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MASTERARBEIT

Titel der Arbeit

"The Symmetry of the q, t-Catalan Numbers"

Verfasser

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Abstract

We introduce the q,t-Catalan numbers as the bivariate generating polynomials of two statistics on Dyck paths. We discuss some of their properties from a combinatorial point of view, e.g. a description by means of different statistics, a recursion, specialisations, and a possible generalisation to parking functions. We then use the fact that the q,t-Catalan numbers are symmetric in q and t to count the partitions of n with k diagonal inversions, and certain classes of "long" Dyck paths.

We proceed to shed some light on the role the q,t-Catalan numbers play in related fields like the representation theory of $\mathfrak{S}_n$, and the theory of symmetric functions and Macdonald polynomials. The final aim is to present a formula for the q,t-Catalan numbers in terms of a sum of rational functions in q and t indexed by integer partitions. We collect and motivate all the results that contribute to the proof of this equality, and do a good deal of the work in detail.

This result can be interpreted in different ways. Depending on the starting point, it is either a proof that the q,t-Catalan numbers are symmetric in q and t, or that the sum of rational functions is a polynomial with non-negative integer coefficients, i.e., lies in $\mathbb{N}[q,t]$. Moreover, it provides a combinatorial interpretation for the Hilbert series of a bigraded representation of the symmetric group which can be regarded as the alternating component of the space of diagonal harmonics.

Zusammenfassung

Wir definieren die q,t-Catalan-Zahlen als bivariate erzeugende Polynome zweier Statistiken auf Dyck-Pfaden. Wir besprechen einige ihrer Eigenschaften von einem kombinatorischen Standpunkt aus gesehen, zum Beispiel eine Beschreibung durch weitere Statistiken, eine Rekursion, Spezialisierungen und eine mögliche Verallgemeinerung auf Parkfunktionen. Wir verwenden die Symmetrie der q,t-Catalan-Zahlen in den Variablen q und t, um die Partitionen von n mit k Diagonalinversionen und bestimmte Klassen von ”‘langen”’ Dyck-Pfaden abzuzählen.

Anschließend beleuchten wir die Rolle, die die q,t-Catalan-Zahlen in verwandten Gebieten, wie der Darstellungstheorie von $\mathfrak{S}_n$ oder der Theorie symmetrischer Funktionen und Macdonald-Polynomen, spielen. Das abschließende Ziel ist, die q,t-Catalan-Zahlen als Summe von rationalen Funktionen in q und t, die durch Zahlpartitionen indiziert sind, darzustellen. Wir sammeln und motivieren alle zum Beweis dieser Identität beitragenden Resultate und erledigen einen guten Teil der Arbeit im Detail.

Das Ergebnis kann unterschiedlich interpretiert werden. In Abhängigkeit vom Standpunkt zeigt es entweder, dass die q,t-Catalan-Zahlen symmetrisch in q und t sind, oder dass die Summe der rationalen Funktionen ein Polynom mit nicht negativen ganzzahligen Koeffizienten ist, also in $\mathbb{N}[q,t]$ liegt. Außerdem ermöglicht es eine kombinatorische Beschreibung der Hilbertreihe einer bigraduierten Darstellung der symmetrischen Gruppe, die als alternierende Komponente des Raumes der diagonalharmonischen Polynome angesehen werden kann.

Contents

1 Outline

A little bit further down the line, in Section 2, we will fix some notation about the basic combinatorial objects we will encounter, that is, permutations and integer partitions. In addition, we introduce the concept of q-generalisation on the basis of q-binomial coefficients and the q-Pochhammer symbol. We will wind up this chapter by proving a q-analogue of Taylor's Theorem.

In Section 3 we shall discuss statistics on lattice paths, and Dyck paths in particular. In this way we will define the q,t-Catalan numbers. We will investigate some of their properties and possible generalisations based on combinatorial manipulations. Most of the result presented in this section can be found in Haglund's book [9]. Towards the end, some of the author's own ideas are stated.

In Section 4 we will outline our strategy of attacking the symmetry theorem. We will proceed to recall some basic notions about algebras, representations, polynomial rings and invariant spaces. In particular, we cite the important structural theorems like Schur's Lemma, the Artin–Wedderburn-Theorem and Maschke's Theorem, which allow a characterisation of the irreducible representations of finite groups. We also give a short introduction to character theory and its role in the representation theory of finite groups. These results can be found in most standard textbooks on representation theory, e.g. [7].

In Section 5 we put the theory to use and survey the symmetric group in detail. The first part outlines the construction of the irreducible characters of the symmetric group following Fulton and Harris [3]. Afterwards, we turn our attention to group actions of permutations on polynomial rings. We define the rings of invariants and coinvariants for the classical action via permutation of variables as well as the diagonal action on polynomials in two sets of variables. We prove some results on their basic structure and conclude the section by moving forwards to formal power series in infinitely many variables.

In Section 6 we will introduce the notion of plethystic calculus which is essential to the modern approach to the theory of symmetric functions. We introduce the plethystic substitution of symmetric functions in the language of algebra homomorphisms, leaning on Loehr and Remmel's approach in [13]. The second part of this section is dedicated to the discussion of Schur functions. These results are mainly taken from Macdonald's book [15].

In Section 7 we will dare to make the step from the rationals to a field of rational functions in q and t. We introduce the Macdonald polynomials which generalise the Schur functions. In this context we will also be able to provide the necessary background to the rational functions which appear in the proof of the symmetry problem. Again most of the results can be found in Macdonald's work [15, 16]. We proceed to develop the necessary tools to give the proof in Section 8. This involves so called Pieri formulas which provide information about the coefficients arising when expanding a symmetric function into a linear combination of either Schur functions or Macdonald polynomials. Finally, we show how the equality indicated in the abstract is obtained from the collected results. In the last two sections we will depend on the article of Haglund and Garsia [4] and related works.

2 A q-Introduction

Throughout this thesis $\mathbb{N} = \{0, 1, 2, \dots\}$ denotes the non-negative integers. For i and j being any two numbers, partitions, etc. we denote the *Kronecker delta* by

$$\delta_{i,j} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Let $n \in \mathbb{N}$, then the *symmetric group* on n letters $\mathfrak{S}_n$ is the group of all bijections $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ together with composition. Its elements are called *permutations* of n, and are sometimes represented by the word $\sigma = \sigma_1 \cdots \sigma_n$ where $\sigma_i := \sigma(i)$ for $1 \le i \le n$. It is well known that the number of permutations in $\mathfrak{S}_n$ is given by $n!$.

A sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ of natural numbers is called a *partition* of n, also denoted by $\lambda \vdash n$, if $\lambda_i \ge \lambda_{i+1}$ for all $i \in \mathbb{N}$ and $|\lambda| := \sum_{i=1}^{\infty} \lambda_i = n$. The elements of the sequence are called the *parts* or the summands of the partition. The *length* of the partition $l(\lambda)$ is defined to be number of positive parts which is clearly finite. We denote the set of partitions of n with length at most k by Π_n^k, and the set of all partitions of n by $\Pi_n = \Pi_n^n$

The cardinalities of the sets Π_n are called *partition numbers* (sequence A000041 in OEIS [18]). Let us denote the number of partitions in $\Pi_n^k - \Pi_n^{k-1}$ of length exactly k by $\pi(n, k)$. Then we note that $|\Pi_n^k| = \pi(n + k, k)$. The cardinalities of the more refined sets Π_n^k constitute sequence A026820 in OEIS [18]. We can compute these numbers exploiting the recursion $|\Pi_n^k| = |\Pi_n^{k-1}| + \pi(n, k) = |\Pi_n^{k-1}| + |\Pi_{n-k}^k|$ for $n \ge k \ge 1$.

The *Young diagram* of a partition λ which we denote by $Y(\lambda)$ is the set of all pairs $(i, j) \in \mathbb{Z}^2$ such that $1 \le i \le l(\lambda)$ and $1 \le j \le \lambda_i$. The elements $x \in Y(\lambda)$ are called the *cells* of the diagram. We notice that a finite set of positive cells $Y \subseteq \mathbb{Z}_+^2$ corresponds to a partition if and only if it contains all cells of the form (i', j') where $1 \le i' \le i$, $1 \le j' \le j$ for all $(i, j) \in Y$.

There are equally adequate concepts for visualising the Young diagram of a partition. Throughout this paper we adopt the English convention. Here, the cells are arranged like the entries of a matrix, that is, the pair (i, j) is the cell in the i-th row and the j-th column (see Figure 1).

The *conjugate* of a partition λ is defined as the partition corresponding to the Young diagram $Y' := \{(j, i) : (i, j) \in Y(\lambda)\}$, and is denoted by λ'. By virtue of the map $\lambda \mapsto \lambda'$ it becomes clear that the number of partitions of n with at most k parts equals the number of partitions of n where every part is at most k. To give an additional example for what can be seen using Young diagrams, we remark that the number of partitions of n whose positive parts are odd and distinct equals the number of partitions of n invariant under conjugation. To see this, let $\lambda = \lambda'$ and μ_1 be the number of cells in the first row or in the first column of λ. Then μ_1 is odd or equals zero. Now, let μ_2 be the number

$\lambda = (6, 4, 3, 3, 1, 1, 1)$ $\lambda' = (7, 4, 4, 2, 1, 1)$

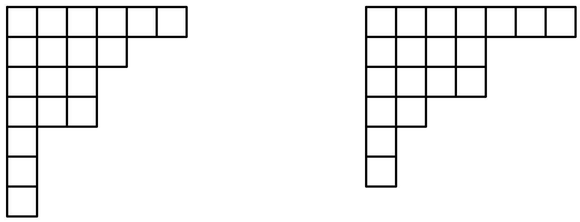


Figure 1: The Young diagram of a partition λ of 19 with 7 parts, and its conjugate λ'.

of cells in the second row of λ weakly to the right of $(2, 2)$ plus the number of cells the second column of λ weakly below $(2, 2)$. Then μ_2 is odd and strictly smaller than μ_1, or $\mu_2 = 0$. The partition $\mu = (\mu_1, \mu_2, \dots)$ has the claimed properties.

We define three different orders on partitions. The *lexicographical order* is defined as $\lambda <_{\text{lex}} \mu$ if and only if the first non-vanishing difference $\lambda_i - \mu_i$ is negative. Secondly, we define the *natural order* as $\lambda \le \mu$ if and only if $\lambda_1 + \cdots + \lambda_i \le \mu_1 + \cdots + \mu_i$ for all $i \in \mathbb{N}$. Lastly, the *inclusion order* is given by $\lambda \subseteq \mu$ if and only if $Y(\lambda) \subseteq Y(\mu)$, or equivalently, if $\lambda_i \le \mu_i$ for all $i \in \mathbb{N}$. The natural order is a partial order including the ordering via inclusion, i.e. $\lambda \subseteq \mu$ implies $\lambda \le \mu$. The lexicographical order is a total order (even a well-order) that includes the natural order. That is $\lambda \le \mu$ already implies $\lambda \le_{\text{lex}} \mu$. Considering the partitions in Figure 1 we have $\lambda < \lambda'$, thus also $\lambda <_{\text{lex}} \lambda'$, but not $\lambda \subseteq \lambda'$.

One of the most basic problems in combinatorics is the enumeration of combinatorial objects of a given size (such as permutations of n, partitions of n, etc...). However, some ways to count are more refined than others. Consider the following statistics on the symmetric group $\mathfrak{S}_n$. The *inversion number* of a permutation is defined as

$$\operatorname{inv}(\sigma) := \left|\left\{(i, j) : 1 \le i < j \le n,\, \sigma(i) > \sigma(j)\right\}\right|,$$

where each such pair (i, j) is called an *inversion* of σ. The *sign* of a permutation is defined as $\operatorname{sgn}(\sigma) := (-1)^{\operatorname{inv}(\sigma)}$. Furthermore, a number $1 \le i \le n-1$ is called a *descent* of σ if $\sigma(i) > \sigma(i+1)$. The *Major index* of a permutation is the sum of all descents

$$\operatorname{maj}(\sigma) := \sum_{1 \le i \le n-1,\, \sigma(i) > \sigma(i+1)} i.$$

Now, if we wish to count permutations, that means we want to find a closed form of

$$\sum_{\sigma \in \mathfrak{S}_n} 1. \tag{2.1}$$

Indeed, as we mentioned above, $n!$ is a satisfiably closed form in this case. But might it not be better to find a closed form of one of the two generating functions

$$g(n;q) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}, \qquad G(n;q) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} \quad ? \tag{2.2}$$

Clearly $g(n;1) = n!$, thus each sum in (2.2) contains strictly more information than the sum in (2.1). In this sense the generating function $g(n;q) \in \mathbb{N}[q]$ generalises $n!$ to a polynomial in q. Given that the generalisation retains some (recursive, arithmetic, algebraic) property of the original, we call such a polynomial a q-analogue. For example, in the case of the factorial we would hope that $g(n;q)$ fulfills some recursion similar to the simple $n!(n+1) = (n+1)!$. Of course this is no mathematically exact definition. There might be more than one possibility to generalise an expression, and it is not necessarily clear which one is better suited. For example $G(n;q)$ might turn out to resemble the factorial more closely (it will not!). In order to handle notions like "closed form" or "retain properties" at least a little bit better, some definitions are needed. For $k,n \in \mathbb{N}$ we define

$$[n]_q := 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}, \tag{2.3}$$

$$[n]_q! := \prod_{i=1}^{n} [n]_q = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}), \tag{2.4}$$

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q := \frac{[n+k]_q!}{[n]_q!\,[k]_q!} = \prod_{i=1}^{k} \frac{(1-q^{n+i})}{(1-q^i)}. \tag{2.5}$$

The symbols in this definition are called the *q-integer*, the *q-factorial*, and the *q-binomial coefficient* for obvious reasons. We are now able to summarise the example above in a theorem due to MacMahon who introduced the Major index [17].

Theorem 2.1. *Let $n \in \mathbb{N}$. Then we have the identities*

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!\,.$$

Proof. We apply induction on n. The case $n = 1$ is trivial for both statistics.

Let $\sigma = \sigma_1 \cdots \sigma_{n+1} \in S_{n+1}$. We define a permutation $\hat\sigma \in \mathfrak{S}_n$ by deleting the letter $n+1$ in the word σ. Clearly, this maps $n+1$ permutations to the same image, and the inversions of σ are just the inversions of $\hat\sigma$ plus inversions of the form $(i, n+1)$. Thus, $\mathrm{inv}(\sigma) = \mathrm{inv}(\hat\sigma) + i$ if and only if $\sigma(n+1) = n+1-i$. Using the induction hypothesis we conclude that

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} q^{\mathrm{inv}(\sigma)} = \sum_{\hat\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\hat\sigma)}(1 + q + \ldots q^n) = [n]_q![n+1]_q = [n+1]_q!\,.$$

To prove the second claim it suffices to show that for each $\tau \in \mathfrak{S}_n$ and each $0 \le i \le n$ there is a permutation $\sigma \in \mathfrak{S}_{n+1}$ such that $\tau = \hat\sigma$ and $\mathrm{maj}(\sigma) - \mathrm{maj}(\tau) = i$. Therefore, let $\tau = \tau_1 \cdots \tau_n \in \mathfrak{S}_n$ have k descents at the positions $1 \le j_1 < \cdots < j_k \le n-1$, and denote by $\tau(m) \in \mathfrak{S}_{n+1}$ the permutation obtained by inserting the letter $n+1$ at the m-th position.

We have $\mathrm{maj}(\tau(n+1)) = \mathrm{maj}(\tau)$. If $m = j_l + 1$ for some $1 \le l \le k$ we have $\mathrm{maj}(\tau(m)) - \mathrm{maj}(\tau) = k - l + 1$. If $1 \le m \le j_1$ then $\mathrm{maj}(\tau(m)) - \mathrm{maj}(\tau)$ takes the values $k+1, \ldots, j_1 + k$. Similarly, if $j_l + 2 \le m \le j_{l+1}$ then $\mathrm{maj}(\tau(m)) - \mathrm{maj}(\tau)$ takes the values $j_l + 2 + k - l, \ldots, j_{l+1} + k - l$. Finally, if $j_k + 2 \le m \le n$, we have $\mathrm{maj}(\tau(m)) - \mathrm{maj}(\tau) = m$. Since all possibilities are exhausted, the claim follows as above. ■

Hence, we do have $g(n;q)[n+1]_q = g(n+1,q)$ which generalises the recursion satisfied by the ordinary factorial in a natural way. Theorem 2.1 also implies that the Major index and the inversion number are equidistributed over the permutations of n. The next thematic step is to consider the bivariate generating function $\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)}$. If this polynomial is symmetric in q and t there must be a bijection $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ interchanging the two statistics. Indeed, Foata and Schützenberger were able to construct an involution with this property in [2].

Of course the problem can be posed the other way around: If a polynomial with non-negative integer coefficients appears in say representation theory or some other abstract branch of mathematics, it is natural to ask for a set of combinatorial objects and a statistic such that the polynomial is the generating function of the combinatorial objects with respect to this statistic. Thus, the question then becomes to find a natural way to realise the said polynomial.

Another useful definition is the *q-Pochhammer symbol* given by

$$(z;q)_k := (1-z)(1-qz)\cdots(1-q^{k-1}z)$$

when $k \ge 1$, and $(z;q)_0 := 1$. Here z denotes another variable over $\mathbb{Q}$. Using the Pochhammer symbol we may rewrite our previous definitions as

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n}, \qquad \begin{bmatrix} n+k \\ k \end{bmatrix}_q = \frac{(q^{n+1};q)_k}{(q;q)_k}. \tag{2.6}$$

Upon investigation one will discover that the class of q-analogues is surprisingly large. There is q-integration and there are q-analogues of the important analytic functions such as cosine and Gamma-function originating in the theory of hypergeometric series. The close connection to combinatorics is hinted at in Theorem 2.1. Concluding this introduction we give one more example which could be considered a q-analogue of Taylor's Theorem approximating functions (polynomials) by their higher derivatives.

Let $K = \mathbb{Q}(q)$, and $f(z) \in K[z]$ be a polynomial in z whose coefficients are rational functions in q. We define the *q-difference operator* as

$$\delta_q f(z) := \frac{f(z) - f(z/q)}{z}. \tag{2.7}$$

In the proof of the following theorem we will need the fact that the set $\{(z;q)_k : k \in \mathbb{N}\}$ is a K-basis of $K[z]$. This is true since it contains exactly one polynomial of degree k for each $k \in \mathbb{N}$.

Theorem 2.2. *Let $K = \mathbb{Q}(q)$ and $f(z) \in K[z]$. Then $f(z)$ has the q-*Taylor expansion *in terms of the basis $\{(z;q)_k : k \in \mathbb{N}\}$ given by*

$$f(z) = \sum_{m=0}^{\infty} (z;q)_m \frac{q^m}{(q;q)_m} \left(\delta_q^m f(z) \Big|_{z=1} \right).$$

Proof. We compute

$$\delta_q (z;q)_k = \frac{(z;q)_k - (z/q;q)_k}{z} = \frac{1-q^k}{q} (z;q)_{k-1}. \tag{2.8}$$

Thus, for all $m \in \mathbb{N}$

$$\delta_q^m (z;q)_k \Big|_{z=1} = \begin{cases} \frac{(q;q)_k}{q^k} & \text{if } k = m, \\ 0 & \text{if } k \neq m. \end{cases} \tag{2.9}$$

But then, applying δ_q^m to both sides of $f(z) = \sum_{k=0}^{\infty} f_k \, (z;q)_k$ and letting $z = 1$ yields

$$f_m = \frac{q^m}{(q;q)_m} \left(\delta_q^m f(z) \Big|_{z=1} \right).$$

■

3 Dyck Paths and the Catalan Numbers

In this section we will review the combinatorial approach to the q,t-Catalan numbers. As mentioned before, Dyck paths are the suitable combinatorial objects to study them. The main theorem in this regard is Theorem 3.5 which we prove at full length. The symmetry theorem (Theorem 3.8) is stated accordingly. Since there is no known combinatorial proof of the symmetry, we will conclude this section by mentioning some continuative topics in this field.

Let $n \in \mathbb{N}$, $a, b \in \mathbb{Z}^2$ and $S \subseteq \mathbb{Z}^2$ be a finite subset. A *lattice path* of *length* n from a to b with steps in S is a finite sequence $(x_0, \ldots, x_n)$ of points $x_i \in \mathbb{Z}^2$ such that $x_0 = a$, $x_n = b$ and $x_i - x_{i-1} \in S$ for all $i = 1, \ldots, n$. The points a and b are called *starting point* and *end point* of the path, respectively. The set S is called the *set of steps*. Let us denote the set of all such paths by $\mathcal{L}^n_{a,b,S}$.

Alternatively, letting $s_i := x_i - x_{i-1}$ every path in $\mathcal{L}^n_{a,b,S}$ is represented by a sequence $(s_1, \ldots, s_n)$ of steps $s_i \in S$ such that $\sum_{i=1}^n s_i = b - a$. Thus, we may identify $\mathcal{L}^n_{a,b,S}$ with a subset of S^n, and in particular, $\mathcal{L}^n_{a,b,S}$ is always finite. Furthermore, if we let $c, d \in \mathbb{Z}^2$

then the sets $\mathcal{L}^n_{a,b,S}$ and $\mathcal{L}^n_{c,d,S}$ are in natural bijection whenever $b-a=d-c$. If we are only interested in counting paths we will therefore assume w.l.o.g. that the starting point equals the origin $(0,0)$, and write $\mathcal{L}^n_{b,S}$ in that case.

Mostly, we will restrict ourselves to the case $S=\{(0,1),(1,0)\}$, where we shall call $N:=(0,1)$ a *North-step* and $E:=(1,0)$ an *East-step*. Whenever we do so we will just write $\mathcal{L}^n_b$ instead of $\mathcal{L}^n_{b,S}$. If $b=(n,k)$, note that the set $\mathcal{L}^m_b$ is nonempty if and only if $b\in\mathbb{N}^2$ and $m=n+k$. More precisely, such a path must consist of exactly n East-steps and k North-steps. To simplify notation further we shall write $\mathcal{L}_b$ instead of $\mathcal{L}^{n+k}_b$, and ignore the empty cases.

Now, let us turn our attention to *Dyck paths*, that is, lattice paths from $a=(0,0)$ to $b=(n,n)$ using only North- and East-steps that do not go below the main diagonal, i.e. if $x=(s_1,s_2,\dots,s_{2n})\in\mathcal{L}_{(n,n)}$ then we demand that there are at least as many North-steps as East-steps in any initial subsequence $(s_1,\dots,s_k)$ where $1\le k\le 2n$. Despite our previous definition of length, we shall adopt the convention of saying a Dyck path from the origin to (n,n) has *length* n. We denote the set of all such Dyck paths by $\mathcal{D}^n$. For an example see Figure 2.

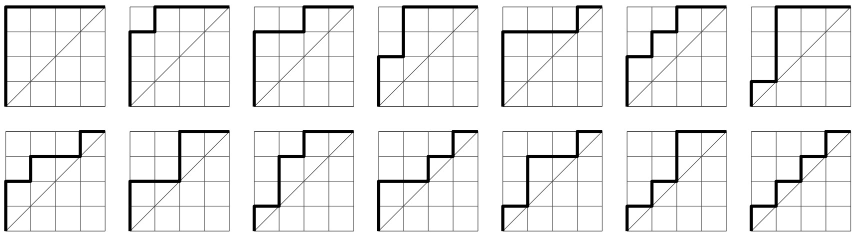


Figure 2: All Dyck paths of length four.

We define the n-th *Catalan number* as $C_n := \frac{1}{n+1}\binom{2n}{n}$. Our first well-known result relates Dyck paths to Catalan numbers:

Proposition 3.1 (i). *Let $n\in\mathbb{N}$. Then $C_n=|\mathcal{D}^n|$, i.e., the number of Dyck paths of length n is given by the n-th Catalan number.*

(ii). *For $n\ge 1$ we have the Catalan recursion $C_n=\sum_{k=1}^n C_{k-1}C_{n-k}$.*

Proof. We prove claim (i) using the so called reflection principle. Generating all lattice paths in $\mathcal{L}_{(n,k)}$ is just a matter of distributing k North-steps over $n+k$ slots, thus we have $|\mathcal{L}_{(n,k)}|=\binom{n+k}{k}$. Now let $x\in\mathcal{L}_{(n,n)}-\mathcal{D}^n$ be a path going below the diagonal. Then we may write $x=(y,E,z)$ where E is the first step below the diagonal (see Figure 3). The path y is a Dyck path, say of length d, while z consists of exactly $n-d$ North-steps and $n-d-1$ East-steps. Let φ be the map that exchanges N and E. Then $\hat{x}:=(y,E,\varphi(z))\in\mathcal{L}_{(n+1,n-1)}$. Since every path in $\mathcal{L}_{(n+1,n-1)}$ crosses the diagonal, the map $x\mapsto\hat{x}$ is a bijection, and we obtain

$$|\mathcal{D}^n|=|\mathcal{L}_{(n,n)}|-|\mathcal{L}_{(n+1,n-1)}|=\binom{2n}{n}-\binom{2n}{n-1}=\frac{1}{n+1}\binom{2n}{n}.$$

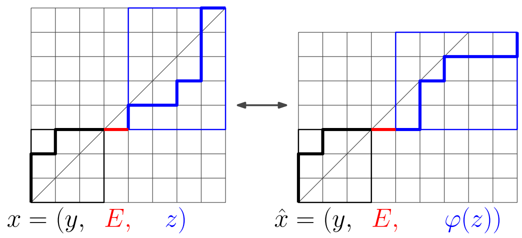


Figure 3: An example where $n = 8$, $d = 3$.

To see (ii), let $n \geq 1$. We construct a bijection from $\mathcal{D}^n$ to the disjoint union $\bigcup_{k=1}^{n-1} \mathcal{D}^{k-1} \times \mathcal{D}^{n-k}$. Any $x \in \mathcal{D}^n$ we may write as $x = (N, y, E, z)$ where E is the East-step before x returns to the diagonal for the first time (see Figure 4). Then for some $1 \leq k \leq n-1$ the paths y and z are Dyck paths of length $k-1$ and $n-k$ respectively, and the map $x \mapsto (y, z)$ is the bijection we were looking for. ∎

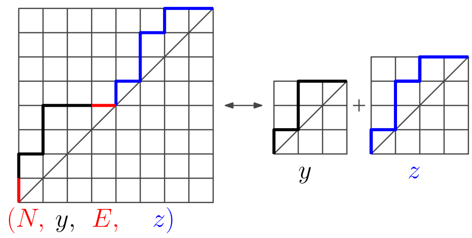


Figure 4: An example where $n = 8$ and $k = 4$.

Our immediate aim is to define a q-analogue of the Catalan numbers, that is, to find a family of polynomials $C_n(q)$ such that $C_n(1) = C_n$, retaining some of the properties of the Catalan numbers. Indeed, we will demonstrate two natural candidates in Proposition 3.4 (which is clearly somewhat in the style of Proposition 3.1).

First, we have to define some statistics on lattice paths (and Dyck paths in particular) whose generating polynomials we will consider. Let $x \in \mathcal{L}_{(n,k)}$ be represented by its North- and East-step sequence $(s_1, \ldots, s_{n+k})$. A pair (i, j) where $1 \leq i < j \leq n+k$ is an *inversion* of x if $s_i = E$ and $s_j = N$, and a *non-inversion* if $s_i = N$ and $s_j = E$. A *descent* of x is a number $1 \leq i \leq n+k$ such that $s_i = E$ and $s_{i+1} = N$. We define the three statistics $\mathrm{inv}, \mathrm{coinv}, \mathrm{maj} : \mathcal{L}_{(n,k)} \to \mathbb{N}$ as follows

$$\begin{aligned} \mathrm{inv}(x) &:= \#\text{ of inversions } (i,j) \text{ of } x, \\ \mathrm{coinv}(x) &:= \#\text{ of non-inversions } (i,j) \text{ of } x, \\ \mathrm{maj}(x) &:= \sum_{i \text{ is a descent}} i. \end{aligned}$$

Note that the every 1×1-square below the lattice path can be naturally associated with a non-inversion. Thus, $\mathrm{coinv}(x)$ gives the total area below x. However, we refrain from calling this statistic "area" in order to avoid confusion with another slightly different statistic a little further down the line.

Proposition 3.2. *Let $n, k \in \mathbb{N}$. Then we have*

$$\sum_{x\in\mathcal{L}_{(n,k)}} q^{\mathrm{inv}(x)} = \sum_{x\in\mathcal{L}_{(n,k)}} q^{\mathrm{coinv}(x)} = \sum_{x\in\mathcal{L}_{(n,k)}} q^{\mathrm{maj}(x)} = \begin{bmatrix} n+k \\ k \end{bmatrix}_q .$$

Proof. First, we notice that the map given by $(s_1,\ldots,s_{n+k}) \mapsto (s_{n+k},\ldots,s_1)$ is an involution on $\mathcal{L}_{(n,k)}$ interchanging inv and coinv, so we may treat only one of the two.

We prove the statement by induction on $n+k$. The cases $n=0$ or $k=0$ are trivial since there is only one path in each case. Thus, let $n \geq 1$ and $k \geq 1$.

Let $x \in \mathcal{L}_{(n,k)}$ and consider how many inversions the last step of x contributes, i.e., inversions of the form $(i, n+k)$. If x ends with a North-step, it gives us n inversions, while there are none if x ends with an East-step. Thus, for $f(n,k;q) := \sum_{x\in\mathcal{L}_{(n,k)}} q^{\mathrm{inv}(x)}$ the induction hypothesis yields

$$\begin{aligned} f(n,k;q) &= f(n-1,k;q) + q^n f(n,k-1;q) \\ &= \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q + q^n \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n+k \\ k \end{bmatrix}_q . \end{aligned}$$

Now, let us take a look at the maj statistic: If x ends with an East-step, we have $\mathrm{maj}(x) = \mathrm{maj}(\hat{x})$ where $\hat{x} \in \mathcal{L}_{(n-1,k)}$ is formed by deleting the last step of x. On the other hand, let $1 \leq d \leq k$, and consider a path $x = (s_1,\ldots,s_{n+k-d-1},E,N,\ldots,N)$ ending with exactly d North-steps. Then, for $\hat{x} = (s_1,\ldots,s_{n+k-d-1}) \in \mathcal{L}_{(n-1,k-d)}$ we have $\mathrm{maj}(x) = \mathrm{maj}(\hat{x}) + n + k - d$.

For $F(n,k;q) := \sum_{x\in\mathcal{L}_{(n,k)}} q^{\mathrm{maj}(x)}$ our observations translate into

$$\begin{aligned} F(n,k;q) &= F(n-1,k;q) + q^n \sum_{d=1}^{k} q^{k-d} F(n-1,k-d;q) \\ &= \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q + q^n \left(q^{k-1} \begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}_q + \cdots + q \begin{bmatrix} n \\ 1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ 0 \end{bmatrix}_q \right). \end{aligned}$$

Finally, using that

$$\begin{bmatrix} n-1 \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ 0 \end{bmatrix}_q \qquad \text{and} \qquad q^l \begin{bmatrix} n+l-1 \\ l \end{bmatrix}_q + \begin{bmatrix} n+l-1 \\ l-1 \end{bmatrix}_q = \begin{bmatrix} n+l \\ l \end{bmatrix}_q$$

for $l \in \mathbb{N}$, the identity above reduces to

$$F(n,k;q) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q + q^n \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n+k \\ k \end{bmatrix}_q .$$

■

We remark that both Theorem 2.1 and Proposition 3.2 can be regarded as special cases of the same theorem about multiset permutations which was already known to MacMahon [17].

Identifying the 1×1 squares within a $n \times k$ rectangle which lie above a lattice path $x \in \mathcal{L}_{(n,k)}$ with the Young diagram of a partition, we obtain a formula for the generating function of partitions whose Young diagrams are subsets of a fixed rectangle.

Corollary 3.3. *Let $k, n \in \mathbb{N}$, and let $P(n,k) := \{\lambda : \lambda \subseteq n^k\}$ be the set of all partitions with at most k parts, and with parts less than or equal to n. Then we have*

$$\sum_{\lambda \in P(n,k)} q^{|\lambda|} = \begin{bmatrix} n+k \\ k \end{bmatrix}_q .$$

■

Another useful way of describing a Dyck path, apart from the step-sequence (or a picture) is the sequence of *row lengths*. That is, a sequence $a = (a_1, \dots, a_n)$ of natural numbers such that $a_1 = 0$ and $0 \le a_{i+1} \le a_i + 1$ for $1 \le i \le n-1$. To obtain the sequence of row lengths of a Dyck path $x \in \mathcal{D}^n$ let a_i be the number of 1×1 squares in the i-th row between x and the diagonal (see Figure 5).

By means of the row lengths we can define two further statistics area, dinv : $\mathcal{D}^n \to \mathbb{N}$ as

$$\begin{aligned} \operatorname{area}(x) &:= \sum_{i=1}^{n} a_i, \\ \operatorname{dinv}(x) &:= \left|\left\{(i,j) : 1 \le i < j \le n,\, a_i = a_j\right\}\right| + \\ &\quad + \left|\left\{(i,j) : 1 \le i < j \le n,\, a_i = a_j + 1\right\}\right|. \end{aligned}$$

A pair (i,j) contributing to dinv is called a *diagonal inversion* or d-inversion, hence the name of the statistic. Note that for any Dyck path $x \in \mathcal{D}^n$ we have $\operatorname{inv}(x) + \operatorname{area}(x) = \binom{n}{2}$. The dinv statistic was suggested by Haiman.

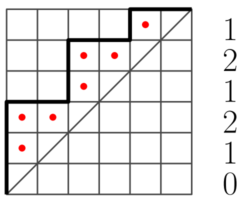


Figure 5: A Dyck path x with row lengths $(0, 1, 2, 1, 2, 1)$, $\operatorname{area}(x) = 7$, and $\operatorname{dinv}(x) = 7$.

Now we are able to give q-analogues of the Catalan numbers as promised. The *MacMahon q-Catalan numbers* are defined as

$$B_n(q) := \sum_{x \in \mathcal{D}^n} q^{\mathrm{maj}(x)}.$$

The *Carlitz–Riordan q-Catalan numbers* are defined as

$$C_n(q) := \sum_{x \in \mathcal{D}^n} q^{\mathrm{area}(x)}.$$

Proposition 3.4 (i). *Let $n \in \mathbb{N}$. Then the MacMahon q-Catalan numbers are given by*

$$\sum_{x \in \mathcal{D}^n} q^{\mathrm{maj}(x)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q.$$

(ii). *For $n \geq 1$ the Carlitz–Riordan q-Catalan numbers fulfil the recursion*

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_{k-1}(q) C_{n-k}(q).$$

Proof. Let $x \in \mathcal{L}_{(n,n)} - \mathcal{D}^n$ and consider the point $z = (z_1, z_2)$ on x minimizing z_1 among all points with maximal $z_1 - z_2$, that is, among the points the farthest below the diagonal (compare to Figure 6). The path x must have an East-step leading to z. Moreover, this East-step cannot be directly preceded by a North-step. Thus, we may construct $\hat{x} \in \mathcal{L}_{(n-1,n+1)}$ by replacing the East-step leading to z by a North-step, and have $\mathrm{maj}(x) = \mathrm{maj}(\hat{x}) + 1$.

Conversely, given any path $\hat{x} \in \mathcal{L}_{(n-1,n+1)}$. By choosing $z' = (z'_1, z'_2)$ such that z'_1 is maximal among all points of $\hat{x}$ with maximal $z'_1 - z'_2$, and replacing the North-step that must follow by an East-step, we obtain the original path. Thus, the map $x \mapsto \hat{x}$ is a bijection between paths in $\mathcal{L}_{(n,n)}$ going below the diagonal and $\mathcal{L}_{(n-1,n+1)}$. Proposition 3.2 establishes

$$\sum_{x \in \mathcal{D}^n} q^{\mathrm{maj}(x)} = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q \begin{bmatrix} 2n \\ n+1 \end{bmatrix}_q = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q.$$

To see (ii) let $n \geq 1$ and $x = (N, y, E, z) \in \mathcal{D}^n$ be decomposed exactly as in Proposition 3.1 (ii) and Figure 4. If y is a Dyck path of length $k-1$ then $\mathrm{area}(x) = (k-1) + \mathrm{area}(y) + \mathrm{area}(z)$, and the claim follows. ∎

Having found two fine candidates for q-generalisations of the Catalan numbers, we shall now define a bivariate polynomial that turns out to contain them both

$$C_n(q,t) := \sum_{x \in \mathcal{D}^n} q^{\mathrm{dinv}(x)} t^{\mathrm{area}(x)}.$$

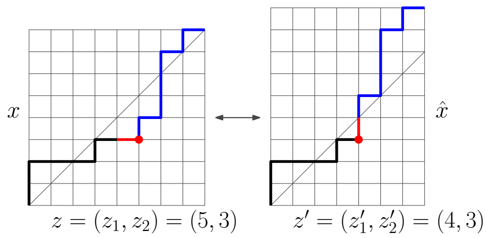


Figure 6: In this example $\mathrm{maj}(x) = 5 + 8 + 10 + 14$ while $\mathrm{maj}(\hat{x}) = 5 + 7 + 10 + 14$.

The first fundamental results on these polynomials, called *q, t-Catalan numbers*, are covered in the next theorem which needs one more preparation, namely the *bounce statistic*.

Let $x \in \mathcal{D}^n$ then we construct the *bounce path* $\hat{x} \in \mathcal{D}^n$ of x in the following way: We start with the empty path, and add as many N-steps as possible without going above x. Next, we add E-steps until we touch the diagonal. We then add a maximal number of N-steps without going above x, add E-steps until we reach the diagonal, and so forth... (see Figure 7). Suppose the bounce path of x returns to the diagonal in the points $(i_1, i_1), \ldots, (i_d, i_d)$ for some $1 \leq d \leq n$. Then the statistic bounce : $\mathcal{D}^n \to \mathbb{N}$ is defined as

$$\mathrm{bounce}(x) := \sum_{j=1}^{d} (n - i_j).$$

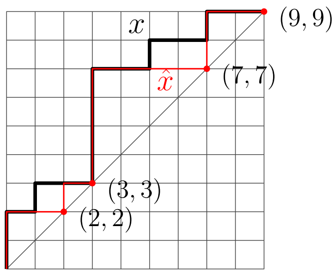


Figure 7: A Dyck path x with its bounce path $\hat{x}$ and $\mathrm{bounce}(x) = (9-2) + (9-3) + (9-7) + (9-9) = 15$.

Note that $\mathrm{bounce}(x)$ is given by $\frac{1}{2}\mathrm{maj}(\hat{x}')$ where $\hat{x}'$ is the reversed bounce path of x. The bounce statistic was first defined by Haglund in [8] in a slightly different form.

Theorem 3.5 (i). *There is a bijection $\zeta : \mathcal{D}^n \to \mathcal{D}^n$ such that for all $x \in \mathcal{D}^n$ we have*

$$(\mathrm{dinv}(x), \mathrm{area}(x)) = (\mathrm{area}(\zeta(x)), \mathrm{bounce}(\zeta(x))).$$

(ii). *Let $n, k \in \mathbb{N}$, $1 \leq k \leq n-1$ and let $\mathcal{D}_k^n$ be the set of Dyck paths of length n that return to the diagonal exactly k times. Then for $F(n,k;q,t) := \sum_{x \in \mathcal{D}_k^n} q^{\mathrm{dinv}(x)} t^{\mathrm{area}(x)}$ we have the recursive formula*

$$F(n,k;q,t) = q^{\binom{k}{2}} t^{n-k} \sum_{d=1}^{n-k} F(n-k,d;q,t) \begin{bmatrix} k+d-1 \\ d \end{bmatrix}_q .$$

The bijection ζ traces back to an article of Andrews, Krattenthaler, Orsina and Papi [1] who already defined the bounce path, but did not consider the statistics dinv or bounce at the time. The recurrence was discovered by Haglund in [8].

Proof (The Bijection). Let $x \in \mathcal{D}^n$ be represented by its row lengths $(a_1, \ldots, a_n)$. Further, let $m := \max_{1 \leq i \leq n}\{a_i\}$ be the maximal row length, and let

$$b_j := |\{i : 1 \leq i \leq n,\, a_i = j\}|$$

for $0 \leq j \leq m$ denote the numbers of rows with a given length.

Now, let $0 \leq j \leq m+1$, and define lattice paths y_j as follows: Reading the vector $(a_1, \ldots, a_n)$ from left to right, whenever we encounter an entry equal to $j-1$ we add an East-step. Whenever we encounter an entry equal to j we add a North-step (compare to Figure 8).

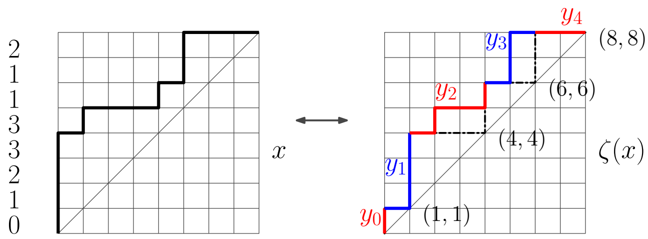


Figure 8: In this example we have $m = 3$, $b = (1,3,2,2)$.

We see that y_0 consists of b_1 North-steps, while y_{m+1} consists of b_m East-steps. For all $1 \leq j \leq m$ the path y_j starts with an East-step and lies in $\mathcal{L}_{(b_j, b_{j+1})}$. This lets us define $\zeta(x) := (y_0, y_1, \ldots, y_{m+1})$.

First of all, $\zeta(x) \in \mathcal{D}^n$ because every initial subsequence of $\zeta(x)$ contains at least as many North-steps as East-steps.

Secondly, let us check that ζ is a bijection. For every Dyck path $z \in \mathcal{D}^n$ there is a unique decomposition into subsequences $z = (y_0, y_1 \ldots, y_{m+1})$, where $m \in \mathbb{N}$ depends on z, such that each y_j starts with an East-step and such that the number of East-steps in y_j equals the number of North-steps in y_{j-1} for all $j \geq 1$. Thus, ζ is bijective because it is injective.

Next, we show that the bounce path of $\zeta(x)$ returns to the diagonal exactly $m+1$ times, namely at the points $\left(\sum_{l=0}^{j-1} b_l, \sum_{l=0}^{j-1} b_l\right)$ for $j = 1, \ldots, m+1$. Clearly, the bounce path of $\zeta(x)$ starts with b_0 North-steps followed by b_0 East-steps. Thus, the first bounce point is indeed (b_0, b_0). Now, we proceed inductively. Assume that there is a bounce point at $\big(\sum_{l=0}^{j-1} b_l, \sum_{l=0}^{j-1} b_l\big)$ for some $1 \le j \le m$. Since $(y_0, \ldots, y_j) \in \mathcal{L}_{(\sum_{l=0}^{j-1} b_l, \sum_{l=0}^{j} b_l)}$, and because y_{j+1} begins with an E-step, we can add exactly b_j North-steps to the bounce path without exceeding $\zeta(x)$. But then the next bounce point is $\big(\sum_{l=0}^{j} b_l, \sum_{l=0}^{j} b_l\big)$ as claimed.

Thus, we have

$$\operatorname{bounce}(\zeta(x)) = \sum_{j=1}^{m} \left(n - \sum_{l=0}^{j-1} b_l \right) = \sum_{j=1}^{m} \left(\sum_{l=j}^{m} b_l \right) = \sum_{j=1}^{m} j b_j = \operatorname{area}(x).$$

Finally, we have

$$\operatorname{area}(\zeta(x)) = \sum_{j=0}^{m} \left(\binom{b_j}{2} + \operatorname{coinv}(y_j) \right) = \operatorname{dinv}(x),$$

where the binomial coefficients correspond to the area of $\zeta(x)$ below its bounce path, respectively the diagonal inversions of rows of equal length in x, and the terms $\operatorname{coinv}(y_j)$ contribute the area between $\zeta(x)$ and its bounce path, respectively the inversions of a row of length j and a row of length $j+1$ in x.

(The Recursion). Let $x \in \mathcal{D}_k^n$ be represented by its row lengths $(a_1, \ldots, a_n)$. The number of a_i that equal zero is exactly k. Let $\hat{x} \in \mathcal{D}^{n-k}$ be the path obtained by reducing each row length of x by one, and then deleting the negative entries (see Figure 9). More precisely, $\hat{x} \in \mathcal{D}_d^{n-k}$ where d is the number of rows of length one in x.

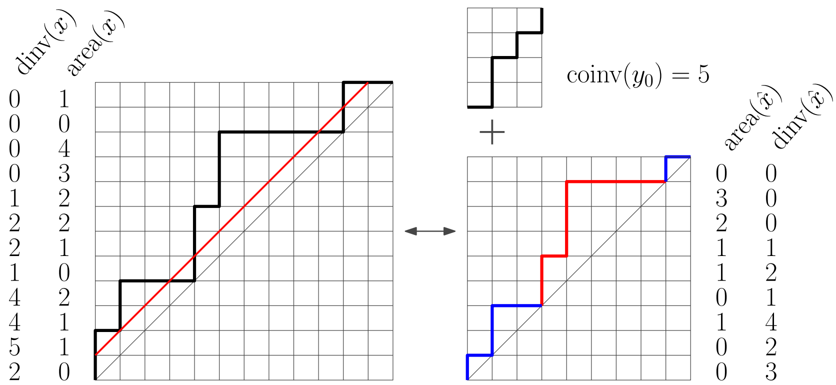


Figure 9: We have $x \in \mathcal{D}_3^{12}$ mapped to $\hat{x} \in \mathcal{D}_4^{9}$ and $y_1 \in \mathcal{L}_{(3,4)}$.

First, we note that $\operatorname{area}(x) = \operatorname{area}(\hat{x}) + n - k$ since we reduced each non-zero row of x by one. Secondly, the d-inversions of $\hat{x}$ correspond to the d-inversions of x which do not involve rows of length zero. Furthermore, for each pair of distinct rows of x of length zero there is one d-inversion of x. Thus, $\operatorname{dinv}(x) = \operatorname{dinv}(\hat{x}) + \binom{k}{2} + \xi$ where ξ counts the d-inversion coming from a row of length one occurring before a row of length zero. As before, we have $\xi = \operatorname{coinv}(y_1)$

Now, let $a' = (a'_1, \ldots, a'_{n-k})$ be the vector of row lengths of $\hat{x}$ after increasing every entry by one. To regain x from a' we have to insert k entries equal to zero, which is possible before an entry equal to one. Fortunately, all the needed information is encoded in y_1: starting at $a'_1 = 1$, and following the path y_1, whenever we encounter an East-step in y_1 we add a zero before the current entry of a'. Whenever we encounter a North-step in y_1 we move to the next entry a'_i that equals one.

Taking into account that y_1 always begins with an East-step, thus really ranges over $\mathcal{L}_{(k-1,d)}$, the map $x \mapsto (\hat{x}, y_1)$ is a bijection from $\mathcal{D}^n_k$ to the disjoint union $\bigcup_{d=1}^{n-k} \mathcal{D}^{n-k}_d \times \mathcal{L}_{(k-1,d)}$. The claim then follows from Proposition 3.2 providing that the q-binomial coefficient arises when $q^{\mathrm{coinv}(y)}$ is summed up with y ranging over $\mathcal{L}_{(k-1,d)}$. ■

Corollary 3.6. *Let $n \in \mathbb{N}$.*

(i). *Then the bijection ζ yields*

$$C_n(q,t) = \sum_{x \in \mathcal{D}^n} q^{\mathrm{area}(x)} t^{\mathrm{bounce}(x)}.$$

(ii). *Furthermore,*

$$\sum_{x \in \mathcal{D}^n} q^{\mathrm{dinv}(x)} = \sum_{x \in \mathcal{D}^n} q^{\mathrm{area}(x)} = \sum_{x \in \mathcal{D}^n} q^{\mathrm{bounce}(x)}.$$

(iii). *For any $1 \leq k \leq n$ let $\mathcal{D}^n_k$ denote the set of Dyck paths of length n that start with exactly k North-steps. Then $F(n,k;q,t) = \sum_{x \in \mathcal{D}^n_k} q^{\mathrm{area}(x)} t^{\mathrm{bounce}(x)}$, and we have the recursion*

$$F(n,k;q,t) = \sum_{d=1}^{n-k} \sum_{y \in \mathcal{D}^{n-k}_d} q^{\binom{k}{2}} t^{n-k} q^{\mathrm{area}(y)} t^{\mathrm{bounce}(y)} \begin{bmatrix} k+d-1 \\ d \end{bmatrix}_q.$$

■

Another consequence of Theorem 3.5 will be needed later on.

Corollary 3.7. *Let $n \in \mathbb{N}$, and k, $\mathcal{D}^n_k$, and $F(n,k;q,t)$ be as in Theorem 3.5 (ii), or alternatively as in Corollary 3.6 (iii). Then we have*

$$t^n C_n(q,t) = F(n+1,1;q,t).$$

■

The following result is the main motivation for this work. As we have seen in Corollary 3.6, the generating functions of the statistics area, dinv and bounce are equal. Indeed, the maps ζ and ζ^{-1} provide bijective rules to transform one statistic into another. However, the q,t-Catalan numbers fulfil a much stronger symmetry property.

Theorem 3.8. *Let $n \in \mathbb{N}$. Then the q,t-Catalan numbers are symmetric in q and t, i.e.*

$$C_n(q,t) = C_n(t,q).$$

Unfortunately, there is no known bijection $\varphi : \mathcal{D}^n \to \mathcal{D}^n$ such that $\operatorname{area}(\varphi(x)) = \operatorname{dinv}(x)$ and $\operatorname{dinv}(\varphi(x)) = \operatorname{area}(x)$. The same goes for area and bounce. In other words, even though Theorem 3.8 guarantees the existence of such a map, no bijective proof of the theorem is known to this date. The only proof we do currently know involves the machinery of symmetric functions and Macdonald polynomials, and is found at the very end of this paper. It should be mentioned that a bijective explanation of the (from a combinatorial point of view) surprising symmetry of the $C_n(q,t)$ would be desirable not only on principle, but it may also shed light on possible decompositions of the corresponding $\mathfrak{S}_n$-module which will be introduced later on.

As indicated above, the q,t-Catalan numbers contain both versions of q-Catalan numbers of Proposition 3.4 as special values. This result was shown by Haglund in [8].

Theorem 3.9. *Let $n \in \mathbb{N}$. Then we have $C_n(q,1) = C_n(1,q) = C_n(q)$, and*

$$q^{\binom{n}{2}} C_n(q, \tfrac{1}{q}) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q .$$

Proof. The first identity is immediate. The second follows by substituting

$$q^{\binom{n}{2}} F(n,k;q,\tfrac{1}{q}) = q^{(k-1)n} \frac{[k]_q}{[n]_q} \begin{bmatrix} 2n-k-1 \\ n-k \end{bmatrix}_q$$

into the recursion in Theorem 3.5 (ii), and Corollary 3.7. For details see [9, Chapter 3 Theorem 3.10 and Corollary 3.10.1]. ■

For now, we shall investigate a possible extension of our three statistics to labelled Dyck paths. A vector $f = (f_1, \ldots, f_n) \in \{1, 2, \ldots, n\}^n$ is called a *parking function* of length n if there exists a permutation $\sigma \in \mathfrak{S}_n$ such that $f_{\sigma(i)} \leq i$ for all $1 \leq i \leq n$. Equivalently, for $1 \leq i \leq n$, there have to be at least i entries of f less than or equal to i. We denote the set of all parking functions of length n by $\mathcal{PF}^n$.

The name "parking function" demands an explanation. Imagine a line of n cars driving through a lane with n parking places. Each car i prefers to park in the f_i-th spot and behaves as follows: It moves to the f_i-th spot and parks if this spot is still unoccupied. Otherwise, it parks at the next available place. This way all cars get to park in the lane if and only if f is a parking function.

Proposition 3.10. *Let $n \in \mathbb{N}$. Then the number of parking functions of length n is given by $(n+1)^{n-1}$.*

Proof. Let $f \in \{1, \ldots, n+1\}^n$ be any function, and suppose that $n+1$ parking places are arranged in a cycle. Each car i moves clockwise to its preferred spot f_i and parks at the first possible spot thereafter. Clearly, every car now finds a place, and exactly one spot remains unused. Furthermore, f is a parking function of length n if and only if the unused spot is $n+1$. By symmetry this means that one out of every $n+1$ functions is a parking function. ■

A *labelled Dyck path* of length n is a pair (x, σ) of a Dyck path $x \in \mathcal{D}^n$ and a permutation $\sigma \in \mathfrak{S}_n$ such that one condition is fulfilled: If we assign the label σ_i to the i-th North step of x, then we demand that successive North-steps that are not separated by an East-step are assigned increasing labels (see Figure 10). In this way, in every row of x we find exactly one label, and $\sigma(i)$ is the label of the i-th row.

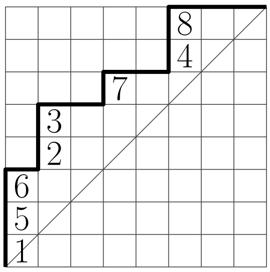


Figure 10: A Dyck path labelled by the permutation $\sigma = 15623748$.

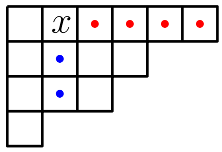
Proposition 3.11. *Let $n \in \mathbb{N}$ then there is a bijection θ from labelled Dyck paths of length n onto parking functions of n.*

Proof. Let (x, σ) be a labelled Dyck path, then we define f_i to be the column in which the label i appears. For example, by this rule Figure 10 yields the vector $(1, 2, 2, 6, 1, 1, 4, 6)$. Since x never crosses the diagonal, the resulting $f \in \{1, \ldots, n\}^n$ must be a parking function. ■

By virtue of the bijection θ we will identify $\mathcal{PF}^n$ with the set of labelled Dyck paths of length n. We define two statistics area, dinv : $\mathcal{PF}^n \to \mathbb{N}$ via

$$\begin{aligned}
\operatorname{area}(x, \sigma) &:= \operatorname{area}(x), \\
\operatorname{dinv}(x, \sigma) &:= \left|\left\{(i, j) : 1 \leq i < j \leq n,\, a_i = a_j,\, \sigma(i) < \sigma(j)\right\}\right| + \\
&\quad + \left|\left\{(i, j) : 1 \leq i < j \leq n,\, a_i = a_j + 1,\, \sigma(i) > \sigma^(j)\right\}\right|,
\end{aligned}$$

where $(a_1, \ldots, a_n)$ are the row lengths of x. The labelled Dyck path in Figure 10 has $\operatorname{dinv}(x, \sigma) = 5+2$, where the contributing non-inversions of σ are $(2, 7), (2, 8), (6, 7), (6, 8)$ and $(7, 8)$, and the contributing inversions are $(4, 6)$ and $(4, 7)$. Note that $\operatorname{dinv}(x, \sigma) \leq \operatorname{dinv}(x)$ for all labelled Dyck paths, and that for every $x \in \mathcal{D}^n$ there is at least one $\sigma \in \mathfrak{S}_n$ such that we have equality.



$a_\lambda(x) = 4$

$l_\lambda(x) = 1$

$a'_\lambda(x) = 1$

$l'_\lambda(x) = 0$

Figure 11: The arm, leg, coarm and coleg of a cell x of the partition $\lambda = (6, 4, 3, 1)$.

We define the following extension of the q, t-Catalan numbers

$$D_n(q,t) := \sum_{(x,\sigma)\in\mathcal{PF}^n} q^{\operatorname{dinv}(x,\sigma)} t^{\operatorname{area}(x,\sigma)}. \tag{3.1}$$

There are equivalent combinatorial descriptions of the $D_n(q, t)$, for example using the bounce statistic, however, we shall leave it at the definition. For more details see Chapter 5 of Haglund's book [9]. It is conjectured, albeit not proven, that the polynomials $D_n(q, t)$ are symmetric in q and t. This would follow from the conjectured fact that $D_n(q, t)$ equals the bivariate Hilbert series of the $\mathfrak{S}_n$-module of diagonal harmonics. The conjecture is due to Haglund and Loehr who defined the polynomials $D_n(q, t)$ in [10]. We will return to this connection with the Hilbert series at the beginning of the last chapter.

The conclusion to this section contains some of the author's own observations. Let $n \in \mathbb{N}$, $\lambda \vdash n$ be a partition, and $x \in Y(\lambda)$ be a cell of the Young diagram of λ. We define the *arm* of x, denoted by $a_\lambda(x)$, as the number of cells in $Y(\lambda)$ in the same row as x and strictly to the right of x (see Figure 11). Similarly, we define the *leg* $l_\lambda(x)$, the *coarm* $a'_\lambda(x)$, and the *coleg* $l'_\lambda(x)$ to be the number of cells strictly below, to the left, and above x respectively.

As in [9, Chapter 3 (3.63)], we define a dinv-statistic on partitions as follows

$$\operatorname{dinv}(\lambda) := \left|\left\{x \in Y(\lambda) : l_\lambda(x) \le a_\lambda(x) \le l_\lambda(x) + 1\right\}\right|.$$

Such a cell x is called a diagonal inversion of λ.

We denote the n-th *staircase partition* by $\delta_n = (n-1, n-2, \dots, 2, 1, 0)$. Incidentally, we notice that $\operatorname{dinv}(\delta_n) = \binom{n}{2}$. There is a natural bijection between Dyck paths of length n and partitions $\lambda \subseteq \delta_n$. This bijection can be made explicit as $\lambda := \delta_n - (a_n, a_{n-1}, \dots, a_1)$, where $(a_1, \dots, a_n)$ are the row lengths of the Dyck path in concern. We call the partition λ defined in this way the *complement* of the Dyck path.

Proposition 3.12. *Let $n \in \mathbb{N}$, $x \in \mathcal{D}^n$ be a Dyck path and $\lambda \subseteq \delta_n$ be the complement of x. Then we have $\operatorname{dinv}(x) = \operatorname{dinv}(\lambda)$, and $\operatorname{area}(x) = \binom{n}{2} - |\lambda|$.*

Proof. The claim about area is trivial. We prove the other claim by induction on n. The statement is true for $n = 1$, thus, let $n \ge 2$.

Suppose x begins with exactly k North-steps, and consider the path $\hat{x} \in \mathcal{D}^{n-1}$ obtained by deleting the first column (see Figure 12). We have $\operatorname{dinv}(x) = \operatorname{dinv}(\hat{x}) + \xi$

where ξ is the number of inversions involving the k-th row of x. Since these diagonal inversions correspond bijectively to the diagonal inversions $(i, 1)$ in the first column of $Y(\lambda)$, the claim follows. ∎

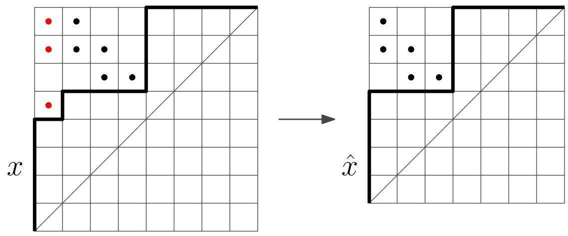


Figure 12: The diagonal inversions of λ are marked by points. The red inversions in the first column of λ correspond to the inversions $(4, 5)$, $(4, 7)$ and $(4, 8)$ of the fourth row of x.

We define an auxiliary statistic on the Dyck paths $\mathcal{D}^n$ as

$$u(x) := \binom{n}{2} - \operatorname{area}(x) - \operatorname{bounce}(x).$$

Clearly, u is never negative which can be seen from the previous proposition together with Theorem 3.5 (i). Suppose φ is a bijection on $\mathcal{D}^n$ exchanging area and bounce, then φ fixes u. In other words, trying to find such a bijection we can restrict ourselves to the subsets $\mathcal{D}^n_{u=k}$ of Dyck paths with $u(x) = k$. Additionally, we may give bounds for the number of Dyck paths $x \in \mathcal{D}^n_{u=k}$ with fixed $\operatorname{bounce}(x) = b$ which are tight when n is large (compared to k and b). For example, it is easy to verify that there is exactly one Dyck path x of length n with $u(x) = 0$ and $\operatorname{bounce}(x) = i$ for each $n \geq 1$ and $i = 0, \ldots, \binom{n}{2}$. The next theorem makes matters a little more precise.

Proposition 3.13. *Let $a, b, k \in \mathbb{N}$.*

(i). *Then there exists an $n(a, k) \in \mathbb{N}$ such that the number of Dyck paths x of length n with $u(x) = k$ and $\operatorname{area}(x) = a$ coincides for all $n \in \mathbb{N}$ such that $n \geq n(a, k)$.*

(ii). *Furthermore, there exists an $N(b, k) \in \mathbb{N}$ such that the number of Dyck paths x of length n with $u(x) = k$ and $\operatorname{bounce}(x) = b$ equals $\pi(k + b, b)$, the number of partitions of k with length at most b, for all $n \in \mathbb{N}$ such that $n \geq N(b, k)$.*

Proof. Consider a Dyck path $x \in \mathcal{D}^n$ with $u(x) = k$ and $\operatorname{area}(x) = a$. Assume (m, m) is not a bounce point of x, then $\operatorname{bounce}(x) \leq \binom{n}{2} - n + m$. It follows that

$$n - m \leq \binom{n}{2} - \operatorname{bounce}(x) = a + k.$$

Thus if $n \geq a + k$, the first $n - a - k$ points on the diagonal must all be bounce points of x. In other words, every "long" path x with $u(x) = k$ and $\operatorname{area}(x) = a$ begins with a "long" alternating sequence of N and E-steps. Therefore, no additional paths arise as n grows.

On the other hand, let $x \in \mathcal{D}^n$ be a Dyck path with $u(x) = k$ and $\mathrm{bounce}(x) = b$. Suppose that x has at least three distinct bounce points, and let m be minimal such that (m, m) is a bounce point of x, as in Figure 13. Let λ be the complement of x. Since (m, m) is a bounce point, λ has length at least $n-m$. Since there is at least a third bounce point between (m, m) and (n, n), we have $\lambda_1 \geq m+1$. Thus, $\mathrm{area}(x) = \binom{n}{2} - |\lambda| \leq \binom{n}{2} - n$ and it follows

$$n \leq \binom{n}{2} - \mathrm{area}(x) = k + b$$

Thus, if $n \geq k+b$ then x has at most two bounce points namely $(n-b, n-b)$ and (n, n). But then the complement of x is a partition of $k+b$ with length exactly b. ■

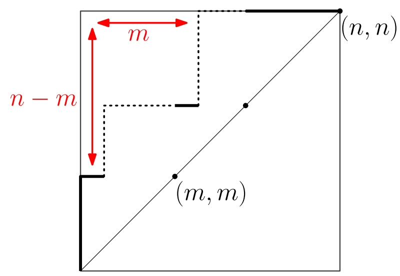


Figure 13: A Dyck path with many bounce points has less area.

Proposition 3.13 can also be stated in terms of "limits" of Dyck paths. For $n \in \mathbb{N}$ let $\psi_n^{(1)} : \mathcal{D}^n \to \mathcal{D}^{n+1}$ denote the map given by $x \mapsto (N, E, x)$, and $\psi_n^{(2)} : D^n \to D^{n+1}$ denote the map defined by $x \mapsto (N, x, E)$. For $i = 1, 2$ let $X^{(i)}$ be the set of all sequences $(x_n)_{n \in \mathbb{N}}$ where $x_n \in \mathcal{D}^n$ such that there exists an $r \in \mathbb{N}$ with $x_{n+1} = \psi_n^{(i)}(x_n)$ for all $n \geq r$. Furthermore, let $\sim$ be the equivalence relation on $X^{(i)}$ given by $(x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}}$ if there exists an $r \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geq r$. Now we define the set of *limit Dyck paths* of type i as

$$\mathcal{D}^{(i)} := \{[x]_\sim : x \in X^{(i)}\}.$$

We denote the limit Dyck paths by $\vec{x} = [(x_n)_n]_\sim$. It is intuitive to think of $\mathcal{D}^{(1)}$ as the union of all Dyck paths starting with at least two North-steps, and of $\mathcal{D}^{(2)}$ as the set of number partitions.

Lemma 3.14. *Let x_0 and y_0 be two Dyck paths, and define two sequences by letting $x_{n+1} := \psi_n^{(1)}(x_n)$ and $y_{n+1} := \psi_n^{(2)}(y_n)$ for all $n \in \mathbb{N}$. Then the sequences $(u(x_n))_n$, $(\mathrm{area}(x_n))_n$ and $(\mathrm{dinv}(y_n))_n$ are constant, and the sequences $(u(y_n))_n$ and $(\mathrm{bounce}(y_n))_n$ become stationary.*

Proof. We have $\mathrm{area}(x_{n+1}) = \mathrm{area}(x_n)$, $\mathrm{bounce}(x_{n+1}) = \mathrm{bounce}(x_n) + n$ and $\mathrm{area}(y_{n+1}) = \mathrm{area}(y_n) + n$. Moreover, $\mathrm{dinv}(y_{n+1}) = \mathrm{dinv}(y_n)$ due to Proposition 3.12, and $\mathrm{bounce}(y_{n+1}) \leq \mathrm{bounce}(y_n)$, where we have equality whenever y_n has only two bounce points. ∎

Hence, for $\vec{x} \in \mathcal{D}^{(1)}$ and $\vec{y} \in \mathcal{D}^{(2)}$, the limits

$$u(\vec{x}) := \lim_{n\to\infty} u(x_n), \qquad \mathrm{area}(\vec{x}) := \lim_{n\to\infty} \mathrm{area}(x_n),$$

$$u(\vec{y}) := \lim_{n\to\infty} u(y_n), \qquad \mathrm{bounce}(\vec{y}) := \lim_{n\to\infty} \mathrm{bounce}(y_n), \qquad \mathrm{dinv}(\vec{y}) := \lim_{n\to\infty} \mathrm{dinv}(y_n).$$

are well defined and finite. Proposition 3.13 and Lemma 3.14 suggest that bounce is the statistic least compatible with the maps $\psi^{(i)}$, but it "converges" to the partition length.

Theorem 3.15. *Let $k \in \mathbb{N}$.*

(i). *For $a \in \mathbb{N}$ the number of limit Dyck paths $\vec{x}$ of type one such that $u(\vec{x}) = k$ and $\mathrm{area}(\vec{x}) = a$ equals $\pi(k+a, a)$.*

(ii). *For $b \in \mathbb{N}$ the number of limit Dyck paths $\vec{y}$ of type two such that $u(\vec{y}) = k$ and $\mathrm{bounce}(\vec{y}) = b$ equals $\pi(k+b, b)$.*

Proof. First, we deduce claim (ii) from Proposition 3.13 (ii). Denote the set of Dyck paths x of length n with $u(x) = k$ and $\mathrm{bounce}(x) = b$ by $S_n \subseteq \mathcal{D}^n$. By Proposition 3.13, we may choose $r \in \mathbb{N}$ large enough such that $|S_n| = \pi(k+b, b)$ and $\psi_n^{(2)}(S_n) \subseteq S_{n+1}$ for all $n \geq r$. Since $\psi_n^{(2)} : S_n \to S_{n+1}$ is injective it is already a bijection for all $n \geq r$.

For any limit Dyck path $\vec{x} \in \mathcal{D}^{(2)}$ we have $u(\vec{x}) = k$ and $\mathrm{bounce}(\vec{x}) = b$ if and only if there exists an $r' \in \mathbb{N}$ such that $x_n \in S_n$ for all $n \geq r'$. It follows that each such $\vec{x}$ is induced by a sequence defined recursively as $x_{r'+i+1} := \psi_{r'+i}^{(2)}(x_{r'+i})$ starting at some $x_{r'} \in S_{r'}$.

Part (i) follows from (ii) and Theorem 3.8. Let $T_n \subseteq \mathcal{D}^n$ denote the set of all Dyck paths x with $u(x) = k$ and $\mathrm{area}(x) = a$. By symmetry we must have $|T_n| = |S_n|$ for all $n \in \mathbb{N}$. By Lemma 3.14, $\psi_n^{(1)}$ maps T_n into T_{n+1}. Therefore, $\psi_n^{(1)}$ is a bijection for sufficiently large n. But then the claim follows as in (ii). ∎

Another version of Theorem 3.15 does not refer to Dyck paths at all.

Theorem 3.16. *Let $k, n \in \mathbb{N}$. Then the number of partitions $\lambda \vdash n+k$ such that $\mathrm{dinv}(\lambda) = k$ equals the number of partitions $\mu \vdash n+k$ with length k. In other words,*

$$\sum_{\lambda \vdash n+k} q^{\mathrm{dinv}(\lambda)} = \sum_{\lambda \vdash n+k} q^{l(\lambda)}.$$

Proof. If λ is the complement of a Dyck path x then λ is also the complement of $\psi_n^{(2)}(x)$. Thus, the complement of a limit Dyck path of type two is well defined. By Proposition 3.12 and Lemma 3.14 we have $|\lambda| = u(\vec{x}) + \mathrm{bounce}(\vec{x})$, and $\mathrm{dinv}(\lambda) = \mathrm{dinv}(\vec{x})$ for all $\vec{x} \in \mathcal{D}^{(2)}$. Since every partition arises as the complement of a unique limit Dyck path, we need to count the number of $\vec{x} \in \mathcal{D}^{(2)}$ with $u(\vec{x}) + \mathrm{bounce}(\vec{x}) = n + k$ and $\mathrm{dinv}(\vec{x}) = k$.

Now, recall the bijection $\zeta_n : \mathcal{D}^n \to \mathcal{D}^n$ from Theorem 3.5 (i). It is not hard to verify that $\zeta_{n+1} \circ \psi_n^{(2)} = \psi_n^{(1)} \circ \zeta_n$. Thus for any sequence $(x_n)_n$ such that $x_{n+1} = \psi_n^{(2)}(x_n)$ for all $n \geq r$, we may define $(y_n)_n := (\zeta_n(x_n))_n$ and obtain

$$y_{n+1} = \zeta_{n+1}(x_{n+1}) = \zeta_{n+1} \circ \psi_n^{(2)}(x_n) = \psi_n^{(1)} \circ \zeta_n(x_n) = \psi_n^{(1)}(y_n).$$

for large n. Thereby, every limit Dyck path $\vec{x}$ of type two corresponds to a well defined limit Dyck path $\zeta(\vec{x})$ of type one, and $\zeta : \mathcal{D}^{(2)} \to \mathcal{D}^{(1)}$ is a bijection. Next, we define a statistic v on $\mathcal{D}^{(2)}$ by $v(\vec{y}) + \operatorname{dinv}(\vec{y}) = u(\vec{y}) + \operatorname{bounce}(\vec{y})$. Thus, we now need to count the limit paths $\vec{x} \in \mathcal{D}^{(2)}$ with $\operatorname{dinv}(\vec{x}) = k$ and $v(\vec{x}) = n$. Because of the equalities $\operatorname{dinv}(\vec{x}) = \operatorname{area}(\zeta(\vec{x}))$ and

$$\begin{aligned}
v(\vec{x}) &= u(\vec{x}) + \operatorname{bounce}(\vec{x}) - \operatorname{dinv}(\vec{x}) \\
&= \lim_{n\to\infty} \binom{n}{2} - \operatorname{area}(x_n) - \operatorname{bounce}(x_n) + \operatorname{bounce}(x_n) - \operatorname{dinv}(x_n) \\
&= \lim_{n\to\infty} \binom{n}{2} - \operatorname{area}(\zeta(x_n)) - \operatorname{bounce}(\zeta(x_n)) \\
&= u(\zeta(\vec{x})),
\end{aligned}$$

the theorem reduces to counting the number of limit paths $\vec{x} \in \mathcal{D}^{(1)}$ of type one such that $\operatorname{area}(\vec{x}) = k$ and $u(\vec{x}) = n$, and follows from Theorem 3.15 (i). ■

The presented proofs of both results, Theorem 3.15 and Theorem 3.16, rely on the symmetry of the q,t-Catalan numbers guaranteed by Theorem 3.8. Using only Proposition 3.13 the enumeration part in Theorem 3.15 (i) is reduced to a mere finiteness statement. However, there is a beautiful bijective proof of Theorem 3.16 due to Loehr and Warrington [14]. In their paper Loehr and Warrington define a family of statistics depending on a real parameter each of which has the same distribution on the partitions of n. The dinv statistic and the partition length arise by letting the real parameter equal one respectively zero. Loehr and Warrington also remark this connection to the q,t-Catalan numbers. Whether their bijection can be exploited to find a bijective proof of the symmetry problem itself remains an interesting question.

That said, computations suggest a stronger version of Proposition 3.13. That is, we conjecture that the lower bounds $n(a,k)$ and $N(b,k)$ in Proposition 3.13 can be replaced by a uniform bound independent of area and bounce, solely depending on u.

Conjecture 3.17. *Let $k \in \mathbb{N}$. Then there exists an $N(k) \in \mathbb{N}$ such that for all $d, n \in \mathbb{N}$, where $n \geq N(k)$ and $0 \leq d \leq \binom{n}{2}$, the number of Dyck paths x of length n with $u(x) = k$ and $\operatorname{area}(x) = d$ coincides with the number of partitions of k with length at most $\min\{d, \binom{n}{2} - k - d\}$. This is also the number of Dyck paths x of length n with $u(x) = k$ and $\operatorname{bounce}(x) = d$.*

Before we leave our trusted Dyck paths behind in favor of a more algebraic approach to the q,t-Catalen numbers, we shall give an example as to why we believe that Conjecture 3.17 is true. Below we find a table for the case $n = 10$. The number in the i-th row and j-th column gives the number of Dyck paths of length ten with $u(x) = i$ and $\text{area}(x) = j$. As we see, for $u(x) = 0, 1, 2, \ldots, 7$ the Dyck paths behave as predicted. For $u(x) = 8$ the numbers are off by one or zero. For $u(x) = 9$ the numbers are off by zero, one or three. As $u(x)$ increases further, things become more unclear.

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
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0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

4 Algebraic Prerequisites

Since the main objective of this work is to explain the proof of Theorem 3.8, our motivation for the upcoming sections will be the development of the necessary theory. To provide the reader with a very short outline of this proof, recall the polynomials $F(n,k;q,t)$ from the previous section. We have seen in Corollary 3.7 that $F(n+1,k;q,t)$ specialises to $C_n(q,t)$ when $k=1$. We will give another function $Q(n,k;q,t)$ which fulfils the same recursion as $F(n,k;q,t)$ does in Theorem 3.5 (ii) and which specialises to the following (at first sight complicated) expression.

$$\sum_{\mu\vdash n} \frac{(1-q)(1-t)\left(\prod_{x\in Y(\mu)} q^{a'_\mu(x)}t^{l'_\mu(x)}\right)^2 \prod'_{x\in Y(\mu)}(1-q^{a'_\mu(x)}t^{l'_\mu(x)})\sum_{x\in Y(\mu)} q^{a'_\mu(x)}t^{l'_\mu(x)}}{\prod_{x\in Y(\mu)}(q^{a_\mu(x)}-t^{l_\mu(x)+1})\prod_{x\in Y(\mu)}(t^{l_\mu(x)}-q^{a_\mu(x)+1})} \tag{4.1}$$

Here $\prod'_{x\in Y(\mu)}$ means that the cell $x=(1,1)$ is omitted in the product. To define the function $Q(n,k;q,t)$ we shall need the terminology of Schur functions and Macdonald polynomials. The proper setting is a sufficiently generalised ring of symmetric functions.

It is not hard to verify that the transition $\mu\mapsto\mu'$ provides the symmetry of (4.1) in q and t.

Example 4.1. We want to show that $C_3(q,t)$ equals the expression in (4.1) when the sum is taken over all partitions of three, that is $\mu\in\{(1,1,1),(2,1),(3)\}$. Firstly, we have

$$C_3(q,t)=q^3+q^2t+qt+qt^2+t^3.$$

On the other hand, the three summands corresponding to $(1,1,1)$, $(2,1)$ and (3) respectively are

$$\begin{aligned}
\frac{(1-q)(1-t)\left(q^0t^{0+1+2}\right)^2(1-t)(1-t^2)(1+t+t^2)}{(1-t^3)(1-t^2)(1-t)(t^2-q)(t-q)(1-q)} &= \frac{t^6(1-t)(1+t+t^2)}{(1-t^3)(t^2-q)(t-q)}\\
\frac{(1-q)(1-t)\left(q^{0+1+0}t^{0+0+1}\right)^2(1-q)(1-t)(1+q+t)}{(q-t^2)(1-t)(1-t)(t-q^2)(1-q)(t-q)} &= \frac{q^2t^2(1+q+t)}{(q-t^2)(t-q^2)}\\
\frac{(1-q)(1-t)\left(q^{0+1+2}t^0\right)^2(1-q)(1-q^2)(1+q+q^2)}{(q^2-t)(q-t)(1-t)(1-q^3)(1-q^2)(1-q)} &= \frac{q^6(1-q)(1+q+q^2)}{(q^2-t)(q-t)(1-q^3)}.
\end{aligned}$$

Indeed, they sum up to

$$\begin{aligned}
&\frac{t^6}{(t-q)(t^2-q)}+\frac{q^2t^2(1+q+t)}{(q-t^2)(t-q^2)}+\frac{q^6}{(q-t)(q^2-t)}\\
&\quad=\frac{t^6(q^2-t)+q^2t^2(1+q+t)(t-q)-q^6(t^2-q)}{(t-q)(t^2-q)(q^2-t)}\\
&\quad=\frac{(t-q)(t^2-q)(q^2-t)(q^3+q^2t+qt+qt^2+t^3)}{(t-q)(t^2-q)(q^2-t)}.
\end{aligned}$$

We notice that the partition $(2,1)$ is self-conjugated and that the corresponding summand is symmetric in q ant t. Similarly, if we exchange q and t in the summand corresponding to $(1,1,1)$ we obtain the summand corresponding to its conjugated partition (3). $\square$

Our strategy is mapped out as follows: We shall study basic notions about the structure of symmetric polynomials (Section 5). We define Schur functions and survey their properties (Section 6). We will replace symmetric polynomials by power series in (multiple sets of) infinitely many variables over a field of rational functions in q and t, and substitute Schur functions by Macdonald polynomials (Section 7). At this point we will need some specific preparatory results, and can finally turn to the proof in Section 8.

We start out by recalling some facts about groups and rings acting on other sets. Our benefits of recalling these concepts are twofold. First of all, the proof we aspire relies heavily on the theory of symmetric polynomials which are defined as the subring of polynomials invariant under the canonic action of the symmetric group. Secondly, the polynomials $C_n(q,t)$ and $D_n(q,t)$ introduced above are known, respectively conjectured, to have close connections to a different representation of the symmetric group.

Let G be a group and X an arbitrary set. The set $\mathfrak{S}_X$ of all bijections from X to X is a group with respect to composition. A *group action* of G on X is an outer binary operation $\bullet : G \times X \to X$ such that $g \bullet (h \bullet x) = (gh) \bullet x$, and $e \bullet x = x$ for all $g, h \in G$ and $x \in X$. Here e denotes the neutral element of G. Equivalently, a group action is a group homomorphism $\phi : G \to \mathfrak{S}_X$. To see this, let $g \bullet x := \phi(g)(x)$, and the required conditions are fulfilled.

The *stabilizer group* of $x \in X$ is defined to be $G_x := \{g \in G : g \bullet x = x\}$ and is, in fact, a subgroup of G. The *orbit* of an element x under G is defined as $G \bullet x := \{g \bullet x : g \in G\}$. The orbits of two elements $x, y \in X$ are either equal or disjoint. Hence, X equals the disjoint union of its orbits. We call $x \in X$ a *fixed point* under the action of G if $G \bullet x = \{x\}$. The set of all fixed points is denoted by X^G. A subset Y of X is called *G-stable* or invariant if $G \bullet Y := \{g \bullet y : g \in G, y \in Y\} \subseteq Y$. For example, X is always invariant and $\{x\}$ is invariant if x is a fixed point.

Now, let R be a ring with unity, and M an Abelian group. The set $\mathrm{End}(M)$ of group endomorphisms on M is a ring via pointwise addition, and multiplication given by composition. We say M is an *R-(left) module* if there is an outer binary operation $\bullet : R \times M \to M$ which is distributive with respect to addition in R and in M, and is compatible with the multiplication in R. That is, we demand $(r+s) \bullet m = r \bullet m + s \bullet m$ and $r \bullet (m+n) = r \bullet m + r \bullet n$ for all $r, s \in R$ and $m, n \in M$, and $1 \bullet m = m$ and $(rs) \bullet m = r \bullet (s \bullet m)$ for all $r, s \in R$ and $m \in M$. Equivalently, the module structure is given by a ring homomorphism $\phi : R \to \mathrm{End}(M)$ mapping $1 \in R$ to the identity id_M.

A subgroup N of M is called *R-submodule* if it is R-stable. For example, let $l_r : R \to R$, $s \mapsto rs$, denote the *left translation* by $r \in R$ on R, and regard $(R,+)$ as an R-module via the map $R \to \mathrm{End}(R,+)$, $r \mapsto l_r$. Then an additive subgroup of R is a submodule if and only if it is a left ideal. An R-module M is called *simple* if its only R-submodules are the trivial ones, namely $\{0\}$ and M itself. An R-module is called *semisimple* if it is the direct sum of simple R-modules. For any field K, a K-module is just a K-vector space which is simple if and only if it has dimension one. On the other hand, this means that every K-module is semisimple. A ring R is called *semisimple* if it is semisimple as an R-module via the canonical module structure given by left multiplication.

Now, let R be a commutative ring with unity, and A a ring with unity. We say A is an *algebra* over R if the additive group $(A, +)$ is an R-module which is also compatible with the multiplication on A, i.e., $r \bullet (ab) = (r \bullet a)b = a(r \bullet b)$ for all $r \in R$ and $a, b \in A$. Equivalently, let $\mathrm{Z}(A) := \{a \in A : ab = ba\ \forall b \in A\}$ denote the *center* of A, then the structure of an R-algebra is given by a ring homomorphism $\phi : R \to \mathrm{Z}(A)$ mapping the multiplicative identity in R to the multiplicative identity in A. If ϕ is injective, it is usual to identify R with $\phi(R) \subseteq A$ and thus write ra instead of $r \bullet a = \phi(r)a$. For example this is the case whenever R is a field.

Let K be a field and G a finite group. A *K-representation* of G is a *linear group action* of G on a K-vector space V, i.e., a group homomorphism $\rho : G \to \mathrm{Aut}_K(V)$.

The dimension of V is also called the *degree* of the representation (ρ, V). If V is finite dimensional, say $\dim_K(V) = n$, then $\rho(G) \leq \mathrm{Gl}(n, K)$ can also be viewed as a linear group. In this case ρ is sometimes called a *matrix representation* of G. Two representations (ρ, V) and (ρ', V') are *isomorphic* if there exists a linear isomorphism $\varphi : V \to V'$ such that $\rho'(g) \circ \varphi = \varphi \circ \rho(g)$ for all $g \in G$. A representation (ρ, V) is called *irreducible* if the only G-invariant linear subspaces of V are $\{0\}$ and V itself.

We recall that a module over a field K is just a K-vector space, thus an algebra over K is a vector space with a K-linear multiplication. With that in mind we define the *group algebra* of G over K, denoted by KG, to be the vector space over K with basis G with multiplication defined as the bilinear extension of the multiplication in G. That is,

$$\Big(\sum_{g\in G} a_g g\Big)\Big(\sum_{h\in G} b_h h\Big) = \sum_{g,h\in G} a_g b_h (gh) = \sum_{g\in G}\Big(\sum_{h\in G} a_{gh^{-1}} b_h\Big) g.$$

It is easy to check that any K-representation (ρ, V) of G corresponds to a KG-module structure on the additive group of V via the natural homomorphism of the K-algebras KG and $\mathrm{End}_K(V)$

$$\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g \rho(g).$$

Sometimes a K-algebra homomorphism $\phi : A \to \mathrm{End}_K(V)$ is also called a *K-algebra representation* of A. Furthermore, we note that irreducible representations correspond to simple KG-modules.

It is convenient to establish many theorems in representation theory by exploiting this analogy, and making use of the theory of semisimple rings. This strategy is founded in Maschke's Theorem.

Theorem 4.2 (Maschke). *Let G be a finite group and K a field such that the characteristic of K does not divide the order of G. Then the group algebra KG is a semisimple K-algebra.*

The structure of semisimple rings is given by the following two results:

Theorem 4.3 (Schur). *Let R be a ring with unity, and M and N simple R-modules.*

(i). *Then* $\mathrm{End}_R(M)$ *is a division ring.*

(ii). *Furthermore,* $\mathrm{Hom}_R(M, N) = \{0\}$ *unless $M \cong N$ are isomorphic as R-modules*

Theorem 4.4 (Artin, Wedderburn). *Let R be a semisimple ring. Then there exist natural numbers $n_1, \dots, n_s$ and division rings $D_1, \dots, D_s$ such that R is isomorphic to the direct sum of matrix rings*

$$R \cong \bigoplus_{i=1}^{s} \mathrm{Mat}(n_i, D_i).$$

The proof of Theorem 4.4 relies on Theorem 4.3 and several results on semisimple rings which are not too laborious to verify. Firstly, a ring R is semisimple if and only if all R-modules are semisimple. All simple R-modules can be shown to be isomorphic to a minimal R-left ideal. Moreover, there are only finitely many isomorphic classes of left ideals of R, and R is the direct sum finitely many minimal left ideals of R.

The D_i in Theorem 4.4 correspond to the R-module endomorphism rings of a system of representatives of the isomorphic classes of simple R-modules. These are division rings due to Theorem 4.3 (i). In particular, the number s gives the number isomorphic classes of simple R-modules, and the numbers n_i give the multiplicity of an isomorphic class in the decomposition of R.

Combining these results we obtain the Fundamental Theorem for representations of finite groups.

Theorem 4.5. *Let G be a finite group and K a field such that the characteristic of K does not divide the order of G.*

(i). *Then there exist only finitely many isomorphic classes of irreducible K-representations of G.*

(ii). *Every irreducible K-representation of G has finite degree.*

(iii). *Furthermore, we have complete reducibility, i.e., each K-representation of G is a direct sum of irreducible representations.*

(iv). *The decomposition of (iii) is unique. More precisely, let (ρ, V) be a finite dimensional K-representation of G, and (ρ_i, V_i) be a system of representatives of the isomorphic classes of irreducible K-representations of G, where $i = 1, \dots, s$. Then there exist unique non-negative integers $m_1, \dots, m_s$ such that*

$$(\rho, V) \cong \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{m_i} (\rho_i, V_i).$$

The numbers m_i in the point (iv) of Theorem 4.5 are called the *multiplicities* of the irreducible representations (ρ_i, V_i) in (ρ, V).

One possible way to “label” the isomorphic classes of irreducible representations of a finite group over a suitable field are the irreducible characters. We shall now give an outline of the beginnings of character theory.

Let G be a finite group and K a field. The *character* of a K-representation (ρ, V) of G is defined to be the map $\chi_\rho : G \to K$ defined by $g \mapsto \mathrm{tr}_K(\rho(g))$, that is, the trace of the linear operator $\rho(g)$. The character of an irreducible representation is called *irreducible character*. Clearly, the characters of G are constant on each conjugacy class of G. Furthermore, $\chi_\rho = \chi_{\rho'}$ whenever $(\rho, V) \cong (\rho', V')$.

Let us consider the case where K is algebraically closed. As above, a closer look at the group algebra is serviceable. Since KG is a finite dimensional K-algebra, so are the division rings D_i in Theorem 4.4. But the only finite dimensional division algebra over an algebraically closed field K is K itself. Thus, we have

$$KG \cong \bigoplus_{i=1}^{s} \mathrm{Mat}(n_i, K)$$

for suitable $n_i \in \mathbb{N}$. The center of a matrix ring over a field K is well known to be isomorphic to K. Hence, the center of the group algebra $\mathrm{Z}(KG)$ is a K-vector space of dimension s, where s equals the number of irreducible K-representations of G.

Let $C(G, K)$ be the K-algebra of *class functions* on G, that is, all functions $f : G \to K$ that are constant on each conjugacy class of G. Clearly, $C(G, K)$ contains the K-characters of G. Let $R(G, K)$ denote the subalgebra of $C(G, K)$ spanned by irreducible characters.

There is a natural K-algebra homomorphism from $C(G, K)$ to KG given by

$$f \mapsto \sum_{g \in G} f(g)g. \tag{4.2}$$

We define a bilinear form $\langle ., . \rangle_\chi : \mathrm{Z}(KG) \times \mathrm{Z}(KG) \to K$ via

$$\Big(\sum_{g \in G} a_g g, \sum_{g \in G} b_g g \Big) \mapsto \sum_{g \in G} a_g b_{g^{-1}}. \tag{4.3}$$

By virtue of the map (4.2) this bilinear form defines an inner product on $R(G, K)$.

Theorem 4.6. *Let G be a finite group and K an algebraically closed field such that the characteristic does not divide the order of G.*

(i). *The number of conjugacy classes of G equals the number of isomorphic classes of irreducible K-representations of G. Furthermore, let $C_1, \dots, C_s$ be the conjugacy classes of G. Then the set*

$$\Big\{ \sum_{g \in C_i} g : 1 \leq i \leq s \Big\}$$

is a K-basis of $\mathrm{Z}(KG)$.

(ii). *Let $\chi_1, \ldots, \chi_s$ be the irreducible K-characters of G. Then the set*

$$\Big\{ \sum_{g \in G} \chi_i(g) g : 1 \leq i \leq s \Big\}$$

is a K-basis of $\mathrm{Z}(KG)$ *that is orthonormal with respect to* (4.3).

(iii). *The K-representations of G of finite degree are determined up to isomorphism by their character. More precisely, let (ρ, V) be a finite dimensional representation with character χ. Then the multiplicity of an irreducible representation (ρ_i, V_i) in (ρ, V) is given by $\langle \chi, \chi_i \rangle$ for $i = 1, \ldots, s$.*

Note that by Theorem 4.6 the map (4.2) induces a linear isomorphism from $R(G, K)$ to $\mathrm{Z}(KG)$. The *regular K-representation* of a group G is KG regarded as a KG-module via left multiplication. The corresponding character χ_{reg} is called the *regular character*. By earlier remarks we have $\chi_{\mathrm{reg}} = \sum_{i=1}^{s} \dim_K(V_i) \chi_i$, where χ_i is the character of the irreducible representation V_i.

We shall be working with polynomial rings and graded rings. Therefore, let R be commutative ring with unity. A *commutative R-algebra* is an algebra A over R that is a commutative ring.

Given a (countable) set X, the *polynomial ring* over R with variables X, denoted by $R[X]$, can be defined as the commutative algebra over R generated by X fulfilling the following universal mapping property: For any commutative R-algebra A and map $f : X \to A$ there exists a unique ring homomorphism $\varphi : R[X] \to A$ such that both triangles in the following diagram commute.

$$\begin{tikzcd} R \arrow[r, "\phi_{R[X]}"] \arrow[dr, "\phi_A"'] & R[X] \arrow[d, "\varphi"] & X \arrow[l, "\iota"'] \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

We will make exhaustive use of the universal mapping property of polynomial rings in our introduction of the plethystic substitution in Section 6. A ring A is called *graded* if it can be decomposed as

$$A = \bigoplus_{k=0}^{\infty} A^k$$

such that $a + b \in A^k$ for all $a, b \in A^k$ and $ab \in A^{k+l}$ for all $a \in A^k$ and $b \in A^l$. Each additive subgroup A^k is called the group of *homogeneous elements* of *degree k*.

An R-algebra A is called *graded algebra* if it is a graded ring, and the homogeneous subgroups A^k are stable, i.e. $RA^k \subseteq A^k$. In that case A^k is an R-submodule of A. For example the polynomial ring $R[X]$ is a graded R-algebra by virtue of the usual degree of polynomials.

Let K be a field and G a group. A K-representation (ρ, V) of G is called *graded* if V is a graded K-algebra and the homogeneous subspaces V^k are invariant under the action of G. If V is a graded K-algebra such that every homogeneous subspace has finite dimension over K, we define its *Hilbert series* as

$$\mathrm{Hilb}_K(V;t) := \sum_{k=0}^{\infty} \dim_K(V^k)t^k.$$

This series, which is also called Hilbert-Poincaré series, fulfils the following elementary properties.

Proposition 4.7. *Let K be a field, and V and W be two graded K-algebras such that the series $\mathrm{Hilb}_K(V;t)$ and $\mathrm{Hilb}_K(W;t)$ exist.*

(i). *Then we have*

$$\mathrm{Hilb}_K(V \oplus W;t) = \mathrm{Hilb}_K(V;t) + \mathrm{Hilb}_K(W;t),$$
$$\mathrm{Hilb}_K(V \otimes_K W;t) = \mathrm{Hilb}_K(V;t)\,\mathrm{Hilb}_K(W;t).$$

(ii). *Suppose that $V \cong K[x_1, \dots, x_n]$ is generated by algebraically independent elements x_i of degree d_i where $i = 1, \dots, n$. Then*

$$\mathrm{Hilb}_K(V;t) = \prod_{i=1}^{n} \frac{1}{1 - t^{d_i}}.$$

Another concept we will encounter is the *inverse limit*, or *projective limit*, of an ordered family of rings. An *inverse system* of rings is a sequence $(R_i)_{i\in\mathbb{N}}$ of rings together with surjective ring homomorphisms $\phi_{m,n} : R_n \to R_m$ called projections for all pairs $m, n \in \mathbb{N}$ such that $m < n$. The inverse limit of such a system is given as the subset

$$\varprojlim_{i\in\mathbb{N}} R_i \subseteq \prod_{i=0}^{\infty} R_i = \{(r_i)_{i\in\mathbb{N}} : r_i \in R_i\}$$

of their Cartesian product consisting of all sequences $(r_i)_{i\in\mathbb{N}}$ satisfying $\phi_{m,n}(r_n) = r_m$ for all $m < n$. It can be shown that this subset is closed under componentwise addition and multiplication. The maps $\phi_n : \varprojlim R_i \to R_n$ given by $(r_i)_{i\in\mathbb{N}} \mapsto r_n$ are called canonic projections.

Alternatively the inverse limit is defined as the ring R together with canonic projections $\phi_n : R \to R_n$ for $n \in \mathbb{N}$ such that the following universal mapping property is satisfied: For every ring S and surjective homomorphisms $\psi_n : S \to R_n$ such that the left hand side of the diagram below commutes for all $m < n$, there exists a unique ring homomorphism $\psi : S \to R$ such that the right side of diagram commutes for all $m < n$.

$$\begin{tikzcd}
 & S \arrow[dl, "\psi_n"'] \arrow[dr, "\psi_m"] & \\
R_n \arrow[rr, "\phi_{m,n}"'] & & R_m
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S \arrow[d, "\psi"] \arrow[ddl, "\psi_n"'] \arrow[ddr, "\psi_m"] & \\
 & R = \varprojlim R_n \arrow[dl, "\phi_n"] \arrow[dr, "\phi_m"'] & \\
R_n \arrow[rr, "\phi_{m,n}"'] & & R_m
\end{tikzcd}$$

More generally, the above definition could be extended to the case where $\mathbb{N}$ is replaced by an arbitrary partially ordered set. However, this will not be needed in our subsequent considerations.

We conclude this section with a very general result on the representations of finite reflection groups. A module M over a ring R is called *free* if M is isomorphic as an R-module to a direct sum of copies of R.

Theorem 4.8 (Chevalley, Shephard, Todd). *Let G be finite a group, and (ρ, V) a complex representation of G such that $\{v_1, \ldots, v_n\}$ is a basis of V. Then the following are equivalent.*

(i). *The group $\rho(G) \leq \mathrm{Gl}(n, \mathbb{C})$ is generated by reflections.*

(ii). *The algebra $\mathbb{C}[v_1, \ldots, v_n]^G$ of invariants is isomorphic (as a $\mathbb{C}$-algebra) to a polynomial algebra over $\mathbb{C}$.*

(iii). *The algebra $\mathbb{C}[v_1, \ldots, v_n]$ is a free module over $\mathbb{C}[v_1, \ldots, v_n]^G$.*

5 Symmetric Functions and Tableaux

We have already been referring to symmetric functions on a few occasions. After presenting some facts about Young tableaux and their connection to the irreducible characters of the symmetric group, it will be time to give the ring of symmetric functions a proper introduction. The main part of this section will be dedicated to this goal. From this point onwards symmetric functions will remain a constant companion and various families of symmetric functions will be the central objects in the proof of Theorem 3.8.

Let $n \in \mathbb{N}$, $\sigma \in \mathfrak{S}_n$ be a permutation, and λ a partition or a vector with n or more entries. We define

$$\sigma \cdot \lambda := (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})$$

The map $\lambda \mapsto \sigma \cdot \lambda$ is an injection from the set Π_n to $\mathbb{N}^n$ for each $\sigma \in \mathfrak{S}_n$. We call the map thus given by the identity permutation the natural embedding, and we will occasionally identify a partition $\lambda \vdash n$ with its image $\mathrm{id} \cdot \lambda \in \mathbb{N}^n$. Clearly, this rule also defines an $\mathfrak{S}_n$ action on the set $\mathbb{N}^n$ preserving the sum vectors. Moreover, each orbit contains exactly one vector with decreasing entries. Let $w \in \mathbb{N}^n$ then we denote the partition corresponding to the decreasing vector in the orbit $\mathfrak{S}_n \cdot w$ by $\tilde{w}$.

Now, let $\lambda \vdash n$ be a partition. A *semistandard Young tableau* of shape λ is a map T from the Young diagram of λ to the natural numbers $\{1, \dots, n\}$, increasing weakly along rows, i.e. $T(i, j) \leq T(i+1, j)$, and strictly along columns, $T(i, j) < T(i, j+1)$ (see Figure 14). Let w_i denote the cardinality of $T^{-1}(i)$ for $i = 1, \dots, n$ then $w = (w_1, \dots, w_n) \in \mathbb{N}^n$ is called the *weight* of the tableau T. The set of all semistandard Young tableaux of shape λ is denoted by $\mathrm{SSYT}(\lambda)$. For any given vector $w \in \mathbb{N}^n$ the subset of all tableaux with shape λ and weight w is denoted by $\mathrm{SSYT}(\lambda, w)$. A *standard Young tableau* is a tableau with weight equal to $1^n := (1, \dots, 1) \in \mathbb{N}^n$. We denote the set of all standard Young tableaux of shape λ by $\mathrm{SYT}(\lambda)$.

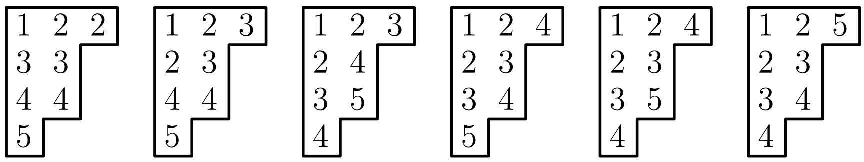

Figure 14: All semistandard Young tableaux of shape $\lambda = (4, 2, 2, 1)$ with weight $w = (1, 2, 2, 2, 1)$. The Kostka number $K_{\lambda, \tilde{w}}$ equals six.

Proposition 5.1. *Let $n \in \mathbb{N}$, $\lambda \vdash n$ be a partition, $w \in \mathbb{N}^n$, and $\sigma \in \mathfrak{S}_n$ be a permutation. Then the sets* $\mathrm{SSYT}(\lambda, w)$ *and* $\mathrm{SSYT}(\lambda, \sigma \cdot w)$ *have equal cardinality.*

Proof. It suffices to consider transpositions $\sigma = (i, i+1)$. We construct a bijection from $\mathrm{SSYT}(\lambda, w)$ to $\mathrm{SSYT}(\lambda, (i, i+1) \cdot w)$ as follows. Let $T \in \mathrm{SSYT}(\lambda, w)$ and $S := T^{-1}(i) \cup T^{-1}(i+1) \subseteq Y(\lambda)$. We define $\hat{T}(x) := T(x)$ if $x \notin S$, or if $x \in S$ and there is another cell $y \in S$ in the same column as x. If x is the only cell of its column that lies in S we change its label from i to $i+1$ or from $i+1$ to i respectively. Afterwards we rearrange the labels in each row such that we have weakly increasing labels to obtain a tableau $\hat{T} \in \mathrm{SSYT}(\lambda, (i, i+1) \cdot w)$ (see Figure 15). The map $T \mapsto \hat{T}$ is clearly an involution. ■

We may therefore define the *Kostka numbers* $K_{\lambda, \mu}$ as the cardinality of $\mathrm{SSYT}(\lambda, w)$, where w is any permutation of μ.

The symmetric group $\mathfrak{S}_n$ acts naturally on the set $\{1, \dots, n\}$ by $\sigma \cdot i := \sigma(i)$ which is the very definition of a permutation. Now, let $\sigma \in \mathfrak{S}_n$ be a permutation. A *cycle* of σ is a directed graph (V, E) where the vertex set $V \subseteq \{1, \dots, n\}$ is an orbit, and an oriented edge $(i, j) \in E$ is drawn if and only if $\sigma(i) = j$. Clearly, σ is fully determined by the union of its cycles. The cardinality of a cycle is defined to be the cardinality of its orbit.

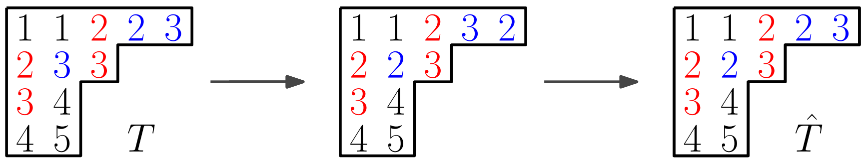

Figure 15: A tableau $T \in \mathrm{SSYT}(\lambda, w)$, and its pendant $\hat{T} \in \mathrm{SSYT}(\lambda, (23)\cdot w)$. The red labels remain unchanged while the blue labels are inverted.

Let the *cycle type* of σ be the partition $\mathrm{z}(\sigma) = (\lambda_1, \lambda_2, \dots) \vdash n$ such that the positive parts λ_i corresponds to the cardinalities of the cycles of σ. For example the permutation $18362457 = (2578)(46) \in \mathfrak{S}_8$ has cycle type $(4, 2, 1, 1)$.

Given a partition λ let m_i denote the number of parts of λ equal to i. We define the important quantity

$$z_\lambda := \prod_{i=1}^{l(\lambda)} i^{m_i}\, m_i!\,.$$

Proposition 5.2. *Let $n \in \mathbb{N}$, and $\lambda \vdash n$ be a partition. Then the number of permutations $\sigma \in \mathfrak{S}_n$ with cycle type λ is given by*

$$\frac{n!}{z_\lambda}.$$

Proof. Consider the following arrangements of n positions

$$(\_)\dots(\_)(\_\,\_)\dots(\_\,\_)(\_\,\_\,\_)\dots(\_\dots\_),$$

where there are m_i cycles of length i such that each cycle corresponds to a part of λ. Clearly, every assignment of the letters $\{1, \dots, n\}$ to these spots produces a permutation σ of the cycle type λ. However, two placements result in the same σ if we permutate cycles of the same length. Thus, we have to divide $n!$ by $m_i!$ for each $1 \le i \le n$. Furthermore, for each individual cycle a shift to the left of each label does not change σ. Hence, we also need to divide by the length of each cycle. The claim follows. ∎

A well known result in the group theory of $\mathfrak{S}_n$ implies that its conjugacy classes are naturally indexed by the set of partitions of n.

Theorem 5.3. *Let $n \in \mathbb{N}$ and $\sigma, \tau \in \mathfrak{S}_n$ be two permutations. Then σ and τ are conjugate to each other if and only if they have the same cycle type.*

By Theorem 4.6 this means that also the irreducible complex characters of the symmetric group are index by partitions. While in general it is not always possible to make the correspondence between irreducible characters and conjugacy classes of a finite group explicit, this is known for the symmetric group $\mathfrak{S}_n$.

Let λ be a partition of n, and let $T \in \mathrm{SYT}(\lambda)$. We define two subgroups of $\mathfrak{S}_n$ as

$$G_T := \Big\{\sigma \in \mathfrak{S}_n : \text{the sets } \{T(i,j) : j = 1, \dots, \lambda_i\} \text{ are invariant under } \sigma \text{ for all } i.\Big\}$$
$$H_T := \Big\{\sigma \in \mathfrak{S}_n : \text{the sets } \{T(i,j) : i = 1, \dots, \lambda'_j\} \text{ are invariant under } \sigma \text{ for all } j.\Big\}$$

The group G_T leaves the columns of T invariant, while H_T leaves the rows of T invariant. We remark that the groups G_{T_1} and G_{T_2} are isomorphic if T_1 and T_2 are standard tableau of the same shape. The same is true for the groups H_{T_1} and H_{T_2}.

Next we define three elements of the group algebra $\mathbb{C}\mathfrak{S}_n$ as

$$a_T := \sum_{\sigma \in G_T} \sigma, \qquad b_T := \sum_{\sigma \in H_T} \operatorname{sgn}(\sigma)\sigma, \qquad c_T := a_T b_T.$$

The c_T are called the *Young symmetrizers*, and allow us to construct the irreducible complex representations of the symmetric groups in the following way:

Theorem 5.4. *Let $n \in \mathbb{N}$, $\lambda \vdash n$ be a partition, and $T \in \mathrm{SYT}(\lambda)$.*

(i). *The element c_T is a scalar multiple of an idempotent element in $\mathbb{C}\mathfrak{S}_n$.*

(ii). *The vector space $V_T := \mathbb{C}\mathfrak{S}_n c_T \leq \mathbb{C}\mathfrak{S}_n$ is an irreducible representation.*

(iii). *Let $\mu \vdash n$ be another partition and $\tilde{T} \in \mathrm{SYT}(\mu)$. Then the two representations V_T and $V_{\tilde{T}}$ obtained in this way are isomorphic if and only if the tableaux T and $\tilde{T}$ have the same shape, i.e. $\lambda = \mu$.*

A proof can be found in [3, Chapter 4].

Example 5.5. Let $\lambda = 1^n$ and $\mu = (n)$. Then the cardinalities of $\mathrm{SYT}(\lambda) = \{T\}$ and $\mathrm{SYT}(\mu) = \{T'\}$ equal one. We have $G_T = \mathfrak{S}_n$, $H_T = \{\mathrm{id}\}$, $G_{T'} = \{\mathrm{id}\}$, and $H_{T'} = \mathfrak{S}_n$, thus

$$c_T = \sum_{g \in \mathfrak{S}_n} g, \qquad c_{T'} = \sum_{g \in \mathfrak{S}_n} \operatorname{sgn}(g) g.$$

We obtain the two one-dimensional complex representations of $\mathfrak{S}_n$, called the *trivial representation* V_T and the *alternating representation* $V_{T'}$. □

From the last point of Theorem 5.4 it follows that all irreducible complex representations of $\mathfrak{S}_n$ are isomorphic to a V_T for some tableau T by exhaustion. It also provides a natural correspondence between partitions in Π_n and the irreducible characters of $\mathfrak{S}_n$. Furthermore, we remark that the Young symmetrizer of a tableau has integer coefficients with respect to the canonical basis $\mathfrak{S}_n$, and all complex irreducible representations can be constructed over the rationals.

Let $X := \{x_1, x_2, x_3, \dots\}$ be an infinite set of variables, and for $n \in \mathbb{N}$ set $X_n := \{x_1, \dots, x_n\}$. Let $\alpha \in \mathbb{N}^n$, then we use the multi-index notation

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

The symmetric group $\mathfrak{S}_n$ acts on the ring $\mathbb{Z}[X_n]$ via permutation of variables. That is, for $f(x_1, \dots, x_n) \in \mathbb{Z}[X_n]$ and $\sigma \in \mathfrak{S}_n$ we define

$$\sigma \cdot f(x_1, \dots x_n) := f(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \tag{5.1}$$

Alternatively, this action can be obtained as the $\mathbb{Z}$-linear extension of the action defined on monomials by $\sigma \cdot x^\alpha := x^{\sigma \cdot \alpha}$. A polynomial $f \in \mathbb{Z}[X_n]$ is called *symmetric* if $f = \sigma \cdot f$ for all permutations $\sigma \in \mathfrak{S}_n$. Let $\Lambda_n := \mathbb{Z}[X_n]^{\mathfrak{S}_n}$ denote the set of symmetric polynomials in n variables. It is easy to confirm that Λ_n is a ring, also called the *ring of symmetric functions* or the ring of invariants of the symmetric group. Since permutation of variables does not affect the degree of a polynomial, a polynomial is symmetric if and only if each of its homogeneous components is symmetric. In other words, Λ_n is a graded subring of $\mathbb{Z}[X_n]$, and we denote by

$$\Lambda_n^k = \Lambda_n \cap \mathbb{Z}[X_n]^k$$

the space of homogeneous symmetric functions in n variables of degree k.

As a first example, we note that for any $f \in \mathbb{Z}[X_n]$ the averaged polynomial

$$\sum_{\sigma \in \mathfrak{S}_n} \sigma \cdot f \in \Lambda_n$$

is symmetric.

The ring $\mathbb{Z}[X_n]$ becomes a $\mathbb{Z}[X_n]$-module either via left multiplication or via partial differentiation given by

$$\partial f(g) := \sum_{\alpha \in \mathbb{N}^n} f_\alpha \frac{\partial^\alpha}{\partial x^\alpha} g \tag{5.2}$$

where $f, g \in \mathbb{Z}[X_n]$, and $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha$. Equivalently, in the language of ring homomorphisms, the module structure is given as the unique extension of the injection $X_n \to \operatorname{End}(\mathbb{Z}[X_n])$, $x_i \mapsto \frac{\partial}{\partial x_i}$ to a ring homomorphism on $\mathbb{Z}[X_n]$.

The *ring of coinvariants*, which is also called the ring of *harmonic* polynomials, is defined as

$$\mathcal{H}_n := \left\{ g \in \mathbb{Z}[X_n] : \partial f(g) = 0 \text{ for all } f \in \Lambda_n^+ \right\}$$

where $\Lambda_n^+ = \bigoplus_{k=1}^\infty \Lambda_n^k$ is the ring of all symmetric polynomials with vanishing constant term. Since $g \in \mathcal{H}_n$ if and only if all its homogeneous components are in $\mathcal{H}_n$, the harmonic polynomials again form a graded subring of $\mathbb{Z}[X_n]$.

Now, let R be a ring that such that the elements $x_i \in X_n$ are algebraically independent over R. Then the action (5.1) naturally extends to $R[X_n] = \mathbb{Z}[X_n] \otimes R$. In analogy to the above, we define $\Lambda_{n,R} = \Lambda_n \otimes R$ as the graded R-algebra of symmetric functions with coefficients in R, and we denote the homogeneous subgroups by $\Lambda_{n,R}^k$. Moreover, we define an $R[X_n]$-module structure on $R[X_n]$ itself, by extending $\partial x_i(x_j) = \delta_{i,j}$ to a homomorphism of R-algebras $\partial : R[X_n] \to \mathrm{End}_R(R[X_n])$, $f \mapsto \partial f$ as in (5.2). We obtain the ring of coinvariants $\mathcal{H}_{n,R} = \mathcal{H}_n \otimes R$ with coefficients in R. Again, $\mathcal{H}_{n,R}$ is a graded R-subalgebra of $R[X_n]$.

In the case where $R = K$ is a field, $K[X_n]$ is a graded K-representation of $\mathfrak{S}_n$, and so is $\mathcal{H}_{n,K}$.

First, we are going to explore bases for Λ_n and $\mathcal{H}_n$. Let $k \in \mathbb{N}$ and $\lambda \vdash k$ a partition with at most n parts. We define the *monomial symmetric functions* in n variables as

$$m_{n,\lambda}(x_1, \ldots, x_n) := \sum_{\alpha \in \mathfrak{S}_n \cdot \lambda} x^\alpha.$$

Clearly $m_{n,\lambda} \in \Lambda_n^k$. Furthermore, since every polynomial in $\mathbb{Z}[X_n]$ is the sum of monomials, the set $\{m_{n,\lambda} : \lambda \vdash k, l(\lambda) \leq n\}$ is a $\mathbb{Z}$-basis of Λ_n^k.

For $0 \leq k \leq n$ we define the *elementary symmetric functions* in n variables as

$$e_{n,k}(x_1, \ldots, x_n) = \sum_{\alpha \in \mathfrak{S}_n \cdot 1^k} x^\alpha = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}.$$

where 1^k denotes the partition consisting of k parts equal to one. Clearly, $e_{n,k}$ lies in Λ_n^k. Furthermore, let Δ_n denote the n-th *Vandermonde determinant*

$$\Delta_n(x_1, \ldots, x_n) := \prod_{1 \leq i < j \leq n} (x_i - x_j) = \det(x_i^{j-1})_{1 \leq i,j \leq n}.$$

The Vandermonde determinant plays an important role in the description of a set of polynomials closely related to the symmetric ones. A polynomial $f \in \mathbb{Z}[X_n]$ is called *alternating* or *anti-symmetric* if for all $\sigma \in \mathfrak{S}_n$ we have

$$\sigma \cdot f = \mathrm{sgn}(\sigma)\, f.$$

Lemma 5.6. *Let $n \in \mathbb{N}$ and $f \in \mathbb{Z}[X_n]$. Then f is alternating if and only if $f = \Delta_n g$ for some symmetric polynomial $g \in \Lambda_n$.*

Proof. First, notice that a permutation of the variables of $\Delta_n(x_1, \ldots, x_n)$ corresponds to a permutation of the rows of the matrix $(x_i^{j-1})_{1 \leq i,j \leq n}$. Thus, $\sigma \cdot \Delta_n = \mathrm{sgn}(\sigma)\Delta_n$ for all $\sigma \in \mathfrak{S}_n$ due to the alternating property of determinants.

Now, let f be alternating and $(i, j) \in \mathfrak{S}_n$ be a transposition. Then

$$f\big|_{x_i = x_j} = (i, j) \cdot \left(f\big|_{x_i = x_j}\right) = \big((i, j) \cdot f\big)\big|_{x_i = x_j} = -f\big|_{x_i = x_j}.$$

Thus, x_i must be a zero of f regarded as a polynomial in x_j, and consequently $x_i - x_j$ divides f. Therefore, also Δ_n divides f. Since f/Δ_n is clearly symmetric, the lemma follows. ■

The sum of two alternating polynomials is again alternating. Hence, by Lemma 5.6 the alternating polynomials form a Λ_n-module generated by Δ_n. We shall now state some major results on the ring of symmetric polynomials involving the elementary symmetric functions defined above.

Theorem 5.7 (Fundamental Theorem of Symmetric Functions). *Let $n \in \mathbb{N}$. Then the following hold:*

(i). *The elementary symmetric functions $\{e_{n,k} : 1 \leq k \leq n\}$ are algebraically independent over $\mathbb{Z}$ (or $\mathbb{C}$) and generate Λ_n (respectively $\Lambda_{n,\mathbb{C}}$) as an algebra.*

(ii). *We have $\mathcal{H}_{n,\mathbb{C}} \cong \mathbb{C}[X_n]/_{\Lambda^+_{n,\mathbb{C}}}$, and the set of partial derivatives of Δ_n is a $\mathbb{C}$-basis for $\mathcal{H}_{n,\mathbb{C}}$.*

(iii). *The symmetric functions $\Lambda_{n,\mathbb{C}}$ act freely on $\mathbb{C}[X_n]$ via multiplication, and the multiplication map $\Lambda_{n,\mathbb{C}} \otimes \mathcal{H}_{n,\mathbb{C}} \to \mathbb{C}[X_n]$, $f \otimes g \mapsto fg$ is a linear isomorphism.*

The result can be found in [7, Chapter 5, Theorems 5.1.3, 5.1.4, and 5.1.8].

Example 5.8 (i). Let $n = 3$ and $\lambda := (4, 2, 1)$. We indicate how

$$m_{3,\lambda} = x_1^4x_2^2x_3 + x_1^4x_2x_3^2 + x_1^2x_2^4x_3 + x_1^2x_2x_3^4 + x_1x_2^4x_3^2 + x_1x_2^2x_3^4$$

can be expressed as a polynomial in the generators

$$e_{3,1} = x_1 + x_2 + x_3, \quad e_{3,2} = x_1x_2 + x_1x_3 + x_2x_3, \quad e_{3,3} = x_1x_2x_3.$$

The lexicographically largest (or dominant) monomial of $m_{3,\lambda}$ is $x_1^4x_2^2x_3$. We approximate $m_{3,\lambda}$ by $e_{3,1}^2e_{3,2}e_{3,3}$ which has the same dominant monimial. Thus, the difference $m_{3,\lambda} - e_{3,1}^2e_{3,2}e_{3,3}$ has a lexicographically smaller dominant term than $m_{3,\lambda}$. Applying induction we obtain an algorithm which yields the desired polynoimal.

(ii). Let us verify that Δ_n is indeed a harmonic polynomial. To do so, we will make use of Lemma 5.6. Let $f \in \Lambda_n^+$ be symmetric with vanishing constant term and $\sigma \in \mathfrak{S}_n$ a permutation. Then

$$\sigma \cdot \partial f(\Delta_n) = \partial(\sigma \cdot f)(\sigma \cdot \Delta_n) = \partial f(\operatorname{sgn}(\sigma)\Delta_n) = \operatorname{sgn}(\sigma)\partial f(\Delta_n).$$

Thus, $\partial f(\Delta_n)$ is alternating and divisible by Δ_n. However, since f has no constant term, the degree of $\partial f(\Delta_n)$ is strictly less than the degree of Δ_n. It follows that $\partial f(\Delta_n) = 0$ as required.

Clearly, also all partial derivatives of Δ_n must be harmonic as a consequence. The fact that these polynomials already form a basis of $\mathcal{H}_{n,\mathbb{C}}$ can be seen through a dimension argument provided by Theorem 5.7 (iii). □

Note that Theorem 5.7 is a special case of Theorem 4.8 where $G = \mathfrak{S}_n$. Using Proposition 4.7 we obtain the Hilbert series of $\mathcal{H}_{n,\mathbb{C}}$.

Corollary 5.9. *Let $n \in \mathbb{N}$. Then the dimensions of the homogeneous subspaces of the ring of coinvariants are given by*

$$\operatorname{Hilb}_{\mathbb{C}}(\mathcal{H}_{n,\mathbb{C}}; q) = \prod_{k=1}^{n} \frac{1 - q^k}{1 - q} = [n]_q!\,.$$

■

In particular, Corollary 5.9 implies $\dim_{\mathbb{C}}(\mathcal{H}_{n,\mathbb{C}}) = n!$. Moreover, we have the following.

Theorem 5.10. *Let $n \in \mathbb{N}$. Then the harmonic polynomials $\mathcal{H}_{n,\mathbb{C}}$ and the regular representation $\mathbb{C}\mathfrak{S}_n$ are isomorphic as representations of the symmetric group.*

Thereby, we have found a graded version of the regular representation of $\mathfrak{S}_n$.

Returning our attention to Λ_n, for $k \in \mathbb{N}$ we define the *complete homogeneous symmetric functions* in n variables as

$$h_{n,k} := \sum_{\lambda \vdash k} m_{n,\lambda}.$$

Proposition 5.11. *Let $k, n \in \mathbb{N}$ such that $k \geq 1$. Then we have the following relations between the elementary and the complete homogeneous symmetric functions*

$$\sum_{i=0}^{n} (-1)^i e_{n,i} h_{n,k-i} = 0. \tag{5.3}$$

Note that here and afterwards we let $e_{n,0} = h_{n,0} = 1$ and treat $e_{n,k}$ and $h_{n,k}$ as zero if $k < 0$.

Proof. The claim follows from the identity $1 = E_n(t)H_n(-t)$ for the generating functions

$$H_n(t) := \sum_{k=0}^{\infty} h_{n,k} t^k = \prod_{i=1}^{n} \frac{1}{1 - x_i t},$$
$$E_n(t) := \sum_{k=0}^{n} e_{n,k} t^k = \prod_{i=1}^{n} (1 + x_i t).$$

■

Let $\omega : \Lambda_n \to \Lambda_n$ be the endomorphism of graded rings such that $\omega(e_{n,k}) = h_{n,k}$ for $k = 1, \ldots, n$. Then by the symmetry of (5.3) the map ω must be an involution, and thus an automorphism.

Corollary 5.12. *Let $n \in \mathbb{N}$. Then the complete homogeneous functions $\{h_{n,k} : 1 \leq k \leq n\}$ are algebraically independent over $\mathbb{Z}$ and generate Λ_n as an algebra.*

■

Given a partition $\lambda \vdash k$ such that each part is less than or equal to n, we define

$$h_{n,\lambda} := \prod_{i=1}^{k} h_{n,\lambda_i}, \qquad e_{n,\lambda} := \prod_{i=1}^{k} e_{n,\lambda_i}.$$

Then Theorem 5.7 (i) and Corollary 5.12 have another consequence.

Corollary 5.13. *Let $k, n \in \mathbb{N}$. Then the sets $\{e_{n,\lambda} : \lambda' \in \Pi_k^n\}$ and $\{h_{n,\lambda} : \lambda' \in \Pi_k^n\}$ are $\mathbb{Z}$-bases for Λ_n^k.*

■

For $k \geq 1$ the *power-sum symmetric functions* in n variables are defined as

$$p_{n,k} = \sum_{i=1}^{n} x_i^k,$$

and for any partition $\lambda \vdash k$ let

$$p_{n,\lambda} := \prod_{i=1}^{l(\lambda)} p_{n,\lambda_i}.$$

These polynomials are connected to the elementary and complete homogeneous symmetric functions by means of the so called Newton identities.

Proposition 5.14. *Let $k, n \in \mathbb{N}$ such that $1 \leq k \leq n$. Then we have*

$$h_{n,k} = \sum_{\lambda \vdash k} \frac{1}{z_\lambda} p_{n,\lambda},$$

$$e_{n,k} = \sum_{\lambda \vdash k} \frac{(-1)^{k-l(\lambda)}}{z_\lambda} p_{n,\lambda}.$$

Proof. The claims will follow from $P_n(-t) = \frac{E_n'(t)}{E_n(t)}$, and $P_n(t) = \frac{H_n'(t)}{H_n(t)}$ respectively, where

$$P_n(t) := \sum_{k=1}^{\infty} p_{n,k} t^{k-1} = \sum_{k=0}^{\infty} \sum_{i=1}^{n} x_i^{k+1} t^k = \sum_{i=1}^{n} \frac{x_i}{1 - x_i t},$$

and differentiation is with respect to the variable t. Formal integration yields

$$H_n(t) = \exp\Big(\sum_{k=1}^{\infty} p_{n,k} \frac{t^k}{k}\Big) = \prod_{k=1}^{\infty} \Big(\sum_{m=0}^{\infty} \frac{1}{m! k^m} p_{n,k}^m t^{km}\Big) = \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda t^n.$$

The second claim is shown by similar computations. ■

Corollary 5.15. *Let $n \in \mathbb{N}$. Then we have the following:*

(i). *The power-sum symmetric functions $\{p_1, \ldots, p_n\}$ are algebraically independent over $\mathbb{Q}$ and generate $\Lambda_{n,\mathbb{Q}}$ as a $\mathbb{Q}$-algebra.*

(ii). *The set $\{p_{n,\lambda} : \lambda' \in \Pi_k^n\}$ is a $\mathbb{Q}$-basis for the homogeneous space $\Lambda_{n,\mathbb{Q}}^k$.*

(iii). *Let λ be a partition, then the involution ω sends $p_{n,\lambda}$ to $(-1)^{|\lambda|-l(\lambda)} p_{n,\lambda}$.*

■

Example 5.16. First, let us understand why the condition on λ' in claim (ii) above is apropriate even though $p_{n,k}$ is defined for all $k \in \mathbb{N}$ in contrast to $e_{n,k}$. For $n = 2$ we easily compute

$$p_{2,(3)} = x_1^3 + x_2^3 = \frac{3}{2}(x_1^2 + x_2^2)(x_1 + x_2) - \frac{1}{2}(x_1 + x_2)^3 = \frac{3}{2}p_{2,(2,1)} - \frac{1}{3}p_{2,(1,1,1)}.$$

However, the relation

$$p_{n,(3)} = \frac{3}{2}p_{n,(2,1)} - \frac{1}{3}p_{n,(1,1,1)}$$

does not hold for all $n \geq 3$ since more mixed terms arise.

Secondly, notice that the $p_{n,\lambda}$ do not form a $\mathbb{Z}$-basis of Λ_n. As we have seen in Proposition 5.14 or even in our little example, rational coefficients are indeed needed. □

Having found three important sets of generators for the ring of symmetric functions, let us go one step further. Since constantly paying attention to the number of variables is somewhat wearisome, it will be desirable to eliminate the problem by introducing infinitely many variables. In light of Corollary 5.15 let us define the *ring of abstract symmetric functions* as $\Lambda := \mathbb{Q}[\{p_k : k \geq 1\}]$. We call the variables p_k the *abstract power-sum symmetric functions*.

To each of these we assign the degree $\deg(p_k) := k$. We obtain a graded ring $\Lambda = \bigoplus_{k=0}^{\infty} \Lambda^k$ where each homogeneous subspace Λ^k has a $\mathbb{Q}$-basis given by $\{p_\lambda : \lambda \vdash k\}$, where

$$p_\lambda := \prod_{i=1}^{\infty} p_{\lambda_i}$$

for any partition λ. Furthermore, for $n \in \mathbb{N}$ we define the *evaluation map* $\phi_n : \Lambda \to \Lambda_{n,\mathbb{Q}}$ via $\phi_n(p_k) := p_{n,k}$ for all $k \geq 1$. The evaluation maps are surjective $\mathbb{Q}$-algebra homomorphisms. Moreover, from Corollary 5.15 it is immediate that for all $k, n \in \mathbb{N}$ such that $k \leq n$ the induced map $\phi_n : \Lambda^k \to \Lambda^k_{n,\mathbb{Q}}$ is an isomorphism of $\mathbb{Q}$-vector spaces. By virtue of these evaluation maps, each homogeneous subspace of the abstract symmetric functions is the inverse limit of the spaces of homogeneous symmetric functions in finitely many indeterminates ordered canonically

$$\Lambda^k \cong \varprojlim_{n \in \mathbb{N}} \Lambda^k_{n,\mathbb{Q}}.$$

This makes Λ the inverse limit of the rings $\Lambda_{n,\mathbb{Q}}$ in the category of graded rings. However, note that Λ is not the inverse limit of the rings $\Lambda_{n,\mathbb{Q}}$ in the category of rings since all elements of Λ have finite degree. For example the formal product

$$\prod_{i=1}^{\infty}(1 + x_i)$$

is not an abstract symmetric function, but it is a limit of symmetric functions in finitely many variables. Given this knowledge, another way to define abstract symmetric functions is as the graded ring of formal power series in infinitely many variables

$$\Lambda \leq \mathbb{Q}[[X]], \tag{5.4}$$

whose degree is bounded and which are invariant under any finite permutation of variables. Starting from this point of view we might chose to define the abstract power-sum symmetric functions via their generating function

$$P(t) := \sum_{k=1}^{\infty} p_k t^{k-1} = \sum_{i=1}^{\infty} \frac{x_i}{1 - x_i t} \tag{5.5}$$

as in Proposition 5.14.

Adopting the results from Proposition 5.14 to the notion of abstract symmetric functions we define the abstract elementary and homogeneous symmetric functions as

$$\begin{aligned} e_k &:= \sum_{\lambda \vdash k} \frac{(-1)^{k-l(\lambda)}}{z_\lambda} p_\lambda, \qquad & h_k &:= \sum_{\lambda \vdash k} \frac{1}{z_\lambda} p_{n,\lambda}, \\ e_\lambda &:= \prod_{i=0}^{\infty} e_{\lambda_i}, \qquad & h_\lambda &:= \prod_{i=0}^{\infty} h_{\lambda_i}. \end{aligned} \tag{5.6}$$

Alternatively, in the context of (5.4), we could also define them via their generating functions

$$E(t) = \sum_{k=0}^{\infty} e_k t^k = \prod_{i=1}^{\infty} (1 + x_i t), \tag{5.7}$$

$$H(t) = \sum_{k=0}^{\infty} h_k t^k = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}. \tag{5.8}$$

For any field K of characteristic zero such that the set $\{p_k : k \geq 1\}$ of abstract power-sum symmetric functions is algebraically independent over K, we let $\Lambda_K := \Lambda \otimes K$. All statements and definitions for Λ have an analogue for Λ_K. For example the evaluation maps ϕ_n point from Λ_K to $\Lambda_{n,K}$, we can interpret Λ_K as a graded subalgebra of $K[[X]]$, and so on...

We conclude this section with the definition of the so called *diagonal action* of the symmetric group. Therefore, let $n \in \mathbb{N}$, and consider the ring $\mathbb{C}[X_n, Y_n]$. The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{C}[X_n, Y_n]$ via

$$\sigma \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) := f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}).$$

In analogy to Theorem 5.7, the corresponding ring of invariants $\mathbb{C}[X_n, Y_n]^{\mathfrak{S}_n}$ is generated by the *polarized power-sums*

$$p_{n,k,l}(x_1, \ldots, x_n, y_1, \ldots, y_n) := \sum_{i=1}^{n} x_i^k y_i^l,$$

where $k, l \in \mathbb{N}$. The ring of diagonal coinvariants or diagonal harmonics is defined as

$$\mathcal{DH}_n := \{f \in \mathbb{C}[X_n, Y_n] : \partial p_{n,k,l}(f) = 0 \text{ for all } k, l \in \mathbb{N} \text{ such that } k + l > 0\}.$$

All of the rings $\mathbb{C}[X_n, Y_n]$, $\mathbb{C}[X_n, Y_n]^{\mathfrak{S}_n}$, and $\mathcal{DH}_n$ are bigraded with respect to the polynomial degrees in x and y. The bivariate Hilbert series

$$\mathrm{Hilb}_{\mathbb{C}}(\mathcal{DH}_n, q, t) = \sum_{i,j=0}^{\infty} \dim_{\mathbb{C}}(V^{i,j})\, q^i t^j,$$

which is in fact a polynomial, has a surprising connection to the generating polynomial of the statistics area and dinv on parking functions introduced earlier. It is conjectured that these two polynomials coincide. Furthermore, results due to Haiman together with what we shall develop in this thesis prove that the bivariate Hilbert series of the subspace of the diagonal harmonics corresponding to the alternating representation of the symmetric group is given by the q, t-Catalan numbers. We shall come back to this subject once more, after establishing some more terminology.

6 Schur Functions and Plethysm

In this section we pursue two goals. First, we will establish the so called plethystic calculus or plethystic substitution which is an important tool for manipulating symmetric functions. Simultaneously, we will describe an algebra which is the proper setting to carry out our proof. Secondly, we will introduce Schur functions and review their connections to the concepts we have encountered thus far. Most notably, we shall cite the Cauchy Identity in Theorem 6.11 and prove a plethystic addition formula for skew Schur functions in Theorem 6.16. These are important technical tools and will play prominent roles in the remaining sections.

Let $n \in \mathbb{N}$, K be a field of characteristic zero and $X = \{x_1, x_2, x_3, \dots\}$, $X_n = \{x_1, \dots, x_n\}$ be sets of variables over K as before. For any vector $f = (f_1, \dots, f_n) \in K[X_n]^n$ of polynomials there is a unique K-algebra endomorphism called the *polynomial substitution* of f sending x_i to f_i. We denote this map by $\psi_f : K[X_n] \to K[X_n]$, that is

$$\psi_f(g(x_1, \dots, x_n)) := g(f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)).$$

A related albeit different notion exists on the ring of abstract symmetric function Λ_K.

Theorem 6.1. *There exists a unique binary operation $\Lambda \times \Lambda \to \Lambda$, $(f, g) \mapsto f \bullet g$ on the ring of abstract symmetric functions that fulfills the following three properties:*

(P1). *For all $m, n \in \mathbb{N}$ we have $p_m \bullet p_n = p_{mn}$.*

(P2). *For all $m \in \mathbb{N}$ the map $L_m : \Lambda \to \Lambda$ given by $L_m(g) := p_m \bullet g$ for all $g \in \Lambda$ is an algebra homomorphism.*

(P3). *For all $g \in \Lambda$ the map $R_g : \Lambda \to \Lambda$ given by $R_g(f) := f \bullet g$ for all $f \in \Lambda$ is an algebra homomorphism.*

Proof. The existence and uniqueness of the operation follow from the repeated use of the universal mapping property of polynomial rings. First, for each $m \in \mathbb{N}$ there exists a unique $\mathbb{Q}$-algebra endomorphism $L_m : \Lambda \to \Lambda$ such that (P1) is satisfied for all $n \in \mathbb{N}$. Secondly, for each $g \in \Lambda$ there exists a unique $\mathbb{Q}$-algebra endomorphism $R_g : \Lambda \to \Lambda$ such that (P2) is satisfied for each $m \in \mathbb{N}$. Thus, the binary operation defined by $f \bullet g := R_g(f)$ for any $f, g \in \Lambda$ fulfills (P1) – (P3), and is uniquely defined by these properties. ∎

Example 6.2 (i). We will compute $e_k \bullet (-p_1)$. The first thing to understand is that plethysm is defined via power-sums, hence, we need to express the examined function in terms of the basis $\{p_\lambda\}$. By (5.6) and using (P3) we have

$$\begin{aligned} e_k \bullet (-p_1) &= \sum_{\lambda \vdash k} \frac{(-1)^{k-l(\lambda)}}{z_\lambda} (p_\lambda \bullet (-p_1)) \\ &= \sum_{\lambda \vdash k} \frac{(-1)^{k-l(\lambda)}}{z_\lambda} \prod_{i=1}^{l(\lambda)} (p_{\lambda_i} \bullet (-p_1)). \end{aligned}$$

Consequently, using (P2) we obtain

$$\begin{aligned} e_k \bullet (-p_1) &= \sum_{\lambda \vdash k} \frac{(-1)^{k-l(\lambda)}}{z_\lambda} \prod_{i=1}^{l(\lambda)} (-(p_{\lambda_i} \bullet p_1)) \\ &= \sum_{\lambda \vdash k} \frac{(-1)^{k-l(\lambda)}}{z_\lambda} (-1)^{l(\lambda)} p_\lambda \\ &= (-1)^k \sum_{\lambda \vdash k} \frac{1}{z_\lambda} p_\lambda. \end{aligned}$$

We conclude that $e_k \bullet (-p_1) = (-1)^k h_k$, due to (5.6). This is a special case of Theorem 6.6 as we shall see later.

(ii). For general $f, g \in \Lambda$ we compute $f \bullet g$ as follows: First, express f and g as a linear combination of the basis given by products of power-sums

$$f = \sum_\mu f_\mu p_\mu \quad , \qquad g = \sum_\nu g_\nu p_\nu .$$

Then we compute

$$\begin{aligned} f \bullet g &= \sum_\mu f_\mu \prod_{i \in \mathbb{N}} (p_{\mu_i} \bullet g) \\ &= \sum_\mu f_\mu \prod_{i \in \mathbb{N}} \left(\sum_\nu g_\nu \prod_{j \in \mathbb{N}} (p_{\mu_i} \bullet p_{\nu_j}) \right) \\ &= \sum_\mu f_\mu \prod_{i \in \mathbb{N}} \left(\sum_\nu g_\nu \prod_{j \in \mathbb{N}} p_{\mu_i \nu_j} \right) , \end{aligned}$$

where we use (P3), (P2) and (P1) respectively. □

Our next result takes care of the most immediate algebraic properties of the plethystic substitution. It is associative and has the neutral element p_1. Moreover, the power-sum symmetric functions commutate with all symmetric functions.

Theorem 6.3. *Let $m, n \in \mathbb{N}$ and $f, g, h \in \Lambda$. Then the plethysm operation fulfills the following properties:*

$$\phi_n(p_m \bullet g) = \phi_n(g)(x_1^m, \ldots, x_n^m) \in \Lambda_{n,\mathbb{Q}} \tag{6.1}$$
$$p_m \bullet h = h \bullet p_m \tag{6.2}$$
$$p_1 \bullet g = g \tag{6.3}$$
$$(f \bullet g) \bullet h = f \bullet (g \bullet h). \tag{6.4}$$

Proof. First we verify these properties for abstract power-sum symmetric functions. For any $k \in \mathbb{N}$

$$\phi_n(p_m \bullet p_k) = \phi_n(p_{mk}) = x_1^{mk} + \cdots + x_n^{mk} = p_{n,k}(x_1^m, \ldots, x_n^m) = \phi_n(p_k)(x_1^m, \ldots, x_n^m).$$

Since both sides of (6.1) are $\mathbb{Q}$-algebra homomorphisms that agree on the power-sum symmetric functions they must be equal. Similarly, (6.2) follows from the fact that L_m and R_{p_m} are $\mathbb{Q}$-algebra endomorphisms that agree on all p_k. The same argument yields (6.3) when we compare L_1 and id_Λ.

For the last claim, we note that for any $i, j, k \in \mathbb{N}$ we have

$$p_i \bullet (p_j \bullet p_k) = p_{ijk} = (p_i \bullet p_j) \bullet p_k.$$

Since the endomorphisms $L_i \circ L_j$ and L_{ij} agree on all p_k, the must agree on Λ, i.e.

$$p_i \bullet (p_j \bullet h) = (p_i \bullet p_j) \bullet h.$$

for all $h \in \Lambda$. But now the same argument implies that $L_i \circ R_h$ equals $R_h \circ L_i$, that is

$$p_i \bullet (g \bullet h) = (p_i \bullet g) \bullet h.$$

for all $g, h \in \Lambda$. Finally, the identity in (6.3) follows by the universal mapping property applied yet again, implying $R_g \circ R_h$ and $R_h \circ R_g$ agree on all $f \in \Lambda$. ■

Example 6.4. Notice that the plethystic substitution is not commutative in general. In Example 6.2 we have seen that $e_k \bullet (-p_1) = (-1)^k h_k$. However, we now know that

$$(-p_1) \bullet e_k = -(p_1 \bullet e_k) = -e_k.$$

□

We will need plethysm in a slightly more general setting. Let therefore $Y := \{y_1, y_2, y_3, \ldots\}$ and $Y_n := \{y_1, \ldots, y_n\}$ be sets of additional variables, and let $K = \mathbb{Q}(q, t)$

be the field of rational functions in q and t over the rationals. We shall be working in the K-algebra

$$A := \Lambda_K \otimes_K \Lambda_K \tag{6.5}$$

where we distinguish abstract symmetric functions in X and in Y.

In the spirit of Theorem 6.1 we define a binary operation $.[.] : \Lambda_K \times A \to A$ by demanding three properties.

(P1'). For all $m, n \in \mathbb{N}$ and $r(q,t) \in \mathbb{Q}(q,t)$ we have

$$p_m[p_n \otimes 1] = p_{mn} \otimes 1, \quad p_m[1 \otimes p_n] = 1 \otimes p_{mn}, \quad p_m[r(q,t)] = r(q^m, t^m).$$

(P2'). For any $m \in \mathbb{N}$ the map $L_m : A \to A$ given by $L_m(g) := p_m[g]$ is a $\mathbb{Q}$-algebra endomorphism.

(P3'). For any $g \in A$ the map $R_g : \Lambda_K \to A$ given by $R_g(f) := f[g]$ is a K-algebra homomorphism.

The notation $f[g]$ is read as "*plethystic substitution* of g in f". For $f \in \Lambda$ we denote $f[p_1 \otimes 1] = f[X]$ and $f[1 \otimes p_1] = f[Y]$. That means, in the sense of Theorem 6.3 (6.3) the map $f \mapsto f[X]$ is the identity on Λ. If $f(x_1, x_2, \ldots; q, t) \in \Lambda_K$, we write $f[X, q, t]$ to remind us of the fact that f is an abstract symmetric function with coefficients depending rationally on the variables q and t. However, by (P3') any term depending only on q and t (in f that is!) are not affected by the plethystic substitution.

Example 6.5 (i). We have

$$\begin{aligned}
p_k[X] &= x_1^k + x_2^k + x_3^k + \cdots = \sum_{i=1}^{\infty} x_i^k, \\
((1-q)p_k)[Y] &= (1-q)(y_1^k + y_2^k + y_3^k + \ldots) = \sum_{i=1}^{\infty} (1-q) y_i^k, \\
p_k[X+Y] &= p_k[X] + p[Y] = \sum_{i=1}^{\infty} (x_i^k + y_i^k), \\
p_k[XY] &= p_k[X] p_k[Y] = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_i^k y_j^k, \\
p_k[\tfrac{X}{(1-q)(1-t)}] &= \sum_{i=1}^{\infty} \frac{x_i^k}{(1-q^k)(1-t^k)} = \frac{p_k[X]}{(1-q^k)(1-t^k)}.
\end{aligned}$$

(ii). In the spirit of Example 6.2 let us express $e_k[-X\frac{t}{q}]$ in terms if the complete homogeneous symmetric functions.

$$\begin{aligned}
e_k[-X\tfrac{t}{q}] &= \sum_{\lambda\vdash k} \frac{(-1)^{k-l(\lambda)}}{z_\lambda} \prod_{i=1}^{l(\lambda)} p_{\lambda_i}[-X\tfrac{t}{q}] \\
&= \sum_{\lambda\vdash k} \frac{(-1)^{k-l(\lambda)}}{z_\lambda} (-1)^{l(\lambda)} \frac{t^k}{q^k} p_\lambda[X] \\
&= (-1)^k \frac{t^k}{q^k} h_k[X]
\end{aligned}$$

□

In analogy to Corollary 5.15 (iii), we define $\omega : \Lambda \to \Lambda$ to be the involutive $\mathbb{Q}$-algebra automorphism given by

$$\omega(p_k) := (-1)^{k-1} p_k.$$

By means of the involution ω we obtain a negation rule which expresses $f[-g]$ in terms of $f[g]$.

Theorem 6.6. *Let $k \in \mathbb{N}$, $f \in \Lambda_K^k$ be homogeneous, and $g \in A$. Then*

$$f[-g] := (-1)^k (\omega(f))[g].$$

Proof. We recall that the set $\{p_\lambda : \lambda \vdash k\}$ is a K-basis of Λ_K^k. Hence, using (P3') and (P2') we have

$$\begin{aligned}
f[-g] = \Big(\sum_{\lambda\vdash k} f_\lambda p_\lambda\Big)[-g] &= \sum_{\lambda\vdash k} f_\lambda \prod_{i=1}^{\infty} p_{\lambda_i}[-g] \\
&= \sum_{\lambda\vdash k} f_\lambda \prod_{i=1}^{\infty} -(p_{\lambda_i}[g]) = \sum_{\lambda\vdash k} f_\lambda \prod_{i=1}^{\infty} (-1)^{\lambda_i} \omega(p_{\lambda_i})[g] \\
&= (-1)^k \sum_{\lambda\vdash k} f_\lambda \omega(p_\lambda)[g] = (-1)^k \omega(f)[g]
\end{aligned}$$

for suitable $f_\lambda \in K$. ■

If the function g is a polynomial in q and t we can say more.

Theorem 6.7. *Let $n \in \mathbb{N}$ and $g = M_1 + \cdots + M_n \in \mathbb{Q}[q,t]$ be the sum of n monic (over $\mathbb{Q}$) monomials, and $f \in \Lambda_K$. Then $f[g] = f(M_1, \ldots, M_n)$, that is the plethysm R_g agrees with the composition $\psi_{(M_1,\ldots,M_n)} \circ \phi_n$ of the evaluation map $\phi_n : \Lambda_K \to \Lambda_{n,K}$ and the polynomial substitution $\psi_{(M_1,\ldots,M_n)} : \Lambda_{n,K} \to \Lambda_{n,K}$ sending x_i to M_i.*

Proof. Let $m \in \mathbb{N}$ then

$$\begin{aligned}
p_m[g] &= M_1(q^m, t^m) + \cdots + M_1(q^m, t^m) = (M_1(q,t))^m + \cdots + (M_n(q,t))^m \\
&= p_{n,m}(M_1, \ldots, M_n).
\end{aligned}$$

Now, once again the result follows from the fact that both R_g and $\psi_{(M_1,\ldots,M_n)} \circ \phi_n$ are K-algebra homomorphisms that agree on the power-sum symmetric functions. ■

The next lemma provides an easy application of Theorem 6.7 above. We shall make use of it in Section 8.

Lemma 6.8. *Let $d, k \in \mathbb{N}$. Then the plethystic substitution of $\frac{1-q^k}{1-q} = 1 + q + \cdots + q^{k-1}$ into the abstract complete homogeneous symmetric function is given by*

$$h_d[\tfrac{1-q^k}{1-q}] = \begin{bmatrix} k+d-1 \\ k \end{bmatrix}_q .$$

Proof. By Theorem 6.7 and the generating function of the $h_{d,k}$ we have

$$\begin{aligned} h_d[\tfrac{1-q^k}{1-q}] &= h_{d,k}(1, q, \ldots, q^{k-1}) \\ &= \langle t^d \rangle \prod_{i=1}^{k} \frac{1}{1 - q^{i-1}t} \\ &= \begin{bmatrix} k+d-1 \\ d \end{bmatrix}_q . \end{aligned}$$

A combinatorial proof using Proposition 3.2 is also possible. ■

Now, let us return to the ring of symmetric functions in n variables Λ_n for one moment. We will introduce another family of symmetric functions indexed by partitions called the Schur functions. Arguably Schur functions are the most important family of symmetric functions due to their connections to both, the representations of the symmetric group and the polynomial representations of the general linear group.

For $k \leq n$ let $\lambda \vdash k$ be a partition and $\delta_n = (n-1, \ldots, 2, 1, 0)$ be the staircase partition. Then we define the *Schur function* in n variables corresponding to λ as

$$s_{n,\lambda}(x_1, \ldots, x_n) := \frac{\det(x_i^{\lambda_j + \delta_j})_{1 \leq i,j \leq n}}{\Delta_n(x_1, \ldots, x_n)}, \tag{6.6}$$

where Δ_n is the Vandermonde determinant. The numerator of the right hand side of (6.6) is an alternating polynomial. Recalling Lemma 5.6, it is therefore the product of the Vandermonde determinant and a symmetric polynomial. Thus, the Schur function is a symmetric polynomial as expected. Moreover, we see that the Schur function is homogeneous since it equals the quotient of homogeneous polynomials. Hence, we have $s_{n,\lambda} \in \Lambda_n^k$.

The following theorem provides two alternative formulas for the Schur functions which the reader finds proven in [15, Chapter I (3.4) and (3.5)].

Theorem 6.9. *Let $n, k \in \mathbb{N}$ such that $k \leq n$, and let $\lambda \vdash k$ be a partition.*

(i). *In terms of the complete homogeneous symmetric functions we have*

$$s_{n,\lambda} = \det(h_{n,\lambda_i - i + j})_{1 \le i,j \le n}. \tag{6.7}$$

(ii). *In term of the elementary symmetric functions we have*

$$s_{n,\lambda} = \det(e_{n,\lambda'_i - i + j})_{1 \le i,j \le n}. \tag{6.8}$$

Since

$$s_{n+1,\lambda}(x_1, \dots, x_n, 0) = s_{n,\lambda}(x_1, \dots, x_n)$$

whenever $|\lambda| \le n$, the Schur functions in finitely many variables determine a class of abstract symmetric functions. More precisely, there exists a unique $s_\lambda \in \Lambda$ such that the evaluation $\phi_n(s_\lambda)$ equals $s_{n,\lambda}$ for each $n \ge |\lambda|$. We shall call s_λ the *abstract Schur function* corresponding to λ.

We remark that for any vector $w \in \mathbb{Z}^n$ such that $w + \delta$ has no negative entries, replacing λ_j by w_j in (6.6) defines a symmetric function $s_{n,w}$. However, if not all numbers $w_i + \delta_j$ are distinct, the resulting determinant is zero. Otherwise, we have $s_{n,w} = \pm s_{n,\tilde{w}}$ where $\tilde{w}$ is a partition.

Applying the involution ω to (6.7) respectively (6.8) we may derive that the set of Schur functions is mapped onto itself. Moreover, if λ consists of a single row or column we rediscover the elementary and complete homogeneous symmetric functions as special cases of the Schur fuctions.

Corollary 6.10. *Let $n \in \mathbb{N}$ such that $1 \le k \le n$.*

(i). *Let $\lambda \vdash n$ be a partition. Then the involution ω sends s_λ to $s_{\lambda'}$.*

(ii). *For $n \in \mathbb{N}$ we have $s_{1^n} = e_n$ and $s_{(n)} = h_n$.* ■

Our next aim is to show that the Schur functions form a particularly nice basis of Λ. Let λ, μ be two partitions. We define a bilinear form $\langle ., . \rangle : \Lambda \times \Lambda$ by

$$\langle p_\lambda, p_\mu \rangle := z_\lambda \delta_{\lambda,\mu}. \tag{6.9}$$

This bilinear form is called the *Hall-inner product*, and it follows from the definition that $\langle ., . \rangle$ is symmetric and positive definite, i.e. is indeed an inner product on Λ. Moreover, the basis of power-sum symmetric polynomials is orthogonal with respect to the Hall inner product. The dual basis is given by $\{\frac{1}{z_\lambda} p_\lambda : \lambda \vdash n, n \in \mathbb{N}\}$. The involution ω is an isometry, i.e. $\langle f, g \rangle = \langle \omega(f), \omega(g) \rangle$ for all $f, g \in \Lambda$.

Motivated by the computations in the proof of Proposition 5.14, we define

$$\Omega[X] := \prod_{i=1}^{\infty} \frac{1}{1 - x_i} = \sum_{\lambda} \frac{p_\lambda[X]}{z_\lambda} = \exp\Big(\sum_{k=1}^{\infty} \frac{p_k[X]}{k} \Big).$$

Then we obtain

$$\Omega[XY] = \exp\Big(\sum_{k=1}^{\infty} \frac{p_k[XY]}{k}\Big) = \exp\Big(\sum_{k=1}^{\infty} \frac{p_k[X]p_k[Y]}{k}\Big) = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j}.$$

However, we remark that $\Omega \in \mathbb{Q}[[X]]$ is a formal power series that does not lie in Λ. Hence, the plethystic substitution in the above deduction is best understood as an abuse of notation. Of course it would be possible to further extend our notion of plethysm to include power series which are limits of abstract symmetric functions.

Theorem 6.11 (Cauchy Identities). *Let $n \in \mathbb{N}$. Then the homogeneous part of $\Omega[XY]$ of degree $2n$ is given by*

$$h_n[XY] = \sum_{\lambda \vdash n} p_\lambda[X] \frac{p_\lambda[Y]}{z_\lambda} = \sum_{\lambda \vdash n} h_\lambda[X] m_\lambda[Y] = \sum_{\lambda \vdash n} s_\lambda[X] s_\lambda[Y]. \tag{6.10}$$

All identities are proven in [15, Chapter I (4.1), (4.2), and (4.3)].

Lemma 6.12. *Let $k \in \mathbb{N}$, and let $\{u_\lambda : \lambda \vdash k\}$ and $\{v_\lambda : \lambda \vdash k\}$ be two $\mathbb{Q}$-bases for Λ^k. Then these bases are dual with respect to* (6.9) *if and only if*

$$h_k[XY] = \sum_{\lambda \vdash k} u_\lambda[X] v_\lambda[Y].$$

For a proof see [15, Chapter I (4.6)].

Given the preparations above it is now easy to proof that the Schur functions form an orthonormal basis of the ring of symmetric functions.

Theorem 6.13 (i). *The expansion of Schur functions into a linear combination of monomial symmetric functions is unit-triangular. That is, let λ be a partition then we have*

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda,\mu} m_\mu \tag{6.11}$$

for suitable $c_{\lambda,\mu} \in \mathbb{Q}$.

(ii). *The Schur functions $\{s_\lambda\}_\lambda$ form an orthonormal basis of Λ with respect to the inner product* (6.9)*, i.e.*

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \tag{6.12}$$

for any two partitions λ and μ.

(iii). *The two properties (i) and (ii) uniquely determine the Schur functions.*

Proof. The first part is proven in [16, Chapter 1 (7.2)]. The second part clearly follows from the first part, Lemma 6.12 and the Cauchy identity (6.10). The uniqueness follows from the Gram–Schmidt orthogonalisation process applied to the monomial symmetric functions. Compare this also to the proof of Theorem 7.4 (iii). ∎

Corollary 6.14. *Let $k, n \in \mathbb{N}$ such that $1 \leq k \leq n$. Then the Schur functions in n variables of degree k form a $\mathbb{Z}$-basis of Λ_n^k.* ∎

Let

$$\tilde{\Omega}[X] := \prod_{i=1}^{\infty}(1+x_i) \qquad \text{and} \qquad \tilde{\Omega}[XY] = \prod_{i,j=1}^{\infty}(1+x_iy_j).$$

Applying the involution ω with respect to the set of variables X to the identities of Theorem 6.11 we obtain another set of identities.

Corollary 6.15 (Dual Cauchy Identities). *Let $n \in \mathbb{N}$. Then the homogeneous part of $\tilde{\Omega}[XY]$ of degree $2n$ is given by*

$$e_n[XY] = \sum_{\lambda \vdash n}(-1)^{n-l(\lambda)}p_\lambda[X]\frac{p_\lambda[Y]}{z_\lambda} = \sum_{\lambda \vdash n} e_\lambda[X]m_\lambda[Y] = \sum_{\lambda \vdash n} s_\lambda[X]s_{\lambda'}[Y]. \tag{6.13}$$

∎

By Theorem 6.13 every symmetric function $f \in \Lambda$ is fully determined by its Schur coefficients, i.e., $\langle f, s_\lambda \rangle$ where λ ranges over all partitions. Let λ and μ be partitions. We define the *abstract skew Schur functions* by letting

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle$$

for all partitions ν. In particular, $s_{\lambda/\emptyset} = s_\lambda$ for all partitions λ. Note that $s_{\lambda/\mu} = 0$ unless $|\mu| + |\nu| = |\lambda|$. In fact, a stronger statement is true. One can show that $s_{\lambda/\mu}$ equals zero unless $\mu \subseteq \lambda$. The next results are on the plethystic calculus of Schur functions which will be important for proving the main result.

Theorem 6.16. *Let λ and ν be partitions such that $\nu \subseteq \lambda$.*

(i). *Then we have the plethystic negation formula for the abstract skew Schur functions*

$$s_{\lambda/\nu}[-X] = (-1)^{(|\lambda|-|\nu|)} s_{\lambda'/\nu'}[X].$$

(ii). *Secondly, we have the plethystic addition formula for the abstract skew Schur functions*

$$s_{\lambda/\mu}[X+Y] = \sum_{\nu \subseteq \mu \subseteq \lambda} s_{\mu/\nu}[X] s_{\lambda/\mu}[Y].$$

Proof. The negation formula is a consequence of Theorem 6.6 and

$$\omega\big(s_{\lambda/\mu}[X]\big) = s_{\lambda'/\mu'}[X].$$

To prove the addition formula, we start by examining that

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_{\nu}$$

is equivalent to

$$s_{\mu}s_{\nu} = \sum_{\lambda} c^{\lambda}_{\mu,\nu} s_{\lambda}$$

because of the definition of the skew Schur functions. Thus, by the Cauchy identity (6.10) we have

$$\sum_{\lambda} s_{\lambda/\mu}[X]s_{\lambda}[Y] = \sum_{\nu} s_{\nu}[X]s_{\mu}[Y]s_{\nu}[Y] = s_{\mu}[Y]\prod_{i,j=1}^{\infty}\frac{1}{1-x_iy_j}.$$

For the moment let $Z = \{Z_1, Z_2, Z_3, \dots\}$ be another set of variables. Then we have

$$\sum_{\lambda,\mu} s_{\lambda/\mu}[X]s_{\mu}[Y]s_{\lambda}[Z] = \prod_{i,j=1}^{\infty}\frac{1}{1-x_iz_j}\cdot\prod_{i,j=1}^{\infty}\frac{1}{1-y_iz_j} = \sum_{\lambda} s_{\lambda}[X+Y]s_{\lambda}[Z].$$

Hence,

$$\sum_{\mu} s_{\lambda/\mu}[X]s_{\mu}[Y] = s_{\lambda}[X+Y].$$

But then, the theorem follows from

$$\sum_{\mu} s_{\lambda/\mu}[X+Y]s_{\mu}[Z] = s_{\lambda}[X+Y+Z] = \sum_{\nu} s_{\lambda/\nu}[X]s_{\nu}[Y+Z] = \sum_{\mu,\nu} s_{\lambda/\nu}[X]s_{\nu/\mu}[Y]s_{\mu}[Z].$$

■

Corollary 6.17. *Let λ and μ be partitions such that $\lambda = (n-k, 1^k)$ is a hook and μ is not a hook. Then we have the two specialisations of the Schur functions*

$$s_{\lambda}[1-q] = (-1)^k q^k(1-q) \qquad \text{and} \qquad s_{\mu}[1-q] = 0.$$

A proof is found in [15, Chapter I (5.11)].

Corollary 6.18. *Let $n \in \mathbb{N}$ and $\lambda \vdash n$ be a partition. Then the abstract elementary symmetric functions and the abstract complete homogeneous symmetric functions fulfil the following properties*

$$\begin{aligned}
e_{\lambda}[-X] &= (-1)^n h_{\lambda}[X] & h_{\lambda}[-X] &= (-1)^n e_{\lambda}[X]\\
e_n[X+Y] &= \sum_{k=0}^{n} h_k[X]h_{n-k}[Y] & h_n[X+Y] &= \sum_{k=0}^{n} e_n[X]e_{n-k}[Y]\\
e_n[X-Y] &= \sum_{k=0}^{n}(-1)^{n-k} h_k[X]e_{n-k}[Y] & h_n[X-Y] &= \sum_{k=0}^{n}(-1)^{n-k} e_k[X]h_{n-k}[Y].
\end{aligned}$$

■

The Schur functions have a combinatorial interpretation in terms of Young tableaux. Let λ and μ be partitions such that $\mu \subseteq \lambda$. We define the *skew shape* or skew diagram λ/μ as $Y(\lambda) - Y(\mu)$. Let $k = |\lambda/\mu|$. A *semistandard skew tableau* of shape λ/μ is a map $T : \lambda/\mu \to \mathbb{N}$ weakly increasing from left to right and strictly increasing from top to bottom (see Figure 16). We define the *weight* $w \in \mathbb{N}^k$ by taking w_i to be the cardinality of $T^{-1}(i)$. The set of all tableau of skew shape λ/μ is denoted by $\mathrm{SSYT}(\lambda/\mu)$, and the subset of tableaux that have a given weight w is denoted by $\mathrm{SSYT}(\lambda/\mu, w)$. A tableau of skew shape λ/μ with weight $w = 1^k$ is called a *standard skew tableau*. Finally, the set of standard skew tableaux of shape λ/μ is denoted by $\mathrm{SYT}(\lambda/\mu)$.

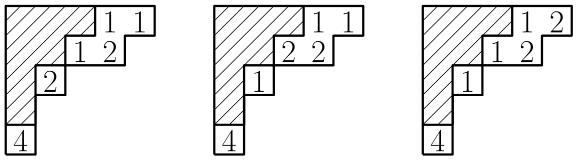

Figure 16: Three skew tableaux in $\mathrm{SSYT}(\lambda/\mu, w)$ where $\lambda = (5, 4, 2, 1, 1)$, $\mu = (3, 2, 1, 1)$ and $w = (3, 2, 0, 1)$.

Theorem 6.19. *Let $k, n \in \mathbb{N}$, and $\lambda \vdash n$ be a partition.*

(i). *For any partition $\mu \vdash n$ recall that $K_{\lambda,\mu}$ denotes the corresponding Kostka number. Then we have*

$$s_\lambda = \sum_{\mu \vdash |\lambda|} K_{\lambda,\mu} m_\mu.$$

(ii). *Let $\mu \subseteq \lambda$ be another partition such that $|\lambda/\mu| = k$. Then*

$$s_{\lambda/\mu}[X] = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} \prod_{c \in \lambda/\mu} x_{T(c)}.$$

Proof. The claim can be proven by extending Theorem 6.16 (ii) to n sets of variables $X^{(n)} := \{x_n\}$ each consisting of only one variable. For details see [15, Chapter I (5.12)]. ■

Furthermore, the connection between the Schur functions and the representation theory of $\mathfrak{S}_k$ is hidden in their expansion in terms of the power-sum symmetric functions.

Theorem 6.20. *For each $n \in \mathbb{N}$, and each partition $\lambda \vdash k$ let χ_λ be the irreducible characters of the symmetric group $\mathfrak{S}_n$ corresponding to λ as described in Theorem 5.4.*

For any partition $\mu \vdash n$ let $\chi_\lambda(\mu)$ be the value taken at elements in the conjugacy class corresponding to the cycle type μ. Then we have

$$s_\lambda = \sum_{\mu \vdash n} \chi_\lambda(\mu) \frac{1}{z_\mu} p_\mu. \tag{6.14}$$

Proof. Again, we only provide a sketch. The full proof can be found in [15, Chapter I (7.6)].

The proof relies on induced representations and the Frobenius reciprocity for irreducible characters, which we did not introduce in this work. We define the product of two characters χ of $\mathfrak{S}_m$ and ρ of $\mathfrak{S}_n$ to be the induced character of $\chi \times \rho$ where $\mathfrak{S}_m \times \mathfrak{S}_n$ is regarded as a subgroup of $\mathfrak{S}_{m+n}$. Thereby, the direct sum $R := \bigoplus_{n=0}^{\infty} R(\mathfrak{S}_n, \mathbb{C})$ becomes a graded $\mathbb{C}$-algebra. One then proves the theorem by constructing an isometric isomorphism $\iota : R \to \Lambda_\mathbb{C}$ with respect to the Hall inner product and the inner product defined on the center of the group algebra $\mathrm{Z}(\mathbb{C}\mathfrak{S}_n)$ in (4.3). ■

7 Macdonald Polynomials and the Pieri Formula

Returning to the $\mathbb{Q}(q,t)$-algebra A defined in (6.5), we will now give an introduction to Macdonald polynomials, a family of (abstract) symmetric functions in X depending on two parameters q and t that generalises the Schur functions. In the second part of this section we are going to develop the remaining notation we need for the proof of the symmetry theorem for the q,t-Catalan numbers. More presicely, we will define modified Macdonald polynomials and adapt some of our results about Schur functions to the new situation.

As before we set $K := \mathbb{Q}(q,t)$. First, we define a q,t-generalisation of the Hall scalar product as $\langle .,. \rangle_{q,t} : \Lambda_K \times \Lambda_K \to K$ by

$$\langle p_\lambda, p_\mu \rangle_{q,t} := z_\lambda(q,t)\delta_{\lambda,\mu} \qquad \text{where} \qquad z_\lambda(q,t) := z_\lambda \prod_{i \in \mathbb{N}} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} = z_\lambda p_\lambda[\tfrac{1-q}{1-t}].$$

Clearly, the abstract power-sum symmetric functions again form an orthogonal basis with respect to this product, which reduces to the Hall inner product of (6.9) when $q = t$. The dual basis is given by $\{z_\lambda(q,t)^{-1} p_\lambda\}_\lambda$ where λ ranges over all partitions.

There is also a q,t-analogue of the involution ω on Λ_K defined as the K-algebra endomorphism induced by

$$\omega_{q,t}(p_k) := (-1)^{k-1} \frac{1-q^k}{1-t^k} p_k = p_k[-\tfrac{1-q}{1-t}].$$

Again, $\omega_{q,t}$ reduces to ω when we let $q = t$. Note that $\omega_{q,t} = \omega_{t,q}^{-1}$, thus it is not an involution!

Proposition 7.1. *Let $f, g \in \Lambda_K$.*

(i). *Then we have*

$$\langle \omega_{t,q}(f), g\rangle_{q,t} = \langle \omega(f), g\rangle.$$

(ii). *The map $\omega_{q,t}$ is self-adjoint, i.e.*

$$\langle \omega_{q,t}(f), g\rangle_{q,t} = \langle f, \omega_{q,t}(g)\rangle_{q,t}$$

■

Just as in the previous section we have a close connection between orthogonality and Cauchy identities. Let

$$\Omega[XY\tfrac{1-t}{1-q}] = \sum_{\lambda} \frac{p_\lambda[X]p_\lambda[Y]}{z_\lambda(q,t)} = \prod_{i,j=1}^{\infty} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

Lemma 7.2. *Let $k \in \mathbb{N}$, and let $\{u_\lambda : \lambda \vdash k\}$ and $\{v_\lambda : \lambda \vdash k\}$ be two K-bases for Λ_K^k. Then these bases are dual with respect to the q,t-Hall scalar product if and only if*

$$h_k[XY\tfrac{1-t}{1-q}] = \sum_{\lambda \vdash k} u_\lambda[X]v_\lambda[Y].$$

The proof is similar to that of Lemma 6.12 and can be found in [15, Chapter VI (2.7)].

The definition of the Macdonald polynomials is not easily done in an explicit way. Instead, they will be defined as the eigenvalues of a certain operator on Λ_K. To this end, let $i, n \in \mathbb{N}$ such that $1 \le i \le n$. Then we define K-linear operators on $K[X_n]$ by

$$\begin{aligned} T_{q,i}f(x_1,\dots,x_n) &:= f(x_1,\dots,qx_i,\dots,x_n), \\ T_{t,i}f(x_1,\dots,x_n) &:= f(x_1,\dots,tx_i,\dots,x_n), \end{aligned}$$

Furthermore, we define an operator $D_n : K[X_n] \to K[X_n]$ by

$$D_n := \sum_{i=1}^{n} \Delta_n^{-1}(T_{t,i}\Delta_n)T_{q,i} = \sum_{i=1}^{n} \Bigg(\prod_{1\le j\le n,\, i\ne j} \frac{tx_i - x_j}{x_i - x_j} \Bigg) T_{q,i}. \tag{7.1}$$

Theorem 7.3. *There exists a K-linear operator $E : \Lambda_K \to \Lambda_K$ with the following properties:*

(E1). *The transformation matrix of E with respect to the basis of monomial symmetric functions is triangular. That is, for any partition λ we have $Em_\lambda = \sum_{\mu\le\lambda} c_{\lambda\mu}m_\mu$ for suitable coefficients $c_{\lambda\mu} \in K$.*

(E2). *The operator E is self-adjoint, that is $\langle Ef, g\rangle_{q,t} = \langle f, Eg\rangle_{q,t}$ for all $f, g \in \Lambda_K$.*

(E3). *All eigenvalues of E are distinct, that is $c_{\lambda\lambda} \ne c_{\mu\mu}$ if $\lambda \ne \mu$.*

Proof. The operator E can be obtained as the inverse limit of operators $E_n : \Lambda_{n,K} \to \Lambda_{n,K}$, where E_n is a slight variation of the operator D_n in (7.1) above. See [15] or [16, Chapter 1 (11.7)] for details. ■

Once we assume the existence of such an operator the Macdonald polynomials can be defined via properties that uniquely determine a family of symmetric functions in Λ_K. More precisely, these properties overdetermined the Macdonald polynomials and the operator E guaranties their existence.

Theorem 7.4 (i). *Let E and $c_{\lambda\mu}$ be as in Theorem 7.3. For each partition λ there exists a unique abstract symmetric function $P_\lambda \in \Lambda_K$ that has an expansion of the form*

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} d_{\lambda\mu} m_\mu \tag{7.2}$$

for suitable $d_{\lambda\mu} \in K$, and

$$EP_\lambda = c_{\lambda\lambda} P_\lambda. \tag{7.3}$$

(ii). *The symmetric functions $\{P_\lambda\}_\lambda$ defined in (i) are orthogonal with respect to the q,t-Hall scalar product. More precisely,*

$$\langle P_\lambda, P_\mu \rangle = c_{\lambda\lambda} \delta_{\lambda,\mu}. \tag{7.4}$$

(iii). *The properties* (7.2) *and* (7.4) *uniquely define a family of abstract symmetric functions.*

Proof. By (E3), the relations $d_{\lambda\lambda} = 1$, and

$$(c_{\lambda\lambda} - c_{\nu\nu}) d_{\lambda\nu} = \sum_{\nu<\mu\leq\lambda} d_{\lambda\mu} c_{\mu\nu} \qquad \lambda \neq \nu$$

uniquely determine elements $d_{\lambda\mu} \in K$. But then we may define $P_\lambda := \sum_{\mu\leq\lambda} d_{\lambda\mu} m_\mu$, and use (E1) to compute

$$\begin{aligned} EP_\lambda := E \sum_{\mu\leq\lambda} d_{\lambda\mu} m_\mu &= \sum_{\mu\leq\lambda} d_{\lambda\mu} \sum_{\nu\leq\mu} c_{\mu\nu} m_\nu \\ &= \sum_{\nu\leq\lambda} \left(\sum_{\nu\leq\mu\leq\lambda} d_{\lambda\mu} c_{\mu\nu} \right) m_\nu \\ &= \sum_{\nu\leq\lambda} \left(c_{\nu\nu} d_{\lambda\nu} + \sum_{\nu<\mu\leq\lambda} d_{\lambda\mu} c_{\mu\nu} \right) m_\nu \\ &= \sum_{\nu\leq\lambda} c_{\lambda\lambda} d_{\lambda\nu} m_\nu = c_{\lambda\lambda} P_\lambda. \end{aligned}$$

Hence, the P_λ satisfy (7.2) and (7.3).

It is well known that the eigenvectors to distinct eigenvalues of self-adjoint operators are orthogonal because

$$c_{\lambda\lambda}\langle P_\lambda, P_\mu\rangle_{q,t} = \langle EP_\lambda, P_\mu\rangle_{q,t} = \langle P_\lambda, EP_\mu\rangle_{q,t} = c_{\mu\mu}\langle P_\lambda, P_\mu\rangle_{q,t}.$$

Thus, the second claim is a consequence of (E2) and (E3).

To see the last claim, we extend the natural order $\leq$ on the partitions to a total order $\preceq$. Then the Gram–Schmidt orthogonalisation applied to the basis consisting of the monomial symmetric functions gives the existence and uniqueness of a family of polynomials $\{P_\lambda\}$ that are orthogonal (7.4) such that each P_λ only depends on lower terms with respect to $\preceq$, i.e.

$$P_\lambda = m_\lambda + \sum_{\mu \prec \lambda} d_{\lambda\mu} m_\mu.$$

Surely the fact that we have an even stronger condition in (7.2) does not alter the uniqueness. ■

The abstract symmetric functions $\{P_\lambda(X, q, t)\}_\lambda$ thus defined are called the *Macdonald polynomials*. When we let $q = t$ the conditions (7.2) and (7.4) reduce to the properties determining the Schur functions in Theorem 6.13.

Corollary 7.5 (i). *Let $k \in \mathbb{K}$. Then the set $\{P_\lambda : \lambda \vdash k\}$ is a K-basis of Λ_K^k.*
(ii). *Let λ be a partition. Then we have $P_\lambda[X; q, q] = s_\lambda[X]$.* ■

Moreover, many other important families of symmetric functions can be rediscovered as special cases of the Macdonald polynomials. Besides the Schur functions, there are the Hall-Littlewood polynomials, the Zonal polynomials, and Jack symmetric functions, to name a few (see [16]).

To simplify further notation we shall now fix some terms which arise frequently in the study of the ring Λ_K. Let $\mu \vdash k$ be a partition. We recall that $x \in Y(\mu)$ is a cell in the Young diagram of μ, and that $a_\mu(x)$, $a'_\mu(x)$, $l_\mu(x)$ and $l'_\mu(x)$ denote its arm, coarm, leg and coleg respectively. Then we define

$$n(\mu) := \sum_{x \in Y(\mu)} l'_\mu(x), \qquad T_\mu(q,t) := t^{n(\mu)} q^{n(\mu')} = \prod_{x \in Y(\mu)} q^{a'_\mu(x)} t^{l'_\mu(x)},$$
$$B_\mu(q,t) := \sum_{x \in Y(\mu)} q^{a'_\mu(x)} t^{l'_\mu(x)}, \qquad \Pi_\mu(q,t) := \prod_{x \in Y(\mu), x \neq (1,1)} (1 - q^{a'_\mu(x)} t^{l'_\mu(x)}).$$

The attentive reader will remember these functions appearing in (4.1). Since we intend to prove the symmetry of the q,t-Catalan numbers by showing that they equal the expression in (4.1), the following symmetries which are obvious from the above definitions should be considered a small but important step in the right direction.

$$T_\mu(q,t) = T_{\mu'}(t,q), \qquad \Pi_\mu(q,t) = \Pi_{\mu'}(t,q), \qquad B_\mu(q,t) = B_{\mu'}(t,q)$$

We also remark that $n(\mu) = \sum_{x\in Y(\mu)} l_\mu(x)$ and thus $T_\mu(q,t) = \prod_{x\in Y(\mu)} q^{a_\mu(x)}t^{l_\mu(x)}$.

Next, we denote by $\{Q_\lambda(X,q,t)\}_\lambda$ the basis of Λ_K that is dual to the Macdonald polynomials. By Lemma 7.2, that means for every $n \in \mathbb{N}$

$$h_n[XY\tfrac{1-t}{1-q}] = \sum_{\lambda\vdash n} P_\lambda[X,q,t]Q_\lambda[Y,q,t].$$

Since the Macdonald polynomials are orthogonal, $Q_\lambda(X,q,t)$ must be a scalar multiple of $P_\lambda(X,q,t)$. We shall denote the arising rational functions by

$$c_\mu(q,t) := \prod_{x\in Y(\mu)} (1-q^{a_\mu(x)}t^{l_\mu(x)+1}), \qquad c'_\mu(q,t) \quad := \prod_{x\in Y(\mu)} (1-q^{a_\mu(x)+1}t^{l_\mu(x)}),$$

and remark the symmetry $c'_\mu(q,t) = c_{\mu'}(t,q)$.

Theorem 7.6. *Let μ be a partition. Then we have*

$$Q_\mu[X,q,t] = \omega_{q,t}(P_{\mu'}[X,q,t]) = \frac{c_\mu(q,t)}{c'_\mu(q,t)}P_\mu[X,q,t].$$

The claim is proven in [15, Chapter VI (5.1) and (6.19)].

For any partition μ we define the so called *integral form* of the Macdonald polynomials as

$$J_\mu[X,q,t] := c_\mu(q,t)P_\mu[X,q,t] = c'_\mu(q,t)Q_\mu[X,q,t].$$

The J_μ are called integral since Macdonald conjectured that their Schur function expansion

$$J_\mu[X,q,t] = \sum_{\lambda\vdash|\mu|} K_{\lambda,\mu}(q,t)s_\lambda[X(1-t)]$$

yields coefficients $K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$. This is known as the Macdonald positivity conjecture which was proven when Marc Haiman demonstrated the $n!$ conjecture [11]. Furthemore, let us define

$$H_\mu[X,q,t] := J_\mu[\tfrac{X}{1-t},q,t] = \sum_{\lambda\vdash|\mu|} K_{\lambda,\mu}(q,t)s_\lambda[X],$$

and the *modified Macdonald polynomials*

$$\tilde{H}_\mu[X,q,t] = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t)s_\lambda[X] := t^{n(\mu)}J_\mu[\tfrac{X}{1-1/t},q,\tfrac{1}{t}] = t^{n(\mu)}\sum_\lambda K_{\lambda,\mu}(q,\tfrac{1}{t})s_\lambda[X].$$

The following technical results are taken from [5] and will be needed later on.

Lemma 7.7. *Let $n \in \mathbb{N}$ and μ be a partition. Then*

$$\tilde{H}_\mu[1,q,t] = 1, \qquad \langle \tilde{H}_\mu[X,q,t], h_n[X]\rangle = 1, \qquad \langle \tilde{H}_\mu[X,q,t], e_n[X]\rangle = T_\mu(q,t). \quad (7.5)$$

The polynomials $\tilde{H}_\lambda[X,q,t]$ fulfil a version of the (dual) Cauchy Identity. The two rational functions appearing in it are related to $c_\mu(t,q)$ and $c'_\mu(q,t)$ and provide the transition from the polynomials $\tilde{H}_\mu[X,q,t]$ to a dual basis. We denote them by

$$\tilde{h}_\mu(q,t) := \prod_{x\in Y(\mu)} (q^{a_\mu(x)} - t^{l_\mu(x)+1}), \qquad \tilde{h}'_\mu(q,t) := \prod_{x\in Y(\mu)} (t^{l_\mu(x)} - q^{a_\mu(x)+1}).$$

First, note that $\tilde{h}_\mu(q,t)$ and $\tilde{h}'_\mu(q,t)$ form the denominator of (4.1). The observation

$$\tilde{h}'_\mu(q,t) = \tilde{h}_{\mu'}(t,q)$$

proves the symmetry of (4.1) in the variables q and t. Secondly, the substitution $q \mapsto \frac{1}{q}$, $t \mapsto \frac{1}{t}$ will turn out to be useful on several occasions. The next result is preparatory in this regard.

Lemma 7.8. *Let $n \in \mathbb{N}$, and $\mu \vdash n$ be a partition. Then we have the identities*

$$\Pi_\mu(\tfrac{1}{q},\tfrac{1}{t}) = (-1)^{n-1}\frac{\Pi_\mu(q,t)}{T_\mu(q,t)}, \quad (7.6)$$

$$\tilde{h}_\mu(\tfrac{1}{q},\tfrac{1}{t}) = (-1)^n \frac{\tilde{h}_\mu(q,t)}{t^n T_\mu(q,t)}, \qquad \tilde{h}'_\mu(\tfrac{1}{q},\tfrac{1}{t}) = (-1)^n \frac{\tilde{h}'_\mu(q,t)}{q^n T_\mu(q,t)}. \quad (7.7)$$

Proof. Elementary computations show that

$$\begin{aligned} T_\mu(q,t)\Pi_\mu(\tfrac{1}{q},\tfrac{1}{t}) &= \prod_{x\in Y(\mu)} (q^{a'_\mu(x)}t^{l'_\mu(x)} - q^{a'_\mu(x)-a'_\mu(x)}t^{l'_\mu(x)-l'_\mu(x)}) \\ &= (-1)^{n-1} \prod_{x\in Y(\mu)} (1 - q^{a'_\mu(x)}t^{l'_\mu(x)+1}), \end{aligned}$$

and

$$\begin{aligned} t^n T_\mu(q,t)\tilde{h}_\mu(\tfrac{1}{q},\tfrac{1}{t}) &= \prod_{x\in Y(\mu)} (q^{a_\mu(x)-a_\mu(x)}t^{l_\mu(x)+1} - q^{l_\mu(x)}t^{l_\mu(x)+1-l_\mu(x)-1}) \\ &= (-1)^n \prod_{x\in Y(\mu)} (q^{l_\mu(x)} - t^{l_\mu(x)+1}) \\ &= (-1)^n \tilde{h}_\mu(q,t). \end{aligned}$$

The last claim follows analogously, or by symmetry

$$\tilde{h}'_\mu(\tfrac{1}{q}, \tfrac{1}{t}) = \tilde{h}_{\mu'}(\tfrac{1}{t}, \tfrac{1}{q}) = (-1)^n \frac{\tilde{h}_{\mu'}(t,q)}{q^n T_{\mu'}(t,q)} = (-1)^n \frac{\tilde{h}'_\mu(q,t)}{q^n T_\mu(q,t)}.$$

■

Now, let us return to the modified Macdonald polynomials and the promised Cauchy identity.

Theorem 7.9. *Let $n \in \mathbb{N}$. Then we have*

$$e_n[\tfrac{XY}{(1-q)(1-t)}] = \sum_{\mu \vdash n} \frac{(-1)^{n-l(\mu)} p_\mu[X] p_\mu[Y]}{z_\mu p_\mu[(1-q)(1-t)]} = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X,q,t] \tilde{H}_\mu[Y,q,t]}{\tilde{h}_\mu(q,t) \tilde{h}'_\mu(q,t)}. \tag{7.8}$$

Proof. The first identity is immediate from Corollary 6.15. By the definition of the polynomials $Q_\mu[X,q,t]$ and Theorem 7.6 we have

$$h_n[XY \tfrac{1-q}{1-t}] = \sum_{\mu \vdash n} P_\mu[X,q,t] Q_\mu[Y,q,t] = \sum_{\mu \vdash n} \frac{c_\mu(q,t)}{c'_\mu(q,t)} P_\mu[X,q,t] P_\mu[Y,q,t].$$

Now, making the plethystic substitutions $X \to \frac{X}{1-t}$ and $Y \to \frac{Y}{1-t}$, and letting $t \to \frac{1}{t}$ yields

$$h_n[-t \tfrac{XY}{(1-q)(1-t)}] = \sum_{\mu \vdash n} \frac{c_\mu(q,\frac{1}{t})}{c'_\mu(q,\frac{1}{t})} P_\mu[\tfrac{X}{1-1/t}, q, t] P_\mu[\tfrac{Y}{1-1/t}, q, t].$$

It is straight forward to verify

$$t^{n(\mu)} c_\mu(q, \tfrac{1}{t}) = (-t)^n \tilde{h}_\mu(q,t), \qquad t^{n(\mu)} c'_\mu(q, \tfrac{1}{t}) = \tilde{h}'_\mu(q,t).$$

Thus, if we apply this to the definition of the modified Macdonald polynomials we obtain

$$\begin{aligned} \frac{c_\mu(q,\frac{1}{t})}{c'_\mu(q,\frac{1}{t})} P_\mu[\tfrac{X}{1-1/t}, q, t] P_\mu[\tfrac{Y}{1-1/t}, q, t] &= \frac{c_\mu(q,\frac{1}{t})}{c'_\mu(q,\frac{1}{t})} \frac{\tilde{H}_\mu[X,q,t] \tilde{H}_\mu[Y,q,t]}{(t^{n(\mu)})^2 (c_\mu(q,\frac{1}{t}))^2} \\ &= (-t)^n \frac{\tilde{H}_\mu[X,q,t] \tilde{H}_\mu[Y,q,t]}{\tilde{h}_\mu(q,t) \tilde{h}'_\mu(q,t)}. \end{aligned}$$

On the other hand, by Theorem 6.6 the left hand side becomes

$$h_n[-t \tfrac{XY}{(1-q)(1-t)}] = (-t)^n \omega(h_n)[\tfrac{XY}{(1-q)(1-t)}] = (-t)^n e_n[\tfrac{XY}{(1-q)(1-t)}].$$

■

Defining yet another scalar product $\langle .,. \rangle_* : \Lambda_K \times \Lambda_K \to K$ via

$$\langle p_\lambda, p_\mu \rangle_* := \delta_{\lambda,\mu} (-1)^{|\lambda| - l(\lambda)} z_\lambda p_\lambda[(1-q)(1-t)], \tag{7.9}$$

and combining the Cauchy identity with Lemma 7.2 we obtain the following result.

Corollary 7.10. *The two sets*

$$\{\tilde{H}_\mu(X,q,t)\}_\mu, \qquad \left\{\frac{\tilde{H}_\mu(X,q,t)}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}\right\}_\mu$$

are dual bases of Λ_K with respect to the $*$-scalar product in (7.9). ■

The $*$-product has a compatible version of the involution ω. For any $f \in \Lambda_K$ set

$$\omega^*(f) := \omega(f[\tfrac{X}{(1-q)(1-t)}]).$$

Lemma 7.11. *Let $g, h \in \Lambda_K$. Then we have $\langle \omega^*(f), g\rangle_* = \langle f, g\rangle$.* ■

At this point, we define an important linear operator $\nabla : \Lambda_K \to \Lambda_K$ via its values taken at the basis of modified Macdonald polynomials, which will play a prominent role in the proof we aim for.

$$\nabla \tilde{H}_\mu[X,q,t] := T_\mu(q,t)\tilde{H}_\mu[X,q,t].$$

We shall see that applying the ∇-operator to the elementary symmetric function e_n provides the link between the theory of Macdonald polynomials and the Catalan numbers.

Another powerful tool for the manipulation of Macdonald polynomials is the following reciprocity. A special case of it will be the starting point of the proof in the next section.

Theorem 7.12 (Koornwinder–Macdonald-Reciprocity). *Let λ and μ be partitions. Then we have*

$$\frac{\tilde{H}_\mu[u(1-q)(1-t)B_\lambda(q,t),q,t]}{\prod_{x\in Y(\mu)}(1-uq^{a'_\mu(x)}t^{l'_\mu(x)})} = \frac{\tilde{H}_\lambda[u(1-q)(1-t)B_\mu(q,t),q,t]}{\prod_{x\in Y(\lambda)}(1-uq^{a'_\lambda(x)}t^{l'_\lambda(x)})}. \tag{7.10}$$

For a proof the reader is referred to [6] or [15, Chapter VI (6.6)].

By cancelling the common factor $(1-u)^{-1}$ in (7.10) and setting $u = 1$ we obtain another variation of this result.

Corollary 7.13. *Let λ and μ be partitions. Then*

$$\frac{\tilde{H}_\mu[(1-q)(1-t)B_\lambda(q,t),q,t]}{\Pi_\mu(q,t)} = \frac{\tilde{H}_\lambda[(1-q)(1-t)B_\mu(q,t),q,t]}{\Pi_\lambda(q,t)}. \tag{7.11}$$

■

We now turn our attention to a family of identities that have been named Pieri formulas because of their resemblance of a result due to Mario Pieri. The original Pieri formula describes the Schur function expansion of the product of an (abstract) elementary or complete homogeneous symmetric functions and a Schur function.

Theorem 7.14. *Let $n \in \mathbb{N}$ and λ be a partition.*
(i). *Then we have*

$$s_\lambda h_n = \sum_\mu s_\mu,$$

where the sum is taken over all partitions μ such that $\lambda \subseteq \mu$, $|\mu/\lambda| = n$ and μ/λ is a horizontal strip, that is, it has no two cells in the same column.
(ii). *Dually, we have*

$$s_\lambda e_n = \sum_\mu s_\mu,$$

where the sum is taken over all partitions μ such that $\lambda \subseteq \mu$, $|\mu/\lambda| = n$ and μ/λ is a vertical strip, i.e., it has no two cells in the same row.

Proof. This can be obtained from Theorem 6.19. The first claim follows from the definition of the skew Schur functions via the Hall scalar product and the duality of the complete homogeneous and the monomial symmetric functions. The second claim follows as usual by applying the involution ω to the first one. ■

In the last part of this section we present, for the sake of completeness and without too many details, some results about generalised Pieri coefficients due to Garsia and Haglund [4]. These formulas will yield the recursion we need. To begin with, there is a Pieri formula for the modified Macdonald polynomials $\tilde{H}_\lambda(X, q, t)$.

Theorem 7.15. *Let $k, n \in \mathbb{N}$, and $\mu \vdash n$ a partition. Then we have*

$$\tilde{H}_\mu[X] h_k[X\tfrac{1}{1-q}] = \sum_\lambda \frac{\prod_{x\in\mu}\left(t^{l_\mu(x)+\mathbb{X}(x\notin B)} - q^{a_\mu(x)+\mathbb{X}(x\in B)}\right)}{\prod_{x\in\lambda}\left(t^{l_\lambda(x)+\mathbb{X}(x\notin B)} - q^{a_\lambda(x)+\mathbb{X}(x\in B)}\right)} \tilde{H}_\lambda[X]$$

where the sum is taken over all partitions $\lambda \vdash n+k$ such that $\mu \subseteq \lambda$ and λ/μ is a vertical strip, B is the union of the columns which contain a cell of λ/μ and $\mathbb{X}$ equals one or zero depending on whether the argument is true or false.

In [15, Chapter VI (6.7)] a proof is given which is closely connected to that of Theorem 7.12.

We introduce two kinds of *generalised Pieri coefficients* for the modified Macdonald polynomials. For any $f \in \Lambda_K$, let $f^\perp$ denote the adjoint operator to multiplication by f with respect to the Hall scalar product. That is, for all $g, h \in \Lambda_K$ we have

$$\langle fg, h\rangle = \langle g, f^\perp h\rangle.$$

Now, for any two partitions λ, μ we define $c^{f\perp}_{\lambda\mu}, d^{f}_{\lambda\mu} \in K$ as the coefficients arising in the expansions

$$f^{\perp}\tilde{H}_{\lambda}(X,q,t) = \sum_{\mu} c^{f\perp}_{\lambda\mu}(q,t)\tilde{H}_{\mu}(X,q,t),$$
$$f\tilde{H}_{\lambda}(X,q,t) = \sum_{\mu} d^{f}_{\lambda\mu}(q,t)\tilde{H}_{\mu}(X,q,t).$$

The relation between the two kinds of Pieri coefficients is given by

Proposition 7.16. *Let $f \in \Lambda_K$, and λ and μ be partitions. Then we have*

$$\frac{c^{f\perp}_{\lambda\mu}(q,t)}{\tilde{h}_{\mu}(q,t)\tilde{h}'_{\mu}(q,t)} = \frac{d^{\omega^*(f)}_{\mu\lambda}(q,t)}{\tilde{h}_{\lambda}(q,t)\tilde{h}'_{\lambda}(q,t)}.$$

Proof. By Corollary 7.10 and Lemma 7.11 we have

$$\begin{aligned}\frac{c^{f\perp}_{\lambda\mu}}{\tilde{h}_{\mu}(q,t)\tilde{h}'_{\mu}(q,t)} &= \langle f^{\perp}\tilde{H}_{\lambda}, \tilde{H}_{\mu}\rangle_{*} = \langle f^{\perp}\tilde{H}_{\lambda}, \omega^{*-1}\tilde{H}_{\mu}\rangle = \langle \tilde{H}_{\lambda}, f\omega^{*-1}\tilde{H}_{\mu}\rangle \\ &= \langle \tilde{H}_{\lambda}, \omega^{*-1}(\omega^{*}(f)\tilde{H}_{\mu})\rangle = \langle \tilde{H}_{\lambda}, \omega^{*}(f)\tilde{H}_{\mu}\rangle_{*} = \frac{d^{\omega^*(f)}_{\mu\lambda}}{\tilde{h}_{\lambda}(q,t)\tilde{h}'_{\lambda}(q,t)}\end{aligned}$$

■

We define the "translation by one" operator $T_1 : \Lambda_K \to \Lambda_K$ by

$$T_1 f[X] := f[X+1].$$

Furthermore, we let $p_k[\epsilon X] := \omega(p_k[-X]) = (-1)^k p_k[X]$, and define a second translation operator as

$$T_{-\epsilon} f[X] := f[X-\epsilon].$$

Lemma 7.17. *Let λ be a partition. Then we have*

$$T_1 = \sum_{k=0}^{\infty} h_k^{\perp}, \qquad T_{-\epsilon} = \sum_{k=0}^{\infty} e_k^{\perp}.$$

Proof. It suffices to show that the respective operators have the same effect on the Schur functions. From Theorem 6.19 and Corollary 6.17 we deduce

$$T_1 s_{\lambda} = s_{\lambda}[X+1] = \sum_{k=0}^{\lambda_1} s_{\lambda/(k)}[X].$$

By Theorem 7.14 we obtain

$$T_1 s_\lambda = \sum_{k=0}^{\infty} h_k^\perp s_\lambda.$$

The second claim is due to similar arguments. ■

The following is a result from [6].

Theorem 7.18. *Let μ be a partition, and $f \in \Lambda_K$. Then we have*

$$\langle f, \tilde{H}_\mu[X+1,q,t]\rangle_* = \Big(\nabla^{-1}(f[X-\epsilon])\Big)[(1-q)(1-t)B_\mu(q,t)-1].$$

Theorem 7.18 enables the first summation formula for Pieri coefficients in [4, Theorem 3.2].

Theorem 7.19. *Let $k,n \in \mathbb{N}$, $\mu \vdash n$ a partition, and $f \in \Lambda_K^k$ a homogeneous symmetric function. Then we have*

$$\sum_{\nu \vdash n-k,\, \nu \subseteq \mu} c_{\mu\nu}^{f\perp} = \Big((\nabla^{-1} \circ \omega^* \circ T_{-\epsilon}) f[X]\Big)[(1-q)(1-t)B_\mu(q,t)-1].$$

Lemma 7.20. *[4, Proposition 3.1]. Let $k \in \mathbb{N}$, $\lambda, \mu \vdash k$ be partitions, and $f \in \Lambda_K^k$. Suppose $f[X,q,t] = \sum_{\nu\vdash k} c_\nu(q,t) s_\nu[X]$ and set $\tilde{f}[X,q,t] := \sum_{\nu \vdash k} c_\nu(\frac{1}{q},\frac{1}{t}) s_\nu[X]$. Then we have*

$$c_{\lambda\mu}^{f\perp}(\tfrac{1}{q},\tfrac{1}{t}) = \frac{T_\lambda(q,t)}{T_\mu(q,t)} c_{\lambda\mu}^{(\omega\tilde{f})\perp}(q,t).$$

Let $\tilde{\nabla}$ denote the operator on Λ_K defined on the basis consisting of the Schur functions by

$$\tilde{\nabla} s_\lambda := (\nabla s_\lambda)\Big|_{q\to\frac{1}{q},\, t \to \frac{1}{t}}.$$

Lemma 7.21. *[4, Proposition 3.3]. We have $\tilde{\nabla} = \omega \circ \nabla^{-1} \circ \omega$.*

Theorem 7.22. *[4, Theorem 3.3]. Let $k,n \in \mathbb{N}$, $\lambda \vdash n$ a partition, and $g \in \Lambda_K^k$ homogeneous. Let $G[X] := \omega\nabla\Big(g[\frac{X+1}{(1-q)(1-t)}]\Big)$, then*

$$\sum_{\mu \vdash n-k,\, \mu\subseteq\lambda} c_{\lambda\mu}^{(\omega g)\perp}(q,t) T_\mu(q,t) = T_\lambda(q,t) G[(1-\tfrac{1}{q})(1-\tfrac{1}{t})B_\lambda(\tfrac{1}{q},\tfrac{1}{t})-1].$$

The following is a major result of Garsia and Haglund in [4, Theorem I.3.].

Theorem 7.23 (Pieri Summation Formula). *Let $k, n \in \mathbb{N}$, $\lambda \vdash n$ a partition, and $g \in \Lambda_K^k$ a homogeneous symmetric function. Then we have*

$$\sum_{\mu \vdash n+k,\, \mu \supseteq \lambda} d_{\lambda\mu}^g(q,t) T_\mu(q,t) \Pi_\mu(q,t) = T_\lambda(q,t) \Pi_\lambda(q,t) \big(\nabla g\big) [(1-q)(1-t) B_\nu(q,t)].$$

8 The Proof

In this last section we shall start out by treating some interesting results and conjectures relating the q,t-Catalan numbers and the polynomials $D_n(q,t)$ to the $\mathbb{C}\mathfrak{S}_n$-module of diagonal harmonics. Afterwards, we will finally turn to the proof of an identity which implies the symmetry theorem (Theorem 3.8) as a corollary. The main ingredients are the Cauchy Identity (Theorem 7.9), the Koornwinder–Macdonald Reciprocity (Theorem 7.12) and the results on Pieri coefficients from the last chapter, most prominently Theorem 7.23. Another tool is the q-Taylor expansion Theorem 2.2 from the very first section.

Let (V, ρ) be a graded complex representation of the symmetric group $\mathfrak{S}_n$ such that each homogeneous subspace is finite dimensional. We define the *Frobenius series* of (V, ρ) as

$$\operatorname{Frob}(V; q) := \sum_{k=0}^{\infty} q^k \sum_{\lambda \vdash n} \langle V_\lambda, V^k \rangle s_\lambda,$$

where $\langle V_\lambda, V^k \rangle$ is the multiplicity of the irreducible representation of $\mathfrak{S}_n$ corresponding to the partition λ in the finite dimensional representation V^k. By equating the Schur coefficient $\langle \operatorname{Frob}(V;q), s_\lambda \rangle$ we obtain the Hilbert series of the representation $W \leq V$ which is the isotypic component of the irreducible representation V_λ in V. Furthermore, it can be shown that the dimension of V_λ equals the coefficient $\langle s_\lambda, h_{1^n} \rangle$. Thus,

$$\langle \operatorname{Frob}(V;q), h_{1^n} \rangle = \operatorname{Hilb}_{\mathbb{C}}(V;q).$$

Given a bigraded representation $V = \bigoplus_{i,j=0}^{\infty} V^{i,j}$ we define a bivariate Frobenius series

$$\operatorname{Frob}(V; q, t) := \sum_{i,j=0}^{\infty} q^i t^j \sum_{\lambda \vdash n} \langle V_\lambda, V^{i,j} \rangle s_\lambda.$$

Using methods from algebraic geometry Haiman showed in [12] the following identity for the bivariate Frobenius series of the space of diagonal harmonics

$$\operatorname{Frob}(\mathcal{DH}_n; q, t) = \sum_{\mu \vdash n} \frac{(T_\mu(q,t))^2 (1-q)(1-t) \tilde{H}_\mu[X,q,t] \Pi_\mu(q,t) B_\mu(q,t)}{\tilde{h}_\mu(q,t) \tilde{h}'_\mu(q,t)}. \tag{8.1}$$

This series has close connections to some of the combinatorial generating functions we considered in Chapter 3. First, the q,t-Catalan numbers correspond to the alternating component $\mathcal{DH}_n^\epsilon$ of the diagonal harmonic polynomials, i.e.

$$C_n(q,t) = \langle \mathrm{Frob}(\mathcal{DH}_n; q,t), s_{1^n}\rangle = \mathrm{Hilb}_{\mathbb{C}}(\mathcal{DH}_n^\epsilon; q,t). \tag{8.2}$$

We are going to prove

$$C_n(q,t) = \sum_{\mu\vdash n} \frac{(T_\mu(q,t))^2(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)} \tag{8.3}$$

below, thus (8.2) will follow assuming (8.1) is given. Secondly, there is a conjectured combinatorial formula for the Hilbert series of the diagonal harmonics given by the generalised q,t-Catalan numbers defined in (3.1) using parking functions

$$D_n(q,t) = \langle \mathrm{Frob}(\mathcal{DH}_n; q,t), h_{1^n}\rangle = \mathrm{Hilb}_{\mathbb{C}}(\mathcal{HD}_n; q,t). \tag{8.4}$$

Our approach to the proof of (8.3) is to provide a recursive structure of the rational function

$$\langle \nabla e_n[X\tfrac{1-q^n}{1-q}], s_{1^n}\rangle \in \mathbb{Q}(q,t)$$

similar to the recursion in Theorem 3.5.

We start out with some auxilliary results. The first one is not hard to verify.

Lemma 8.1. *Let $k\in\mathbb{N}$. Then we have*

$$(-z;q)_k = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q q^{\binom{i}{2}} z^i.$$

Our next lemma is proven in [15, Chapter VI (4.8)].

Lemma 8.2. *Let $k\in\mathbb{N}$ and $\mu=(k,0,\dots)$ be the partition of k of length one. Then*

$$\tilde{H}_\mu[X,q,t] = (q;q)_k h_k[\tfrac{X}{1-q}]. \tag{8.5}$$

Moreover, we need the following result from [5].

Proposition 8.3. *Let μ be a partition. Then*

$$\omega(\tilde{H}_\mu[X,\tfrac{1}{q},\tfrac{1}{t}]) = \frac{\tilde{H}_\mu[X,q,t]}{T_\mu(q,t)}. \tag{8.6}$$

We will now use the Macdonald Reciprocity to prove a specialisation taken from [4].

Lemma 8.4. *Let $k \in \mathbb{N}$, and μ be a partition. Then we have*

$$\tilde{H}_\mu[(1-t)(1-q^k), q, t] = \Pi_\mu(q,t) h_k[(1-t)B_\mu(q,t)](1-q^k). \tag{8.7}$$

Proof. Let $\lambda = (k)$ be the partition of k with one part. It is easily checked that

$$B_{(k)} = 1+q+\cdots+q^{k-1} = \frac{1-q^k}{1-q} \quad \text{and} \quad \Pi_{(k)} = (1-q)(1-q^2)\cdots(1-q^{k-1}) = (q;q)_{k-1}.$$

Then Corollary 7.13 becomes

$$\frac{\tilde{H}_\mu[(1-t)(1-q^k), q, t]}{\Pi_\mu(q,t)} = \frac{\tilde{H}_{(k)}[(1-q)(1-t)B_\mu(q,t), q, t]}{(q;q)_{k-1}},$$

and the claim follows immediately from Lemma 8.2. ■

Combining Lemma 8.4 with the Cauchy Identity for the modified Macdonald polynomials we are able to give a first alternative expression for the Schur coefficient of the ∇e_n.

Theorem 8.5. *Let $k, n \in \mathbb{N}$. Then we have the identities*

$$\langle \nabla e_n[X\tfrac{1-q^k}{1-q}], s_{1^n}\rangle = (1-q^k)\sum_{\mu\vdash n} \frac{(T_\mu(q,t))^2 \Pi_\mu(q,t) h_k[(1-t)B_\mu(q,t)]}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}, \tag{8.8}$$

$$\langle \nabla h_n[X\tfrac{1-q^k}{1-q}], s_{1^n}\rangle = q^{k(n-1)}(1-q^k)(-t)^{n-k}\sum_{\mu\vdash n} \frac{(T_\mu(q,t))^2 \Pi_\mu(q,t) e_k[(1-t)B_\mu(\frac{1}{q},\frac{1}{t})]}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}. \tag{8.9}$$

Proof. First we let $Y = (1-t)(1-q^k)$ in Theorem 7.9 to obtain

$$e_n[X\tfrac{1-q^k}{1-q}] = \sum_{\mu\vdash n} \frac{\tilde{H}_\mu[X,q,t]\tilde{H}_\mu[(1-t)(1-q^k),q,t]}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}.$$

Using Lemma 8.4 we get

$$e_n[X\tfrac{1-q^k}{1-q}] = (1-q^k)\sum_{\mu\vdash n} \frac{\tilde{H}_\mu[X,q,t]\Pi_\mu(q,t)h_k[(1-t)B_\mu(q,t)]}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}. \tag{8.10}$$

Applying the nabla operator and equating the coefficient of s_{1^n} on both sides, (8.8) follows from Lemma 7.7.

To deduce (8.9), we substitute $q \to \frac{1}{q}$ and $t \to \frac{1}{t}$ in (8.10), and apply ω to both sides. On the one hand we have

$$\omega\left(e_n[X\tfrac{1-(1/q)^k}{1-(1/q)}]\right) = h_n[X\tfrac{1}{q^{k-1}}\tfrac{1-q^k}{1-q}] = \frac{1}{q^{n(k-1)}}h_n[X\tfrac{1-q^k}{1-q}].$$

On the other hand, using Lemma 7.8 and Proposition 8.3, we obtain

$$\begin{aligned}\omega\left(e_n[X\tfrac{1-(1/q)^k}{1-(1/q)}]\right) &= (1-\tfrac{1}{q^k})\sum_{\mu\vdash n}\frac{\omega(\tilde{H}_\mu[X,\frac{1}{q},\frac{1}{t}])\Pi_\mu(\frac{1}{q},\frac{1}{t})h_k[(1-\frac{1}{t})B_\mu(\frac{1}{q},\frac{1}{t})]}{\tilde{h}_\mu(\frac{1}{q},\frac{1}{t})\tilde{h}'_\mu(\frac{1}{q},\frac{1}{t})}\\ &= (1-\tfrac{1}{q^k})\sum_{\mu\vdash n}\frac{\frac{\tilde{H}_\mu[X,q,t]}{T_\mu(q,t)}(-1)^{n-1}\frac{\Pi_\mu(q,t)}{T_\mu(q,t)}h_k[(1-\frac{1}{t})B_\mu(\frac{1}{q},\frac{1}{t})]}{(-1)^n\frac{\tilde{h}_\mu(q,t)}{t^nT_\mu(q,t)}(-1)^n\frac{\tilde{h}'_\mu(q,t)}{q^nT_\mu(q,t)}}.\end{aligned}$$

Finally, by Theorem 6.6 we have

$$h_k[(1-\tfrac{1}{t})B_\mu(\tfrac{1}{q},\tfrac{1}{t})] = \frac{1}{t^k}h_k[-(1-t)B_\mu(\tfrac{1}{q},\tfrac{1}{t})] = \frac{1}{t^k}(-1)^k e_k[(1-t)B_\mu(\tfrac{1}{q},\tfrac{1}{t})].$$

All this amounts to

$$\begin{aligned}h_n[X\tfrac{1-q^k}{1-q}] &= q^{n(k-1)}(1-\tfrac{1}{q^k})q^nt^n(-1)^{n-1}\frac{(-1)^k}{t^k}\sum_{\mu\vdash n}\frac{\tilde{H}_\mu[X,q,t]\Pi_\mu(q,t)e_k[(1-t)B_\mu(\frac{1}{q},\frac{1}{t})]}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}\\ &= q^{(n-1)k}(1-q^k)(-t)^{n-k}\sum_{\mu\vdash n}\frac{\tilde{H}_\mu[X,q,t]\Pi_\mu(q,t)e_k[(1-t)B_\mu(\frac{1}{q},\frac{1}{t})]}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)},\end{aligned}$$

and the theorem follows by applying the nabla operator and equating the Schur coefficient as before. ■

Theorem 8.6 (A First Recursion). *Let $d, n \in \mathbb{N}$ such that $1 \le d \le n$. Then we have*

$$\begin{aligned}&q^{\binom{d}{2}}\langle\nabla e_{n-d}[X\tfrac{1-q^d}{1-q}], s_{1^{n-d}}\rangle\\ &\quad = \frac{q^d}{t^{n-d}}\sum_{i=1}^{d}\begin{bmatrix}d\\ i\end{bmatrix}_q\frac{t^{n-i}}{q^i}q^{\binom{i}{2}}(1-q^i)\sum_{\mu\vdash n}\frac{(T_\mu(q,t))^2\Pi_\mu(q,t)}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}e_i[(1-t)B_\mu(\tfrac{1}{q},\tfrac{1}{t})].\end{aligned} \tag{8.11}$$

In [4, Chapter 4] this is deduced from Theorem 8.5 by means of the Pieri summation formulas in the previous section.

Corollary 8.7. *Let $d, n \in \mathbb{N}$ such that $1 \le d \le n$. Then*

$$\sum_{k=0}^{d}\begin{bmatrix}d\\ k\end{bmatrix}_q\frac{q^{\binom{d}{2}}}{q^{kn}}(-1)^{n-k}\langle\nabla h_n[X\tfrac{1-q^k}{1-q}], s_{1^n}\rangle = \frac{t^{n-d}}{q^d}q^{\binom{d}{2}}\langle\nabla e_{n-d}[X\tfrac{1-q^d}{1-q}], s_{1^{n-d}}\rangle. \tag{8.12}$$

Proof. This follows immediately after inserting equation (8.9) from Theorem 8.5 in the right hand side of (8.11) from Theorem 8.6. We obtain

$$\begin{aligned} q^{\binom{d}{2}}\langle \nabla e_{n-d}[X\tfrac{1-q^d}{1-q}], s_{1^{n-d}}\rangle &= \frac{q^d}{t^{n-d}}\sum_{k=1}^{d}\begin{bmatrix} d\\ k\end{bmatrix}_q \frac{t^{n-k}}{q^k} q^{\binom{k}{2}}(1-q^k)\frac{\langle \nabla h_n[X\tfrac{1-q^k}{1-q}], s_{1^n}\rangle}{q^{k(n-1)}(1-q^k)(-t)^{n-k}} \\ &= \frac{q^d}{t^{n-d}}\sum_{k=1}^{d}\begin{bmatrix} d\\ k\end{bmatrix}_q \frac{(-1)^{n-k}}{q^{kn}} q^{\binom{k}{2}}\langle \nabla h_n[X\tfrac{1-q^k}{1-q}], s_{1^n}\rangle. \end{aligned}$$

■

Our next aim is to simplify the notation by eliminating the sum in the left hand side of Corollary 8.7. Recall the q-difference operator δ_q defined by $f(z) \mapsto \frac{1}{z}(f(z) - f(z/q))$. We may rewrite this as

$$\delta_q = \frac{1}{z}(1 - S_q),$$

where S_q denotes the *q-shift operator* given by $f(z) \mapsto f(\frac{z}{q})$.

Theorem 8.8. *Let $d, n \in \mathbb{N}$ such that $1 \le d \le n$. Then we have*

$$\delta_q^d \langle \nabla e_n[X\tfrac{1-z}{1-q}], s_{1^n}\rangle\Big|_{z=1} = \frac{t^{n-d}}{q^d} q^{\binom{d}{2}}\langle \nabla e_{n-d}[X\tfrac{1-q^d}{1-q}], s_{1^{n-d}}\rangle. \tag{8.13}$$

Proof. First, we notice that for any two partitions μ, ν and a polynomial $f(q,t,z) \in \mathbb{Q}(q,t)[z]$ we have

$$\begin{aligned} \langle \nabla \Big(f(q,t,z) s_\mu[X]\Big), s_\nu[X]\rangle &= f(q,t,z)\langle \nabla s_\mu[X], s_\nu[X]\rangle, \quad \text{and} \\ \delta_q \Big(f(q,t,z) s_\mu[X]\Big)\Big|_{z=1} &= s_\mu[X]\ \delta_q f(q,t,z)|_{z=1}. \end{aligned}$$

It follows that

$$\delta_q^d \langle \nabla e_n[X\tfrac{1-z}{1-q}], s_{1^n}\rangle\Big|_{z=1} = \langle \nabla \left(\delta_q^d e_n[X\tfrac{1-z}{1-q}]\Big|_{z=1}\right), s_{1^n}\rangle.$$

Since S_q is $\mathbb{Q}(q,t)$ linear, and $S_q \frac{1}{z} = \frac{1}{z} q S_q$, we have

$$\delta_q^d = \left(\frac{1}{z}(1-S_q)\right)^d = \frac{1}{z^d}(1-S_q)(1-qS_q)\cdots(1-q^{d-1}S_q) = \frac{1}{z^d}(S_q;q)_d,$$

and hence, by Lemma 8.1,

$$\delta_q^d = \frac{1}{z^d}\sum_{i=0}^{d}\begin{bmatrix} d\\ i\end{bmatrix}_q q^{\binom{i}{2}}(-1)^i (S_q)^i.$$

But then we compute

$$\begin{aligned}\delta_q^d e_n[X\tfrac{1-z}{1-q}]\Big|_{z=1} &= \sum_{i=0}^{d}\begin{bmatrix}d\\ i\end{bmatrix}_q q^{\binom{i}{2}}(-1)^i e_n[X\tfrac{1-1/q^i}{1-q}]\\ &= \sum_{i=0}^{d}\begin{bmatrix}d\\ i\end{bmatrix}_q q^{\binom{i}{2}}(-1)^i e_n[-X\tfrac{1}{q^i}\tfrac{1-q^i}{1-q}]\\ &= \sum_{i=0}^{d}\begin{bmatrix}d\\ i\end{bmatrix}_q \frac{q^{\binom{i}{2}}}{q^{in}}(-1)^{n-i}h_n[X\tfrac{1-q^i}{1-q}],\end{aligned}$$

and the claim follows from Corollary 8.7 after applying the nabla operator and equating the coefficient of s_{1^n} on both sides. ∎

As indicated above, we will now use Theorem 2.2 to derive the desired recursion.

Theorem 8.9 (The Right Recursion). *Let $k, n \in \mathbb{N}$. Then we have the recursion*

$$\langle \nabla e_n[X\tfrac{1-q^k}{1-q}], s_{1^n}\rangle = \sum_{d=1}^{n}\begin{bmatrix}k+d-1\\ d\end{bmatrix}_q t^{n-d}q^{\binom{d}{2}}\langle \nabla e_{n-d}[X\tfrac{1-q^d}{1-q}], s_{1^{n-d}}\rangle. \tag{8.14}$$

Proof. Since

$$\begin{bmatrix}k+d-1\\ d\end{bmatrix}_q = \frac{(q^k;q)_d}{(q;q)_d},$$

we may rewrite (8.14) as

$$\langle \nabla e_n[X\tfrac{1-q^k}{1-q}], s_{1^n}\rangle = \sum_{d=1}^{n}\frac{(q^k;q)_d}{(q;q)_d}t^{n-d}q^{\binom{d}{2}}\langle \nabla e_{n-d}[X\tfrac{1-q^d}{1-q}], s_{1^{n-d}}\rangle. \tag{8.15}$$

Since this equality holds for all $k \in \mathbb{N}$ the substitution $q^k \to z$ yields

$$\langle \nabla e_n[X\tfrac{1-z}{1-q}], s_{1^n}\rangle = \sum_{d=1}^{n}(z;q)_d\frac{q^d}{(q;q)_d}\frac{t^{n-d}}{q^d}q^{\binom{d}{2}}\langle \nabla e_{n-d}[X\tfrac{1-q^d}{1-q}], s_{1^{n-d}}\rangle. \tag{8.16}$$

which is an equivalent polynomial identity in z of degree n. But now, by Theorem 2.2 we have

$$\langle \nabla e_n[X\tfrac{1-q^k}{1-q}], s_{1^n}\rangle = \sum_{d=1}^{n}(z;q)_d\frac{q^d}{(q;q)_d}\left(\delta_q^d\langle \nabla e_n[X\tfrac{1-z}{1-q}], s_{1^n}\rangle\right)\Big|_{z=1}. \tag{8.17}$$

Comparing the coefficients of $(z;q)_d\frac{q^d}{(q;q)_d}$ in the right sides of (8.16) and (8.17), the claim follows from Theorem 8.8. ∎

Theorem 8.10. *Let $n \in \mathbb{N}$. Then*

$$\langle \nabla e_n, s_{1^n} \rangle = C_n(q,t).$$

Proof. By Corollary 3.7 we have $C_n(q,t) = t^{-n}F(n+1,1;q,t)$, where the polynomials $F(n,k;q,t)$ fulfil the recursion

$$F(n,k;q,t) = q^{\binom{n}{2}} t^{n-k} \sum_{d=1}^{n-k} F(n-k,d;q,t) \begin{bmatrix} k+d-1 \\ d \end{bmatrix}_q .$$

Now, let $Q(n,k;q,t) := q^{\binom{k}{2}} t^{n-k} \langle \nabla e_{n-k}[X\frac{1-q^k}{1-q}], s_{1^{n-k}} \rangle$. Then by Theorem 8.9 we have

$$q^{\binom{k}{2}} t^{n-k} \sum_{d=1}^{n-k} Q(n-k,d;q,t) \begin{bmatrix} k+d-1 \\ d \end{bmatrix}_q \tag{8.18}$$

$$= q^{\binom{k}{2}} t^{n-k} \sum_{d=1}^{n-k} \begin{bmatrix} k+d-1 \\ d \end{bmatrix}_q q^{\binom{d}{2}} t^{n-k-d} \langle \nabla e_{n-k-d}[X\tfrac{1-q^d}{1-q}], e_{n-k-d} \rangle \tag{8.19}$$

$$= q^{\binom{k}{2}} t^{n-k} \langle \nabla e_{n-k}[X\tfrac{1-q^k}{1-q}], s_{1^{n-k}} \rangle = Q(n,k;q,t). \tag{8.20}$$

Since F and Q fulfill the same initial conditions $F(n,0;q,t) = Q(n,0;q,t) = \delta_{n,0}$ they must be equal. But then we have

$$C_n(q,t) = t^{-n}Q(n+1,1;q,t) = \langle \nabla e_n[X], s_{1^n} \rangle.$$

■

Theorem 8.11. *Let $n \in \mathbb{N}$. Then*

$$C_n(q,t) = \sum_{\mu \vdash n} \frac{(1-q)(1-t)(T_\mu(q,t))^2 \Pi_\mu(q,t) B_\mu(q,t)}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}.$$

Proof. This now follows immediately from Theorem 8.10 and Theorem 8.5 by letting $k = 1$ in (8.8). ■

Thus, we have finally succeeded in establishing (8.3). Theorem 3.8 and equation (8.2) follow accordingly.

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1994 bis 1998:	Besuch der Volkschule Dr.-Albert-Geßmann-Gasse.
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