, d/b, d/c \\ d, aq^{1-n}/e, de/bc \end{matrix}; q, q\right]. \tag{VI.38} $$

This is not the only the only way to obtain Theorem VI.37 from (VI.2). We may also interchange other parameters, two by two, first, equate the ${}_4\phi_3$ sides as before, and then relabel parameters. In the classical case, due to symmetry all these cases lead to the same identity, up to a trivial relabelling of parameters. However, in our multivariable generalizations of Watson's formula, some of its symmetry is broken, and we obtain many different generalizations of Sears' transformation. Note that because of the broken symmetry in our ${}_{10}\phi_9$ transformations, we had also obtained, as limiting cases, different multivariable generalizations of Watson's transformation formula.

But it is also possible to obtain Sears' ${}_4\phi_3$ transformation, (VI.38), from Bailey's ${}_{10}\phi_9$ transformation, (V.3), directly. Replace b by aq/b and e by aq/e in (V.3), and then take the limit $a \to 0$. After relabelling of parameters, $b \mapsto d$, $d \mapsto b$, and $f \mapsto a$, we obtain (VI.38). Again, different limits lead to Sears' transformation, where distinct identities might be obtained in the multidimensional cases.

A complete list of all possible Sears' transformations would be even longer than a list including all Watson's transformations. For illustration, we only give two multidimensional Sears' transformations, both of which follow from Theorem V.15.

THEOREM VI.39 (AN A_r SEARS' TRANSFORMATION). *Let $x_1, \ldots, x_r$, a, b, c, d, and e be indeterminate, let $n_1, \ldots, n_r$ be nonnegative integers with $r \geq 1$, and suppose that none of the denominators in* (VI.40) *vanish. Then*

$$ \sum_{\substack{0 \leq k_i \leq n_i \\ i=1,2,\ldots,r}} \Bigg(\prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \right) \prod_{i=1}^{r} \frac{(abcq/dex_i;q)_{|\mathbf{k}|-k_i}}{(b/x_i;q)_{|\mathbf{k}|-k_i}} $$
$$ \times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(b/x_i;q)_{|\mathbf{k}|}}{(abcq^{1-n_i}/dex_i;q)_{|\mathbf{k}|}} $$
$$ \times \prod_{i=1}^{r} \frac{(cx_i;q)_{k_i}}{(ex_i;q)_{k_i}} \cdot \frac{(a;q)_{|\mathbf{k}|}}{(d;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^{r} i\,k_i} \Bigg) $$
$$ = \prod_{i=1}^{r} \frac{(ex_i/a;q)_{n_i}\,(dex_i/bc;q)_{n_i}}{(ex_i;q)_{n_i}\,(dex_iq/abc;q)_{n_i}} $$

$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,2,\dots,r}} \left(\prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{i=1}^r \frac{(aq/ex_i;q)_{|\mathbf{k}|-k_i}}{(d/cx_i;q)_{|\mathbf{k}|-k_i}} \right.$$
$$\times \prod_{i,j=1}^r \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^r \frac{(d/cx_i;q)_{|\mathbf{k}|}}{(aq^{1-n_i}/ex_i;q)_{|\mathbf{k}|}}$$
$$\left. \times \prod_{i=1}^r \frac{(dx_i/b;q)_{k_i}}{(dex_i/bc;q)_{k_i}} \cdot \frac{(a;q)_{|\mathbf{k}|}}{(d;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^r i k_i} \right). \quad \text{(VI.40)}$$

PROOF. We first replace b by aq/b and e by aq/e in (V.16), and then take the limit $a \to 0$. After relabelling of parameters, $b \mapsto d$, $d \mapsto b$, and $f \mapsto a$, we obtain (VI.40). □

THEOREM VI.41 (A D_r SEARS' TRANSFORMATION). *Let $x_1, \dots, x_r$, a, b, c, d, and e be indeterminate, let $n_1, \dots, n_r$ be nonnegative integers with $r \ge 1$, and suppose that none of the denominators in* (VI.42) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\dots,r}} \left(\prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r} (dx_ix_j;q)_{k_i+k_j} \right.$$
$$\times \prod_{i=1}^r \frac{(abcq/dex_i;q)_{|\mathbf{k}|-k_i}}{(abcq^{1-n_i}/dex_i;q)_{|\mathbf{k}|}} \prod_{i,j=1}^r \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(dx_ix_j;q)_{k_i}}$$
$$\left. \times \prod_{i=1}^r \frac{(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}}{(ex_i;q)_{k_i}} \cdot (a;q)_{|\mathbf{k}|}\, q^{\sum_{i=1}^r i k_i} \right)$$
$$= \prod_{i=1}^r \frac{(ex_i/a;q)_{n_i}\,(dex_i/bc;q)_{n_i}}{(ex_i;q)_{n_i}\,(dex_iq/abc;q)_{n_i}}$$
$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,2,\dots,r}} \left(\prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r} (dx_ix_j;q)_{k_i+k_j} \right.$$
$$\times \prod_{i=1}^r \frac{(aq/ex_i;q)_{|\mathbf{k}|-k_i}}{(aq^{1-n_i}/ex_i;q)_{|\mathbf{k}|}} \prod_{i,j=1}^r \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(dx_ix_j;q)_{k_i}}$$
$$\left. \times \prod_{i=1}^r \frac{(dx_i/b;q)_{k_i}\,(dx_i/c;q)_{k_i}}{(dex_i/bc;q)_{k_i}} \cdot (a;q)_{|\mathbf{k}|}\, q^{\sum_{i=1}^r i k_i} \right). \quad \text{(VI.42)}$$

PROOF. We first replace d by aq/d and e by aq/e in (V.16), and then take the limit $a \to 0$. After relabelling, $f \mapsto a$, we obtain (VI.42). □

CHAPTER VII

More multidimensional summation theorems

In this chapter, which can be read independently from the previous ones, we derive multiple generalizations of many of the classical basic hypergeometric summation formulas. These extensions of a new, natural, kind of multidimensional basic hypergeometric series are also associated to root systems of classical type. We proceed by combining the classical (one-dimensional) summation formulas with certain determinant evaluations. Our theorems include A_r extensions of Ramanujan's bilateral ${}_1\psi_1$ sum, C_r extensions of Bailey's very-well-poised ${}_6\psi_6$ summation, and a C_r extension of Jackson's very-well-poised balanced ${}_8\phi_7$ summation formula. We also derive multidimensional extensions, associated to the classical root systems of type A_r, B_r, C_r, and D_r, respectively, of Chu's bilateral transformation formula for basic hypergeometric series of Gasper-Karlsson-Minton type. Limiting cases of our series identities include multidimensional generalizations of many of the most important summation theorems of the classical theory of basic hypergeometric series. As explicit examples we provide C_r terminating and nonterminating ${}_6\phi_5$ summation formulas, and an A_r extension of the q-Pfaff-Saalschütz summation. This chapter is part of a forthcoming paper [68].

Recently, Gustafson and Krattenthaler [32, Theorem 1.15] discovered an A_r (or equivalently $U(r+1)$) extension of Ramanujan's bilateral ${}_1\psi_1$ sum. The work in their paper involved A_r series of a new kind. In particular, Gustafson and Krattenthaler's A_r ${}_1\psi_1$ sum is different from Milne's [52].

Before we review Ramanujan's ${}_1\psi_1$ sum and Gustafson and Krattenthaler's extension thereof we introduce the Gasper-Rahman notation

$$(a_1,\dots,a_m;q)_k \equiv (a_1;q)_k\,(a_2;q)_k\cdots(a_m;q)_k,$$

where the rising q-factorial $(a;q)_k$ is defined in (IV.3), for simplifying our displays in this chapter.

Ramanujan's classical ${}_1\psi_1$ summation formula (see [35] or [25, (5.2.1)]) reads

THEOREM VII.1 (RAMANUJAN'S CLASSICAL ${}_1\psi_1$ SUM). *Let a, b, and z be indeterminate, and suppose that none of the denominators in* (VII.2) *vanish. Then*

$$\sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(b;q)_k} z^k = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \tag{VII.2}$$

provided the series terminates or $|q| < 1$ and $|b/a| < |z| < 1$.

REMARK VII.3. Ramanujan's bilateral ${}_1\psi_1$ summation formula is one of the most fundamental formulas of the theory of basic hypergeometric series. A simple and elegant proof was given by Ismail [38] who noted that Theorem VII.1 is an immediate consequence of the q-binomial theorem (which is the $b = q$ case of Theorem VII.1), and analytic continuation.

Gustafson and Krattenthaler's A_r ${}_1\psi_1$ summation formula [32, Theorem 1.15] can be stated as follows. Here and in the following we use the notation $|\mathbf{k}| = k_1 + k_2 + \cdots + k_r$.

THEOREM VII.4 (GUSTAFSON AND KRATTENTHALER). *Let $x_1, \ldots, x_r$, a, b, and z be indeterminate, let $r \geq 1$, and suppose that none of the denominators in* (VII.5) *vanish. Then*

$$\sum_{k_1,\ldots,k_r=-\infty}^{\infty} \left(\prod_{1\le i<j\le r} \left(\frac{1 - q^{k_i-k_j} x_i/x_j}{1 - x_i/x_j} \right) \prod_{i=1}^{r} \frac{(a;q)_{k_i}}{(b;q)_{k_i}} \cdot z^{|\mathbf{k}|} q^{\sum_{i=1}^{r}(i-r)k_i} \right)$$
$$= \prod_{i=1}^{r} \frac{(q, b/a, azq^{1-i}, q^i/az; q)_\infty}{(b, q/a, zq^{1-i}, bq^{i-1}/az; q)_\infty}, \tag{VII.5}$$

provided the series terminates or $|b/a| < |z| < |q|^{r-1} < 1$.

Gustafson and Krattenthaler proved Theorem VII.4 by a combination of the classical $r = 1$ case (VII.2) and the Vandermonde determinant evaluation. One of the main purposes of this chapter is to provide some more A_r extensions of Ramanujan's ${}_1\psi_1$ summation formula. But we are also able to derive C_r extensions of Bailey's bilateral very-well-poised ${}_6\psi_6$ summation, and Jackson's balanced very-well-poised ${}_8\phi_7$ summation formula, respectively. Furthermore, at the end of this chapter, we combine determinant evaluations with Chu's [18] remarkable bilateral transformation formula of Gasper-Karlsson-Minton type [23], [42], [60], [25, sec. 1.9] to deduce multiple versions of Chu's identity. It is suprising that we obtain identities associated to various root systems of classical type, and we may also employ different bases, $q_1, \ldots, q_r$, in our series (see Theorem VII.46). It is also possible to obtain other multiple extensions of Chu's transformation formula by using other determinants in our derivation.

Proceeding by essentially the same method as Gustafson and Krattenthaler in the proof of their A_r ${}_1\psi_1$ summation theorem our derivations in this chapter require certain determinant evaluations which are more general than the classical Vandermonde determinant evaluation. One of the determinant evaluations we utilize, Lemma A.22, comes from a determinant lemma [44, Lemma 34] (see Lemma A.24 in Appendix A) which has been successfully used by Krattenthaler in the computation of generating functions for plane partitions and tableaux. This determinant lemma has also been used by Gessel and Krattenthaler [28] for deriving several A_r basic hypergeometric series identities. Independently, a special case of Lemma A.22 was involved in [77] in the computation of biorthogonal rational functions.

The particular technique in this chapter has already been used in [33] to provide new proofs and generalizations of the A_r extensions of Heine's ${}_2\phi_1$ transformations which have been discovered in [32]. In fact, the reading of [33] was the starting point of the author's investigations for identities of this kind of series. We believe that the method of this chapter, which is entirely elementary, will be useful for proving other multidimensional series identities as well [68].

Our A_r extensions of Ramanujan's ${}_1\psi_1$ summation formula, (VII.2), are the following.

Theorem VII.6 (A_r ${}_1\psi_1$ summations). *Let $x_1,\dots,x_r$, $a_1,\dots,a_r$, $b_1\dots,b_r$, $z_1,\dots,z_r$, a, b, and z be indeterminate, let $r\ge 1$, and suppose that none of the denominators in* (VII.7), (VII.8), *or* (VII.9) *vanish. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(ax_i;q)_{k_i}}{(b_i;q)_{k_i}}\cdot z^{|\mathbf{k}|}\,q^{\sum_{i=1}^r(i-r)\,k_i}\right)$$
$$=q^{-\binom{r}{2}}\prod_{1\le i<j\le r}\left(\frac{b_i/x_i-b_j/x_j}{1/x_i-1/x_j}\right)\prod_{i=1}^{r}\frac{(q,b_i/ax_i,ax_iz,q/ax_iz;q)_\infty}{(b_i,q/ax_i,zq^{1-i},b_i/ax_iz;q)_\infty},\quad\text{(VII.7)}$$

provided the series terminates or $|b_i/ax_i|<|z|<|q|^{r-1}<1$ for $i=1,\dots,r$,

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(a_i;q)_{k_i}}{(bx_i;q)_{k_i}}\cdot z^{|\mathbf{k}|}\,q^{\sum_{i=1}^r(i-r)\,k_i}\right)$$
$$=\prod_{1\le i<j\le r}\left(\frac{1-a_jx_i/a_ix_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(q,bx_i/a_i,a_izq^{1-i},q^i/a_iz;q)_\infty}{(bx_i,q/a_i,zq^{1-i},bx_i/a_iz;q)_\infty},\quad\text{(VII.8)}$$

provided the series terminates or $|bx_i/a_i|<|z|<|q|^{r-1}<1$ for $i=1,\dots,r$,

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(ax_i;q)_{k_i}}{(bx_i;q)_{k_i}}\,z_i^{k_i}\cdot q^{\sum_{i=1}^r(i-r)\,k_i}\right)$$

$$= \prod_{1\le i<j\le r}\left(\frac{1-z_j/z_i}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(q,bq^{1-i}/a,ax_iz_iq^{1-i},q^i/ax_iz_i;q)_\infty}{(bx_i,q/ax_i,z_iq^{1-r},b/az_i;q)_\infty}, \tag{VII.9}$$

provided the series terminates or $|b/a|<|z_i|<|q|^{r-1}<1$ *for* $i=1,\dots,r$.

PROOF. We start with the sum

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(a_i;q)_{k_i}}{(b_i;q)_{k_i}}z_i^{k_i}\cdot q^{\sum_{i=1}^{r}(i-r)\,k_i}\right), \tag{VII.10}$$

and specialize the parameters a_i, b_i, z_i later. We have

$$\begin{aligned}q^{\sum_{i=1}^{r}(i-r)\,k_i}\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)&=\prod_{1\le i<j\le r}\left(\frac{q^{-k_i}/x_i-q^{-k_j}/x_j}{1/x_i-1/x_j}\right)\\&=\prod_{1\le i<j\le r}(1/x_i-1/x_j)^{-1}\det_{1\le s,t\le r}\left(\left(q^{-k_s}/x_s\right)^{r-t}\right),\end{aligned}$$

the last equation due to the Vandermonde determinant evaluation. Hence we may write (VII.10), when multiplied by $\prod_{1\le i<j\le r}(1/x_i-1/x_j)$, as

$$\det_{1\le s,t\le r}\left(x_s^{t-r}\sum_{k_s=-\infty}^{\infty}\frac{(a_s;q)_{k_s}}{(b_s;q)_{k_s}}\left(z_sq^{t-r}\right)^{k_s}\right).$$

Now, to the sum inside the determinant we apply Ramanujan's classical ${}_1\psi_1$ summation, (VII.2), with $a\mapsto a_s$, $b\mapsto b_s$, and $z\mapsto z_sq^{t-r}$. Thus we obtain

$$\det_{1\le s,t\le r}\left(x_s^{t-r}\frac{(q,b_s/a_s,a_sz_sq^{t-r},q^{1+r-t}/a_sz_s;q)_\infty}{(b_s,q/a_s,z_sq^{t-r},b_sq^{r-t}/a_sz_s;q)_\infty}\right).$$

Now, by using linearity of the determinant with respect to rows, we take some factors out of the determinant and obtain

$$\prod_{i=1}^{r}\frac{(q,b_i/a_i,a_iz_i,q/a_iz_i;q)_\infty}{(b_i,q/a_i,z_i,b_i/a_iz_i;q)_\infty}\det_{1\le s,t\le r}\left(\left(\frac{x_s}{a_s}\right)^{t-r}\frac{(b_s/a_sz_s;q)_{r-t}}{(q/z_s;q)_{r-t}}\right). \tag{VII.11}$$

The determinant in (VII.11) cannot be evaluated in closed form in general. But we can choose different specializations of the parameters a_s, b_s, and z_s, for $s=1,\dots,r$, for which the determinant can be reduced to a product by means of Lemma A.22.

The simplest choice is $a_s\equiv a$, $b_s\equiv b$, and $z_s\equiv z$. In this case the determinant in (VII.11) equals

$$a^{\binom{r}{2}}\prod_{i=1}^{r}\frac{(b/az;q)_{r-i}}{(q/z;q)_{r-i}}\det_{1\le s,t\le r}\left(x_s^{t-r}\right),$$

and the last determinant can be reduced to $\prod_{1\le i<j\le r}(1/x_i - 1/x_j)$. Substituting these calculations and performing some other elementary manipulations, we easily recover (VII.5).

By choosing different specializations of the parameters in (VII.11) we will now prove the cases (VII.7), (VII.8), and (VII.9) of our theorem.

To prove (VII.7), we set $a_s = ax_s$ and $z_s \equiv z$. In this case the determinant in (VII.11) equals

$$a^{\binom{r}{2}} \prod_{i=1}^{r} (q/z;q)_{r-i}^{-1} \det_{1\le s,t\le r} \left((b_s/ax_s z;q)_{r-t}\right).$$

The determinant can be evaluated by means of Lemma A.22 with $X_s \mapsto b_s/x_s$, $A \mapsto 1/az$, $B \mapsto 0$, and $C \mapsto 0$. Subsequently, substituting our calculations and performing some other elementary manipulations leads to (VII.7).

To prove (VII.8), we set $b_s = bx_s$ and $z_s \equiv z$. In this case the determinant in (VII.11) equals

$$\prod_{i=1}^{r} (q/z;q)_{r-i}^{-1} \det_{1\le s,t\le r} \left(\left(\frac{x_s}{a_s}\right)^{t-r} (bx_s/a_s z;q)_{r-t}\right). \tag{VII.12}$$

The determinant can be evaluated by means of a limiting case of Lemma A.22. Namely, first multiply both sides of the $C \mapsto 0$ case of (A.23) with $(-B)^{\binom{r}{2}} q^{\binom{r}{3}}$ and then let $B \to \infty$ in the resulting identity to see that

$$\det_{1\le s,t\le r} \left(X_s^{t-r} (AX_s;q)_{r-t}\right) = \prod_{1\le i<j\le r} (1/X_i - 1/X_j). \tag{VII.13}$$

Now evaluate the determinant in (VII.12) by the $X_s \mapsto x_s/a_s$ and $A \mapsto b/z$ case of (VII.13). Substituting our calculations and performing some elementary manipulations, we easily deduce (VII.8).

To prove (VII.9), we set $a_s = ax_s$ and $b_s \equiv bx_s$. In this case the determinant in (VII.11) equals

$$a^{\binom{r}{2}} \det_{1\le s,t\le r} \left(\frac{(b/az_s;q)_{r-t}}{(q/z_s;q)_{r-t}}\right).$$

The determinant can be evaluated by means of Lemma A.22 with $X_s \mapsto 1/z_s$, $A \mapsto b/a$, $B \mapsto q$, and $C \mapsto 0$. Finally, substituting our calculations and performing some elementary manipulations, we arrive at (VII.9). $\square$

REMARK VII.14. There are two other cases where the determinant in (VII.11) can be evaluated in closed form.

We may set $b_s = bx_s$ and $z_s \equiv zx_s/a_s$. In this case the determinant in (VII.11) equals

$$\prod_{i=1}^{r}(b/z;q)_{r-i} \det_{1\le s,t\le r}\left(\left(\frac{a_s}{x_s}\right)^{r-t}(qa_s/x_sz;q)_{r-t}^{-1}\right). \tag{VII.15}$$

The determinant can be evaluated by means of a limiting case of Lemma A.22. Namely, first multiply both sides of the $C \mapsto 0$ case of (A.23) with $(-A)^{-\binom{r}{2}}q^{-\binom{r}{3}}$ and then let $A \to \infty$ in the resulting identity to see that

$$\det_{1\le s,t\le r}\left(X_s^{r-t}(BX_s;q)_{r-t}^{-1}\right) = \prod_{i=1}^{r}(BX_i;q)_{r-1}^{-1}\prod_{1\le i<j\le r}(X_i - X_j). \tag{VII.16}$$

Now evaluate the determinant in (VII.15) by the $X_s \mapsto a_s/x_s$ and $B \mapsto q/z$ case of (VII.16). Substituting our calculations and performing some other elementary manipulations leads to the following identity:

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(a_i;q)_{k_i}}{(bx_i;q)_{k_i}}\left(\frac{zx_i}{a_i}\right)^{k_i}\cdot q^{\sum_{i=1}^{r}(i-r)k_i}\right)$$
$$= \prod_{1\le i<j\le r}\left(\frac{x_i/a_i - x_j/a_j}{x_i - x_j}\right)\prod_{i=1}^{r}\frac{(q, bx_i/a_i, zx_iq^{1-r}, q^r/zx_i;q)_\infty}{(bx_i, q/a_i, zx_iq^{1-r}/a_i, bq^{i-1}/z;q)_\infty}, \tag{VII.17}$$

provided the series terminates or $|bx_i/a_i| < |z| < |q|^{r-1} < 1$ for $i = 1,\dots,r$.

Actually, (VII.17) is equivalent to (VII.7), since (VII.17) can be obtained by doing the replacements $k_i \mapsto -k_i$, $x_i \mapsto 1/x_i$, $b_i \mapsto q/a_i$, for $i = 1,\dots,r$, $a \mapsto q/b$, and $z \mapsto q^{r-1}b/z$ in (VII.7), and some elementary manipulations.

On the other hand, if we set $a_s = ax_s$ and $z_s \equiv zb_s/x_s$ in (VII.11) the determinant equals

$$a^{\binom{r}{2}}\prod_{i=1}^{r}(1/az;q)_{r-i} \det_{1\le s,t\le r}\left((qx_s/b_sz;q)_{r-t}^{-1}\right). \tag{VII.18}$$

The determinant can be evaluated by means of the $A, C \to 0$ limiting case of Lemma A.22, reading

$$\det_{1\le s,t\le r}\left((BX_s;q)_{r-t}^{-1}\right) = B^{\binom{r}{2}}q^{2\binom{r}{3}}\prod_{i=1}^{r}(BX_i;q)_{r-1}^{-1}\prod_{1\le i<j\le r}(X_i - X_j). \tag{VII.19}$$

Now evaluate the determinant in (VII.18) by the $X_s \mapsto x_s/b_s$ and $B \mapsto q/z$ case of (VII.19). Again, substituting our calculations and performing some other elementary manipulations we arrive at the following summation:

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(ax_i;q)_{k_i}}{(b_i;q)_{k_i}}\left(\frac{zb_i}{x_i}\right)^{k_i}\cdot q^{\sum_{i=1}^{r}(i-r)k_i}\right)$$
$$=\prod_{1\le i<j\le r}\left(\frac{1-b_jx_i/b_ix_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(q,b_i/ax_i,azb_iq^{1-i},q^i/azb_i;q)_\infty}{(b_i,q/ax_i,zb_iq^{1-r}/x_i,q^{i-1}/az;q)_\infty},\quad\text{(VII.20)}$$

provided the series terminates or $|b_i/ax_i|<|z|<|q|^{r-1}<1$ for $i=1,\dots,r$.

As in the case above we have not obtained a new identity, since (VII.20) is equivalent to (VII.8), where (VII.20) can be obtained by doing the replacements $k_i\mapsto -k_i$, $x_i\mapsto 1/x_i$, $a_i\mapsto q/b_i$, for $i=1,\dots,r$, $b\mapsto q/a$, and $z\mapsto q^{r-1}a/z$ in (VII.8), and some elementary manipulations.

Remark VII.21. The choice $b_i\equiv qa_i$ in (VII.10) is a special case of the summation formula implied by Theorem VII.46, where we even could have started with different bases q_i in the series (VII.10). In this case most of the factors in (VII.11) cancel.

Before we state the C_r extensions of Bailey's ${}_6\psi_6$ summation formula that we are going to prove, it might be convenient to recall Bailey's original ${}_6\psi_6$ summation. This is (cf. [9], [25, (5.3.1)])

Theorem VII.22 (Bailey's classical ${}_6\psi_6$ sum). *Let a, b, c, d, and e be indeterminate, and suppose that none of the denominators in* (VII.23) *vanish. Then*

$$\sum_{k=-\infty}^{\infty}\frac{1-aq^{2k}}{1-a}\frac{(b,c,d,e;q)_k}{(aq/b,aq/c,aq/d,aq/e;q)_k}\left(\frac{qa^2}{bcde}\right)^k$$
$$=\frac{(aq,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de,q,q/a;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,qa^2/bcde;q)_\infty},\quad\text{(VII.23)}$$

provided the series terminates or $|q|<1$ and $|qa^2/bcde|<1$.

Remark VII.24. Andrews [1] discusses some applications of Bailey's very-well-poised ${}_6\psi_6$ summation formula to number theory.

Our C_r extensions of Bailey's bilateral very-well poised ${}_6\psi_6$ summation formula are the following.

Theorem VII.25 (C_r ${}_6\psi_6$ summations). *Let $x_1,\dots,x_r$, $e_1,\dots,e_r$, a, b, c, d, and e be indeterminate, let $r\ge 1$, and suppose that none of the denominators in* (VII.26) *or* (VII.27) *vanish. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{q^{-k_i}/x_i-q^{-k_j}/x_j}{1/x_i-1/x_j}\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right)\prod_{i=1}^{r}\left(\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2}\right)\right.$$

$$\times \prod_{i=1}^{r} \frac{(bx_i, cx_i, dx_i, e_ix_i; q)_{k_i}}{(ax_iq/b, ax_iq/c, ax_iq/d, ax_iq/e_i; q)_{k_i}} \left(\frac{a^2q}{bcde_i}\right)^{k_i}\Bigg)$$
$$= a^{\binom{r}{2}} \prod_{1\le i<j\le r} \left(\frac{1/e_i - 1/e_j}{1/x_i - 1/x_j} \frac{1}{1-ax_ix_j}\right) \prod_{i=1}^{r} \frac{(ax_i^2q, aq^{2-i}/bc, aq^{2-i}/bd; q)_\infty}{(ax_iq/b, ax_iq/c, ax_iq/d; q)_\infty}$$
$$\times \prod_{i=1}^{r} \frac{(aq/be_i, aq^{2-i}/cd, aq/ce_i, aq/de_i, q, q/ax_i^2; q)_\infty}{(ax_iq/e_i, q/bx_i, q/cx_i, q/dx_i, q/e_ix_i, a^2q^{2-r}/bcde_i; q)_\infty}, \quad \text{(VII.26)}$$

provided the series terminates or $|q| < 1$ *and* $|a^2q^{2-r}/bcde_i| < 1$ *for* $i = 1, \dots, r$,

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le r} \left(\frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i - 1/x_j} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right) \prod_{i=1}^{r} \left(\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2}\right)$$
$$\times \prod_{i=1}^{r} \frac{(bx_i, cx_i, dx_i/e_i, e_ix_i; q)_{k_i}}{(ax_iq/b, ax_iq/c, ae_ix_iq/d, ax_iq/e_i; q)_{k_i}} \left(\frac{a^2q}{bcd}\right)^{k_i}\Bigg)$$
$$= a^{\binom{r}{2}} \prod_{1\le i<j\le r} \left(\frac{1/e_i - 1/e_j}{1/x_i - 1/x_j} \frac{1-e_ie_j/d}{1-ax_ix_j}\right) \prod_{i=1}^{r} \frac{(ax_i^2q, aq^{2-i}/bc, ae_iq/bd; q)_\infty}{(ax_iq/b, ax_iq/c, ae_ix_iq/d; q)_\infty}$$
$$\times \prod_{i=1}^{r} \frac{(aq/be_i, ae_iq/cd, aq/ce_i, aq/d, q, q/ax_i^2; q)_\infty}{(ax_iq/e_i, q/bx_i, q/cx_i, e_iq/dx_i, q/e_ix_i, a^2q^{2-i}/bcd; q)_\infty}, \quad \text{(VII.27)}$$

provided the series terminates or $|q| < 1$ *and* $|a^2q^{2-r}/bcd| < 1$.

Proof. We start with the sum

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le r} \left(\frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i - 1/x_j} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right) \prod_{i=1}^{r} \left(\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2}\right)$$
$$\times \prod_{i=1}^{r} \frac{(bx_i, c_ix_i, d_ix_i, e_ix_i; q)_{k_i}}{(ax_iq/b, ax_iq/c_i, ax_iq/d_i, ax_iq/e_i; q)_{k_i}} \left(\frac{a^2q}{bc_id_ie_i}\right)^{k_i}\Bigg), \quad \text{(VII.28)}$$

and specialize the parameters c_i, d_i later. We have

$$\prod_{1\le i<j\le r} \left(\frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i - 1/x_j} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right) = \prod_{1\le i<j\le r} [(1/x_j - 1/x_i)(1-ax_ix_j)]^{-1}$$
$$\times (a/b)^{\binom{r}{2}} q^{-\binom{r}{3}} \det_{1\le s,t\le r} \left((bq^{-k_s}/ax_s; q)_{r-t} \, (bx_sq^{k_s}; q)_{r-t} \right),$$

due to the $X_s \mapsto q^{-k_s}/x_s$, $A \mapsto b/a$, $B \mapsto 0$, and $C \mapsto a$ case of Lemma A.22.

Hence, using the elementary identities [25, (I.13) and (I.18)]

$$(bq^{-k_s}/ax_s;q)_{r-t} = \frac{(b/ax_s;q)_{r-t}\,(ax_sq/b;q)_{k_s}}{(ax_sq^{1-r+t}/b;q)_{k_s}}\,q^{(t-r)k_s}$$

and

$$(bx_sq^{k_s};q)_{r-t} = \frac{(bx_s;q)_{r-t}\,(bx_sq^{r-t};q)_{k_s}}{(bx_s;q)_{k_s}},$$

we may write (VII.28), when multiplied by $\prod_{1\le i<j\le r}[(1/x_j-1/x_i)(1-ax_ix_j)]$, as

$$(a/b)^{\binom r2}\,q^{-\binom r3}\det_{1\le s,t\le r}\Bigg((b/ax_s;q)_{r-t}\,(bx_s;q)_{r-t}\sum_{k_s=-\infty}^{\infty}\frac{1-ax_s^2q^{2k_s}}{1-ax_s^2}$$
$$\cdot\frac{(bx_sq^{r-t},c_sx_s,d_sx_s,e_sx_s;q)_{k_s}}{(ax_sq^{1-r+t}/b,ax_sq/c_s,ax_sq/d_s,ax_sq/e_s;q)_{k_s}}\left(\frac{a^2q^{1+t-r}}{bc_sd_se_s}\right)^{k_s}\Bigg).$$

Now, to the sum inside the determinant we apply Bailey's classical ${}_6\psi_6$ summation, (VII.23), with $a\mapsto ax_s^2$, $b\mapsto bx_sq^{r-t}$, $c\mapsto c_sx_s$, $d\mapsto d_sx_s$, and $e\mapsto e_sx_s$. Thus we obtain

$$(a/b)^{\binom r2}\,q^{-\binom r3}\det_{1\le s,t\le r}\Bigg((b/ax_s;q)_{r-t}\,(bx_s;q)_{r-t}\,\frac{(ax_s^2q,aq^{1-r+t}/bc_s;q)_\infty}{(ax_sq^{1-r+t}/b,ax_sq/c_s;q)_\infty}$$
$$\cdot\frac{(aq^{1-r+t}/bd_s,aq^{1-r+t}/be_s,aq/c_sd_s,aq/c_se_s,aq/d_se_s,q,q/ax_s^2;q)_\infty}{(ax_sq/d_s,ax_sq/e_s,q^{1-r+t}/bx_s,q/c_sx_s,q/d_sx_s,q/e_sx_s,a^2q^{1-r+t}/bc_sd_se_s;q)_\infty}\Bigg).$$

Now, by using linearity of the determinant with respect to rows, we take some factors out of the determinant and obtain

$$\prod_{i=1}^r\frac{(ax_i^2q,aq/bc_i,aq/bd_i,aq/be_i,aq/c_id_i,aq/c_ie_i,aq/d_ie_i,q,q/ax_i^2;q)_\infty}{(ax_iq/b,ax_iq/c_i,ax_iq/d_i,ax_iq/e_i,q/bx_i,q/c_ix_i,q/d_ix_i,q/e_ix_i,qa^2/bc_id_ie_i;q)_\infty}$$
$$\times(a/b)^{\binom r2}\,q^{-\binom r3}\det_{1\le s,t\le r}\left(\frac{(bc_s/a;q)_{r-t}\,(bd_s/a;q)_{r-t}\,(be_s/a;q)_{r-t}}{(bc_sd_se_s/a^2;q)_{r-t}}\right).\qquad\text{(VII.29)}$$

To evaluate the determinant in (VII.29) we choose different specializations of the parameters c_s, d_s, for $s=1,\dots,r$, for which the determinant can be reduced to a product by means of Lemma A.22.

One choice is $c_s\equiv c$, and $d_s\equiv d$. In this case the determinant in (VII.29) equals

$$\prod_{i=1}^r[(bc/a;q)_{r-i}\,(bd/a;q)_{r-i}]\det_{1\le s,t\le r}\left(\frac{(be_s/a;q)_{r-t}}{(bcde_s/a^2;q)_{r-t}}\right).$$

The determinant can be evaluated by means of Lemma A.22 with $X_s \mapsto e_s$, $A \mapsto b/a$, $B \mapsto bcd/a^2$, and $C \mapsto 0$. Subsequently, substituting our calculations and performing some other elementary manipulations leads to (VII.26).

The other choice is $c_s \equiv c$, and $d_s \equiv d/e_s$. In this case the determinant in (VII.29) equals

$$\prod_{i=1}^{r} \frac{(bc/a;q)_{r-i}}{(bcd/a^2;q)_{r-i}} \det_{1\le s,t\le r} \left((bd/ae_s;q)_{r-t}\,(be_s/a;q)_{r-t}\right).$$

The determinant can be evaluated by means of Lemma A.22 with $X_s \mapsto e_s$, $A \mapsto b/a$, $B \mapsto 0$, and $C \mapsto d$. Finally, substituting our calculations and performing some other elementary manipulations leads to (VII.27). □

REMARK VII.30. Other multivariate extensions, associated to root systems, of the very-well-poised ${}_6\psi_6$ summation formula were derived by Gustafson [31] using difference equations. He used these higher-dimensional ${}_6\psi_6$ summations to obtain, by specialization and limits, the Macdonald identities for the affine root systems of classical type.

We state our C_r extension of Jackson's balanced very-well poised ${}_8\phi_7$ summation formula.

THEOREM VII.31 ((DEDICATED TO TEJASI*) A C_r JACKSON'S SUM). *Let $x_1, \dots, x_r$, and a, b, c, and d be indeterminate, let N be a nonnegative integer, let $r \ge 1$, and suppose that none of the denominators in* (VII.32) *vanish. Then,*

$$\sum_{k_1,\dots,k_r=0}^{N} \Bigg(\prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\,\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \right) \prod_{i=1}^{r} \left(\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2} \right)$$
$$\times \prod_{i=1}^{r} \frac{(ax_i^2, bx_i, cx_i, dx_i, a^2x_iq^{2-r+N}/bcd, q^{-N};q)_{k_i}}{(q, ax_iq/b, ax_iq/c, ax_iq/d, bcdx_iq^{r-1-N}/a, ax_i^2q^{1+N};q)_{k_i}} \cdot q^{\sum_{i=1}^r i\,k_i} \Bigg)$$
$$= \prod_{1\le i<j\le r} \left(\frac{1-ax_ix_jq^N}{1-ax_ix_j} \right) \prod_{i=1}^{r} \frac{(ax_i^2q, aq^{2-i}/bc, aq^{2-i}/bd, aq^{2-i}/cd;q)_N}{(aq^{2-r}/bcdx_i, ax_iq/d, ax_iq/c, ax_iq/b;q)_N}. \qquad \text{(VII.32)}$$

REMARK VII.33. Note that all summation indices on the left hand side of (VII.32) have the same range. Summing over a cube is strange but essential for the determinant in our derivation of (VII.32) to simplify. In fact, the series

$$\sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,r}} \Bigg(\prod_{1\le i<j\le r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\,\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \right) \prod_{i=1}^{r} \left(\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2} \right)$$

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$$\left.\times \prod_{i=1}^{r} \frac{(ax_i^2, bx_i, cx_i, dx_i, a^2x_iq^{2-r+N_i}/bcd, q^{-N_i};q)_{k_i}}{(q, ax_iq/b, ax_iq/c, ax_iq/d, bcdx_iq^{r-1-N_i}/a, ax_i^2q^{1+N_i};q)_{k_i}} \cdot q^{\sum_{i=1}^r i\,k_i}\right)$$

does not factor, except for $N_1 = N_2 = \cdots = N_r$. To our knowledge, such a phenomenon has not occurred so far with terminating multiple series associated to root systems.

Unfortunately, we cannot use Theorem VII.31 for deriving a multiple ${}_{10}\phi_9$ transformation, analogously as we derived some new ${}_{10}\phi_9$ transformations in chapter V.

Proof of Theorem VII.31. We have

$$q^{\sum_{i=1}^r (i-r)\,k_i} \prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\,\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right)$$

$$= \prod_{1\le i<j\le r}\left(\frac{q^{-k_i}/x_i-q^{-k_j}/x_j}{1/x_i-1/x_j}\,\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right)$$

$$= \prod_{1\le i<j\le r}[(1/x_j-1/x_i)(1-ax_ix_j)]^{-1}\prod_{i=1}^{r}\frac{(q^{2-r-k_i}/cx_i;q)_{r-1}\,(ax_iq^{2-r+k_i}/c;q)_{r-1}}{(aq^{2-r}/bc;q)_{i-1}\,(bq^{2+r-2i}/c;q)_{i-1}}$$

$$\times (a/b)^{\binom r2}\, q^{-\binom r3} \det_{1\le s,t\le r}\left(\frac{(bq^{-k_s}/ax_s;q)_{r-t}\,(bx_sq^{k_s};q)_{r-t}}{(q^{2-r-k_s}/cx_s;q)_{r-t}\,(ax_sq^{2-r+k_s}/c;q)_{r-t}}\right),$$

due to the $X_s\mapsto q^{-k_s}/x_s$, $A\mapsto b/a$, $B\mapsto q^{2-r}/c$, and $C\mapsto a$ case of Lemma A.22. Hence, using some elementary identities from [25, Appendix I], we may write the left hand side of (VII.32), when multiplied by $\prod_{1\le i<j\le r}[(1/x_j-1/x_i)(1-ax_ix_j)]$, as

$$(a^2/bc^2)^{\binom r2}\, q^{-3\binom r3} \prod_{i=1}^{r}[(aq^{2-r}/bc;q)_{i-1}\,(bq^{2+r-2i}/c;q)_{i-1}]^{-1}$$

$$\times \det_{1\le s,t\le r}\Bigg((b/ax_s, bx_s;q)_{r-t}\,(c/ax_s, cx_s;q)_{t-1}\sum_{k_s=0}^{N}\frac{1-ax_s^2q^{2k_s}}{1-ax_s^2}$$

$$\left.\cdot\frac{(ax_s^2, bx_sq^{r-t}, cx_sq^{t-1}, dx_s, a^2x_sq^{2-r+N}/bcd, q^{-N};q)_{k_s}}{(q, ax_sq^{1-r-t}/b, ax_sq^{2-t}/c, ax_sq/d, bcdx_sq^{r-1-N}/a, ax_i^2q^{1+N};q)_{k_s}}\,q^{k_s}\right).$$

Now, to the sum inside the determinant we apply Jackson's classical ${}_8\phi_7$ summation, Theorem IV.8, with the replacements $a\mapsto ax_s^2$, $b\mapsto bx_sq^{r-t}$, $c\mapsto cx_sq^{t-1}$, $d\mapsto dx_s$, and $n\mapsto N$. Thus we obtain

$$(a^2/bc^2)^{\binom r2}\, q^{-3\binom r3} \prod_{i=1}^{r}[(aq^{2-r}/bc;q)_{i-1}\,(bq^{2+r-2i}/c;q)_{i-1}]^{-1}$$

$$\times \det_{1\le s,t\le r}\Bigg((b/ax_s, bx_s;q)_{r-t}\,(c/ax_s, cx_s;q)_{t-1} \cdot \frac{(ax_s^2q, aq^{2-r}/bc, aq^{1-r+t}/bd, aq^{2-t}/cd;q)_N}{(aq^{2-r}/bcdx_s, ax_sq/d, ax_sq^{2-t}/c, ax_sq^{1-r+t}/b;q)_N}\Bigg).$$

Now, we take some factors out of the determinant and obtain

$$(a^3/bc^4)^{\binom{r}{2}}\, q^{3N\binom{r}{2}-7\binom{r}{3}} \prod_{i=1}^{r} \frac{(cq^{-N}/ax_i;q)_{r-1}\,(cx_i;q)_{r-1}}{(aq^{2-r}/bc;q)_{i-1}\,(bq^{2+r-2i}/c;q)_{i-1}} \times \prod_{i=1}^{r} \frac{(ax_i^2q, aq^{2-r}/bc, aq^{1-r+i}/bd, aq^{2-i}/cd;q)_N}{(aq^{2-r}/bcdx_i, ax_iq/d, ax_iq/c, ax_iq/b;q)_N} \times \det_{1\le s,t\le r}\left(\frac{(bq^{-N}/ax_s;q)_{r-t}\,(bx_s;q)_{r-t}}{(q^{2-r}/cx_s;q)_{r-t}\,(ax_sq^{2-r+N}/c;q)_{r-t}}\right). \quad \text{(VII.34)}$$

The determinant can be evaluated by means of Lemma A.22 with $X_s \mapsto 1/x_s$, $A \mapsto bq^{-N}/a$, $B \mapsto q^{2-r}/c$, and $C \mapsto aq^N$. Subsequently, substituting our calculations and performing some other elementary manipulations leads to (VII.32). □

We state an important limiting case of Theorem VII.31, namely an A_r extension of Jackson's [39] q-analog of the Pfaff-Saalschütz formula [25, (1.7.2)], a summation theorem for a terminating and balanced ${}_3\phi_2$ series. The $q = 1$ case, the ${}_3F_2$ summation theorem [25, (1.7.1)], was originally found by Pfaff [62], and was rediscovered by Saalschütz [65].

THEOREM VII.35 (AN A_r q-PFAFF-SAALSCHÜTZ SUM). *Let $x_1, \dots, x_r$, and a, b, and c be indeterminate, let N be a nonnegative integer, let $r \ge 1$, and suppose that none of the denominators in* (VII.36) *vanish. Then,*

$$\sum_{k_1,\dots,k_r=0}^{N} \left(\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{i=1}^{r} \frac{(ax_i, bx_i, q^{-N};q)_{k_i}}{(q, cx_i, abx_iq^{r-N}/c;q)_{k_i}} \cdot q^{\sum_{i=1}^r i\,k_i}\right) = \prod_{i=1}^{r} \frac{(cq^{1-i}/a, cq^{1-i}/b;q)_N}{(cx_i, cq^{1-r}/abx_i;q)_N}. \quad \text{(VII.36)}$$

PROOF. The proof is just as in the classical one-dimensional case. Take the limit $a \to 0$ after replacing d by aq/d in (VII.32). Finally, relabel $c \mapsto a$ and $d \mapsto c$ in the resulting identity to obtain (VII.36). □

Other important specializations of our Theorems VII.25 and VII.31 are nonterminating and terminating C_r ${}_6\phi_5$ summations.

THEOREM VII.37 (C_r NONTERMINATING ${}_6\phi_5$ SUMMATIONS). *Let $x_1, \ldots, x_r$, a, b, c, and d be indeterminate, let $r \geq 1$, and suppose that none of the denominators in* (VII.38) *or* (VII.39) *vanish. Then*

$$\sum_{k_1,\ldots,k_r=0}^{\infty} \left(\prod_{1\leq i<j\leq r} \left(\frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i - 1/x_j} \, \frac{1 - ax_ix_jq^{k_i+k_j}}{1 - ax_ix_j} \right) \prod_{i=1}^{r} \left(\frac{1 - ax_i^2q^{2k_i}}{1 - ax_i^2} \right) \right.$$
$$\left. \times \prod_{i=1}^{r} \frac{(ax_i^2, bx_i, cx_i, dx_i; q)_{k_i}}{(q, ax_iq/b, ax_iq/c, ax_iq/d; q)_{k_i}} \left(\frac{aq}{bcdx_i} \right)^{k_i} \right)$$
$$= \prod_{1\leq i<j\leq r} (1 - ax_ix_j)^{-1} \prod_{i=1}^{r} \frac{(ax_i^2q, aq^{2-i}/bc, aq^{2-i}/bd, aq^{2-i}/cd; q)_\infty}{(aq^{2-r}/bcdx_i, ax_iq/d, ax_iq/c, ax_iq/b; q)_\infty}, \quad \text{(VII.38)}$$

provided the series terminates or $|q| < 1$ and $|aq^{2-r}/bcdx_i| < 1$ for $i = 1, \ldots, r$,

$$\sum_{k_1,\ldots,k_r=0}^{\infty} \left(\prod_{1\leq i<j\leq r} \left(\frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i - 1/x_j} \, \frac{1 - ax_ix_jq^{k_i+k_j}}{1 - ax_ix_j} \right) \prod_{i=1}^{r} \left(\frac{1 - ax_i^2q^{2k_i}}{1 - ax_i^2} \right) \right.$$
$$\left. \times \prod_{i=1}^{r} \frac{(ax_i^2, bx_i, cx_i, d; q)_{k_i}}{(q, ax_iq/b, ax_iq/c, ax_i^2q/d; q)_{k_i}} \left(\frac{aq}{bcd} \right)^{k_i} \right)$$
$$= \prod_{1\leq i<j\leq r} \frac{(1 - ax_ix_j/d)}{(1 - ax_ix_j)} \prod_{i=1}^{r} \frac{(ax_i^2q, aq^{2-i}/bc, ax_iq/bd, ax_iq/cd; q)_\infty}{(aq^{2-i}/bcd, ax_i^2q/d, ax_iq/c, ax_iq/b; q)_\infty}, \quad \text{(VII.39)}$$

provided the series terminates or $|q| < 1$ and $|aq^{2-r}/bcd| < 1$.

PROOF. For (VII.38), let $e_i \equiv ax_i$ in (VII.26), or, equivalently, $N \to \infty$ in (VII.32). For (VII.39), let $e_i \equiv ax_i$ and $d \mapsto ad$ in (VII.27). $\square$

THEOREM VII.40 (A C_r TERMINATING ${}_6\phi_5$ SUMMATION). *Let $x_1, \ldots, x_r$, a, b, and c be indeterminate, let N be a nonnegative integer, let $r \geq 1$, and suppose that none of the denominators in* (VII.41) *vanish. Then*

$$\sum_{k_1,\ldots,k_r=0}^{N} \left(\prod_{1\leq i<j\leq r} \left(\frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i - 1/x_j} \, \frac{1 - ax_ix_jq^{k_i+k_j}}{1 - ax_ix_j} \right) \prod_{i=1}^{r} \left(\frac{1 - ax_i^2q^{2k_i}}{1 - ax_i^2} \right) \right.$$
$$\left. \times \prod_{i=1}^{r} \frac{(ax_i^2, bx_i, cx_i, q^{-N}; q)_{k_i}}{(q, ax_iq/b, ax_iq/c, ax_i^2q^{1+N}; q)_{k_i}} \left(\frac{aq^{1+N}}{bc} \right)^{k_i} \right)$$
$$= \prod_{1\leq i<j\leq r} \frac{(1 - ax_ix_jq^N)}{(1 - ax_ix_j)} \prod_{i=1}^{r} \frac{(ax_i^2q, aq^{2-i}/bc; q)_N}{(ax_iq/b, ax_iq/c; q)_N}, \quad \text{(VII.41)}$$

provided the series terminates or $|q| < 1$ and $|aq^{2+N-r}/bc| < 1$.

PROOF. Let $d \mapsto q^{-N}$ in (VII.39), or, equivalently, $d \to \infty$ in (VII.32). $\square$

If we specialize our above theorems further, we obtain A_r extensions of various important classical summation theorems such as the q-Gauß, q-Chu-Vandermonde, and q-binomial theorem.

Finally, we give multiple extensions of Chu's [18, eq. (15)] general formula transforming a bilateral ${}_{p+2}\psi_{p+2}$ series of Gasper-Karlsson-Minton type into a multiple of a unilateral (one-sided) ${}_{p+2}\phi_{p+1}$ series of Gasper-Karlsson-Minton type. (We say that a ${}_{p+2}\psi_{p+2}$ series is of Gasper-Karlsson-Minton type if there are p upper parameters $a_1, \dots, a_p$ and p lower parameters $b_1, \dots, b_p$ such that each a_i differs from b_i by a nonnegative integer power of q, i.e. $a_i = b_i q^{m_i}$, $m_i \geq 0$, [25, sec. 1.9].) Our observation is quite interesting, namely, not only that we may employ several bases $q_1, \dots, q_r$ in the multiple series but our calculations can also be carried out using various "Vandermonde determinants", corresponding to the associated root systems. Further multilateral transformations may be deduced by employing other determinants in our series. Our intention is merely to give an idea how such calculations work.

Chu's general bilateral transformation formula reads

THEOREM VII.42 (CHU). *Let a, c, d, z, and $b_1, \dots, b_p$ be indeterminate, let N be an integer and $m_1, \dots, m_p$ nonnegative integers, and suppose that none of the denominators in* (VII.43) *vanish. Then*

$$\sum_{k=-\infty}^{\infty} \frac{(a, z, b_1 q^{m_1}, b_2 q^{m_2}, \dots, b_p q^{m_p}; q)_k}{(c, qdz, b_1, b_2, \dots, b_p; q)_k} \left(q^{1-N}/a\right)^k$$
$$= z^N \frac{(qz/a, c/z, qd, q; q)_\infty}{(qdz, q/z, q/a, c; q)_\infty} \prod_{j=1}^{p} \frac{(b_j/z; q)_{m_j}}{(b_j; q)_{m_j}}$$
$$\times \sum_{k=0}^{\infty} \frac{(1/d, qz/c, qz/b_1, qz/b_2, \dots, qz/b_p; q)_k}{(q, qz/a, q^{1-m_1}z/b_1, q^{1-m_2}z/b_2, \dots, q^{1-m_p}z/b_p; q)_k} \left(cdq^{N-|\mathbf{m}|}\right)^k, \quad \text{(VII.43)}$$

provided the series terminate or $|q| < 1$ and $|q/a| < |q^N| < |q^{|\mathbf{m}|}/cd|$. Here, we have used the notation $|\mathbf{m}| = m_1 + \dots + m_p$.

REMARK VII.44. Note, that when $d = 1$, the sum on the right hand side of (VII.43) reduces to a single term, and hence, (VII.43) reduces to a summation formula. On the other hand, when $c = q$ we obtain a ${}_{p+2}\phi_{p+1}$ transformation (where the $N = |\mathbf{m}|$ case was derived in [23]). Jim Haglund [34] has stumbled over the $c = q$ case of (VII.43) via rook theory. He has also noticed that (VII.43) can be obtained by specializing a general transformation formula for bilateral series, due to Slater [70], [25, (5.4.4)].

We give multiple generalizations of (VII.43) associated to the root systems A_r, B_r, C_r, and D_r of classical type (cf. [15]).

The determinant evaluations we use in these cases are listed in the following lemma. We remark that the evaluations are basically the Weyl denominator factorizations of type A_r, B_r, C_r, and D_r, respectively (cf. [22, Lemma 24.3]).

LEMMA VII.45. *The following determinant evaluations hold*

$$\det_{1\le i,j\le r}\left(x_i^{r-j}\right) = \prod_{1\le i<j\le r}(x_i - x_j) \tag{A}$$

$$\det_{1\le i,j\le r}\left(x_i^{r-j} - x_i^{r+j-1}\right) = \prod_{1\le i<j\le r}[(x_i - x_j)(1 - x_i x_j)]\prod_{i=1}^{r}(1 - x_i) \tag{B}$$

$$\det_{1\le i,j\le r}\left(x_i^{r-j} - x_i^{r+j}\right) = \prod_{1\le i<j\le r}[(x_i - x_j)(1 - x_i x_j)]\prod_{i=1}^{r}(1 - x_i^2) \tag{C}$$

$$\frac{1}{2}\cdot\det_{1\le i,j\le r}\left(x_i^{r-j} + x_i^{r+j-2}\right) = \prod_{1\le i<j\le r}[(x_i - x_j)(1 - x_i x_j)]. \tag{D}$$

PROOF. Identities (A), (B), (C), and (D) are readily proved by the standard argument that proves Vandermonde-type determinant evaluations. Namely, first it is verified that the determinant on the left hand side vanishes whenever one of the factors of the right hand side vanishes. Then one checks that both sides are polynomials in $x_1, \dots, x_r$ of the same total degree. Hence, the determinant equals a constant times the right hand side. The constant is easily computed by comparing coefficients of a "maximal" or "minimal" monomial. $\square$

Let G denote A_r, B_r, C_r, or D_r. Besides, let $\Delta_G[x_1, \dots, x_r]$ denote the product side of the G-Vandermonde as displayed in Lemma VII.45. Now we can state a general multilateral transformation formula for basic hypergeometric series associated to G.

THEOREM VII.46. *Let a_i, c_i, d_i, z_i, and $b_{i1}, \dots, b_{ip}$ be indeterminate, let N_i be integers and $m_{i1}, \dots, m_{ip}$ nonnegative integers, for $i = 1, \dots, r$, and suppose that none of the denominators in* (VII.47) *vanish. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\frac{\Delta_G\left[q_1^{-k_1}/x_1, \dots, q_r^{-k_r}/x_r\right]}{\Delta_G[1/x_1, \dots, 1/x_r]}\right.$$
$$\left.\times\prod_{i=1}^{r}\frac{(a_i, z_i, b_{i1}q_i^{m_{i1}}, \dots, b_{ip}q_i^{m_{ip}}; q_i)_{k_i}}{(c_i, q_i d_i z_i, b_{i1}, \dots, b_{ip}; q_i)_{k_i}}\left(q_i^{1-N_i}/a_i\right)^{k_i}\right)$$

$$= \prod_{i=1}^{r} z_i^{N_i} \frac{(q_i z_i/a_i, c_i/z_i, q_i d_i, q_i; q_i)_\infty}{(q_i d_i z_i, q_i/z_i, q_i/a_i, c_i; q_i)_\infty} \prod_{i=1}^{r}\prod_{j=1}^{p} \frac{(b_{ij}/z_i; q_i)_{m_{ij}}}{(b_{ij}; q_i)_{m_{ij}}}$$

$$\times \frac{\Delta_G\left[z_1/x_1, \dots, z_r/x_r\right]}{\Delta_G\left[1/x_1, \dots, 1/x_r\right]} \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg(\frac{\Delta_G\left[q_1^{k_1} z_1/x_1, \dots, q_r^{k_r} z_r/x_r\right]}{\Delta_G\left[z_1/x_1, \dots, z_r/x_r\right]}$$

$$\times \prod_{i=1}^{r} \frac{(1/d_i, q_i z_i/c_i, q_i z_i/b_{i1}, \dots, q_i z_i/b_{ip}; q_i)_{k_i}}{(q_i, q_i z_i/a_i, q_i^{1-m_{i1}} z_i/b_{i1}, \dots, q_i^{1-m_{ip}} z_i/b_{ip}; q_i)_{k_i}} \left(c_i d_i q_i^{N_i - |\mathbf{m}_i|}\right)^{k_i} \Bigg), \tag{VII.47}$$

provided the series terminate or converge (where we have used the notation $|\mathbf{m}_i| = m_{i1} + \cdots + m_{ip}$*, for* $i = 1, \dots, r$*).*

Sketch of proof. We start with the left hand side of (VII.47). Using Lemma VII.45, we substitute the corresponding determinant for $\Delta_G\left[q_1^{-k_1}/x_1, \dots, q_r^{-k_r}/x_r\right]$, and by using linearity of the determinant with respect to rows, we write the whole series as a single determinant. Now we transform the terms in the determinant by Theorem VII.42. In the resulting determinant, again by using linearity of the determinant with respect to rows, we take some factors out and are left with a determinant for which we can apply Lemma VII.45 again, obtaining the right hand side of (VII.47). It is an easy exercise to verify that for the respective determinants of Lemma VII.45 the described calculations indeed work out well and yield (VII.47). □

Remark VII.48. It is also possible to extend (VII.43) by using other determinants, instead of the respective G-Vandermonde determinants of Lemma VII.45, in the analysis of our above sketch of proof.

For example, we can extend Theorem VII.46 by using the determinantal definition of Schur functions (cf. [22]). Given a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of length $\le r$ (cf. [3]), the Schur function S_λ is defined by

$$S_\lambda(x_1, \dots, x_r) := \frac{\det_{1\le i,j\le r}\left(x_i^{r+\lambda_j - j}\right)}{\det_{1\le i,j\le r}\left(x_i^{r-j}\right)}.$$

We can extend this definition to "Schur functions" associated to root systems of classical type. Given a root system G of classical type, i.e. G is A_r, B_r, C_r, or D_r, respectively, and a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of length $\le r$, then define $S_{G;\lambda}$ (this definition is *not* standard) by

$$S_{A_r;\lambda}(x_1, \dots, x_r) := S_\lambda(x_1, \dots, x_r), \tag{SA}$$

$$S_{B_r;\lambda}(x_1,\dots,x_r) := \frac{\det\limits_{1\le i,j\le r}\left(x_i^{r+\lambda_j-j}-x_i^{r+j-\lambda_j-1}\right)}{\det\limits_{1\le i,j\le r}\left(x_i^{r-j}-x_i^{r+j-1}\right)}, \tag{SB}$$

$$S_{C_r;\lambda}(x_1,\dots,x_r) := \frac{\det\limits_{1\le i,j\le r}\left(x_i^{r+\lambda_j-j}-x_i^{r+j-\lambda_j}\right)}{\det\limits_{1\le i,j\le r}\left(x_i^{r-j}-x_i^{r+j}\right)}, \tag{SC}$$

$$S_{D_r;\lambda}(x_1,\dots,x_r) := \frac{\det\limits_{1\le i,j\le r}\left(x_i^{r+\lambda_j-j}-x_i^{r+j-\lambda_j-2}\right)}{\det\limits_{1\le i,j\le r}\left(x_i^{r-j}-x_i^{r+j-2}\right)}, \tag{SD}$$

compare with Lemma VII.45. With this determinantal definition, we can add an arbitrary partition λ of length $\le r$, in the formulation of Theorem VII.46: It is easily verified that (VII.47) also holds if we replace all G-Vandermonde determinants occurring on both sides in the series of (VII.47) by their multiplication with the corresponding function $S_{G;\lambda}$,

$$\Delta_G[y_1,\dots,y_r] \mapsto \Delta_G[y_1,\dots,y_r]\cdot S_{G;\lambda}(y_1,\dots,y_r),$$

for the respective values of $y_1,\dots,y_r$, where all other expressions on both sides of (VII.47) remain unchanged.

APPENDIX A

Some determinant lemmas

Here we provide three determinant lemmas which we needed for proving our theorems. The first two lemmas, which we utilized in the proofs of Theorems III.7 and III.29, are interesting generalizations of the classical Vandermonde determinant evaluation

$$\det_{1\le i,j\le r}\left(x_i^{r-j}\right)=\prod_{1\le i<j\le r}(x_i-x_j),$$

and the "symplectic" Vandermonde determinant evaluation

$$\det_{1\le i,j\le r}\left(x_i^{r-j}-x_i^{r+j}\right)=\prod_{1\le i<j\le r}(x_i-x_j)\prod_{1\le i\le j\le r}(1-x_ix_j),$$

respectively. The third lemma of this appendix proved to be useful for deriving multiple series identities in chapter VII.

Lemma A.1. *Let $x_1,\dots,x_r$, $y_1,\dots,y_r$, a, and c be indeterminate. Then*

$$\det_{1\le i,j\le r}\left(x_i^{r+1-j}-a^{r+1-j}\frac{(x_i-c/\prod_{s=1}^r y_s)}{(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_i-y_s)}{(a-y_s)}\right)$$
$$=\frac{(a-c/\prod_{j=1}^r x_j)}{(a-c/\prod_{j=1}^r y_j)}\prod_{i=1}^r\frac{(a-x_i)}{(a-y_i)}\prod_{i=1}^r x_i\prod_{1\le i<j\le r}(x_i-x_j).\quad\text{(A.2)}$$

Proof. In the determinant on the left side of (A.2) we take x_i out of the i-th row, $i=1,\dots,r$, and a^{r-j} out of the j-th column, $j=1,\dots,r$, obtaining

$$a^{\binom r2}\prod_{i=1}^r x_i\det_{1\le i,j\le r}\left(\left(\frac{x_i}{a}\right)^{r-j}-\frac{a(x_i-c/\prod_{s=1}^r y_s)}{x_i(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_i-y_s)}{(a-y_s)}\right).$$

In the last determinant we subtract the r-th column from all other columns. We are left with entries $(x_i/a)^{r-j}-1$ for $i=1,\dots,r$ and $j=1,\dots,r-1$, but the r-th column remains unchanged,

$$1-\frac{a(x_i-c/\prod_{s=1}^r y_s)}{x_i(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_i-y_s)}{(a-y_s)}\quad\text{for}\quad i=1,\dots,r.$$

Next we expand the determinant along the last column, to get

$$\sum_{k=1}^r(-1)^{r+k}\left(1-\frac{a(x_k-c/\prod_{s=1}^r y_s)}{x_k(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_k-y_s)}{(a-y_s)}\right)\det_{\substack{1\le i\le r,i\ne k\\ 1\le j\le r-1}}\left(\left(\frac{x_i}{a}\right)^{r-j}-1\right).$$

In the minors we take $(x_i/a-1)$ out of the i-th row

$$\det_{\substack{1\le i\le r,i\ne k\\ 1\le j\le r-1}}\left(\left(\frac{x_i}{a}\right)^{r-j}-1\right)=\prod_{\substack{i=1\\ i\ne k}}^r\left(\frac{x_i}{a}-1\right)\det_{\substack{1\le i\le r,i\ne k\\ 1\le j\le r-1}}\left(\sum_{s=0}^{r-1-j}\left(\frac{x_i}{a}\right)^s\right). \tag{A.3}$$

Now, the determinant on the right side of (A.3) can be reduced to the Vandermonde determinant

$$\det_{\substack{1\le i\le r,i\ne k\\ 1\le j\le r-1}}\left(\left(\frac{x_i}{a}\right)^{r-1-j}\right),$$

and therefore simplifies to

$$\prod_{\substack{1\le i<j\le r\\ i,j\ne k}}\left(\frac{x_i}{a}-\frac{x_j}{a}\right).$$

Substituting our calculations, we arrive at

$$\det_{1\le i,j\le r}\left(x_i^{r+1-j}-a^{r+1-j}\frac{(x_i-c/\prod_{s=1}^r y_s)}{(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_i-y_s)}{(a-y_s)}\right)$$
$$=\prod_{i=1}^r(a-x_i)\prod_{i=1}^r x_i\prod_{1\le i<j\le r}(x_i-x_j)$$
$$\times\sum_{k=1}^r\left(1-\frac{a(x_k-c/\prod_{s=1}^r y_s)}{x_k(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_k-y_s)}{(a-y_s)}\right)(a-x_k)^{-1}\prod_{\substack{i=1\\ i\ne k}}^r(x_k-x_i)^{-1}. \tag{A.4}$$

We are done if we can show that the sum in (A.4) equals

$$\frac{(a-c/\prod_{j=1}^r x_j)}{(a-c/\prod_{j=1}^r y_j)}\frac{1}{\prod_{i=1}^r(a-y_i)}.$$

This is accomplished by splitting the sum and applying the partial fraction decomposition

$$\prod_{i=1}^{r}\frac{(t-a_i)}{(t-b_i)}=1+\sum_{j=1}^{r}\frac{\prod\limits_{i=1}^{r}(b_j-a_i)}{(t-b_j)\prod\limits_{\substack{i=1\\ i\neq j}}^{r}(b_j-b_i)}, \tag{A.5}$$

and the equivalent formula

$$\prod_{i=1}^{r}\frac{(t-a_i)}{(t-b_i)}=\prod_{i=1}^{r}\frac{a_i}{b_i}+\sum_{j=1}^{r}\frac{t\prod\limits_{i=1}^{r}(b_j-a_i)}{(t-b_j)\,b_j\prod\limits_{\substack{i=1\\ i\neq j}}^{r}(b_j-b_i)}, \tag{A.6}$$

(which can be obtained from (A.5) by the replacements $t\to 1/t$, $a_i\to 1/a_i$, $b_i\to 1/b_i$, for $i=1,\dots,r$,) appropriately to its parts. Namely, we write the sum on the right side of (A.4) as

$$\begin{aligned}
\sum_{k=1}^{r}&\left(1-\frac{a(x_k-c/\prod_{s=1}^r y_s)}{x_k(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^{r}\frac{(x_k-y_s)}{(a-y_s)}\right)\frac{1}{(a-x_k)}\frac{1}{\prod\limits_{\substack{i=1\\ i\neq k}}^{r}(x_k-x_i)}\\
&=\sum_{k=1}^{r}\frac{1}{(a-x_k)\prod\limits_{\substack{i=1\\ i\neq k}}^{r}(x_k-x_i)}\\
&\quad-\frac{a}{(a-c/\prod_{j=1}^r y_j)\prod\limits_{i=1}^{r}(a-y_i)}\sum_{k=1}^{r}\frac{\prod\limits_{i=1}^{r}(x_k-y_i)}{(a-x_k)\prod\limits_{\substack{i=1\\ i\neq k}}^{r}(x_k-x_i)}\\
&\quad+\frac{c/\prod_{j=1}^r y_j}{(a-c/\prod_{j=1}^r y_j)\prod\limits_{i=1}^{r}(a-y_i)}\sum_{k=1}^{r}\frac{a\prod\limits_{i=1}^{r}(x_k-y_i)}{(a-x_k)x_k\prod\limits_{\substack{i=1\\ i\neq k}}^{r}(x_k-x_i)}.
\end{aligned}\tag{A.7}$$

The first expression on the right hand side of (A.7) can be summed by the partial fraction decomposition (A.6) with $a_i=0$, $t\to 1/t$, and $b_i\to 1/b_i$, for $i=1,\dots,r$, and reduces to

$$\sum_{k=1}^{r} \frac{1}{(a-x_k)\prod\limits_{\substack{i=1\\ i\neq k}}^{r}(x_k-x_i)} = \frac{1}{\prod\limits_{i=1}^{r}(a-x_i)}, \tag{A.8}$$

the second by the partial fraction decomposition (A.5),

$$\begin{aligned} &\frac{a}{(a-c/\prod_{j=1}^{r}y_j)\prod\limits_{i=1}^{r}(a-y_i)} \sum_{k=1}^{r} \frac{\prod\limits_{i=1}^{r}(x_k-y_i)}{(a-x_k)\prod\limits_{\substack{i=1\\ i\neq k}}^{r}(x_k-x_i)} \\ &\qquad = \frac{a}{(a-c/\prod_{j=1}^{r}y_j)\prod\limits_{i=1}^{r}(a-y_i)} \left(\prod_{i=1}^{r}\frac{(a-y_i)}{(a-x_i)} - 1\right), \end{aligned} \tag{A.9}$$

and the third can be summed by (A.6),

$$\begin{aligned} &\frac{c/\prod_{j=1}^{r}y_j}{(a-c\prod_{j=1}^{r}y_j)\prod\limits_{i=1}^{r}(a-y_i)} \sum_{k=1}^{r} \frac{a\prod\limits_{i=1}^{r}(x_k-y_i)}{(a-x_k)\,x_k\prod\limits_{\substack{i=1\\ i\neq k}}^{r}(x_k-x_i)} \\ &\qquad = \frac{c/\prod_{j=1}^{r}y_j}{(a-c/\prod_{j=1}^{r}y_j)\prod\limits_{i=1}^{r}(a-y_i)} \left(\prod_{i=1}^{r}\frac{(a-y_i)}{(a-x_i)} - \prod_{i=1}^{r}\frac{y_i}{x_i}\right). \end{aligned} \tag{A.10}$$

Simplifying (A.7) by means of (A.8), (A.9), and (A.10), we get

$$\begin{aligned} &\sum_{k=1}^{r}\left(1 - \frac{a(x_k - c/\prod_{s=1}^{r}y_s)}{x_k(a-c/\prod_{s=1}^{r}y_s)}\prod_{s=1}^{r}\frac{(x_k-y_s)}{(a-y_s)}\right)(a-x_k)^{-1}\prod_{\substack{i=1\\ i\neq k}}^{r}(x_k-x_i)^{-1} \\ &\quad = \prod_{i=1}^{r}\frac{1}{(a-x_i)} - \frac{a}{(a-c/\prod_{j=1}^{r}y_j)\prod\limits_{i=1}^{r}(a-y_i)}\left(\prod_{i=1}^{r}\frac{(a-y_i)}{(a-x_i)} - 1\right) \\ &\qquad + \frac{c/\prod_{j=1}^{r}y_j}{(a-c/\prod_{j=1}^{r}y_j)\prod\limits_{i=1}^{r}(a-y_i)}\left(\prod_{i=1}^{r}\frac{(a-y_i)}{(a-x_i)} - \prod_{i=1}^{r}\frac{y_i}{x_i}\right) \\ &\qquad\qquad = \frac{(a-c/\prod_{j=1}^{r}x_j)}{(a-c/\prod_{j=1}^{r}y_j)}\frac{1}{\prod\limits_{i=1}^{r}(a-y_i)}, \end{aligned}$$

which completes the proof of Lemma A.1. $\square$

LEMMA A.11. *Let $x_1, \dots, x_r$, $y_1, \dots, y_r$, and c be indeterminate. Then*

$$\det_{1\le i,j\le r}\left(x_i^{r+1-j} - x_i^j \frac{(x_i - c/\prod_{s=1}^r y_s)}{(1 - x_i c/\prod_{s=1}^r y_s)} \prod_{s=1}^r \frac{(x_i - y_s)}{(1 - x_i y_s)}\right)$$
$$= \prod_{i=1}^r \frac{(1 - y_i c/\prod_{j=1}^r y_j)}{(1 - x_i c/\prod_{j=1}^r y_j)} \prod_{i=1}^r (1 - x_i^2) \prod_{i=1}^r x_i$$
$$\times \prod_{i,j=1}^r (1 - x_i y_j)^{-1} \prod_{1\le i<j\le r} [(x_i - x_j)(1 - x_i x_j)(1 - y_i y_j)]. \quad \text{(A.12)}$$

PROOF. Proving (A.12), we will use a completely different method than in the proof of Lemma A.1. In the determinant on the left side of (A.12) we take $x_i(x_i c - \prod_{s=1}^r y_s)^{-1} \prod_{s=1}^r (1 - x_i y_s)^{-1}$ out of the i-th row, $i = 1, \dots, r$, obtaining

$$\det_{1\le i,j\le r}\left(x_i^{r+1-j} - x_i^j \frac{(x_i - c/\prod_{s=1}^r y_s)}{(1 - x_i c/\prod_{s=1}^r y_s)} \prod_{s=1}^r \frac{(x_i - y_s)}{(1 - x_i y_s)}\right)$$
$$= \prod_{i=1}^r \frac{x_i}{(x_i c - \prod_{j=1}^r y_j)} \prod_{i,j=1}^r (1 - x_i y_j)^{-1} \cdot \Delta(c, \mathbf{x}, \mathbf{y}),$$

where $\Delta(c, \mathbf{x}, \mathbf{y})$ is the determinant

$$\det_{1\le i,j\le r}\left(x_i^{r-j}(x_i c - \prod_{s=1}^r y_s) \prod_{s=1}^r (1 - x_i y_s) - x_i^{j-1}(c - x_i \prod_{s=1}^r y_s) \prod_{s=1}^r (x_i - y_s)\right). \tag{A.13}$$

Thus, in order to establish the lemma, we have to show that

$$\Delta(c, \mathbf{x}, \mathbf{y}) = \prod_{i=1}^r (y_i c - \prod_{j=1}^r y_j) \prod_{i=1}^r (1 - x_i^2) \prod_{1\le i<j\le r} [(x_i - x_j)(1 - x_i x_j)(1 - y_i y_j)].$$

We will do this by identifying all factors using a polynomial argument.

We see that $\Delta(c, \mathbf{x}, \mathbf{y})$ is a polynomial in c, x_i, y_i $(i = 1, \dots, r)$ of maximal degree $(7r^2 - r)/2$. Now observe that if $x_{i_1} = x_{i_2}$, for $i_1 \ne i_2$, two rows in the determinant (A.13) are equal, hence $\prod_{1\le i<j\le r}(x_i - x_j)$ must divide $\Delta(c, \mathbf{x}, \mathbf{y})$. Next suppose $x_{i_1} = 1/x_{i_2}$ for some $i_1 \ne i_2$. In this case the i_1-th row is $-x_{i_2}^{-2r}$ times the i_2-th row which implies that $\prod_{1\le i<j\le r}(1 - x_i x_j)$ also divides $\Delta(c, \mathbf{x}, \mathbf{y})$. If $x_i = 1$ or $x_i = -1$ then all entries of the i-th row are zero, so $\prod_{i=1}^r (1 - x_i^2)$ divides $\Delta(c, \mathbf{x}, \mathbf{y})$.

The remaining factors of $\Delta(c, \mathbf{x}, \mathbf{y})$ are a bit more delicate to establish. For each special case we will succeed in specifying nontrivial linear combinations of the columns that vanish. Suppose $y_k = 1/y_l$ for some $k \ne l$. Taking $-(1 -$

$x_iy_k)(1-x_i/y_k)\prod_{s\neq k,l} y_s$ out of the i-th row of (A.13), for all $i=1,\dots,r$, we obtain the determinant

$$\det_{1\le i,j\le r}\left(x_i^{r-j}(1-x_ic/\prod_{s\neq k,l}y_s)\prod_{s\neq k,l}(1-x_iy_s)-x_i^{j-1}(x_i-c/\prod_{s\neq k,l}y_s)\prod_{s\neq k,l}(x_i-y_s)\right). \tag{A.14}$$

We expand the entries of this determinant in terms of the elementary symmetric functions (see [49, p.19])

$$e_m(y_1,\dots,\hat{y}_k,\dots,\hat{y}_l,\dots,y_r,c/\textstyle\prod_{s\neq k,l}y_s), \tag{A.15}$$

of order m with $r-1$ arguments, $\hat{y}_k$ and $\hat{y}_l$ indicating that the variables y_k,y_l are omitted. Namely, if we write $e_m(\mathbf{y}^{(k,l)})$ for the elementary symmetric function (A.15) for short, (A.14) can be written as

$$\det_{1\le i,j\le r}\left(\sum_{m=0}^{r-1}(-1)^m e_m(\mathbf{y}^{(k,l)})\left(x_i^{r-j+m}-x_i^{j-1+r-1-m}\right)\right). \tag{A.16}$$

To prove that this determinant vanishes we show that the columns of (A.16) are linearly dependent. As the coefficients for the linear combination we choose $(-1)^{j-1}e_{j-1}(\mathbf{y}^{(k,l)})$, for $j=1,\dots,r$. Then we have

$$\begin{aligned}\sum_{j=1}^{r}(-1)^{j-1}e_{j-1}(\mathbf{y}^{(k,l)})\sum_{m=0}^{r-1}(-1)^m e_m(\mathbf{y}^{(k,l)})\left(x_i^{r-j+m}-x_i^{j-1+r-1-m}\right)\\ =\sum_{j=0}^{r-1}\sum_{m=0}^{r-1}(-1)^{j+m}e_j(\mathbf{y}^{(k,l)})e_m(\mathbf{y}^{(k,l)})\left(x_i^{r-j-1+m}-x_i^{j+r-1-m}\right)=0.\end{aligned} \tag{A.17}$$

That the sum equals 0 is because it is a double sum in j and m with terms that are skew symmetric in j and m. Hence we have proved that $\prod_{1\le i<j\le r}(1-y_iy_j)$ divides $\Delta(c,\mathbf{x},\mathbf{y})$.

Now suppose $c=\prod_{s\neq k}y_s$ for some $k=1,\dots,r$. If we take $-(1-x_iy_k)(1-x_i/y_k)\prod_{s=1}^r y_s$ out of the i-th row of (A.13) for all $i=1,\dots,r$, we obtain the determinant

$$\det_{1\le i,j\le r}\left(x_i^{r-j}\prod_{s\neq k}(1-x_iy_s)-x_i^{j-1}\prod_{s\neq k}(x_i-y_s)\right). \tag{A.18}$$

We expand the entries of this determinant in terms of the elementary symmetric functions

$$e_m(y_1,\dots,\hat{y}_k,\dots,y_r), \tag{A.19}$$

of order m with $r-1$ arguments, $\hat{y}_k$ indicating that the variable y_k is omitted. Namely, if we write $e_m(\mathbf{y}^{(k)})$ for the elementary symmetric function (A.19) for short, (A.18) can be written as

$$\det_{1\le i,j\le r}\left(\sum_{m=0}^{r-1}(-1)^m e_m(\mathbf{y}^{(k)})\left(x_i^{r-j+m}-x_i^{j-1+r-1-m}\right)\right). \tag{A.20}$$

To prove that this determinant vanishes we show that the columns of (A.20) are linearly dependent. Here the coefficients $(-1)^{j-1}e_{j-1}(\mathbf{y}^{(k)})$ for $j=1,\dots,r$ do the job (compare with (A.17)). Hence $\prod_{1\le i\le r}(c-\prod_{s\ne i}y_s)$ divides $\Delta(c,\mathbf{x},\mathbf{y})$.

Now suppose $y_k=0$ for some $k=1,\dots,r$. If we take $(-x_ic)$ out of the i-th row of (A.13) for all $i=1,\dots,r$, we obtain the determinant (A.18), and we can proceed as above. I.e., we have also shown that $\prod_{1\le i\le r}y_i$ divides $\Delta(c,\mathbf{x},\mathbf{y})$.

Collecting all factors of $\Delta(c,\mathbf{x},\mathbf{y})$ that we have identified so far, we now know that

$$\begin{aligned}\Delta(c,\mathbf{x},\mathbf{y})=\prod_{i=1}^r\big(y_ic-\textstyle\prod_{j=1}^r y_j\big)\displaystyle\prod_{i=1}^r(1-x_i^2)&\\ \times\prod_{1\le i<j\le r}[(x_i-x_j)(1-x_ix_j)(1-y_iy_j)]\cdot p(c,\mathbf{x},\mathbf{y}),&\end{aligned} \tag{A.21}$$

where $p(c,\mathbf{x},\mathbf{y})$ is some polynomial in c,x_i,y_i $(i=1,\dots,r)$. But the degree of the factors we already identified amounts to $(7r^2-r)/2$, which is the same degree as that of $\Delta(c,\mathbf{x},\mathbf{y})$. Thus the polynomial p has to be a constant which is easily seen to be 1, since the coefficients of $c^0\prod_{i=1}^r(x_i^{r-i}y_i^r)$ in $\Delta(c,\mathbf{x},\mathbf{y})$ and in the product on the right side of (A.21) both equal $(-1)^r$. $\square$

Our calculations in the proofs of our multidimensional summation theorems in chapter VII required the following Lemma.

LEMMA A.22. *Let $X_1,\dots,X_r$, A, B, and C be indeterminate. Then there holds*

$$\begin{aligned}\det_{1\le s,t\le r}\left(\frac{(AX_s;q)_{r-t}\,(AC/X_s;q)_{r-t}}{(BX_s;q)_{r-t}\,(BC/X_s;q)_{r-t}}\right)&=\prod_{1\le i<j\le r}[(X_j-X_i)\,(1-C/X_iX_j)]\\ &\times A^{\binom{r}{2}}\,q^{\binom{r}{3}}\prod_{i=1}^r\frac{(B/A;q)_{i-1}\,(ABCq^{2r-2i};q)_{i-1}}{(BX_i;q)_{r-1}\,(BC/X_i;q)_{r-1}}.\end{aligned} \tag{A.23}$$

This determinant evaluation follows easily from a determinant lemma of Krattenthaler [44, Lemma 34], which we state here without proof.

Let the *degree* of a *Laurent polynomial* $\sum_{i=M}^N a_ix^i$, $M,N\in\mathbb{Z}$, $a_i\in\mathbb{R}$ and $a_N\ne 0$, be defined by $\deg p:=N$.

LEMMA A.24 (KRATTENTHALER). *Let* $X_1, X_2, \dots, X_r$, $A_2, A_3, \dots, A_r$, *and* C *be indeterminates. If* $p_0, p_1, \dots, p_{r-1}$ *are Laurent polynomials with* $\deg p_j \le j$ *and* $p_j(C/X) = p_j(X)$ *for* $j = 0, 1, \dots, r-1$, *then*

$$\det_{1\le s,t\le r}\left((A_r + X_s)\cdots(A_{t+1}+X_s)(A_r + C/X_s)\cdots(A_{t+1}+C/X_s)\cdot p_{t-1}(X_s)\right)$$
$$= \prod_{1\le i<j\le r}(X_i - X_j)(1 - C/X_iX_j)\prod_{i=1}^{r} A_i^{i-1}\prod_{i=1}^{r} p_{i-1}(-A_i), \quad \text{(A.25)}$$

with the convention that empty products (like $(A_r + X_s)\cdots(A_{t+1}+X_s)$ *for* $t = r$*) are equal to 1. (The indeterminate* A_1*, which occurs at the right side of* (A.25)*, in fact is superfluous since it occurs in the argument of a constant polynomial.)*

PROOF OF LEMMA A.22. By taking some factors out of the determinant, we write the left side of (A.23) as

$$A^{2\binom{r}{2}}\, q^{2\binom{r}{3}} \prod_{i=1}^{r}\Big((BX_i;q)_{r-1}^{-1}(BC/X_i;q)_{r-1}^{-1}\Big)$$
$$\times \det_{1\le s,t\le r}\Big((1/A - X_s)\cdots(q^{1+t-r}/A - X_s)(1/A - C/X_s)\cdots(q^{1-r+t}/A - C/X_s)$$
$$\cdot\,(BX_sq^{r-t};q)_{t-1}\,(BCq^{r-t}/X_s;q)_{t-1}\Big). \quad \text{(A.26)}$$

Now we apply Lemma A.24 with the replacements $X_s \mapsto -X_s$, $A_t \mapsto q^{t-r}/A$, $C \mapsto C$, and $p_j(X) \mapsto (-BXq^{r-1-j};q)_j\cdot(-BCq^{r-1-j}/X;q)_j$. A few simplifications give the product side of (A.23). $\square$

REMARK A.27. The special case $X_s \equiv q^s$ of Lemma A.22 is equivalent to a determinant evaluation which J. A. Wilson [77] utilized to compute biorthogonal rational functions.

APPENDIX B

Background information

Here we state a simplification lemma and the multidimensional Jackson's summations we needed in the proofs of our multiple ${}_{10}\phi_9$ transformations.

LEMMA B.1.

$$\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-q^{l_i-l_j}x_i/x_j}\right)\prod_{i,j=1}^{r}\frac{(q^{l_i-k_j}x_i/x_j;q)_{k_i-l_i}}{(q^{1+l_i-l_j}x_i/x_j;q)_{k_i-l_i}}$$
$$=(-1)^{|\mathbf{k}|-|\mathbf{l}|}\,q^{-\binom{|\mathbf{k}|-|\mathbf{l}|}{2}}\,q^{-\sum_{i=1}^{r}i\,(k_i-l_i)}.$$

REMARK B.2. Lemma B.1 is equivalent to Lemma 4.3 of [55], where it is proved by some elementary manipulations.

THEOREM B.3 ((MILNE) AN A_r JACKSON'S SUM). *Let $x_1,\ldots,x_r$, a, b, c, and d be indeterminate, let $n_1,\ldots,n_r$ be nonnegative integers with $r\ge 1$, and suppose that none of the denominators in* (B.4) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\ldots,r}}\Bigg(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times\prod_{i,j=1}^{r}\frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+n_i};q)_{|\mathbf{k}|}}$$
$$\times\prod_{i=1}^{r}\frac{(dx_i;q)_{k_i}\,(a^2x_iq^{1+|\mathbf{n}|}/bcd;q)_{k_i}}{(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}}$$
$$\times\frac{(b;q)_{|\mathbf{k}|}\,(c;q)_{|\mathbf{k}|}}{(aq/d)_{|\mathbf{k}|}\,(bcdq^{-|\mathbf{n}|}/a;q)_{|\mathbf{k}|}}\,q^{\sum_{i=1}^{r}i\,k_i}\Bigg)$$

$$= \frac{(aq/bd;q)_{|\mathbf{n}|}\,(aq/cd;q)_{|\mathbf{n}|}}{(aq/d;q)_{|\mathbf{n}|}\,(aq/bcd;q)_{|\mathbf{n}|}} \prod_{i=1}^{r} \frac{(ax_iq;q)_{n_i}\,(ax_iq/bc;q)_{n_i}}{(ax_iq/b;q)_{n_i}\,(ax_iq/c;q)_{n_i}}. \tag{B.4}$$

REMARK B.5. Theorem B.3 appeared as Theorem 6.14 in [52] with different notation. It is stated almost identical to the above form in [59, A12].

THEOREM B.6. ((DENIS AND GUSTAFSON, MILNE AND LILLY) A C_r JACKSON'S SUM). *Let $x_1,\ldots,x_r$, a, b, c, and d be indeterminate, let $n_1,\ldots,n_r$ be nonnegative integers with $r \geq 1$, and suppose that none of the denominators in* (B.7) *vanish. Then*

$$\sum_{\substack{0\leq k_i\leq n_i\\ i=1,2,\ldots,r}} \Bigg(\prod_{i=1}^{r} \left(\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2} \right) \prod_{1\leq i<j\leq r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\, \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \right)$$
$$\times \prod_{i=1}^{r} \frac{(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}\,(dx_i;q)_{k_i}\,(a^2x_iq^{1+|\mathbf{n}|}/bcd;q)_{k_i}}{(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\,(bcdx_iq^{-|\mathbf{n}|}/a;q)_{k_i}}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(ax_ix_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq^{1+n_j};q)_{k_i}}\; q^{\sum_{i=1}^{r} i\,k_i} \Bigg)$$
$$= \prod_{1\leq i<j\leq r} (ax_ix_jq;q)^{-1}_{n_i+n_j} \prod_{i,j=1}^{r} (ax_ix_jq;q)_{n_i}$$
$$\times \frac{(aq/bc;q)_{|\mathbf{n}|}\,(aq/bd;q)_{|\mathbf{n}|}\,(aq/cd;q)_{|\mathbf{n}|}}{\prod_{i=1}^{r} \left[(ax_iq/b;q)_{n_i}\,(ax_iq/c;q)_{n_i}\,(ax_iq/d;q)_{n_i}\,(aq^{1+|\mathbf{n}|-n_i}/bcdx_i;q)_{n_i}\right]}. \tag{B.7}$$

REMARK B.8. Theorem B.6 appeared as Theorem 4.1 in [19] with different notation. In a form very similar to ours above, with relabeled parameters and slightly rearranged product side, it appeared (independently from [19]) as Theorem 6.13 in [58].

THEOREM B.9 ((BHATNAGAR) A D_r JACKSON'S SUM). *Let $x_1,\ldots,x_r$, a, b, c, and d be indeterminate, let $n_1,\ldots,n_r$ be nonnegative integers with $r \geq 1$, and suppose that none of the denominators in* (B.10) *vanish. Then*

$$\sum_{\substack{0\leq k_i\leq n_i\\ i=1,2,\ldots,r}} \Bigg(\prod_{i=1}^{r} \left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i} \right) \prod_{1\leq i<j\leq r} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j} \right)$$
$$\times \prod_{1\leq i<j\leq r} (ax_ix_jq/d;q)_{k_i+k_j} \prod_{i,j=1}^{r} (ax_ix_jq/d;q)^{-1}_{k_i}$$

$$
\times \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+n_i};q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|-k_i}}
$$

$$
\left.\times \frac{\prod_{i=1}^{r}\left[(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}\,(a^2x_iq^{1+|\mathbf{n}|}/bcd;q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(bcdq^{-|\mathbf{n}|}/a;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^{r} i\,k_i}\right)
$$

$$
= \prod_{1\le i<j\le r} (ax_ix_jq/d;q)_{n_i+n_j} \prod_{i,j=1}^{r} (ax_ix_jq/d;q)_{n_i}^{-1}
$$

$$
\times \frac{\prod_{i=1}^{r}\left[(ax_iq;q)_{n_i}\,(ax_iq/cd;q)_{n_i}\,(ax_iq/bd;q)_{n_i}\,(aq^{1+|\mathbf{n}|-n_i}/bcx_i;q)_{n_i}\right]}{(aq/b;q)_{|\mathbf{n}|}\,(aq/c;q)_{|\mathbf{n}|}\,(aq/bcd;q)_{|\mathbf{n}|}}. \tag{B.10}
$$

Remark B.11. Theorem B.9 appears in [12] where it has been derived by combining a D_r ${}_3\phi_2$ summation and a D_r very-well-poised ${}_6\phi_5$ summation theorem.

Theorem B.9 is also equivalent to Theorem B.12 since it follows from reversing the sum in (B.13) and relabelling parameters.

Theorem B.12 (A D_r Jackson's sum). *Let $x_1,\dots,x_r$, a, b, c, and d be indeterminate, let $n_1,\dots,n_r$ be nonnegative integers with $r\ge 1$, and suppose that none of the denominators in* (B.13) *vanish. Then*

$$
\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\dots,r}} \left(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\right.
$$

$$
\times \prod_{1\le i<j\le r} (a^2x_ix_jq/bcd;q)_{k_i+k_j}^{-1} \prod_{i,j=1}^{r} (a^2x_ix_jq^{1+n_j}/bcd;q)_{k_i}
$$

$$
\times \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(bcd/ax_i;q)_{|\mathbf{k}|-k_i}}{(ax_iq^{1+n_i};q)_{|\mathbf{k}|}\,(bcdq^{-n_i}/ax_i;q)_{|\mathbf{k}|}}
$$

$$
\left.\times \frac{(b;q)_{|\mathbf{k}|}\,(c;q)_{|\mathbf{k}|}\,(d;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}\left[(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\right]}\, q^{\sum_{i=1}^{r} i\,k_i}\right)
$$

$$
= \prod_{i=1}^{r} \frac{(ax_iq;q)_{n_i}\,(ax_iq/bc;q)_{n_i}\,(ax_iq/bd;q)_{n_i}\,(ax_iq/cd;q)_{n_i}}{(ax_iq/bcd;q)_{n_i}\,(ax_iq/d;q)_{n_i}\,(ax_iq/c;q)_{n_i}\,(ax_iq/b;q)_{n_i}}. \tag{B.13}
$$

Remark B.14. Theorem B.12 is just Theorem IV.22 with slightly relabeled parameters.

Theorem B.12 is also equivalent to Theorem B.9 since it follows from reversing the sum in (B.10) and relabelling parameters.

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