DISSERTATION

Titel der Dissertation

Catalan combinatorics of crystallographic root systems

Verfasser

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CONTENTS

SUMMARY

The present thesis contains four main contributions to the Catalan combinatorics of crystallographic root systems.

The first is a uniform bijection $\mathcal{A}$ that generalises the bijection $\mathcal{A}_{GMV}$ defined by Gorsky, Mazin and Vazirani for the affine symmetric group in their study of a rational generalisation of the Hilbert series of the space of diagonal harmonics.

The second is a uniform bijection ζ that generalises the bijection ζ_{HL} defined by Haglund and Loehr, also in the context of diagonal harmonics.

The third is a proof of a conjecture of Armstrong relating floors and ceilings of the m-Shi arrangement. In this proof, a bijection that provides more refined enumerative information is introduced.

The fourth is a proof of the $H = F$ correspondence, originally conjectured by Chapoton and then generalised by Armstrong. The $H = F$ correspondence describes a way of transforming a refined enumeration of the faces of the m-cluster complex $\mathsf{Assoc}_{\Phi}^{(m)}$ to a refined enumeration of the set $\mathsf{NN}_{\Phi}^{(m)}$ of m-generalised nonnesting partitions by means of an invertible change of variables. The proof uses a uniform bijection together with a case-by-case verification.

ZUSAMMENFASSUNG

Die vorliegende Dissertation enthält vier hauptsächliche Beiträge zur Catalan-Kombinatorik der kristallographischen Wurzelsysteme.

Der erste ist eine einheitliche Anderson-Abbildung $\mathcal{A}$ die die kombinatorische Anderson-Abbildung $\mathcal{A}_{GMV}$ verallgemeinert. Die kombinatorische Anderson-Abbildung wurde von Gorsky, Mazin und Vazirani für die affine symmetrische Gruppe definiert um eine rationale Verallgemeinerung der Hilbertreihe des Raumes der diagonalen harmonischen Polynome zu erhalten.

Der zweite ist eine einheitliche Zeta-Abbildung ζ die die kombinatorische Zeta-Abbildung ζ_{HL} verallgemeinert. Die kombinatorische Zeta-Abbildung wurde von Haglund und Loehr definiert, auch im Kontext der diagonalen harmonischen Polynome.

Der dritte ist der Beweis einer Vermutung von Armstrong die die Böden und die Decken von dominanten Regionen des m-Shi-Gefüges in Verbindung setzt. In dem Beweis wird eine Bijektion eingeführt die noch feinere Abzählungen ermöglicht.

Der vierte ist der Beweis der $H = F$ Korrespondenz. Diese wurde zuerst von Chapoton vermutet und später durch Armstrong verallgemeinert. Sie beschreibt eine invertierbare Variablensubstitution die eine verfeinerte Abzählung der Seiten des Cluster-Komplexes $\mathsf{Assoc}_\Phi^{(m)}$ in eine verfeinerte Abzählung der Menge $\mathsf{NN}_\Phi^{(m)}$ der m-nichtschachtelnden Partitionen umwandelt. Der Beweis verwendet eine einheitliche Bijektion sowie eine Fallunterscheidung.

1 INTRODUCTION

1.1 Catalan combinatorics

1.1.1 *Catalan numbers*

One of the most famous number sequences in combinatorics is the sequence of *Catalan numbers* $1, 1, 2, 5, 14, 42, 132, \dots$ given by the formula

$$\mathsf{Cat}_n := \frac{1}{2n+1}\binom{2n+1}{n}.$$

A vast variety of combinatorial objects are counted by the Catalan numbers and may thus justly be called *Catalan objects*. Many of them have been collected by Stanley [Sta]. Some Catalan objects are

(Assoc) triangulations of a convex $(n+2)$-gon,

(NC) noncrossing partitions of $[n] := \{1, 2, \dots, n\}$,

(NN) nonnesting partitions of $[n]$, and

($\check{Q}$) increasing parking functions of length $[n]$.

1.1.2 *Coxeter-Catalan numbers*

In algebraic combinatorics, a common theme is to take combinatorial objects and view them as emerging from or sitting inside some algebraic structure. Doing this may reveal further structure or suggest possible generalisations.

In our case, the four Catalan objects mentioned in Section 1.1.1 may be seen as objects associated with the symmetric group S_n and its root system, which is of Dynkin type A_{n-1}. This makes it possible to generalise each of them to all irreducible crystallographic root systems Φ. For background on crystallographic root systems refer to Chapter 2. These generalisations are

(Assoc) maximal sets of pairwise compatible almost positive roots,

(NC) minimal factorisations of a *Coxeter element* c into two *Weyl group* elements,

(NN) order filters in the root poset of Φ, and

($\check{Q}$) orbits of the action of the Weyl group W on the *finite torus* $\check{Q}/(h+1)\check{Q}$.

Surprisingly, the Catalan numbers survive these generalisations: each of these objects is counted by the same number Cat_Φ, the *Coxeter-Catalan number* of Φ. It is defined as

$$\mathsf{Cat}_\Phi := \frac{1}{|W|}\prod_{i=1}^{r}(h+1+e_i).$$

Here r is the *rank* of Φ, W is its Weyl group, h is its *Coxeter number* and $e_1, e_2, \dots, e_r$ are its *exponents*. When Φ is of Dynkin type A_{n-1}, the Coxeter-Catalan number Cat_Φ equals the classical Catalan number Cat_n.

1.1.3 *Fuß-Catalan numbers*

Further generalisations of Coxeter-Catalan objects are found by introducing a positive integer m as a *Fuß parameter*. This gives rise to the following *Fuß-Catalan objects*:

(Assoc) maximal sets of pairwise compatible m-coloured almost positive roots,

(NC) minimal factorisations of a Coxeter element c into $(m+1)$ Weyl group elements,

(NN) geometric chains of m order filters in the root poset of Φ, and

($\check{Q}$) W-orbits of the finite torus $\check{Q}/(mh+1)\check{Q}$.

All of these are counted by the *Fuß-Catalan number*

$$\mathsf{Cat}_{\Phi}^{(m)} := \frac{1}{|W|}\prod_{i=1}^{r}(mh+1+e_i).$$

They specialise to the corresponding Coxeter-Catalan objects in the case where $m=1$.

If Φ is of type A_{n-1}, the Fuß-Catalan number $\mathsf{Cat}_{\Phi}^{(m)}$ equals the *classical Fuß-Catalan number*

$$\mathsf{Cat}_{n}^{(m)} := \frac{1}{(m+1)n+1}\binom{(m+1)n+1}{n}.$$

One underlying philosophy of the field of Fuß-Catalan combinatorics is that *uniform theorems ask for uniform proofs*. That is, when a statement holds for all irreducible crystallographic root systems, one should try to prove it without appealing to their classification.

Given this philosophy, it is helpful to divide the Fuß-Catalan objects into two different worlds: the *noncrossing world* containing (Assoc) and (NC) and the *nonnesting world* containing (NN) and ($\check{Q}$). There are uniform bijections known between the objects within each world, but none between objects of different worlds.

Each world has its own unique flavour: in the noncrossing world, there are *Cambrian recurrences* and generalisations to noncrystallographic root systems [STW15]. On the other hand, in the nonnesting world there is a *uniform* proof of the fact that the Fuß-Catalan objects are indeed counted by the Fuß-Catalan number $\mathsf{Cat}_{\Phi}^{(m)}$. Our focus in this thesis will be on the nonnesting world.

1.1.4 *Rational Catalan numbers*

A further generalisation of Fuß-Catalan numbers are *rational Catalan numbers*. For any irreducible crystallographic root system Φ and a positive integer p relatively prime to the Coxeter number h of Φ define the rational Catalan number

$$\mathsf{Cat}_{p/\Phi} := \frac{1}{|W|}\prod_{i=1}^{r}(p+e_i).$$

It reduces to the Fuß-Catalan number $\mathsf{Cat}_{\Phi}^{(m)}$ when $p=mh+1$. The rational Catalan numbers count

($\check{Q}$) W-orbits of the finite torus $\check{Q}/p\check{Q}$.

For the (Assoc), (NC) and (NN) Fuß-Catalan objects there is no satisfactory generalisation to the rational Catalan level yet. However, (Assoc) and (NC) rational Catalan objects have been proposed for type A_{n-1} [ARW13].

If Φ is of type A_{n-1}, the rational Catalan number $\mathsf{Cat}_{p/\Phi}$ equals the *rational (p/n)-Catalan number*

$$\mathsf{Cat}_{p/n} := \frac{1}{n+p}\binom{n+p}{n}.$$

1.2 DIAGONAL HARMONICS

A different generalisation of the Catalan numbers are the (q,t)-Catalan numbers $\mathsf{Cat}_n(q,t)$. They are polynomials in q and t that were defined by Garsia and Haiman as the bigraded Hilbert series of the space of diagonal harmonic alternants of the symmetric group S_n [GH96]. There are two equivalent combinatorial interpretations of them: we have

$$\mathsf{Cat}_n(q,t) = \sum_P q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} = \sum_P q^{\mathrm{area}(P)} t^{\mathrm{bounce}(P)},$$

where both sums are over all *Dyck paths* P of length n and dinv, area and bounce are three statistics on Dyck paths [Hag08, Corollary 3.2.1]. There is a bijection ζ_H on Dyck paths due to Haglund that sends dinv to area and area to bounce, proving the second equality [Hag08, Theorem 3.15]. In particular we have the specialisation $\mathsf{Cat}_n(1,1) = \mathsf{Cat}_n$.

It is natural to also consider the Hilbert series $\mathcal{DH}_n(q,t)$ of the space of *all* diagonal harmonics of S_n. This also has two equivalent (conjectural) combinatorial interpretations: we have

$$\mathcal{DH}_n(q,t) = \sum_{(P,\sigma)\in\mathcal{PF}_n} q^{\mathrm{dinv}'(P,\sigma)} t^{\mathrm{area}(P,\sigma)} = \sum_{(w,D)\in\mathcal{D}_n} q^{\mathrm{area}'(w,D)} t^{\mathrm{bounce}(w,D)},$$

where $\mathcal{PF}_n$ is the set of *parking functions* of length n viewed as *vertically labelled Dyck paths* and $\mathcal{D}_n$ is the set of *diagonally labelled Dyck paths* of length n. There is a bijection ζ_{HL} due to Haglund and Loehr [Hag08, Theorem 5.6] that maps $\mathcal{PF}_n$ to $\mathcal{D}_n$ and sends the bistatistic $(\mathrm{dinv}', \mathrm{area})$ to $(\mathrm{area}', \mathrm{bounce})$, demonstrating the second equality.

Armstrong has given a third equivalent (conjectural) combinatorial interpretation of $\mathcal{DH}_n(q,t)$ as a sum over the *Shi alcoves* of the root system of type A_{n-1} [Arm13]. This allowed him to generalise to the Fuß-Catalan level by defining a similar combinatorial Hilbert series as a sum over *m-Shi alcoves*. This connects to the nonnesting Fuß-Catalan combinatorics, since there is a natural bijection between geometric chains of m order filters in the root poset and *dominant* m-Shi alcoves [Ath05].

Gorsky, Mazin and Vazirani generalised further to the rational Catalan level by defining an analogous combinatorial Hilbert series as a sum over the set $\widetilde{S}_n^p$ of *p-stable affine permutations* for any p relatively prime to n [GMV14]. When $p = mn+1$, these correspond to the m-Shi alcoves. In their construction, they defined the *Anderson map* $\mathcal{A}_{GMV}$ as a bijection from $\widetilde{S}_n^p$ to the set of *rational* p/n*-parking functions* $\mathcal{PF}_{p/n}$.

1.3 STRUCTURE OF THE THESIS

The thesis is structured as follows.

In Chapter 2 we give an overview of the theory of crystallographic root systems and their Weyl groups. We also introduce the corresponding affine root systems and affine Weyl groups. All the material in this chapter is well-known, so the expert reader may choose to skip it and only refer to it as needed.

In Chapter 3 we generalise the Anderson map $\mathcal{A}_{GMV}$ of Gorsky, Mazin and Vazirani to an Anderson map $\mathcal{A}$ that is defined uniformly for all irreducible crystallographic root systems. It is a bijection from the set $\widetilde{W}^p$ of *p-stable affine Weyl group elements* to the finite torus $\check{Q}/p\check{Q}$.

In Chapter 4 we introduce the *m-Shi arrangement*. This hyperplane arrangement is central to this thesis and to the field of nonnesting Fuß-Catalan combinatorics as a whole. We use the Anderson map $\mathcal{A}$ in the special case where $p = mh+1$ to recover a number of known enumerative properties of it.

In Chapter 5 we use the Anderson map $\mathcal{A}$ in the case where $p = mh + 1$ to obtain a uniform generalisation of the *zeta map* ζ_{HL} of Haglund and Loehr to all irreducible crystallographic root systems and to the Fuß-Catalan level of generality.

In Chapter 6 we prove a conjecture of Armstrong [Arm09, Conjecture 5.1.24] relating floors and ceilings of dominant regions of the m-Shi arrangement. We do this by introducing a bijection that provides even more refined enumerative information.

In Chapter 7 we prove the $H = F$ correspondence, a close enumerative correspondence between the (NN) and (Assoc) Fuß-Catalan objects that was originally conjectured by Chapoton at the Coxeter-Catalan level of generality [Cha06, Conjecture 6.1] and later generalised to the Fuß-Catalan level by Armstrong [Arm09, Conjecture 5.3.1].

2 BASIC NOTIONS

In this chapter we provide an introduction to crystallographic root systems, their Weyl groups and associated hyperplane arrangements. All the content of this chapter is well-known. A good reference is [Hum90].

2.1 ROOT SYSTEMS AND REFLECTION GROUPS

In the structure theory of Lie algebras, crystallographic root systems naturally arise as the sets of weights of the adjoint representation of a semisimple complex Lie algebra. But they are also interesting objects to study in their own right, and can be defined abstractly without referring to Lie algebras as follows.

2.1.1 *Definition of a root system*

Let V be a finite dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$. A *root system* is a finite set $\Phi \subseteq V$ of nonzero vectors (called *roots*) such that:

1. $\mathbb{R}\Phi = V$,
2. For $\alpha \in \Phi$, we have $\Phi \cap \mathbb{R}\{\alpha\} = \{\alpha, -\alpha\}$, and
3. For $\alpha, \beta \in \Phi$, we have $\beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \in \Phi$.

We call a root system *crystallographic* if it also satisfies

4. For $\alpha, \beta \in \Phi$, we have $2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

The most interesting property here is the third: For $\alpha \in \Phi$, consider the linear endomorphism of V defined by

$$s_\alpha : x \mapsto x - 2\frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

This is the reflection through the linear hyperplane

$$H_\alpha := \{x \in V : \langle x, \alpha \rangle = 0\}.$$

In particular it is an orthogonal transformation of V and the third property of root systems just says that it permutes the roots in Φ. Let $T := \{s_\alpha : \alpha \in \Phi\}$ be the set of all reflections through linear hyperplanes orthogonal to the roots in Φ. We are led to consider the *reflection group* $W := \langle T \rangle$ generated by all these reflections. It acts on the root system Φ.

In this thesis, we restrict our attention to crystallographic root systems. In that case, we call W the *Weyl group* of Φ.

Example. Let n be a positive integer and let e_i be the i-th vector in the standard basis for $\mathbb{R}^n$ for $i \in [n]$. The crystallographic root system of type A_{n-1} is defined as the set of roots

$$\Phi := \{e_i - e_j : i, j \in [n], i \neq j\}$$

spanning the vector space

$$V := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0\}.$$

The reflection $s_{e_i - e_j} \in T$ acts on V by exchanging the i-th with the j-th coordinate. Thus the Weyl group $W = \langle T \rangle$ is the symmetric group S_n acting on V by permuting coordinates.

2.1.2 *Positive roots and simple roots*

Let Φ be a crystallographic root system and choose any linear hyperplane H in V that does not contain any root. The vector space V is then divided into two parts by H, pick one of them and call all roots in it *positive*. Then Φ can be written as a disjoint union of the set of positive roots Φ^+ and the set of negative roots $-\Phi^+$.

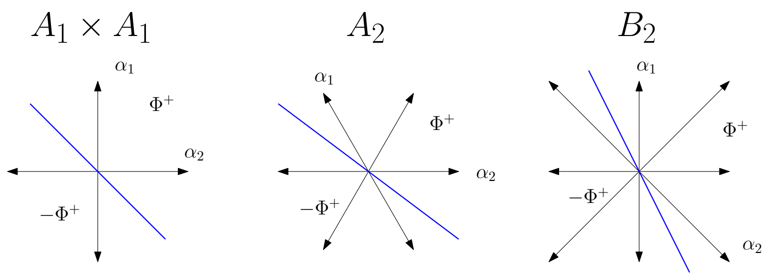

Figure 2.1.1: Three crystallographic root systems in two dimensions. For each root system, the blue hyperplane separates the positive roots in Φ^+ from the negative roots in $-\Phi^+$ and the simple roots in Δ are labelled α_1 and α_2.

Those positive roots that cannot be written as a sum of other positive roots are called *simple*. The set Δ of simple roots is a basis of V, and if a positive root is written in terms of this basis, all coefficients are nonnegative integers. Thus the number r of simple roots equals the dimension of V. It is called the *rank* of the root system. We will sometimes choose to index our set of simple roots as $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$.

From now on, we assume that a fixed choice of the set of positive roots Φ^+ (and thus also of the set of simple roots Δ) has been made for Φ. It turns out that for any two choices Φ_1^+, Φ_2^+ for the set of positive roots there exists a unique $w \in W$ with $w(\Phi_1^+) = \Phi_2^+$. Thus it is inessential which positive system we chose.

Example. As a set of positive roots for the root system of type A_{n-1} we choose

$$\Phi^+ := \{e_i - e_j : 1 \leq i < j \leq n\}.$$

The corresponding system of simple roots is

$$\Delta = \{e_i - e_{i+1} : i \in [n-1]\}.$$

We set $\alpha_i := e_i - e_{i+1}$ for any $i \in [n-1]$.

2.1.3 *Dynkin diagrams and the classification*

The *Dynkin diagram* of a root system Φ is a (multi)graph whose vertices are the simple roots of Φ. Two distinct simple roots $\alpha, \beta \in \Delta$ are connected by $d_{\alpha\beta} := \frac{4\langle\alpha,\beta\rangle^2}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}$ edges. Notice that by the crystallographic property this number is a (nonnegative) integer, and by the Cauchy-Schwarz inequality it is less than 4. If α and β are connected by a multiple edge, then they have different lengths, and we orient the multiple edge towards the shorter root.

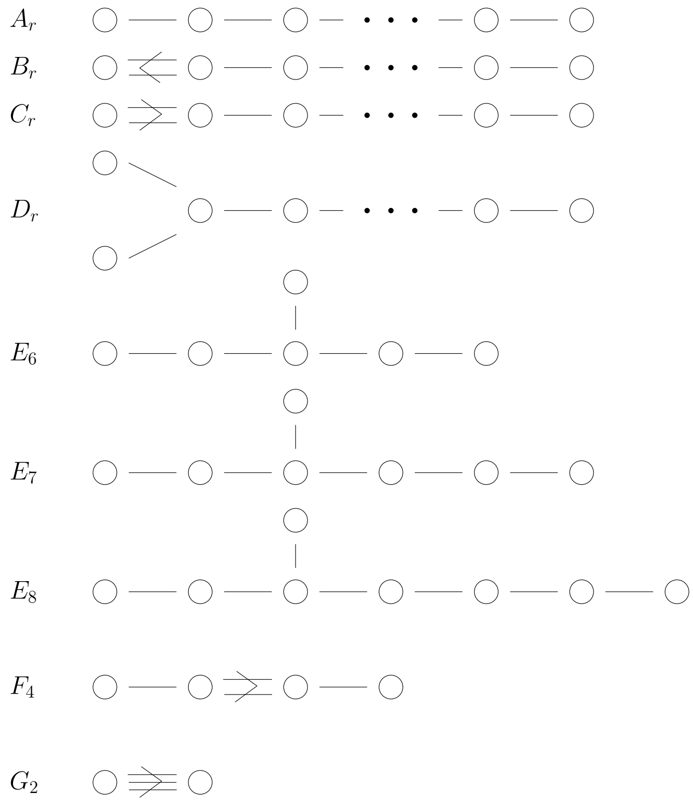

Figure 2.1.2: The list of all connected Dynkin diagrams. The subscript indicates the number of vertices, or equivalently the rank of the corresponding root system.

Theorem 2.1.1. *A crystallographic root system is determined up to isomorphism by its Dynkin diagram. The only connected Dynkin diagrams are those given in Figure 2.1.2.*

We call those root systems that have a connected Dynkin diagram *irreducible*. Any root system Φ can be written as a disjoint union of irreducible root systems $\Phi_1, \Phi_2, \ldots, \Phi_l$ whose Dynkin diagrams are the connected components of the Dynkin diagram of Φ in such a way that for $i \neq j$ every root in Φ_i is orthogonal to every root in Φ_j.

The classification of irreducible root systems by their Dynkin diagrams given in Theorem 2.1.1 is a powerful tool for proving results concerning root systems. To prove that all root systems have some property, it is often enough to check that property just for irreducible root systems, which one can do by checking it for each type in Figure 2.1.2. However, such case-checking is less illuminating than a proof that does not use the classification, so we will prefer *uniform* proofs whenever possible.

The Weyl group W of Φ is generated by the set of *simple reflections* $S := \{s_\alpha : \alpha \in \Delta\}$. We will sometimes index the set of simple reflections as $S = \{s_1, s_2, \ldots, s_r\}$ where $s_i := s_{\alpha_i}$ for $i \in [r]$. Denote the identity element of a group by e. Clearly every simple reflection is an involution, that is $s_\alpha^2 = e$ for all $s_\alpha \in S$. Since for $\alpha, \beta \in \Delta$ the angle between the linear hyperplanes H_α and H_β is $\theta_{\alpha\beta} := \cos^{-1}\left(\frac{|\langle\alpha,\beta\rangle|}{|\alpha||\beta|}\right) = \cos^{-1}(\frac{1}{2}\sqrt{d_{\alpha\beta}})$, the product $s_\alpha s_\beta$ is a rotation by $2\theta_{\alpha\beta}$. It turns out that $m_{\alpha\beta} := \frac{2\pi}{2\theta_{\alpha\beta}} = \pi/\theta_{\alpha\beta}$ is always an integer, so we have $(s_\alpha s_\beta)^{m_{\alpha\beta}} = e$. See Table 1.

2.1.4 *Weyl groups as Coxeter groups*

In fact, every relation between the generators in S is a consequence of the basic relations $s_\alpha^2 = e$ and $(s_\alpha s_\beta)^{m_{\alpha\beta}} = e$ for $s_\alpha, s_\beta \in S$. Equivalently, the pair (W, S) forms a *Coxeter system* with

$d_{\alpha\beta}$	$\theta_{\alpha\beta}$	$m_{\alpha\beta}$
0	$\pi/2$	2
1	$\pi/3$	3
2	$\pi/4$	4
3	$\pi/6$	6

Table 1: From the Dynkin diagram to the Coxeter matrix

Coxeter matrix $(m_{\alpha\beta})_{\alpha,\beta\in\Delta}$.

We may write any $w \in W$ as a *word* in the generators in S, that is we can write $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ with $s_{i_j} \in S$ for all $j \in [l]$. A word of w of minimal length is called a *reduced word* for w. The *length* $l(w)$ of w is the length of any reduced word for w.

Example. For the root system of type A_{n-1}, the simple reflections $s_i := s_{\alpha_i} = s_{e_i-e_{i+1}}$ where $i \in [n-1]$ act on V by exchanging adjacent coordinates. Together they generate the Weyl group $W = S_n$. The simple reflections $s_1, s_2, \ldots, s_{n-1}$ satisfy the Coxeter relations

1. $s_i^2 = e$ for all $i \in [n-1]$
2. $(s_is_{i+1})^3 = e$ for all $i \in [n-2]$ and
3. $(s_is_j)^2 = e$ if $|i-j| > 1$.

Every other relation between the generators is a consequence of these.

If $J \subseteq S$, we call $W_J := \langle s_\alpha : \alpha \in J\rangle$ the *standard parabolic subgroup* of W corresponding to J. Every left coset wW_J of W_J in W has a unique representative w' of minimum length. Furthermore, w' is the only element u of wW_J such that $u(J) \subseteq \Phi^+$. A *parabolic subgroup* of W is any subgroup that is conjugate to W_J for some J.

2.2 HYPERPLANE ARRANGEMENTS

For $k \in \mathbb{Z}$ and $\alpha \in \Phi$, define the affine hyperplane

$$H_\alpha^k := \{x \in V : \langle x, \alpha\rangle = k\}.$$

2.2.1 *The Coxeter arrangement*

The *Coxeter arrangement* is the central hyperplane arrangement in V given by all the linear hyperplanes $H_\alpha^0 = H_\alpha$ for $\alpha \in \Phi^+$. The complement of this arrangement falls apart into connected components which we call *chambers*. The Weyl group W acts simply transitively on the chambers. Thus we define the *dominant chamber* by

$$C := \{x \in V : \langle x, \alpha\rangle > 0 \text{ for all } \alpha \in \Delta\}$$

and write any chamber as wC for a unique $w \in W$. The length $l(w)$ of w equals the number of hyperplanes of the Coxeter arrangement that separate wC from C.

2.2.2 *The affine Coxeter arrangement and the affine Weyl group*

From now on we will assume that Φ is irreducible. The *root order* on Φ^+ is the partial order defined by $\alpha \leq \beta$ if and only if $\beta - \alpha$ can be written as a sum of positive roots. The set of positive roots Φ^+ with this partial order is called the *root poset* of Φ. It has a unique maximal

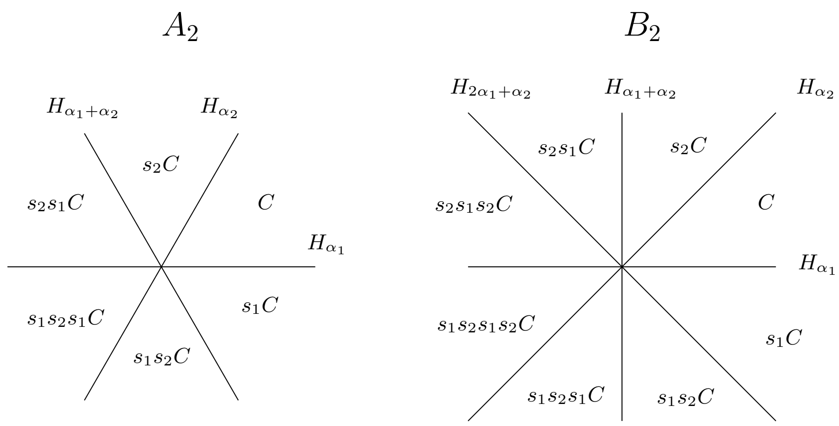

Figure 2.2.1: The Coxeter arrangements of types A_2 and B_2.

element, the *highest root* $\tilde{\alpha}$.

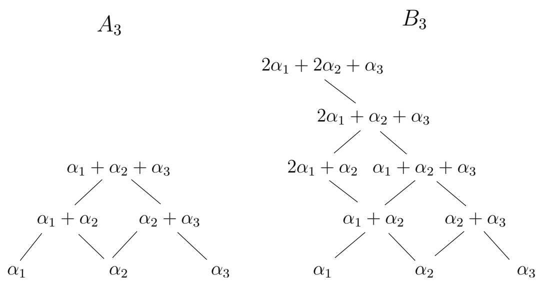

Figure 2.2.2: The Hasse diagrams of the root posets of types A_3 and B_3.

The *affine Coxeter arrangement* is the affine hyperplane arrangement in V given by all the affine hyperplanes H_α^k for $\alpha \in \Phi$ and $k \in \mathbb{Z}$. The complement of this arrangement falls apart into connected components which are called *alcoves*. They are all isometric simplices. We call an alcove *dominant* if it is contained in the dominant chamber. Define s_α^k as the reflection through the affine hyperplane H_α^k. That is,

$$s_\alpha^k(x) := x - 2\frac{\langle x, \alpha\rangle - k}{\langle \alpha, \alpha\rangle}\alpha.$$

We will also write s_α for the linear reflection s_α^0.

Let the *affine Weyl group* $\widetilde{W}$ be the group of affine automorphisms of V generated by all the reflections through hyperplanes in the affine Coxeter arrangement, that is

$$\widetilde{W} := \langle s_\alpha^k : \alpha \in \Phi \text{ and } k \in \mathbb{Z}\rangle.$$

The affine Weyl group $\widetilde{W}$ acts simply transitively on the alcoves of the affine Coxeter arrangement. Thus we define the *fundamental alcove* by

$$A_\circ := \{x \in V : \langle x, \alpha\rangle > 0 \text{ for all } \alpha \in \Delta \text{ and } \langle x, \tilde{\alpha}\rangle < 1\}$$

and write any alcove of the affine Coxeter arrangement as $\widetilde{w}A_\circ$ for a unique $\widetilde{w} \in \widetilde{W}$.

If we define $\widetilde{S} := S \cup \{s_{\tilde{\alpha}}^1\}$, then $(\widetilde{W}, \widetilde{S})$ is a Coxeter system. In particular, we may write any $\widetilde{w} \in \widetilde{W}$ as a word in the generators in $\widetilde{S}$, called reduced if its length is minimal, and define the length $l(\widetilde{w})$ of $\widetilde{w}$ as the length of any reduced word for it. If $\widetilde{w} \in W \subseteq \widetilde{W}$, then its length in

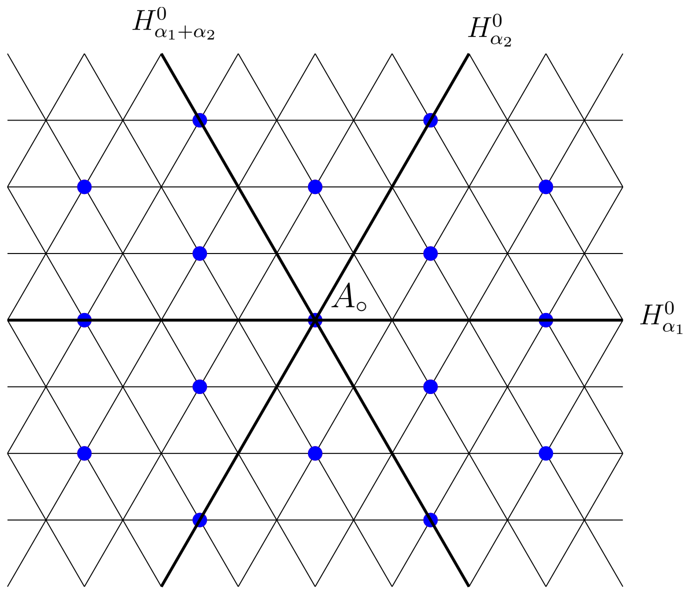

Figure 2.2.3: The affine Coxeter arrangement of type A_2. The points in the coroot lattice $\check{Q}$ are marked as blue dots.

terms of the generators in $\widetilde{S}$ equals its length in terms of the generators in S, so we are free to use the same notation for both lengths. The length $l(\widetilde{w})$ of $\widetilde{w} \in \widetilde{W}$ is also the number of hyperplanes of the affine Coxeter arrangement separating $\widetilde{w}A_\circ$ from $A_\circ$.

For a root $\alpha \in \Phi$, its *coroot* is defined as $\alpha^\vee = 2\frac{\alpha}{\langle \alpha, \alpha \rangle}$. The set $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ is itself an irreducible crystallographic root system, called the *dual root system* of Φ. Clearly $\Phi^{\vee\vee} = \Phi$.

The *root lattice* Q of Φ is the lattice in V spanned by all the roots in Φ. The *coroot lattice* $\check{Q}$ of Φ is the lattice in V spanned by all the coroots in $\Phi^\vee$. It is not hard to see that $\widetilde{W}$ acts on the coroot lattice. To any $\mu \in \check{Q}$, there corresponds the translation

$$\begin{aligned} t_\mu : V &\to V \\ x &\mapsto x + \mu. \end{aligned}$$

If we identify $\check{Q}$ with the corresponding group of translations acting on the affine space V then we may write $\widetilde{W} = W \ltimes \check{Q}$ as a semidirect product. In particular, we may write any $\widetilde{w} \in \widetilde{W}$ as $\widetilde{w} = wt_\mu$ for unique $w \in W$ and $\mu \in \check{Q}$.

For an alcove $\widetilde{w}A_\circ$ and a root $\alpha \in \Phi$ there is a unique integer k such that $k < \langle x, \alpha \rangle < k+1$ for all $x \in \widetilde{w}A_\circ$. We denote this integer by $k(\widetilde{w}, \alpha)$. We call the tuple $(k(\widetilde{w}, \alpha))_{\alpha \in \Phi^+}$ the *address* of the alcove $\widetilde{w}A_\circ$.

Notice that we have $k(\widetilde{w}, -\alpha) = -k(\widetilde{w}, \alpha) - 1$ and $k(w\widetilde{w}, w(\alpha)) = k(\widetilde{w}, \alpha)$ for all $\alpha \in \Phi$ and $w \in W$. Also note that if $k(\widetilde{w}, \alpha) = k(\widetilde{w}', \alpha)$ for all $\alpha \in \Phi^+$, then $\widetilde{w} = \widetilde{w}'$. The number of hyperplanes orthogonal to a root $\alpha \in \Phi^+$ that separate $\widetilde{w}A_\circ$ from $A_\circ$ is exactly $|k(\widetilde{w}, \alpha)|$. Thus we have $l(\widetilde{w}) = \sum_{\alpha \in \Phi^+} |k(\widetilde{w}, \alpha)|$.

2.2.3 *Polyhedra*

A *half-space* in V is subset of the form

$$\mathcal{H} = \{x \in V : \lambda(x) < l\}$$

for some linear functional λ in the dual space V^* of V and a real number $l \in \mathbb{R}$. A *polyhedron* in V is any nonempty subset of V that is defined as the intersection of a finite number of half-spaces. We will also consider the closure $\overline{P}$ of any polyhedron P a polyhedron.

Any hyperplane corresponding to an irredundant inequality defining a polyhedron P is considered a *wall* of P. Its intersection with $\overline{P}$ is called a *facet* of P. A wall of P is called a *floor* of P if it does not contain the origin and separates P from the origin. A wall of P that does not contain the origin and does not separate P from the origin is called a *ceiling* of P.

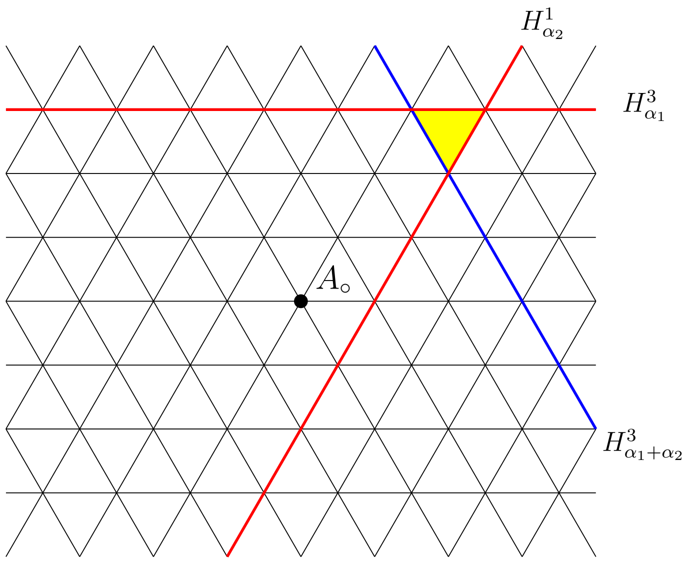

Figure 2.2.4: The yellow alcove has a single floor, namely $H^3_{\alpha_1+\alpha_2}$. Its ceilings are $H^3_{\alpha_1}$ and $H^1_{\alpha_2}$.

2.2.4 *Affine roots*

We may understand $\widetilde{W}$ in terms of its action on the set of *affine roots* $\widetilde{\Phi}$ of Φ. To do this, let δ be a formal variable and define $\widetilde{V} := V \oplus \mathbb{R}\delta$. Define the set of affine roots as

$$\widetilde{\Phi} := \{\alpha + k\delta : \alpha \in \Phi \text{ and } k \in \mathbb{Z}\}.$$

If $\widetilde{w} \in \widetilde{W}$, write it as $\widetilde{w} = wt_\mu$ for unique $w \in W$ and $\mu \in \check{Q}$ and define

$$\widetilde{w}(\alpha + k\delta) = w(\alpha) + (k - \langle \mu, \alpha \rangle)\delta.$$

This defines an action of $\widetilde{W}$ on $\widetilde{\Phi}$. It imitates the action of $\widetilde{W}$ on the half-spaces of V defined by the hyperplanes of the affine Coxeter arrangement. To see this, define the half-space

$$\mathcal{H}_{\alpha+k\delta} := \{x \in V : \langle x, \alpha \rangle > -k\}.$$

Then for $\widetilde{w} \in \widetilde{W}$ we have $\widetilde{w}(\mathcal{H}_{\alpha+k\delta}) = \mathcal{H}_{\beta+l\delta}$ if and only if $\widetilde{w}(\alpha + k\delta) = \beta + l\delta$. Define the set of *positive affine roots* as

$$\widetilde{\Phi}^+ := \{\alpha + k\delta : \alpha \in \Phi^+ \text{ and } k \geq 0\} \cup \{\alpha + k\delta : \alpha \in -\Phi^+ \text{ and } k > 0\},$$

the set of affine roots corresponding to half-spaces that contain $A_\circ$. So $\widetilde{\Phi}$ is the disjoint union of $\widetilde{\Phi}^+$ and $-\widetilde{\Phi}^+$. Define the set of *simple affine roots* as

$$\widetilde{\Delta} := \Delta \cup \{-\tilde{\alpha} + \delta\},$$

the set of affine roots corresponding to half-spaces that contain $A_\circ$ and share one of its walls. We will also write $\alpha_0 := -\tilde{\alpha} + \delta$.

For $\widetilde{w} \in \widetilde{W}$, we say that $\alpha + k\delta \in \widetilde{\Phi}^+$ is an *inversion* of $\widetilde{w}$ if $\widetilde{w}(\alpha + k\delta) \in -\widetilde{\Phi}^+$. Define

$$\mathsf{Inv}(\widetilde{w}) := \widetilde{\Phi}^+ \cap \widetilde{w}^{-1}(-\widetilde{\Phi}^+)$$

as the set of inversions of $\widetilde{w}$.

Lemma 2.2.1. *The positive affine root $\alpha + k\delta \in \widetilde{\Phi}^+$ is an inversion of $\widetilde{w}$ if and only if the hyperplane H_α^{-k} separates $\widetilde{w}^{-1}A_\circ$ from $A_\circ$.*

Proof. If $\alpha + k\delta \in \widetilde{\Phi}^+ \in \mathsf{Inv}(\widetilde{w})$, then $A_\circ \subseteq \mathcal{H}_{\alpha+k\delta}$ and $A_\circ \nsubseteq \widetilde{w}(\mathcal{H}_{\alpha+k\delta})$. Thus $\widetilde{w}^{-1}A_\circ \nsubseteq \mathcal{H}_{\alpha+k\delta}$ and therefore H_α^{-k} separates $\widetilde{w}^{-1}A_\circ$ from $A_\circ$.

Conversely, if $\alpha + k\delta \in \widetilde{\Phi}^+$ and H_α^{-k} separates $\widetilde{w}^{-1}A_\circ$ from $A_\circ$, then $A_\circ \subseteq \mathcal{H}_{\alpha+k\delta}$ and $\widetilde{w}^{-1}A_\circ \nsubseteq \mathcal{H}_{\alpha+k\delta}$. Therefore $A_\circ \nsubseteq \widetilde{w}(\mathcal{H}_{\alpha+k\delta})$ and thus $\widetilde{w}(\alpha + k\delta) \in -\widetilde{\Phi}^+$. So $\alpha + k\delta \in \mathsf{Inv}(\widetilde{w})$. $\square$

Lemma 2.2.2. *If $\alpha + k\delta \in \widetilde{\Phi}^+$, $k > 0$ and $\widetilde{w} \in \widetilde{W}$, then $\widetilde{w}^{-1}(\alpha + k\delta) \in -\widetilde{\Delta}$ if and only if H_α^{-k} is a floor of $\widetilde{w}A_\circ$.*

Proof. For the forward implication, suppose $\alpha + k\delta \in \widetilde{\Phi}^+$, $k > 0$, $\widetilde{w} \in \widetilde{W}$ and $\widetilde{w}^{-1}(\alpha + k\delta) \in -\widetilde{\Delta}$. Then by Lemma 2.2.1 the hyperplane H_α^{-k} separates $\widetilde{w}A_\circ$ from $A_\circ$. But we also have that $\widetilde{w}^{-1}(\mathcal{H}_{\alpha+k\delta})$ shares a wall with $A_\circ$, so H_α^{-k} is a wall of $\widetilde{w}A_\circ$. Thus it is a floor of $\widetilde{w}A_\circ$.

Conversely, if $\alpha + k\delta \in \widetilde{\Phi}^+$ and H_α^{-k} is a floor of $\widetilde{w}A_\circ$, then by Lemma 2.2.1, $\widetilde{w}^{-1}(\alpha + k\delta) \in -\widetilde{\Phi}^+$, so $A_\circ \nsubseteq \widetilde{w}^{-1}(\mathcal{H}_{\alpha+k\delta})$. But since $\mathcal{H}_{\alpha+k\delta}$ shares a wall with $\widetilde{w}A_\circ$, $\widetilde{w}^{-1}(\mathcal{H}_{\alpha+k\delta})$ shares a wall with $A_\circ$. So $\widetilde{w}^{-1}(\alpha + k\delta) \in -\Delta$. $\square$

2.2.5 *The height of roots*

For $\alpha \in \Phi$, we write it in terms of the basis of simple roots as $\alpha = \sum_{i=1}^r a_i \alpha_i$ and define its *height* $\mathrm{ht}(\alpha) := \sum_{i=1}^r a_i$ as the sum of the coefficients. Notice that $\mathrm{ht}(\alpha) > 0$ if and only if $\alpha \in \Phi^+$ and $\mathrm{ht}(\alpha) = 1$ if and only if $\alpha \in \Delta$. The highest root $\tilde{\alpha}$ is the unique root in Φ of maximal height. We define the *Coxeter number* of Φ as $h := 1 + \mathrm{ht}(\tilde{\alpha})$.

> **Example.** The highest root of the root system of type A_{n-1} is $\tilde{\alpha} = \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}$. It has height $n - 1$. So the Coxeter number h of the root system of type A_{n-1} equals n.

For any integer t, write Φ_t for the set of roots of height t. Then in particular we have $\Phi_1 = \Delta$ and $\Phi_{h-1} = \{\tilde{\alpha}\}$.

Define the *height* of an affine root $\alpha + k\delta$ as $\mathrm{ht}(\alpha + k\delta) = \mathrm{ht}(\alpha) + kh$. So $\mathrm{ht}(\alpha + k\delta) > 0$ if and only if $\alpha + k\delta \in \widetilde{\Phi}^+$ and $\mathrm{ht}(\alpha + k\delta) = 1$ if and only if $\alpha + k\delta \in \widetilde{\Delta}$. For any integer t, write $\widetilde{\Phi}_t$ for the set of affine roots of height t.

3 THE ANDERSON MAP

In this chapter, which is based on [Thi15], we generalise the Anderson map $\mathcal{A}_{GMV}$ of Gorsky, Mazin and Vazirani [GMV14] to all irreducible crystallographic root systems. This can be seen as a step towards a uniform theory of rational Catalan combinatorics. We start by expounding the theory of *classical* rational Catalan combinatorics associated with type A_{n-1}.

3.1 CLASSICAL RATIONAL CATALAN COMBINATORICS

3.1.1 *Rational Catalan numbers and rational Dyck paths*

For a positive integer n and another positive integer p relatively prime to n, the *rational (p,n)-Catalan number* is defined as

$$\mathsf{Cat}_{p/n} := \frac{1}{n+p}\binom{n+p}{n}.$$

These are generalisations of the classical Fuß-Catalan numbers: for a positive integer m, the rational Catalan number $\mathsf{Cat}_{mn+1/n}$ equals the classical Fuß-Catalan number $\mathsf{Cat}_n^{(m)}$. It was proven by Bizley [Biz54] that $\mathsf{Cat}_{p/n}$ counts the number of *rational p/n-Dyck paths*. These are lattice paths in $\mathbb{Z}^2$ consisting of North and East steps that go from $(0,0)$ to (p,n) and never go below the diagonal $y = \frac{n}{p}x$ of rational slope.

For a (p,n)-Dyck path P and $i \in [n]$, let P_i be the x-coordinate of the i-th North step of P. We may identify the path P with the (weakly) increasing tuple of nonnegative integers $(P_1, P_2, \dots, P_n)$. The condition that P lies above the diagonal $y = \frac{n}{p}x$ translates to either

$$P_i \leq \frac{p}{n}(i-1)$$

for all $i \in [n]$, or equivalently

$$\#\{i : P_i < l\} \geq \frac{nl}{p} \tag{3.1.1}$$

for all $l \in [p]$.

We call the full lattice squares (boxes) between a p/n-Dyck path P and the diagonal $y = \frac{n}{p}x$ its *area squares*. The number of them in the i-th row from the bottom is the *area* $a_i := \lfloor \frac{p}{n}(i-1) \rfloor - P_i$ of that row. We call the tuple $(a_1, a_2, \dots, a_n)$ the *area vector* of P.

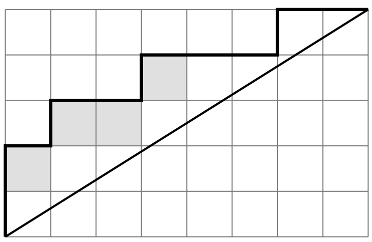

Figure 3.1.1: The rational 8/5-Dyck path $P = (0,0,1,3,6)$ with its area squares marked in gray. It has area vector $(0,1,2,1,0)$.

Rational p/n-Dyck paths were used by Anderson [And02], who provided a bijection between them and a certain set of number partitions called *(p, n)-cores*. The Anderson map $\mathcal{A}_{GMV}$ of Gorsky, Mazin and Vazirani, which will be defined later, may be seen as an extension of that bijection.

3.1.2 *Rational parking functions*

Equation (3.1.1) suggests the following generalisation of rational Dyck paths. A *rational p/n-parking function* is any tuple $(f_1, f_2, \ldots, f_n)$ of n nonnegative integers such that

$$\#\{i : f_i < l\} \geq \frac{nl}{p}$$

for all $l \in [p]$. Thus rational p/n-Dyck paths correspond to *increasing* rational p/n-parking functions.

> **Example.** The tuple $f = (6, 0, 1, 0, 3)$ is a rational $8/5$-parking function. Its increasing rearrangement is the rational $8/5$-Dyck path $(0, 0, 1, 3, 6)$.

An important property of rational parking functions is the following folklore theorem. Let $\mathcal{PF}_{p/n}$ be the set of p/n-parking functions.

Theorem 3.1.1. *$\mathcal{PF}_{p/n}$ is a set of representatives for the cosets of the cyclic subgroup generated by $(1, 1, \ldots, 1)$ in the abelian group $\mathbb{Z}_p^n$.*

Proof. We represent elements of $\mathbb{Z}_p$ as tuples $f = (f_1, f_2, \ldots, f_n)$ with $f_i \in \{0, 1, \ldots, p-1\}$ for all $i \in [n]$. Define

$$\mathsf{Sum}(f) := \sum_{i=1}^{n} f_i \in \mathbb{N}.$$

We claim that every coset H of the cyclic subgroup generated by $(1, 1, \ldots, 1)$ in $\mathbb{Z}_p^n$ has a unique representative with minimal Sum. To see this, we prove the stronger statement that all representatives of H have different Sum. Suppose that $f \in H$, $l \in [p]$ and

$$\mathsf{Sum}\,(f - l(1, 1, \ldots, 1)) = \mathsf{Sum}(f).$$

We calculate that

$$\mathsf{Sum}\,(f - l(1, 1, \ldots, 1)) = \mathsf{Sum}(f) - nl + p \cdot \#\{i \in [n] : f_i - l < 0\}, \tag{3.1.2}$$

so $-nl + p \cdot \#\{i \in [n] : f_i - l < 0\} = 0$. Thus p divides nl. Since p is relatively prime to n, this implies that p divides l. Therefore $f - l(1, 1, \ldots, 1) = f$.

To finish the proof, we claim that $f \in H$ has minimal Sum if and only if f is a p/n-parking function. To see this, first suppose that $f \in H$ has minimal Sum. Then $\mathsf{Sum}\,(f - l(1, 1, \ldots, 1)) \geq \mathsf{Sum}(f)$ for all $l \in [p]$. From Equation (3.1.2) we deduce that $-nl + p \cdot \#\{i \in [n] : f_i - l < 0\} \geq 0$ for all $l \in [p]$, or equivalently $\#\{i \in [n] : f_i < l\} \geq \frac{nl}{p}$ for all $l \in [p]$. So f is a p/n-parking function. Reversing the argument shows that if f is a p/n-parking function then it has minimal Sum in the coset containing it. $\square$

Corollary 3.1.2 ([ALW14, Corollary 4]). $|\mathcal{PF}_{p/n}| = p^{n-1}$.

The symmetric group S_n naturally acts on $\mathcal{PF}_{p/n}$ by permuting coordinates. Every orbit of this action contains exactly one increasing p/n-parking function. Thus the S_n-orbits on $\mathcal{PF}_{p/n}$ are naturally indexed by p/n-Dyck paths. In particular the number of S_n-orbits on $\mathcal{PF}_{p/n}$ is $\mathsf{Cat}_{p/n}$.

3.1.3 *Vertically labelled Dyck paths*

It is natural to introduce a combinatorial model for p/n-parking functions in terms of *vertically labelled p/n-Dyck paths*. In fact, this is how rational parking functions were originally defined in [ALW14].

An index $i \in [n]$ is called a *rise* of a p/n-Dyck path P if the i-th North step of P is followed by another North step. Equivalently, i is a rise of P if $P_i = P_{i+1}$. A vertically labelled p/n-Dyck path is a pair (P, σ) of a p/n-Dyck path P and a permutation $\sigma \in S_n$ such that whenever i is a rise of P we have $\sigma(i) < \sigma(i+1)$. We think of σ as labeling the North steps of P and say that the rise i is labelled $(\sigma(i), \sigma(i+1))$.

The bijection from p/n-parking functions to vertically labelled p/n-Dyck paths works as follows: For a p/n-parking function $f = (f_1, f_2, \ldots, f_n)$, let $(P_1, P_2, \ldots, P_n)$ be its increasing rearrangement and let P be the corresponding p/n-Dyck path. So P encodes the values that f takes, with multiplicities. In order to recover f, we also need to know their preimages under f. Thus for all $l \in \{0, 1, \ldots, p\}$ we label the North steps of P with x-coordinate equal to l by the preimages of l under f. If there are multiple North steps with the same x-coordinate we label them increasingly from bottom to top. Let $\sigma \in S_n$ be the permutation that maps $i \in [n]$ to the label of the i-th North step of P. Then (P, σ) is the vertically labelled p/n-Dyck path corresponding to f.

The inverse bijection is simple: if (P, σ) is the vertically labelled p/n-Dyck path corresponding to the p/n-parking function f then $f = \sigma \cdot (P_1, P_2, \ldots, P_n)$. We will often use this bijection implicitly and do not distinguish between rational p/n-parking functions and their combinatorial interpretation as vertically labelled p/n-Dyck paths.

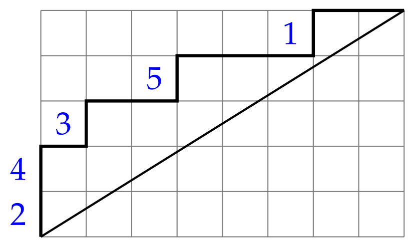

Figure 3.1.2: The vertically labelled 8/5-Dyck path (P, σ) with $P = (0, 0, 1, 3, 6)$ and $\sigma = 24351$. It corresponds to the 8/5-parking function $(6, 0, 1, 0, 3)$. Its only rise is 1, and it is labelled $(2, 4)$.

In terms of vertically labelled Dyck paths, the natural S_n-action on $\mathcal{PF}_{p/n}$ is realized by defining for $\tau \in S_n$

$$\tau \cdot (P, \sigma) = (P, (\tau\sigma)'),$$

where $(P, (\tau\sigma)')$ comes from labelling the North steps of P with $\tau\sigma$ and then sorting the labels in each column increasingly from bottom to top.

3.2 THE AFFINE SYMMETRIC GROUP

Let Φ be a root system of type A_{n-1}. We choose

$$\begin{aligned}\Phi &= \{e_i - e_j : i,j \in [n], i \neq j\},\\ \Phi^+ &= \{e_i - e_j : i,j \in [n], i < j\},\\ \Delta &= \{e_i - e_{i+1} : i \in [n-1]\},\\ V &= \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0\}, \text{ and}\\ \check{Q} &= \{(x_1, x_2, \dots, x_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 0\}.\end{aligned}$$

The Weyl group W is the symmetric group S_n acting on V by permuting coordinates, the rank of Φ is $r = n-1$ and the Coxeter number is $h = n$.

The affine Weyl group $\widetilde{W}$ also has a combinatorial model as $\widetilde{S}_n$, the set of *affine permutations* of period n. These are the bijections $\tilde{\sigma} : \mathbb{Z} \to \mathbb{Z}$ with

$$\tilde{\sigma}(l+n) = \tilde{\sigma}(l) + n \text{ for all } l \in \mathbb{Z} \text{ and}$$

$$\sum_{i=1}^{n} \tilde{\sigma}(i) = \binom{n+1}{2}.$$

The affine symmetric group is generated by the *affine simple transpositions* $\widetilde{s}_0, \widetilde{s}_1, \dots, \widetilde{s}_{n-1}$ that act on $\mathbb{Z}$ by

$$\begin{aligned}\widetilde{s}_j(l) &= l+1 \text{ for } l \equiv j \pmod{n},\\ \widetilde{s}_j(l) &= l-1 \text{ for } l \equiv j+1 \pmod{n}, \text{ and}\\ \widetilde{s}_j(l) &= l \text{ otherwise.}\end{aligned}$$

To identify the affine Weyl group $\widetilde{W}$ with $\widetilde{S}_n$ we index its generating set as $\widetilde{S} = \{s_0, s_1, \dots, s_{n-1}\}$, where $s_i = s_{e_i - e_{i+1}}$ for $i \in [n-1]$ and $s_0 = s^1_{e_1 - e_n}$. Here $e_1 - e_n = \tilde{\alpha}$ is the highest root of Φ. The generators $s_0, s_1, \dots, s_{n-1}$ of $\widetilde{W}$ satisfy the same relations as the generators $\widetilde{s}_0, \widetilde{s}_1, \dots, \widetilde{s}_{n-1}$ of $\widetilde{S}_n$, so sending $s_i \mapsto \widetilde{s}_i$ for $i = 0, 1 \dots, n-1$ defines an isomorphism from $\widetilde{W}$ to $\widetilde{S}_n$. We use this identification extensively and do not distinguish between elements of the affine Weyl group and the corresponding affine permutations.

Since $\widetilde{w} \in \widetilde{S}_n$ is uniquely defined by its values on $[n]$, we sometimes write it in *window notation* as $\widetilde{w} = [\widetilde{w}(1), \widetilde{w}(2), \dots, \widetilde{w}(n)]$.

For $\widetilde{w} \in \widetilde{S}_n$, write $\widetilde{w}(i) = w(i) + n\mu_i$ with $w(i) \in [n]$ for all $i \in [n]$. Then $w \in S_n$, $\mu := (\mu_1, \mu_2, \dots, \mu_n) \in \check{Q}$ and $\widetilde{w} = wt_\mu \in \widetilde{W}$.

Example. Consider the affine permutation $\widetilde{w} \in \widetilde{S}_4$ with window $[-3, 10, 4, -1]$. We can write $(-3, 10, 4, -1) = (1, 2, 4, 3) + 4(-1, 2, 0, -1)$, so we have $w = 1243 \in S_4$ and $\mu = (-1, 2, 0, -1) \in \check{Q}$. So $\widetilde{w} = wt_\mu = s_3 t_{(-1,2,0,-1)} \in \widetilde{W}$.

In order to avoid notational confusion, we use a "$\cdot$" for the action of $\widetilde{W}$ on V. So in particular $\widetilde{w}(0)$ is the affine permutation $\widetilde{w}$ evaluated at $0 \in \mathbb{Z}$, whereas $\widetilde{w} \cdot 0$ is the affine Weyl group element $\widetilde{w}$ applied to $0 \in V$.

3.3 ABACI

For any affine permutation $\widetilde{w}$, we consider the set

$$\Delta_{\widetilde{w}} := \{l \in \mathbb{Z} : w_R(l) > 0\} = \widetilde{w}^{-1}(\mathbb{Z}_{>0}).$$

We define its *abacus diagram* $\mathsf{A}(\Delta_{\widetilde{w}})$ as follows: draw n *runners*, labelled $1, 2, \ldots, n$ from left to right, with runner i containing all the integers congruent to i modulo n arranged in increasing order from top to bottom. We say that the k-th *level* of the abacus contains the integers $(k-1)n + i$ for $i \in [n]$ and arrange the runners in such a way that the integers of the same level are on the same horizontal line.

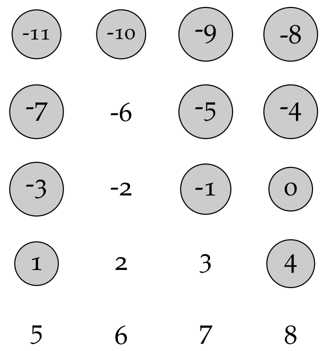

Figure 3.3.1: The balanced 4-flush abacus $\mathsf{A}(\Delta_{\widetilde{w}})$ for $\widetilde{w} = [-3, 10, 4, -1]$. Note that the values in the window of $\widetilde{w}^{-1} = [5, -6, 8, 3]$ are the lowest gaps (that is, the gaps with the lowest level on each runner) of $\mathsf{A}(\Delta_{\widetilde{w}})$. The levels of the runners of $\mathsf{A}(\Delta_{\widetilde{w}})$ are $1, -2, 0, 1$. Thus $\widetilde{w}^{-1} \cdot 0 = (1, -2, 0, 1)$.

We circle the elements of $\mathbb{Z} \backslash \Delta_{\widetilde{w}}$ and call them *beads*, whereas we call the elements of $\Delta_{\widetilde{w}}$ *gaps*. Notice that the fact that $\widetilde{w}(l+n) = \widetilde{w}(l) + n > \widetilde{w}(l)$ for all $l \in \mathbb{Z}$ implies that whenever $l \in \Delta_{\widetilde{w}}$ then also $l + n \in \Delta_{\widetilde{w}}$. We say that $\Delta_{\widetilde{w}}$ is *n-invariant*. Thus the abacus $\mathsf{A}(\Delta_{\widetilde{w}})$ is *n-flush*, that is whenever l is a gap then all the $l + kn$ for $k \in \mathbb{Z}_{>0}$ below it are also gaps. Or equivalently whenever l is a bead then so are all the $l - kn$ for $k \in \mathbb{Z}_{>0}$ above it.

For an n-flush abacus A define $\mathsf{level}_i(\mathsf{A})$ to be the highest level of a bead on runner i in A for $i \in [n]$. Define the integer tuple

$$\mathsf{levels}(\mathsf{A}) = (\mathsf{level}_1(\mathsf{A}), \mathsf{level}_2(\mathsf{A}), \ldots, \mathsf{level}_n(\mathsf{A})).$$

The following theorem is well-known.

Theorem 3.3.1. *For $\widetilde{w} \in \widetilde{S}_n$, we have $\mathsf{levels}(\mathsf{A}(\Delta_{\widetilde{w}})) = \widetilde{w}^{-1} \cdot 0$.*

Proof. Note that $\mathsf{levels}(\mathsf{A}(\Delta_e)) = 0$ and

$$\begin{aligned}
\mathsf{levels}(\mathsf{A}(\Delta_{\widetilde{w}s_j})) &= \mathsf{levels}(\mathsf{A}((\widetilde{w}s_j)^{-1}(\mathbb{Z}_{>0}))) \\
&= \mathsf{levels}(\mathsf{A}(s_j\widetilde{w}^{-1}(\mathbb{Z}_{>0}))) \\
&= \mathsf{levels}(\mathsf{A}(s_j(\Delta_{\widetilde{w}}))) \\
&= s_j \cdot \mathsf{levels}(\mathsf{A}(\Delta_{\widetilde{w}}))
\end{aligned}$$

for $\widetilde{w} \in \widetilde{S}_n$ and $j = 0, 1, \ldots, n-1$. Thus the result follows by induction on the length $l(\widetilde{w})$ of $\widetilde{w}$. □

In particular $\mathsf{levels}(\mathsf{A}(\Delta_{\widetilde{w}})) \in \check{Q}$, so the sum of the levels of $\mathsf{A}(\Delta_{\widetilde{w}})$ is zero. We call such an abacus *balanced*.

Let $M_{\widetilde{w}}$ be the minimal element of $\Delta_{\widetilde{w}}$ (that is, the smallest gap of $\mathsf{A}(\Delta_{\widetilde{w}})$) and define $\widetilde{\Delta}_{\widetilde{w}} = \Delta_{\widetilde{w}} - M_{\widetilde{w}}$. This is also an n-invariant set, so we form its n-flush abacus $\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}})$. This is a *normalized* abacus, that is its smallest gap is 0.

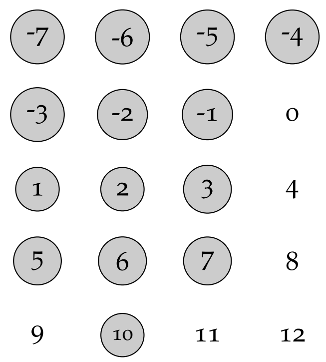

Figure 3.3.2: The normalized 4-flush abacus $\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}})$ for $\widetilde{w} = [-3, 10, 4, -1]$. Here $M_{\widetilde{w}} = -6$.

Remark. It is easy to see that if Δ is an n-invariant set with $\mathsf{levels}(\mathsf{A}(\Delta)) = (x_1, x_2, \ldots, x_n)$, then $\mathsf{levels}(\mathsf{A}(\Delta + 1)) = (x_n + 1, x_1, x_2, \ldots, x_{n-1})$. Thus we define the bijection

$$\begin{aligned} g : \mathbb{Z}^n &\to \mathbb{Z}^n \\ (x_1, x_2, \ldots, x_n) &\mapsto (x_n + 1, x_1, x_2, \ldots, x_{n-1}) \end{aligned}$$

and get that

$$\mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}})) = g^{-M_{\widetilde{w}}} \cdot \mathsf{levels}(\mathsf{A}(\Delta_{\widetilde{w}})). \tag{3.3.1}$$

In particular $\sum_{i=1}^{n} \mathsf{level}_i(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}})) = -M_{\widetilde{w}}$.

3.4 p-STABLE AFFINE PERMUTATIONS

For a positive integer p relatively prime to n, we define the set of *p-stable affine permutations* $\widetilde{S}_n^p$ as [GMV14, Definition 2.13]

$$\widetilde{S}_n^p := \{\widetilde{w} \in \widetilde{S}_n : \widetilde{w}(i + p) > \widetilde{w}(i) \text{ for all } i \in \mathbb{Z}\}.$$

If $\widetilde{w}$ is p-stable, then $\Delta_{\widetilde{w}}$ is p-invariant in addition to being n-invariant. So the n-flush abacus $\mathsf{A}(\Delta_{\widetilde{w}})$ is also p-flush, that is whenever l is a gap then so is $l + kp$ for all $k \in \mathbb{Z}_{>0}$.

Example. The affine permutation $\widetilde{w} = [-3, 10, 4, -1]$ is 9-stable. Thus its balanced abacus $\mathsf{A}(\Delta_{\widetilde{w}})$ is 9-flush in addition to being 4-flush.

3.5 THE COMBINATORIAL ANDERSON MAP

We are now ready to describe the *Anderson map* $\mathcal{A}_{GMV}$ defined by Gorsky, Mazin and Vazirani [GMV14, Section 3.1]. It is a bijection from the set of p-stable affine permutations $\widetilde{S}_n^p$ to the set of p/n-parking functions.

We use English notation, so for us a *Young diagram* is a finite set of square boxes that is left-justified and top-justified. Take $\widetilde{w} \in \widetilde{S}_n^p$. As in Section 3.3 we consider the set

$$\Delta_{\widetilde{w}} := \{i \in \mathbb{Z} : \widetilde{w}(i) > 0\}$$

and let $M_{\widetilde{w}}$ be its minimal element. Let $\widetilde{\Delta}_{\widetilde{w}} := \Delta_{\widetilde{w}} - M_{\widetilde{w}}$. In contrast to [GMV14], we shall use $\widetilde{\Delta}_{\widetilde{w}}$ in place of $\Delta_{\widetilde{w}}$ and therefore also have a different labelling of $\mathbb{Z}^2$.

View the integer lattice $\mathbb{Z}^2$ as the set of square boxes. Define the rectangle

$$R_{p,n} := \{(x,y) \in \mathbb{Z}^2 : 0 \leq x < p, 0 \leq y < n\}$$

and label $\mathbb{Z}^2$ by the linear function

$$l(x,y) := -n - nx + py.$$

Define the Young diagram

$$D_{\widetilde{w}} := \{(x,y) \in R_{p,n} : l(x,y) \in \widetilde{\Delta}_{\widetilde{w}}\}$$

and let $P_{\widetilde{w}}$ be the path that defines its lower boundary. It is a p/n-Dyck path. Label its i-th North step by $\sigma(i) := \widetilde{w}(l_i + M_{\widetilde{w}})$, where l_i is the label of the rightmost box of $D_{\widetilde{w}}$ in the i-th row from the bottom (or the label of $(-1, i-1)$ if its i-th row is empty). Then we have that $\sigma \in S_n$ and $(P_{\widetilde{w}}, \sigma)$ is a vertically labelled p/n-Dyck path. We define $\mathcal{A}_{GMV} := (P_{\widetilde{w}}, \sigma)$.

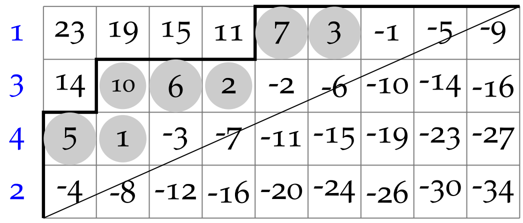

Figure 3.5.1: The vertically labelled $9/4$-Dyck path $\mathcal{A}_{GMV}(\widetilde{w})$ for $\widetilde{w} = [-3, 10, 4, -1]$. It has area vector $(0,2,3,2)$ and labelling $\sigma = 2431$. It corresponds to the $9/4$-parking function $(4,0,1,0)$. The positive beads of the normalized abacus $\mathrm{A}(\widetilde{\Delta}_{\widetilde{w}})$ are shaded in gray.

Theorem 3.5.1 ([GMV14, Theorem 3.4]). *The Anderson map $\mathcal{A}_{GMV}$ is a bijection from $\widetilde{S}_n^p$ to the set of vertically labelled p/n-Dyck paths.*

3.6 THE UNIFORM ANDERSON MAP

In this section, we will generalise the Anderson map $\mathcal{A}_{GMV}$ to a *uniform* Anderson map $\mathcal{A}$ that is defined for all irreducible crystallographic root systems Φ. It is a bijection from the set $\widetilde{W}^p$ of *p-stable* affine Weyl group elements to the *finite torus* $\check{Q}/p\check{Q}$. We will proceed in several steps, all of which have already appeared in the literature in some form.

3.6.1 *p-stable affine Weyl group elements*

Let Φ be any irreducible crystallographic root system and let $\widetilde{W}$ be its affine Weyl group. We say that $\widetilde{w} \in \widetilde{W}$ is *p-stable* if it has no inversions of height p. That is, $\widetilde{w}$ is p-stable if $\widetilde{w}(\widetilde{\Phi}_p) \subseteq \widetilde{\Phi}^+$. We denote the set of p-stable affine Weyl group elements by $\widetilde{W}^p$.

Define

$$^p\widetilde{W} := \{\widetilde{w}^{-1} : \widetilde{w} \in \widetilde{W}^p\}$$

as the set of inverses of p-stable affine Weyl group elements. We call these elements *p-restricted*. Recall from Lemma 2.2.1 that an affine root $\alpha + k\delta \in \widetilde{\Phi}^+$ is an inversion of $\widetilde{w} \in \widetilde{W}$ if and only if the corresponding hyperplane H_α^{-k} separates $\widetilde{w}^{-1}A_\circ$ from $A_\circ$. Thus $\widetilde{w} \in {}^p\widetilde{W}$ if and only if no hyperplane corresponding to an affine root of height p separates $\widetilde{w}A_\circ$ from $A_\circ$.

Write $p = ah + b$, where a and b are nonnegative integers and $0 < b < h$. Then we have

$$\begin{aligned}\widetilde{\Phi}_p &= \{\alpha + a\delta : \alpha \in \Phi_b\} \cup \{\alpha + (a+1)\delta : \alpha \in \Phi_{b-h}\} \\ &= \{\alpha + a\delta : \alpha \in \Phi_b\} \cup \{-\alpha + (a+1)\delta : \alpha \in \Phi_{h-b}\}\end{aligned}$$

Thus the hyperplanes corresponding to affine roots of height p are those of the form H_α^{-a} with $\alpha \in \Phi_b$ and those of the form $H_{-\alpha}^{-(a+1)} = H_\alpha^{a+1}$ for $\alpha \in \Phi_{h-b}$. We define the *Sommers region* as the region in V bounded by these hyperplanes:

$$\mathcal{S}_\Phi^p := \{x \in V : \langle x, \alpha\rangle > -a \text{ for all } \alpha \in \Phi_b \text{ and } \langle x, \alpha\rangle < a+1 \text{ for all } \alpha \in \Phi_{h-b}\}.$$

We will later see that this region is in fact a simplex. For now we make the following observation.

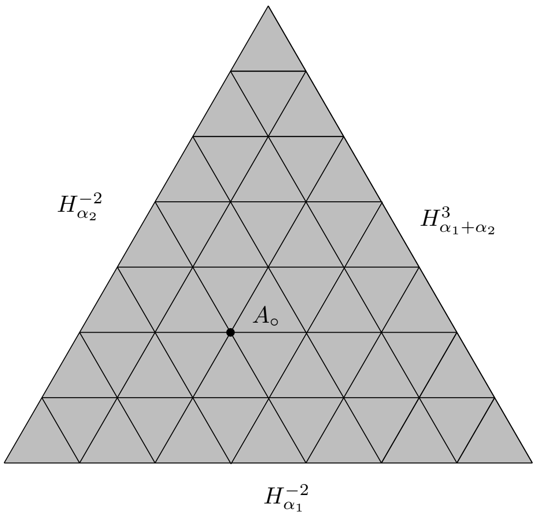

Figure 3.6.1: The 49 alcoves in $\mathcal{S}_\Phi^7$ for Φ of type A_2.

Lemma 3.6.1. *We have $\widetilde{w} \in {}^p\widetilde{W}$ if and only if $\widetilde{w}A_\circ \subseteq \mathcal{S}_\Phi^p$.*

Proof. We have $\widetilde{w} \in {}^p\widetilde{W}$ if and only if no hyperplane corresponding to an affine root of height p separates $\widetilde{w}A_\circ$ from $A_\circ$. But those hyperplanes are exactly the hyperplanes bounding the Sommers region $\mathcal{S}_\Phi^p$, and $A_\circ$ is inside $\mathcal{S}_\Phi^p$. □

3.6.2 *From the Sommers region to the dilated fundamental alcove*

In this subsection, we will see that the Sommers region $\mathcal{S}_\Phi^p$ is in fact a simplex. We will even show the following stronger statement: there is a unique element $\widetilde{w}_p \in \widetilde{W}$ such that $\widetilde{w}_p(\mathcal{S}_\Phi^p) = pA_\circ$. This will require some preparation. We follow and expand on ideas in [Fan05], some of which are due to Lusztig.

The coweight lattice and $\check{\rho}$

Define $\{\check{\omega}_1, \check{\omega}_2, \ldots, \check{\omega}_r\}$ as the dual basis to the basis $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ of simple roots. Then $\check{\omega}_1, \check{\omega}_2, \ldots, \check{\omega}_r$ are called the *fundamental coweights*. They generate the *coweight lattice*

$$\check{\Lambda} := \{x \in V : \langle x, \alpha\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}.$$

The coroot lattice $\check{Q}$ is a sublattice of $\check{\Lambda}$. We define the *index of connection* as

$$f := [\check{\Lambda} : \check{Q}].$$

Define $\check{\rho} := \frac{1}{2}\sum_{\alpha\in\Phi^+}\check{\alpha}$. Index the set of simple roots of Φ as $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$. It is well-known that any simple reflection s_i sends $\check{\alpha}_i$ to $-\check{\alpha}_i$ and permutes the set of all other positive coroots. Thus

$$\begin{aligned} s_{\alpha_i}(\check{\rho}) &= \frac{1}{2}\left(-\check{\alpha}_i + \sum_{\alpha\in\Phi^+\setminus\{\alpha_i\}}\check{\alpha}\right) \\ &= \frac{1}{2}\left(\sum_{\alpha\in\Phi^+}\check{\alpha}\right) - \check{\alpha}_i \\ &= \check{\rho} - \check{\alpha}_i. \end{aligned}$$

So we have $\langle\check{\rho}, \alpha_i\rangle = 1$ for all $\alpha_i \in \Delta$. Thus we can write $\check{\rho} = \sum_{i=1}^r \check{\omega}_i \in \check{\Lambda}$. Furthermore we have $\langle\check{\rho}, \alpha\rangle = \mathrm{ht}(\alpha)$ for all $\alpha \in \Phi$. We will need the following lemma due to Kostant.

Lemma 3.6.2 ([LP12, Lemma 3.6]). *Every alcove $\widetilde{w}A_\circ$ contains exactly one point in $\frac{1}{h}\check{\Lambda}$. For the fundamental alcove $A_\circ$, this point is $\frac{\check{\rho}}{h}$.*

Proof. We have $\check{\rho} = \sum_{i=1}^n \check{\omega}_i \in \check{\Lambda}$, so $\frac{\check{\rho}}{h} \in \frac{1}{h}\check{\Lambda}$. We also have that $\langle\frac{\check{\rho}}{h}, \alpha\rangle = \mathrm{ht}(\alpha)/h \in (0,1)$ for all $\alpha \in \Phi^+$. Thus $\frac{\check{\rho}}{h}$ lies in $A_\circ$—in fact, it is the only element in $A_\circ \cap \frac{1}{h}\check{\Lambda}$.

Indeed, suppose that $\nu \in A_\circ \cap \frac{1}{h}\check{\Lambda}$. Then for all $\alpha_i \in \Delta$ we have $\langle\nu, \alpha_i\rangle = a_i/h$ for some $a_i \in \mathbb{Z}_+$. But we also have $\langle\nu, \tilde{\alpha}\rangle = (\sum_{i=1}^n a_i c_i)/h < 1$, so $a_i = 1$ for all $i \in [n]$ and thus $\nu = \frac{\check{\rho}}{h}$.

Since $\widetilde{W}$ acts on $\frac{1}{h}\check{\Lambda}$, there is exactly one element of $\frac{1}{h}\check{\Lambda}$ in any alcove $\widetilde{w}A_\circ$. □

Theorem 3.6.3. *For p relatively prime to h, there exists a unique element $\widetilde{w}_p = t_\mu w \in \widetilde{W}$ with*

$$p\frac{\check{\rho}}{h} = \widetilde{w}_p\left(\frac{\check{\rho}}{h}\right).$$

Proof. For all $\alpha \in \Phi^+$, we have that

$$\left\langle p\frac{\check{\rho}}{h}, \alpha\right\rangle = p\frac{\mathrm{ht}(\alpha)}{h} \notin \mathbb{Z},$$

since p is relatively prime to h and h does not divide $\mathrm{ht}(\alpha)$. Thus $p\frac{\check{\rho}}{h}$ lies on no hyperplane of the affine Coxeter arrangement, so it is contained in some alcove $\widetilde{w}_pA_\circ$. Since $p\frac{\check{\rho}}{h} \in \frac{1}{h}\check{\Lambda}$ we have that $p\frac{\check{\rho}}{h} = \widetilde{w}_p(\frac{\check{\rho}}{h})$ by Lemma 3.6.2. □

Theorem 3.6.4. *The affine Weyl group element $\widetilde{w}_p = t_\mu w$ maps $\mathcal{S}_\Phi^p$ bijectively to $pA_\circ$.*

Proof. We calculate that

$$\frac{\mathrm{ht}(\alpha)}{h} = \left\langle\frac{\check{\rho}}{h}, \alpha\right\rangle = \left\langle w\left(\frac{\check{\rho}}{h}\right), w(\alpha)\right\rangle = \left\langle p\frac{\check{\rho}}{h} - \mu, w(\alpha)\right\rangle = p\frac{\mathrm{ht}(w(\alpha))}{h} - \langle\mu, w(\alpha)\rangle.$$

Thus $\mathrm{ht}(\alpha) = p\mathrm{ht}(w(\alpha)) - h\langle\mu, w(\alpha)\rangle$. Again write $p = ah + b$ with $a, b \in \mathbb{Z}_{\geq 0}$ and $0 < b < h$. So reducing modulo h we get $\mathrm{ht}(\alpha) \equiv b\mathrm{ht}(w(\alpha)) \mod h$. Thus $\mathrm{ht}(\alpha) \equiv b \mod h$ if and only if $\mathrm{ht}(w(\alpha)) \equiv 1 \mod h$. So

$$w(\Phi_b \cup \Phi_{b-h}) = \Phi_1 \cup \Phi_{1-h} = \Delta \cup \{-\tilde{\alpha}\}.$$

For $\alpha \in \Delta$, we have

$$\frac{\mathrm{ht}(w^{-1}(\alpha))}{h} = p\frac{\mathrm{ht}(\alpha)}{h} - \langle\mu, \alpha\rangle = \frac{p}{h} - \langle\mu, \alpha\rangle.$$

Now $\mathrm{ht}(w^{-1}(\alpha))$ equals either b or $b - h$, so $\langle\mu, \alpha\rangle = a$ if $w^{-1}(\alpha) \in \Phi^+$ and $\langle\mu, \alpha\rangle = a + 1$ if $w^{-1}(\alpha) \in -\Phi^+$. Comparison with [Fan05, Section 2.3] ($w = w'$, $\mu = \nu$) gives the result. □

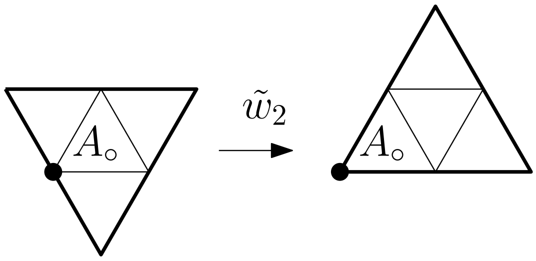

Figure 3.6.2: For the root system of type A_2, the affine Weyl group element $\widetilde{w}_2 = s^1_{\alpha_1+\alpha_2}$ maps the Sommers region $\mathcal{S}^2_\Phi$ to the dilated fundamental alcove $2A_\circ$.

In fact, the property of $\widetilde{w}_p$ given in Theorem 3.6.4 defines it uniquely.

Theorem 3.6.5. *$\widetilde{w}_p$ is the unique $\widetilde{w} \in \widetilde{W}$ with $\widetilde{w}(\mathcal{S}^p_\Phi) = pA_\circ$.*

Proof. It remains to show that if $\widetilde{w} \in \widetilde{W}$ and $\widetilde{w}(pA_\circ) = pA_\circ$ then $\widetilde{w} = e$ is the identity.

The fundamental alcove $A_\circ$ has a vertex at 0, and its other vertices are $\frac{1}{c_i}\check{\omega}_i$ for $i \in [r]$. Define L as the lattice generated by $\{\frac{1}{c_i}\check{\omega}_i : i \in [r]\}$. Then

$$[L : \check{Q}] = [L : \check{\Lambda}][\check{\Lambda} : \check{Q}] = c_1c_2\cdots c_r f.$$

Now a **case-by-case** check using the classification of irreducible crystallographic root systems reveals that every prime that divides either f or some c_i also divides the Coxeter number h. So since p is relatively prime to h it is also relatively prime to $[L : \check{Q}]$ [Som97, Remark 3.6]. This implies that the map

$$\begin{aligned} L/\check{Q} &\to L/\check{Q} \\ x + \check{Q} &\mapsto px + \check{Q} \end{aligned}$$

is invertible. Since 0 is the only vertex of $A_\circ$ in the coroot lattice, $\frac{1}{c_i}\check{\omega}_i \notin \check{Q}$ for all $i \in [r]$, so we also have $p\frac{1}{c_i}\check{\omega}_i \notin \check{Q}$ for all $i \in [r]$. Thus 0 is the only vertex of $pA_\circ$ that is in $\check{Q}$.

If $\widetilde{w} \in \widetilde{W}$ and $\widetilde{w}(pA_\circ) = pA_\circ$, then $\widetilde{w} \cdot 0 \in \check{Q}$ must be a vertex of $pA_\circ$. Thus $\widetilde{w} \cdot 0 = 0$. So $\widetilde{w} \in W$. Since $\widetilde{w}(pA_\circ) = pA_\circ$ we must have $\widetilde{w}C = C$, therefore $\widetilde{w} = e$ as required. □

Corollary 3.6.6. $|\widetilde{W}^p| = |{}^p\widetilde{W}| = p^r$.

Proof. By Lemma 3.6.1 $|\widetilde{W}^p| = |{}^p\widetilde{W}|$ is the number of alcoves in the Sommers region $\mathcal{S}^p_\Phi$. By Theorem 3.6.4 this equals the number of alcoves in $pA_\circ$. But the volume of $pA_\circ$ is p^r times that of $A_\circ$, so it contains p^r alcoves. □

3.6.3 *From the dilated fundamental alcove to the finite torus*

We follow a remark in [Som97, Section 6]. Define the p-dilated affine Weyl group as

$$\widetilde{W}_p := W \ltimes p\check{Q}.$$

Let $I_p := \{\widetilde{w} \in \widetilde{W} : \widetilde{w}A_\circ \subseteq pA_\circ\}$. Since $\overline{A_\circ}$ is a fundamental domain for the action of $\widetilde{W}$ on V, $p\overline{A_\circ}$ is a fundamental domain for the action of $\widetilde{W}_p$ on V. Therefore I_p is a set of right coset representatives of $\widetilde{W}_p$ in $\widetilde{W}$.

Another set of right coset representatives of $\widetilde{W}_p$ in $\widetilde{W}$ is any set of (the translations corresponding to) representatives of the finite torus $\check{Q}/p\check{Q}$. Thus we get a bijection from I_p to (the translations corresponding to a set of representatives of) $\check{Q}/p\check{Q}$ by sending an element $\widetilde{w} \in I_p$

to the translation that represents the same coset of $\widetilde{W}_p$ in $\widetilde{W}$. Explicitly, for $\widetilde{w} = wt_\mu$, this is given by $t_\mu = t_{-\widetilde{w}^{-1}\cdot 0}$. So the map

$$\begin{aligned} I_p &\to \check{Q}/p\check{Q} \\ \widetilde{w} &\mapsto -\widetilde{w}^{-1} \cdot 0 + p\check{Q} \end{aligned}$$

is a bijection.

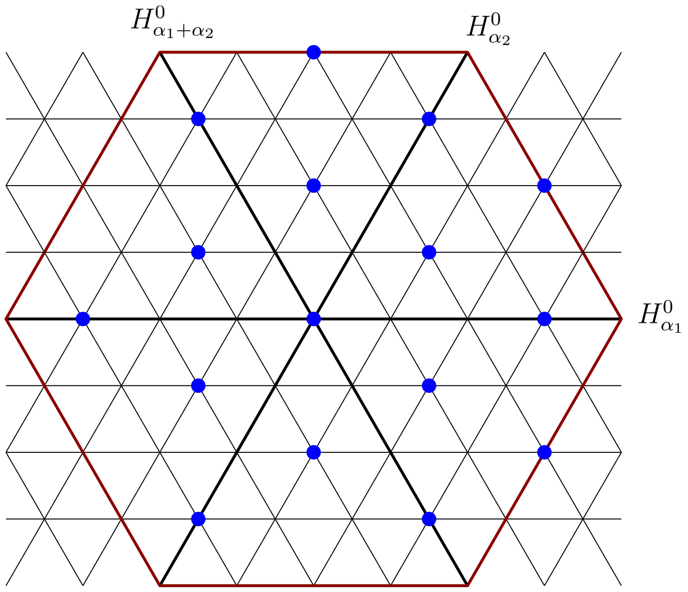

Figure 3.6.3: The blue dots are a natural set of representatives for the 16 elements of the finite torus $\check{Q}/4\check{Q}$ of the root system of type A_2.

3.6.4 *Putting it all together*

We are now ready to define the uniform Anderson map $\mathcal{A}$ as

$$\begin{aligned} \mathcal{A} : \widetilde{W}^p &\to \check{Q}/p\check{Q} \\ \widetilde{w} &\mapsto \widetilde{w}\widetilde{w}_p^{-1} \cdot 0 + p\check{Q} \end{aligned}$$

Theorem 3.6.7. *The Anderson map $\mathcal{A}$ is a bijection.*

Proof. We start with $\widetilde{w} \in \widetilde{W}^p$. We take its inverse $\widetilde{w}^{-1} \in {}^p\widetilde{W}$. By Lemma 3.6.1, we have $\widetilde{w}^{-1}A_\circ \subseteq \mathcal{S}^p_\Phi$. So by Theorem 3.6.4, we have $\widetilde{w}_p\widetilde{w}^{-1}A_\circ \subseteq pA_\circ$. That is $\widetilde{w}_p\widetilde{w}^{-1} \in I_p$. So as in Section 3.6.3 we map it to

$$-(\widetilde{w}_p\widetilde{w}^{-1})^{-1} \cdot 0 + p\check{Q} = -\widetilde{w}\widetilde{w}_p^{-1} \cdot 0 + p\check{Q} \in \check{Q}/p\check{Q}.$$

At the end we multiply by -1 to change sign. Each of the steps is bijective, so $\mathcal{A}$ is a bijection. $\square$

3.6.5 *The stabilizer of $\mathcal{A}(\widetilde{w})$*

At this point we prove a somewhat technical result about $\mathcal{A}$ that will be of use in Chapter 5. The Weyl group W acts on the coroot lattice $\check{Q}$ and its dilation $p\check{Q}$, so also on the finite torus $\check{Q}/p\check{Q}$. For $\mu + p\check{Q} \in \check{Q}/p\check{Q}$ we define its *stabilizer* as

$$\mathsf{Stab}(\mu + p\check{Q}) := \{w \in W : w(\mu + p\check{Q}) = \mu + p\check{Q}\}.$$

Theorem 3.6.8. *For $\widetilde{w} \in \widetilde{W}^p$ the stabilizer of $\mathcal{A}(\widetilde{w}) \in \check{Q}/p\check{Q}$ is generated by $\{s_\beta : \beta \in \widetilde{w}(\widetilde{\Phi}_p) \cap \Phi\}$.*

To prove this, we will need the following result due to Haiman.

Lemma 3.6.9 ([Hai94, Lemma 7.4.1]). *The set $p\overline{A_\circ} \cap \check{Q}$ is a system of representatives for the orbits of the W-action on $\check{Q}/p\check{Q}$. The stabilizer of an element of $\check{Q}/p\check{Q}$ represented by $\mu \in p\overline{A_\circ} \cap \check{Q}$ is generated by the reflections through the linear hyperplanes parallel to the walls of $p\overline{A_\circ}$ that contain μ.*

Proof of Theorem 3.6.8. Suppose $\widetilde{w} \in \widetilde{W}^p$. Observe first that this implies $\widetilde{w}(\widetilde{\Phi}_p) \subseteq \widetilde{\Phi}^+$, so $\widetilde{w}(\widetilde{\Phi}_p) \cap \Phi \subseteq \Phi^+$. Write $\widetilde{w}\widetilde{w}_p^{-1} = ut_{-\mu}$. We have $\widetilde{w}_p\widetilde{w}^{-1}A_\circ \subseteq pA_\circ$, so $\mu = \widetilde{w}_p\widetilde{w}^{-1} \cdot 0 \in p\overline{A_\circ} \cap \check{Q}$. We wish to show that the stabilizer of

$$\mathcal{A}(\widetilde{w}) = \widetilde{w}\widetilde{w}_p^{-1} \cdot 0 + p\check{Q} = -u(\mu) + p\check{Q}$$

in $\check{Q}/p\check{Q}$ is generated by $\{s_\beta : \beta \in \widetilde{w}(\widetilde{\Phi}_p) \cap \Phi\}$.

First observe that $\widetilde{w}_p$ maps the walls of $\mathcal{S}_\Phi^p$ to the walls of $pA_\circ$. In terms of affine roots this means that

$$\widetilde{w}_p(\widetilde{\Phi}_p) = \Delta \cup \{-\tilde{\alpha} + p\delta\}.$$

Now calculate that for $\beta \in \Phi^+$ we have the following equivalences:

$$\begin{aligned}
&\beta \in \widetilde{w}(\widetilde{\Phi}_p) \\
&\Leftrightarrow \widetilde{w}^{-1}(\beta) \in \widetilde{\Phi}_p \\
&\Leftrightarrow \widetilde{w}_p\widetilde{w}^{-1}(\beta) \in \widetilde{w}_p(\widetilde{\Phi}_p) = \Delta \cup \{-\tilde{\alpha} + p\delta\} \\
&\Leftrightarrow \beta = \widetilde{w}\widetilde{w}_p^{-1}(\alpha) \text{ for some } \alpha \in \Delta \text{ or } \beta = \widetilde{w}\widetilde{w}_p^{-1}(-\tilde{\alpha} + p\delta) \\
&\Leftrightarrow \beta = u(\alpha) \text{ and } \langle \mu, \alpha \rangle = 0 \text{ for some } \alpha \in \Delta \text{ or } \beta = u(-\tilde{\alpha}) \text{ and } \langle \mu, \tilde{\alpha} \rangle = p.
\end{aligned}$$

Here we used $\widetilde{w}\widetilde{w}_p^{-1} = ut_{-\mu}$ and the definition of the action of $\widetilde{W}$ on $\widetilde{\Phi}$. Combining this with Lemma 3.6.9 we get

$$\begin{aligned}
&\mathsf{Stab}(\widetilde{w}\widetilde{w}_p^{-1} \cdot 0 + p\check{Q}) \\
&= \mathsf{Stab}(-u(\mu) + p\check{Q}) \\
&= u\mathsf{Stab}(\mu + p\check{Q})u^{-1} \\
&= u\langle s_\alpha : \mu \text{ lies in a wall of } p\overline{A_\circ} \text{ orthogonal to } \alpha \rangle u^{-1} \\
&= u\langle s_{u^{-1}(\beta)} : \beta \in \widetilde{w}(\widetilde{\Phi}_p) \cap \Phi \rangle u^{-1} \\
&= \langle s_\beta : \beta \in \widetilde{w}(\widetilde{\Phi}_p) \cap \Phi \rangle,
\end{aligned}$$

as required. □

3.7 THE COMBINATORIAL ANDERSON MAP AND THE UNIFORM ANDERSON MAP

It remains to relate the uniform Anderson map $\mathcal{A}$ defined in Section 3.6 to the combinatorial Anderson map $\mathcal{A}_{GMV}$ defined in Section 3.5. So, for this section, let Φ be a root system of type A_{n-1}.

First note that the set of p-stable affine Weyl group elements $\widetilde{W}^p$ coincides with the set of p-stable affine permutations $\widetilde{S}_n^p$ [GMV14, Section 2.3]. It remains to relate the finite torus $\check{Q}/p\check{Q}$ to the set $\mathcal{PF}_{p/n}$ of rational p/n-parking functions.

3.7.1 *Parking functions and the finite torus*

We follow [Ath05, Section 5.1]. First recall that

$$\check{Q} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 0\}.$$

The natural projection

$$\bmod p : \check{Q} \to \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_p^n : \sum_{i=1}^{n} x_i = 0\}$$

has kernel $p\check{Q}$. Futhermore, since n and p are relatively prime, the natural projection

$$\bmod (1,1,\ldots,1) : \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_p^n : \sum_{i=1}^{n} x_i = 0\} \to \mathbb{Z}_p^n/(1,1,\ldots,1)$$

is a bijection to the set $\mathbb{Z}_p^n/(1,1,\ldots,1)$ of cosets of the cyclic subgroup of $\mathbb{Z}_p^n$ generated by $(1,1,\ldots,1)$. Thus if $\pi_{\check{Q}} := \bmod (1,1,\ldots,1) \circ \bmod p$, then

$$\pi_{\check{Q}} : \check{Q}/p\check{Q} \to \mathbb{Z}_p^n/(1,1,\ldots,1)$$

is a well-defined bijection.

Recall from Theorem 3.1.1 that the set of rational p/n-parking functions is a set of representatives for $\mathbb{Z}_p^n/(1,1,\ldots,1)$, so the natural projection

$$\pi_{\mathcal{PF}} : \mathcal{PF}_{p/n} \to \mathbb{Z}_p^n/(1,1,\ldots,1)$$

is a bijection.

Note that $W = S_n$ naturally acts on $\check{Q}/p\check{Q}$, $\mathcal{PF}_{p/n}$ and $\mathbb{Z}_p^n/(1,1,\ldots,1)$ and that both $\pi_{\check{Q}}$ and $\pi_{\mathcal{PF}}$ are isomorphisms with respect to these actions. So we define $\chi := \pi_{\mathcal{PF}}^{-1} \circ \pi_{\check{Q}}$ as the natural S_n-isomorphism from $\check{Q}/p\check{Q}$ to $\mathcal{PF}_{p/n}$.

3.7.2 *The Anderson maps are equivalent*

The following theorem interprets the combinatorial Anderson map $\mathcal{A}_{GMV}$ as the uniform Anderson map $\mathcal{A}$ specialised to type A_{n-1}.

$$\begin{tikzcd}
 & \check{Q}/p\check{Q} \arrow[dd, "\chi"] \arrow[dr, "\pi_{\check{Q}}"] & \\
\widetilde{W}^p = \widetilde{S}_n^p \arrow[ur, "\mathcal{A}"] \arrow[dr, "\mathcal{A}_{GMV}"'] & & \mathbb{Z}_p^n/(1,1,\ldots,1) \\
 & \mathcal{PF}_{p/n} \arrow[ur, "\pi_{\mathcal{PF}}"'] &
\end{tikzcd}$$

Figure 3.7.1: Theorem 3.7.1 as a commutative diagram of bijections.

Theorem 3.7.1. *Suppose Φ is of type A_{n-1} and p is a positive integer relatively prime to n. Then*

$$\pi_{\check{Q}} \circ \mathcal{A} = \pi_{\mathcal{PF}} \circ \mathcal{A}_{GMV}.$$

Proof. Let $\widetilde{w} \in \widetilde{W}^p = \widetilde{S}_n^p$. Refer to Section 3.5 for the construction of $\mathcal{A}_{GMV}(\widetilde{w})$. We employ the same notation as in that section here.

We first consider the case where $\widetilde{w} = \widetilde{w}_p$, as defined in Theorem 3.6.3. Since

$$\widetilde{w}_p(\widetilde{\Phi}_p) = \Delta \cup \{-\tilde{\alpha} + p\delta\} \subseteq \widetilde{\Phi}^+$$

we indeed have $\widetilde{w}_p \in \widetilde{W}^p$. From [GMV14, Lemma 2.16] we get that its inverse is

$$\widetilde{w}_p^{-1} = [p-c, 2p-c, \dots, np-c]$$

where $c = \frac{(p-1)(n+1)}{2}$. Since $\Delta_{\widetilde{w}_p} = \widetilde{w}_p^{-1}(\mathbb{Z}_{>0})$ the set of lowest gaps of the runners of the balanced abacus $\mathsf{A}(\Delta_{\widetilde{w}_p})$ is

$$\{\widetilde{w}_p^{-1}(1), \widetilde{w}_p^{-1}(2) \dots \widetilde{w}_p^{-1}(n)\} = \{p-c, 2p-c, \dots, np-c\}.$$

Thus the set of lowest gaps of the runners of the normalized abacus $\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})$ is

$$\{0, p, 2p, \dots, (n-1)p\}.$$

This is exactly the set of labels of $(-1, i-1)$ for $i \in [n]$. Thus all the labels in $R_{p,n}$ are beads in $\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})$. Therefore $D_{\widetilde{w}_p}$ is empty and $\mathcal{A}_{GMV}(\widetilde{w}_p) = (P_{\widetilde{w}_p}, \sigma) = (0, 0, \dots, 0)$.

For $x, y \in \mathbb{Z}^n$ write $x \equiv y$ if the projections of x and y into $\mathbb{Z}_p^n/(1, 1, \dots, 1)$ agree. Then $\equiv$ is compatible both with addition and with the S_n-action on $\mathbb{Z}^n$. The set of lowest gaps of the runners of the abacus $\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p} + p)$ is $\{p, 2p, \dots, np\}$. Thus

$$\mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p} + p)) = \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})) + (0, 0, \dots, 0, p) \equiv \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})).$$

We also have

$$\mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p} + n)) = \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})) + (1, 1, \dots, 1) \equiv \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})).$$

In terms of the bijection g from Section 3.3, this means that

$$g^p \cdot \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})) \equiv \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p}))$$

and

$$g^n \cdot \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})) \equiv \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})).$$

Since p and n are coprime, this implies that

$$g \cdot \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})) \equiv \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})). \tag{3.7.1}$$

Now take any $\widetilde{w} \in \widetilde{S}_n^p$. Note that the labels of boxes in the i-th row of the Young diagram $D_{\widetilde{w}}$ from the bottom (those with y-coordinate $i-1$) are those congruent to $p(i-1)$ modulo n. Thus we define the permutation $\tau \in S_n$ by

$$\tau(i) \equiv p(i-1) \bmod n$$

for all $i \in [n]$. The fact that p is relatively prime to n implies that this indeed gives a permutation of n.

Let P_i be the number of boxes on the i-th row of $D_{\widetilde{w}}$ from the bottom. This is the number of gaps of $\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}})$ on runner $\tau(i)$ that are in $R_{p,n}$. Equivalently, it is the number of gaps of $\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}})$ on runner $\tau(i)$ that are smaller than the smallest gap on runner $\tau(i)$ of $\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})$. Thus

$$P_i = \mathsf{level}_{\tau(i)}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})) - \mathsf{level}_{\tau(i)}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}})),$$

that is

$$(P_1, P_2, \dots, P_n) = \tau^{-1} \cdot \left[\mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})) - \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}}))\right]. \tag{3.7.2}$$

Now we start looking at the labelling σ of the p/n-Dyck path $P_{\widetilde{w}}$. We have for $i \in [n]$

$$\sigma(i) := \widetilde{w}(l_i + M_{\widetilde{w}}) \equiv \widetilde{w}(\tau(i) + M_{\widetilde{w}}) \bmod n.$$

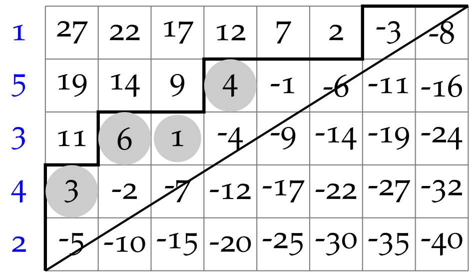

Figure 3.7.2: The vertically labelled 8/5-Dyck path $\mathcal{A}_{GMV}(\widetilde{w})$ for $\widetilde{w} = [0, 7, -2, 6, 4]$. In this case we have $\tau = 53142$ and $M_{\widetilde{w}} = -3$. The positive beads of the normalized abacus $\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}})$ are shaded in gray.

Define $r \in S_n$ by $r(i) \equiv i + 1 \bmod n$. Write $\widetilde{w} = wt_{-\mu}$ with $w \in W = S_n$ and $\mu \in \check{Q}$, simultaneously viewing w as an affine permutation in $\widetilde{S}_n$ also. Then

$$w(r^{M_{\widetilde{w}}}(\tau(i))) \equiv w(\tau(i) + M_{\widetilde{w}}) \equiv \widetilde{w}(\tau(i) + M_{\widetilde{w}}) \equiv \sigma(i) \bmod n.$$

Since $\sigma(i)$ and $w(r^{M_{\widetilde{w}}}(\tau(i)))$ are congruent modulo n and both in $[n]$, they are equal. Thus

$$\sigma = w \circ r^{M_{\widetilde{w}}} \circ \tau. \tag{3.7.3}$$

Now we calculate

$$\begin{aligned}
\mathcal{A}_{GMV}(\widetilde{w}) &= (P_{\widetilde{w}}, \sigma) \\
&= \sigma \cdot (P_1, P_2, \ldots, P_n) \\
&= (w \circ r^{M_{\widetilde{w}}} \circ \tau) \cdot \tau^{-1} \cdot \left[\mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})) - \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}}))\right] \\
&\equiv (w \circ r^{M_{\widetilde{w}}}) \cdot \left[g^{M_{\widetilde{w}_p} - M_{\widetilde{w}}} \cdot \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}_p})) - \mathsf{levels}(\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}}))\right] \\
&= (w \circ r^{M_{\widetilde{w}}}) \cdot \left[g^{-M_{\widetilde{w}}} \cdot \mathsf{levels}(\mathsf{A}(\Delta_{\widetilde{w}_p})) - g^{-M_{\widetilde{w}}} \cdot \mathsf{levels}(\mathsf{A}(\Delta_{\widetilde{w}}))\right] \\
&= (w \circ r^{M_{\widetilde{w}}}) \cdot \left[r^{-M_{\widetilde{w}}} \cdot \left[\mathsf{levels}(\mathsf{A}(\Delta_{\widetilde{w}_p})) - \mathsf{levels}(\mathsf{A}(\Delta_{\widetilde{w}}))\right]\right] \\
&= w \cdot (\widetilde{w}_p^{-1} \cdot 0 - \widetilde{w}^{-1} \cdot 0) \\
&= w \cdot (\widetilde{w}_p^{-1} \cdot 0 - \mu) \\
&= wt_{-\mu} \widetilde{w}_p^{-1} \cdot 0 \\
&= \widetilde{w} \widetilde{w}_p^{-1} \cdot 0 \\
&\equiv \mathcal{A}(\widetilde{w}).
\end{aligned}$$

Here we used Equation (3.7.3), Equation (3.7.2), Equation (3.7.1), Equation (3.3.1) and Theorem 3.3.1, in that order. $\square$

3.8 RATIONAL COXETER-CATALAN NUMBERS

We wish to extend the rational p/n-Catalan combinatorics uniformly to other root systems. The rational p/n-Catalan number counts p/n-Dyck paths. These paths index the S_n-orbits on $\mathcal{PF}_{p/n}$. Section 3.7.1 implies that the S_n-orbits on $\mathcal{PF}_{p/n}$ correspond to W-orbits on the finite torus $\check{Q}/p\check{Q}$ for the root system of type A_{n-1}. Thus we will look at the W-action on the finite torus $\check{Q}/p\check{Q}$ for any irreducible crystallographic root system Φ and any positive integer p relatively prime to the Coxeter number h of Φ. This was studied in [Som97]. To understand the results therein, we need to intruduce some further concepts.

3.8.1 The partition lattice

Let $\mathcal{H}$ be a set of hyperplanes in a finite dimensional vector space U. Let $\mathcal{L} = \mathcal{L}(\mathcal{H}) := \{\bigcap_{H\in A} H : A \subseteq \mathcal{H}\}$ be the set of all intersections of hyperplanes in $\mathcal{H}$, including the ambient space U. Partially order $\mathcal{L}$ by reverse inclusion, and define the *Möbius function* μ on pairs of elements of $\mathcal{L}$ by

$$\begin{aligned} \mu(X,X) &= 1, \\ \sum_{X\le Z\le Y} \mu(X,Z) &= 0, \text{ for } X < Y, \\ \mu(X,Y) &= 0, \text{ for } X \not\le Y. \end{aligned}$$

Then the *characteristic polynomial* of $\mathcal{L}$ is the polynomial in t defined by

$$\chi(\mathcal{L},t) := \sum_{X\in\mathcal{L}} \mu(U,X)t^{\dim(X)}.$$

We take $U = V$ as the ambient space of a crystallographic root system Φ and $\mathcal{H} = \{H_\alpha : \alpha \in \Phi^+\}$ as the set of hyperplanes of the Coxeter arrangement. We call $\mathcal{L} = \mathcal{L}(\mathcal{H})$ the *partition lattice* of Φ.

The characteristic polynomial can be written as

$$\chi(\mathcal{L},t) = \prod_{i=1}^{r}(t-e_i),$$

where $e_1 \le e_2 \le \ldots \le e_r$ are positive integers, called the *exponents* of $\mathcal{L}$ [OS80]. We may also call them the exponents of Φ.

Example. The exponents of the root system of type A_{n-1} are $1, 2, \ldots, n-1$.

For any $X \in \mathcal{L}$ define

$$\mathcal{H}^X = \{X \cap H : H \in \mathcal{H} \text{ and } X \not\subseteq H\},$$

a set of hyperplanes in the ambient space X. Let $\mathcal{L}^X := \mathcal{L}(\mathcal{H}^X)$ be the corresponding poset and let $\chi(\mathcal{L}^X, t)$ be its characteristic polynomial. It factors as

$$\chi(\mathcal{L}^X,t) = \prod_{i=1}^{d}(t-e_i(X)),$$

where $d = \dim(X)$ and the positive integers $e_1(X) \le e_2(X) \le \ldots \le e_d(X)$ are the *exponents* of $\mathcal{L}^X$ [OS83, Theorem 1.4].

The following result by Barcelo and Ihrig [BI99, Theorem 3.1] relates the partition lattice to the parabolic subgroups of W.

Theorem 3.8.1 ([Arm09, Theorem 5.1.9]). *The map*

$$X \mapsto \mathsf{Iso}(X) := \{w \in W \mid w \cdot x = x \text{ for all } x \in X\}$$

is a poset isomorphism from $\mathcal{L}$ to the set of parabolic subgroups of W ordered by inclusion, with inverse

$$W' \mapsto \mathsf{Fix}(W') := \{x \in V \mid w \cdot x = x \text{ for all } w \in W'\}.$$

3.8.2 The number of orbits of the finite torus

The W-orbits on the finite torus $\check{Q}/p\check{Q}$ were counted by Haiman.

Theorem 3.8.2 ([Hai94, Theorem 7.4.4]). *For a positive integer p relatively prime to h, the number of W-orbits on the finite torus $\check{Q}/p\check{Q}$ is*

$$\mathsf{Cat}_{p/\Phi} := \frac{1}{|W|}\prod_{i=1}^{r}(p+e_i).$$

We call these numbers $\mathsf{Cat}_{p/\Phi}$ the *rational Coxeter-Catalan numbers* of Φ. They specialise to the rational p/n-Catalan numbers $\mathsf{Cat}_{p/n}$ when Φ is of type A_{n-1}.

3.8.3 Types of orbits

Define a *type* $\mathcal{T}$ to be a conjugacy class of subgroups of W. The type of a subgroup $W' \subseteq W$ is the conjugacy class containing it. We call a type $\mathcal{T}$ *parabolic* if it contains a standard parabolic subgroup, or equivalently if it consists of parabolic subgroups. If W acts transitively on a set O, then we define the *type* of O to be the type of the stabilizer of any $x \in O$. This is well-defined, since the stabilizers of any two elements of O are conjugate.

Define a *W-set* as a set together with a W-action on it. An isomorphism between W-sets is a bijection that commutes with the action of W on both sets. It is easy to see that two transitive W-sets are isomorphic if and only if they have the same type. Thus two W-sets are isomorphic if and only if they have the same number of orbits of type $\mathcal{T}$ for every $\mathcal{T}$.

Theorem 3.8.3 ([Som97, Propositions 4.1 and 5.1]). *If p is relatively prime to h, the stabilizer of any element of $\check{Q}/p\check{Q}$ is a parabolic subgroup of W. For any parabolic type $\mathcal{T}$ and any $W' \in \mathcal{T}$, the number of W-orbits of $\check{Q}/p\check{Q}$ of type $\mathcal{T}$ is*

$$\mathsf{Krew}_{p/\Phi}(\mathcal{T}) := \frac{1}{[N(W'):W']}\chi(\mathcal{L}^X, p),$$

where $X = \mathsf{Fix}(W')$ and $N(W') := \{w \in W : wW'w^{-1} = W'\}$ is the normalizer of W' in W.

The number $\mathsf{Krew}_{p/\Phi}(\mathcal{T})$ is well-defined, since it is readily seen to be independent of the choice of representative $W' \in \mathcal{T}$. We call the numbers $\mathsf{Krew}_{p/\Phi}(\mathcal{T})$ the *rational Kreweras numbers* of Φ. Note that $\mathsf{Krew}_{p/\Phi}(\mathcal{T})$ is a polynomial in p.

Example. If $W = S_n$ is the Weyl group of the root system Φ of type A_{n-1}, any parabolic subgroup of W is conjugate to a unique *Young subgroup*

$$W' := S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_l} \subseteq S_n$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a *partition* of n, that is a (weakly) decreasing sequence of positive integers whose sum is n. For $i \in [n]$ we define m_i as $\{j \in [l] : \lambda_j = i\}$, the number of parts of λ of size i. We set $m_\lambda := m_1!m_2!\cdots m_n!$. So $[N(W'):W'] = m_\lambda$.

From [OS83, Proposition 2.1] we see that, if $X = \mathsf{Fix}(W')$, then $\mathcal{L}^X$ is of type A_{l-1}, so its exponents are $1, 2, \ldots, l-1$ and

$$\chi(\mathcal{L}^X, t) = \prod_{i=1}^{l-1}(t-i).$$

Thus for the type $\mathcal{T}$ of W' we have

$$\begin{aligned}\mathsf{Krew}_{p/\Phi}(\mathcal{T}) &= \frac{1}{[N(W'):W']}\chi(\mathcal{L}^X, p)\\ &= \frac{1}{m_\lambda}\prod_{i=1}^{l-1}(p-i)\\ &= \frac{(p-1)!}{m_\lambda(p-l)!}.\end{aligned}$$

For the particular case where $p = n+1$, we get

$$\mathsf{Krew}_{n+1/\Phi}(\mathcal{T}) = \frac{n!}{m_\lambda(n-l+1)!}.$$

These are the *classical Kreweras numbers* found by Kreweras in his study of classical noncrossing partitions [Kre72, Théorème 4]. This justifies our choice of terminology.

3.9 DOMINANT p-STABLE AFFINE WEYL GROUP ELEMENTS

We say that $\widetilde{w} \in \widetilde{W}$ is *dominant* if $\widetilde{w}A_\circ$ is contained in the dominant chamber. In this section, we will study the set $\widetilde{W}^p_{\text{dom}}$ of dominant p-stable affine Weyl group elements. The following lemma is well-known.

Lemma 3.9.1. *$\widetilde{w} \in \widetilde{W}$ is dominant if and only if for all $w \in W$ we have $\mathsf{Inv}(\widetilde{w}) \subseteq \mathsf{Inv}(w\widetilde{w})$.*

Proof. Suppose $\widetilde{w} \in \widetilde{W}$ is dominant. Then no hyperplane of the (linear) Coxeter arrangement separates $\widetilde{w}A_\circ$ from $A_\circ$. So by Lemma 2.2.1 $\mathsf{Inv}(\widetilde{w}^{-1}) \cap \Phi^+ = \emptyset$. Equivalently

$$\widetilde{w}(\widetilde{\Phi}^+) \cap -\Phi^+ = \emptyset.$$

So if $\alpha + k\delta \in \mathsf{Inv}(\widetilde{w})$, then $\widetilde{w}(\alpha+k\delta) \in -\widetilde{\Phi}^+\backslash(-\Phi^+)$. Thus $w\widetilde{w}(\alpha+k\delta) \in -\widetilde{\Phi}^+$ and $\alpha + k\delta \in \mathsf{Inv}(w\widetilde{w})$.

Conversely, suppose $\widetilde{w} \in \widetilde{W}$ and $\mathsf{Inv}(\widetilde{w}) \subseteq \mathsf{Inv}(w\widetilde{w})$ for all $w \in W$. Then for $\alpha + k\delta \in \mathsf{Inv}(\widetilde{w})$ we have $w\widetilde{w}(\alpha+k\delta) \in -\widetilde{\Phi}^+$ for all $w \in W$, so $\widetilde{w}(\alpha+k\delta) \in -\widetilde{\Phi}^+\backslash(-\Phi^+)$. Thus

$$\widetilde{w}(\widetilde{\Phi}^+) \cap -\Phi^+ = \emptyset,$$

or equivalently $\mathsf{Inv}(\widetilde{w}^{-1}) \cap \Phi^+ = \emptyset$. So by Lemma 2.2.1 no hyperplane of the (linear) Coxeter arrangement separates $\widetilde{w}A_\circ$ from $A_\circ$ and thus $\widetilde{w}A_\circ$ is dominant. □

Example. If Φ is of type A_{n-1}, so that $\widetilde{W} = \widetilde{S}_n$, then the dominant affine Weyl group elements $\widetilde{w}$ are exactly those whose inverse is *affine Grassmanian*, that is those that satisfy

$$\widetilde{w}^{-1}(1) < \widetilde{w}^{-1}(2) < \cdots < \widetilde{w}^{-1}(n).$$

Lemma 3.9.2. *If $\widetilde{w} \in \widetilde{W}^p$ and $w \in W$ such that $\widetilde{w}A_\circ \subseteq wC$, then $w^{-1}\widetilde{w} \in \widetilde{W}^p_{\text{dom}}$.*

Proof. Clearly $w^{-1}\widetilde{w}$ is dominant. By Lemma 3.9.1 we have $\mathsf{Inv}(w^{-1}\widetilde{w}) \subseteq \mathsf{Inv}(\widetilde{w})$, so

$$\mathsf{Inv}(w^{-1}\widetilde{w}) \cap \widetilde{\Phi}_p = \emptyset,$$

thus $w^{-1}\widetilde{w} \in \widetilde{W}^p$. □

The following theorem is the generalisation of [Ath05, Theorem 4.2] to the rational Coxeter-Catalan level.

Theorem 3.9.3. *The map*

$$\begin{aligned} \rho : \widetilde{W}^p_{\text{dom}} &\to p\overline{A_\circ} \cap \check{Q} \\ \widetilde{w} &\mapsto \widetilde{w}_p \widetilde{w}^{-1} \cdot 0 \end{aligned}$$

is a bijection.

Proof. We claim that the map $\widetilde{w} \mapsto \widetilde{w}^{-1} \cdot 0$ is a bijection from $\widetilde{W}^p_{\text{dom}}$ to $\overline{\mathcal{S}^p_\Phi} \cap \check{Q}$. By Theorem 3.6.1 its image really is in $\overline{\mathcal{S}^p_\Phi} \cap \check{Q}$. To see that it is surjective, note that if $\mu \in \overline{\mathcal{S}^p_\Phi} \cap \check{Q}$, then $\mu = \widetilde{w} \cdot 0$ for some $\widetilde{w} A_\circ \subseteq \mathcal{S}^p_\Phi$, which by Lemma 3.6.1 means $\widetilde{w}^{-1} \in \widetilde{W}^p$. Say $\widetilde{w}^{-1} A_\circ \in wC$, so that $w^{-1}\widetilde{w}^{-1} \in \widetilde{W}^p_{\text{dom}}$ by Lemma 3.9.2. Then $\mu = (w^{-1}\widetilde{w}^{-1})^{-1} \cdot 0$ as required. To see that it is injective, note that if $\widetilde{w}_1^{-1} \cdot 0 = \widetilde{w}_2^{-1} \cdot 0$, then $\widetilde{w}_1 = w\widetilde{w}_2$ for some $w \in W$. If $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{W}^p_{\text{dom}}$ this implies that $w = e$ and thus $\widetilde{w}_1 = \widetilde{w}_2$.

To complete the proof, it suffices to note that $\widetilde{w}_p$ is a bijection from $\overline{\mathcal{S}^p_\Phi} \cap \check{Q}$ to $p\overline{A_\circ} \cap \check{Q}$. □

Corollary 3.9.4. $|\widetilde{W}^p_{\text{dom}}| = \mathsf{Cat}_{p/\Phi}$.

Proof. By Theorem 3.9.3, $|\widetilde{W}^p_{\text{dom}}| = |p\overline{A_\circ} \cap \check{Q}|$. By Lemma 3.6.9, this is the number of W-orbits on $\check{Q}/p\check{Q}$, which equals $\mathsf{Cat}_{p/\Phi}$ by Theorem 3.8.2. □

4

THE M-SHI ARRANGEMENT

In this chapter, which is based on [Thi15], we introduce the *m-Shi arrangement* of an irreducible crystallographic root system Φ, a hyperplane arrangement in V whose study forms a major part of nonnesting Fuß-Catalan combinatorics in general and of this thesis in particular. There is a close link between this chapter and the previous one: the regions of the m-Shi arrangement can be indexed by their *minimal alcoves*, which correspond to the $(mh+1)$-stable affine Weyl group elements, the Fuß-Catalan specialisation ($p = mh+1$) of the set $\widetilde{W}^p$ considered in Chapter 3.

4.1 THE SHI ARRANGEMENT

The *Shi arrangement* is the affine hyperplane arrangement given by all the hyperplanes H_α^d for $\alpha \in \Phi^+$ and $d = 0, 1$. It was first introduced in [Shi87b] and arose from the study of the Kazhdan-Lusztig cells of the affine Weyl group of type $\widetilde{A}_{n-1}$. The complement of these hyperplanes, we call them *Shi hyperplanes*, falls apart into connected components which we call the *regions* of the Shi arrangement, or Shi regions for short. We call a Shi region *dominant* if it is contained in the dominant chamber.

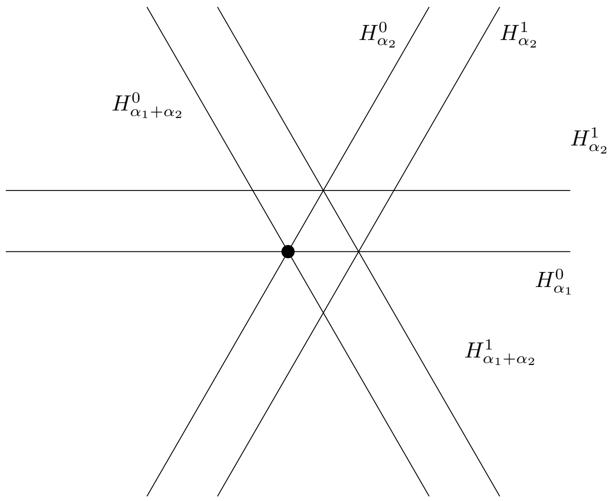

Figure 4.1.1: The Shi arrangement of the root system of type A_2. It has 16 regions.

An *ideal* in the root poset is a subset $I \subseteq \Phi^+$ such that whenever $\alpha \in I$ and $\beta \leq \alpha$, then $\beta \in I$. Dually, we define an *order filter* as a subset $J \subseteq \Phi^+$ such that, whenever $\alpha \in J$ and $\alpha \leq \beta$, then $\beta \in J$. For a dominant Shi region R define

$$\phi(R) := \{\alpha \in \Phi^+ : \langle x, \alpha \rangle > 1 \text{ for all } x \in R\}.$$

It is easy to see that $\phi(R)$ is an order filter in the root poset of Φ. In fact, ϕ even defines a bijection between the set of dominant Shi regions and the set of order filters in the root poset [Shi97, Theorem 1.4].

4.2 THE m-EXTENDED SHI ARRANGEMENT

For a positive integer m, the *m-extended Shi arrangement*, or simply m-Shi arrangement, is the affine hyperplane arrangement given by all the hyperplanes H_α^k for $\alpha \in \Phi^+$ and $-m < k \leq m$. We call them *m-Shi hyperplanes*. The complement of these hyperplanes falls apart into connected components, which we call the *regions* of the m-Shi arrangement, or m-Shi regions for short.

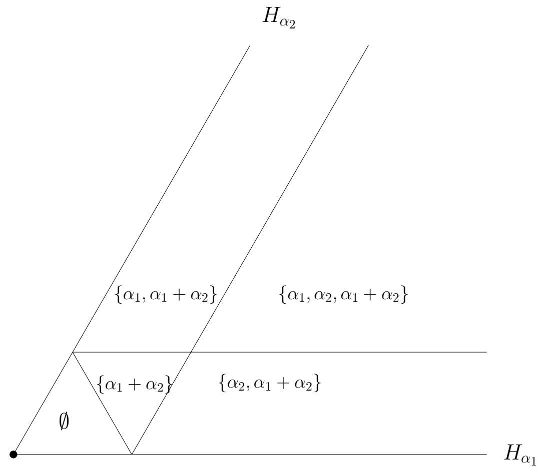

Figure 4.1.2: The 5 dominant Shi regions of the root system of type A_2 together with their corresponding order filters in the root poset.

Notice that the 1-Shi arrangement is exactly the Shi arrangement introduced in Section 4.1.

Following [Ath05], we will encode dominant m-Shi regions by *geometric chains of ideals* or equivalently *geometric chains of order filters*. Suppose $\mathcal{I} = (I_1, I_2, \ldots, I_m)$ is an ascending (multi)chain of m ideals in the root poset of Φ, that is $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_m$. Setting $J_i := \Phi^+ \backslash I_i$ for $i \in [m]$ and $\mathcal{J} := (J_1, J_2, \ldots, J_m)$ gives us the corresponding descending chain of order filters. That is, we have $J_1 \supseteq J_2 \supseteq \ldots \supseteq J_m$. The ascending chain of ideals $\mathcal{I}$ and the corresponding descending chain of order filters $\mathcal{J}$ are both called *geometric* if the following conditions are satisfied simultaneously.

1. $(I_i + I_j) \cap \Phi^+ \subseteq I_{i+j}$ for all $i, j \in \{0, 1, \ldots, m\}$ with $i + j \leq m$, and

2. $(J_i + J_j) \cap \Phi^+ \subseteq J_{i+j}$ for all $i, j \in \{0, 1, \ldots, m\}$.

Here we set $I_0 := \varnothing$, $J_0 := \Phi^+$ and $J_i := J_m$ for $i > m$. We call $\mathcal{I}$ and $\mathcal{J}$ *positive* if $\Delta \subseteq I_m$, or equivalently $\Delta \cap J_m = \varnothing$.

Example. For example, the chain of 2 order filters $\mathcal{J} = (\{\alpha_1 + \alpha_2\}, \{\alpha_1 + \alpha_2\})$ in the root system of type A_2 is not geometric, since the corresponding chain of ideals $\mathcal{I} = (\{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_2\})$ has $\alpha_1, \alpha_2 \in I_1$, but $\alpha_1 + \alpha_2 \notin I_{1+1} = I_2$.

If R is a dominant m-Shi region define $\theta(R) := (I_1, I_2, \ldots, I_m)$ and $\phi(R) := (J_1, J_2, \ldots, J_m)$, where

$$I_i := \{\alpha \in \Phi^+ \mid \langle x, \alpha \rangle < i \text{ for all } x \in R\} \text{ and}$$

$$J_i := \{\alpha \in \Phi^+ \mid \langle x, \alpha \rangle > i \text{ for all } x \in R\}$$

for $i \in \{0, 1, \ldots, m\}$. It is not difficult to verify that $\theta(R)$ is a geometric chain of ideals and that $\phi(R)$ is the corresponding geometric chain of order filters.

In fact θ is a bijection from dominant m-Shi regions to geometric chains of ideals. Equivalently ϕ is a bijection from dominant m-Shi regions to geometric chains of order filters [Ath05, Theorem 3.6].

4.3 MINIMAL ALCOVES OF m-SHI REGIONS

Any alcove of the affine Coxeter arrangement is contained in a unique m-Shi region. We will soon see that for any m-Shi region R there is a unique alcove $\widetilde{w}_R A_\circ \subseteq R$ such that for all $\widetilde{w} A_\circ \subseteq R$ and all $\alpha \in \Phi^+$ we have

$$|k(\widetilde{w}_R, \alpha)| \leq |k(\widetilde{w}, \alpha)|.$$

We call $\widetilde{w}_R A_\circ$ the *minimal alcove* of R. We say that an alcove $\widetilde{w} A_\circ$ is an *m-Shi alcove* if it is the minimal alcove of the m-Shi region containing it. We define $\mathsf{Alc} = \{\widetilde{w}_R A_\circ : R \text{ is an } m\text{-Shi region}\}$ to be the set of m-Shi alcoves.

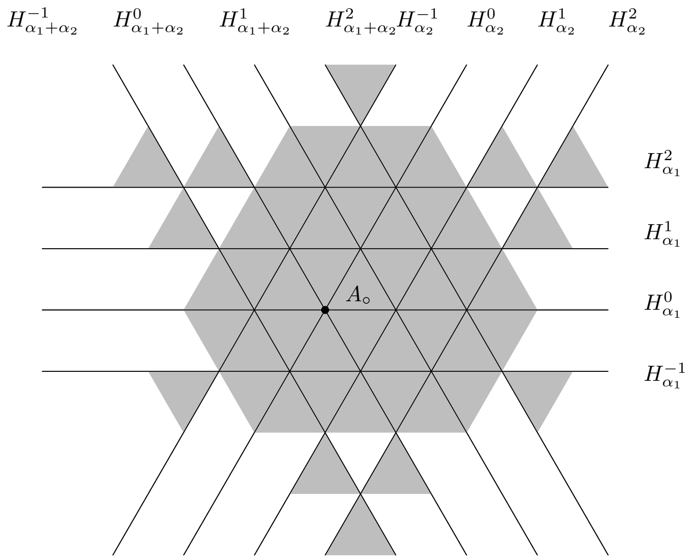

Figure 4.3.1: The 49 minimal alcoves of the 2-Shi arrangement of type A_2.

4.3.1 *The address of a dominant m-Shi alcove*

We first concentrate on dominant m-Shi regions and their minimal alcoves. The following lemma from [Shi87a, Theorem 5.2] gives necessary and sufficient conditions for a tuple $(k_\alpha)_{\alpha \in \Phi^+}$ to be the address of some alcove $\widetilde{w} A_\circ$.

Lemma 4.3.1 ([Ath05, Lemma 2.3]). *Suppose that for each $\alpha \in \Phi^+$ we are given some integer k_α. Then there exists $\widetilde{w} \in \widetilde{W}$ with $k(\widetilde{w}, \alpha) = k_\alpha$ for all $\alpha \in \Phi^+$ if and only if*

$$k_\alpha + k_\beta \leq k_{\alpha+\beta} \leq k_\alpha + k_\beta + 1$$

for all $\alpha, \beta \in \Phi^+$ with $\alpha + \beta \in \Phi^+$.

For a geometric chain of order filters $\mathcal{J} = (J_1, J_2, \ldots, J_m)$ and $\alpha \in \Phi^+$, define

$$k_\alpha(\mathcal{J}) = \max\{k_1 + k_2 + \ldots + k_l : \alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_l \text{ and } \alpha_i \in J_{k_i} \text{ for all } i \in [l]\}$$

where $k_i \in \{0, 1, \ldots, m\}$ for all $i \in [l]$.

It turns out that the integer tuple $(k_\alpha(\mathcal{J}))_{\alpha \in \Phi^+}$ satisfies the conditions of Lemma 4.3.1 [Ath05, Corollary 3.4], so there is a unique $\widetilde{w} \in \widetilde{W}$ with

$$k(\widetilde{w}, \alpha) = k_\alpha(\mathcal{J}) \text{ for all } \alpha \in \Phi^+.$$

The alcove $\widetilde{w} A_\circ$ is exactly the minimal alcove $\widetilde{w}_R A_\circ$ of the dominant m-Shi region $R := \phi^{-1}(\mathcal{J})$ corresponding to $\mathcal{J}$ [Ath05, Proposition 3.7].

4.3.2 *Floors of dominant m-Shi regions and alcoves*

The floors of a dominant m-Shi region R can be seen in the corresponding geometric chain of order filters $\mathcal{J} := \phi(R)$ as follows. If k is a positive integer, a root $\alpha \in \Phi^+$ is called a *rank k indecomposable element* of a geometric chain of order filters $\mathcal{J} = (J_1, J_2, \ldots, J_m)$ if the following hold:

1. $k_\alpha(\mathcal{J}) = k$,
2. $\alpha \notin J_i + J_j$ for any $i, j \in \{0, 1, \ldots, m\}$ with $i + j = k$ and
3. if $\alpha + \beta \in J_t$ and $k_{\alpha+\beta}(J) = t \leq m$ for some $\beta \in \Phi^+$, then $\beta \in J_{t-k}$.

The following theorem relates the indecomposable elements of $\mathcal{J}$ to the floors of R and $\widetilde{w}_R A_\circ$.

Theorem 4.3.2 ([Ath05, Theorem 3.11])**.** *If R is a dominant m-Shi region, $\mathcal{J} = \phi(R)$ is the corresponding geometric chain of order filters and $\alpha \in \Phi^+$, then the following are equivalent:*

1. *α is a rank k indecomposable element of $\mathcal{J}$,*
2. *H_α^k is a floor of R, and*
3. *H_α^k is a floor of $\widetilde{w}_R A_\circ$.*

4.3.3 *m-Shi regions and alcoves in other chambers*

The following easy lemma, generalising [ARR15, Lemma 10.2], describes what the m-Shi arrangement looks like in each chamber.

Lemma 4.3.3. *For $w \in W$, the m-Shi hyperplanes that intersect the chamber wC are exactly those of the form $H_{w(\alpha)}^k$ where $\alpha \in \Phi^+$ and either $1 \leq k < m$ or $k = m$ and $w(\alpha) \in \Phi^+$.*

Proof. If an m-Shi hyperplane H_β^k with $\beta \in \Phi$ and $1 \leq k \leq m$ intersects wC, then there is some $x \in wC$ with $\langle x, \beta \rangle = k$. So $w^{-1}(x) \in C$ and $\langle w^{-1}(x), w^{-1}(\beta) \rangle = k > 0$, thus $\alpha := w^{-1}(\beta) \in \Phi^+$. If $k = m$, then $\beta = w(\alpha) \in \Phi^+$ since otherwise H_β^k is not an m-Shi hyperplane.

Conversely, if $\alpha \in \Phi^+$ and either $1 \leq k < m$ or $k = m$ and $w(\alpha) \in \Phi^+$, then $H_{w(\alpha)}^k$ is an m-Shi hyperplane. Take $x \in C$ with $\langle x, \alpha \rangle = k$. Then $w(x) \in wC$ and $\langle w(x), w(\alpha) \rangle = k$, so $H_{w(\alpha)}^k$ intersects wC. □

We are now ready for our first main theorem about minimal alcoves of m-Shi regions, which we will use frequently and without mention. It is already known for dominant regions [Ath05, Proposition 3.7, Theorem 3.11].

Theorem 4.3.4. *Every region R of the m-Shi arrangement contains a unique minimal alcove $\widetilde{w}_R A_\circ$. That is, for any $\alpha \in \Phi^+$ and $\widetilde{w} \in \widetilde{W}$ such that $\widetilde{w} A_\circ \subseteq R$, we have $|k(\widetilde{w}_R, \alpha)| \leq |k(\widetilde{w}, \alpha)|$. The floors of $\widetilde{w}_R A_\circ$ are exactly the floors of R.*

Proof. The concept of the proof is as follows. Start with an m-Shi region R contained in the chamber wC. Consider $R_{\mathrm{dom}} := w^{-1} R \subseteq C$. This is not in general an m-Shi region, but it contains a unique m-Shi region $R_{\min}$ that is closest to the origin. We take its minimal alcove $\widetilde{w}_{\min} A_\circ$ and find that $w\widetilde{w}_{\min} A_\circ$ is the minimal alcove of R.

Suppose R is an m-Shi region contained in the chamber wC. Let $R_{\mathrm{dom}} := w^{-1} R \subseteq C$. Notice that R_{dom} need not itself be an m-Shi region. By Lemma 4.3.3, the walls of R are of the form $H_{w(\alpha)}^k$ where $\alpha \in \Phi^+$ and either $0 \leq k < m$ or $k = m$ and $w(\alpha) \in \Phi^+$. Thus the walls of $R_{\mathrm{dom}} = w^{-1} R$ are of the form H_α^k with $\alpha \in \Phi^+$ and either $0 \leq k < m$ or $k = m$ and $w(\alpha) \in \Phi^+$. In particular, they are m-Shi hyperplanes. The only m-Shi hyperplanes H that may intersect R_{dom} are those such that $w(H)$ is not an m-Shi hyperplane, that is those of the form H_α^m with

$w(\alpha) \in -\Phi^+$.

Now suppose R' is a dominant m-Shi region and $\mathcal{J}' = \phi(R')$ is the corresponding geometric chain of order filters. Then $R' \subseteq R_{\text{dom}}$ if and only if for every m-Shi hyperplane H_α^k, whenever all of R_{dom} is on one side of H_α^k, then all of R' is on the same side of it. Equivalently, $R' \subseteq R_{\text{dom}}$ precisely when for all $1 \leq k \leq m$ and $\alpha \in \Phi^+$ we have $\alpha \in J_k'$ if $\langle x, \alpha \rangle > k$ for all $x \in R_{\text{dom}}$, and $\alpha \in I_k'$ if $\langle x, \alpha \rangle < k$ for all $x \in R_{\text{dom}}$.

Let $\mathcal{J} = (J_1, J_2, \ldots, J_m)$ be the chain of order filters with $\alpha \in J_k$ if and only if $\langle x, \alpha \rangle > k$ for all $x \in R_{\text{dom}}$. To see that $\mathcal{J}$ is geometric, first note that if $\alpha \in J_i$, $\beta \in J_j$ and $\alpha + \beta \in \Phi^+$, then $\langle x, \alpha + \beta \rangle = \langle x, \alpha \rangle + \langle x, \beta \rangle > i + j$ for all $x \in R_{\text{dom}}$, so $\alpha + \beta \in J_{i+j}$. Let R' be some m-Shi region contained in R_{dom} and let $\mathcal{J}' = \phi(R')$ be the corresponding geometric chain of order filters. Then R' and R_{dom} are on the same side of every m-Shi hyperplane that does not intersect R_{dom}, so in particular $J_k = J_k'$ for $1 \leq k < m$. Whenever $\alpha \in J_m$, then $\langle x, \alpha \rangle > m$ for all $x \in R_{\text{dom}}$, so $\alpha \in J_m'$. Thus $J_m \subseteq J_m'$. If $i + j \leq m$, assume without loss of generality that $i, j > 0$, so that $i, j < m$ and

$$(I_i + I_j) \cap \Phi^+ = (I_i' + I_j') \cap \Phi^+ \subseteq I_{i+j}' \subseteq I_{i+j},$$

since $\mathcal{J}'$ is geometric. This shows that $\mathcal{J}$ is geometric. Thus there is a dominant region $R_{\min} = \phi^{-1}(\mathcal{J})$. We clearly have $\alpha \in J_k$ if $\langle x, \alpha \rangle > k$ for all $x \in R_{\text{dom}}$, and whenever $\langle x, \alpha \rangle < k$ for all $x \in R_{\text{dom}}$, then $\alpha \in I_k' \subseteq I_k$. Thus $R_{\min} \subseteq R_{\text{dom}}$. Observe that $k_\alpha(\mathcal{J}) \leq k_\alpha(\mathcal{J}')$ for all $\alpha \in \Phi^+$. Also note that $\langle x, \alpha \rangle > k_\alpha(\mathcal{J})$ for all $x \in R_{\text{dom}}$.

Let $\widetilde{w}_{\min} A_\circ$ be the minimal alcove of $R_{\min}$ [Ath05]. Thus we have $k(\widetilde{w}_{\min}, \alpha) = k_\alpha(\mathcal{J})$ for all $\alpha \in \Phi^+$. So if $\widetilde{w} A_\circ$ is any alcove contained in R_{dom}, say $\widetilde{w} A_\circ \subseteq R'$ for some m-Shi region $R' \subseteq R_{\text{dom}}$, then if $\mathcal{J}' = \phi(R')$ we have $k(\widetilde{w}_{\min}, \alpha) = k_\alpha(\mathcal{J}) \leq k_\alpha(\mathcal{J}') \leq k(\widetilde{w}, \alpha)$ for all $\alpha \in \Phi^+$.

So if we define $\widetilde{w}_R := w\widetilde{w}_{\min}$, then $\widetilde{w}_R A_\circ \subseteq R$ and $k(\widetilde{w}_R, \alpha) = k(\widetilde{w}_{\min}, w^{-1}(\alpha))$ for all $\alpha \in \Phi$. If $\widetilde{w} A_\circ$ is any alcove contained in R, $\alpha \in \Phi^+$ and $w^{-1}(\alpha) \in \Phi^+$, then

$$k(\widetilde{w}, \alpha) = k(w^{-1}\widetilde{w}, w^{-1}(\alpha)) \geq k(\widetilde{w}_{\min}, w^{-1}(\alpha)) = k(\widetilde{w}_R, \alpha),$$

since $w^{-1}\widetilde{w} A_\circ \subseteq R_{\text{dom}}$. Note that in this case $k(\widetilde{w}_R, \alpha) = k(\widetilde{w}_{\min}, w^{-1}(\alpha)) \geq 0$, since $w^{-1}(\alpha) \in \Phi^+$ and $\widetilde{w}_{\min} A_\circ$ is dominant. If instead $w^{-1}(\alpha) \in -\Phi^+$, then

$$\begin{aligned} k(\widetilde{w}, \alpha) = k(w^{-1}\widetilde{w}, w^{-1}(\alpha)) &= -k(w^{-1}\widetilde{w}, -w^{-1}(\alpha)) - 1 \\ &\leq -k(\widetilde{w}_{\min}, -w^{-1}(\alpha)) - 1 = k(\widetilde{w}_{\min}, w^{-1}(\alpha)) = k(\widetilde{w}_R, \alpha). \end{aligned}$$

Note that in this case, $k(\widetilde{w}_R, \alpha) = -k(\widetilde{w}_{\min}, -w^{-1}(\alpha)) - 1 < 0$. So either way we have $|k(\widetilde{w}_R, \alpha)| \leq |k(\widetilde{w}, \alpha)|$.

Suppose H_α^k is a floor of $\widetilde{w}_R A_\circ$. Then it is the only hyperplane separating $s_\alpha^k \widetilde{w}_R A_\circ$ from $\widetilde{w}_R A_\circ$. Thus $k(s_\alpha^k \widetilde{w}_R, \beta) = k(\widetilde{w}_R, \beta)$ for all $\beta \neq \pm\alpha$ and $|k(s_\alpha^k \widetilde{w}_R, \alpha)| = |k(\widetilde{w}_R, \alpha)| - 1$. Since $\widetilde{w}_R A_\circ$ is the minimal alcove of R this implies that $s_\alpha^k \widetilde{w}_R A_\circ$ is not contained in R. Thus H_α^k must be an m-Shi hyperplane, and therefore a floor of R.

Suppose H_α^k is a floor of R_{dom}, where $\alpha \in \Phi^+$. Then we claim that α is a rank k indecomposable element of $\mathcal{J} = \phi(R_{\min})$. To see this, first note that $\langle x, \alpha \rangle > k$ for all $x \in R_{\text{dom}}$, so $\alpha \in J_k$. Also, $\langle x, \alpha \rangle < k + 1$ for some $x \in R_{\text{dom}}$, so $k_\alpha(\mathcal{J}) = k$. Suppose $\alpha = \beta + \gamma$ with $\beta \in J_i$ and $\gamma \in J_j$ and $i + j = k$. Then $\langle x, \beta \rangle > i$ and $\langle x, \gamma \rangle > j$ imply that $\langle x, \alpha \rangle > k$ for $x \in R_{\text{dom}}$, so H_α^k does not support a facet of R, a contradiction. If $\alpha + \beta \in J_t$ and $k_{\alpha+\beta}(\mathcal{J}) = t \leq m$ for some $\beta \in \Phi^+$ then we have $\langle x, \alpha + \beta \rangle > t$ for all $x \in R_{\text{dom}}$ so we cannot have $\langle x, \beta \rangle < t - k$ for all $x \in R_{\text{dom}}$, since together they would imply that $\langle x, \alpha \rangle > k$ for all $x \in R_{\text{dom}}$, so H_α^k would not support a facet of R. Since $t - k < m$, the hyperplane H_β^{t-k} does not intersect R_{dom}, so this implies that $\langle x, \beta \rangle > t - k$ for all $x \in R_{\text{dom}}$, so $\beta \in J_{t-k}$. This verifies the claim. From the fact that α is a rank k indecomposable element of $\mathcal{J}$ it follows that H_α^k is a floor of $\widetilde{w}_{\min} A_\circ$ by Theorem 4.3.2.

Now suppose that H_α^k is a floor of R. Then $H_{w^{-1}(\alpha)}^k$ is a floor of R_{dom} and thus a floor of $\widetilde{w}_{\min}A_\circ$. So H_α^k is a floor of $\widetilde{w}_R A_\circ = w\widetilde{w}_{\min}A_\circ$. □

4.3.4 *m-Shi alcoves and $(mh+1)$-stable affine Weyl group elements*

The following lemma characterises the m-Shi alcoves. It is a straightforward generalisation of [Shi87b, Proposition 7.3].

Lemma 4.3.5. *An alcove $\widetilde{w}A_\circ$ is an m-Shi alcove if and only if all its floors are m-Shi hyperplanes.*

Proof. The forward implication is immediate from Theorem 4.3.4.

For the backward implication, we prove the contrapositive: we show that every alcove that is not an m-Shi alcove has a floor that is not an m-Shi hyperplane. So suppose $\widetilde{w}A_\circ$ is an alcove contained in an m-Shi region R, and $\widetilde{w} \neq \widetilde{w}_R$. Consider the set

$$\begin{aligned} K = \{x \in V \mid\ & k(\widetilde{w}_R,\alpha) < \langle x,\alpha\rangle < k(\widetilde{w},\alpha)+1 \text{ for all } \alpha \in \Phi \text{ with } k(\widetilde{w}_R,\alpha) \geq 0 \\ & \text{and } k(\widetilde{w},\alpha) < \langle x,\alpha\rangle < k(\widetilde{w}_R,\alpha)+1 \text{ for all } \alpha \in \Phi \text{ with } k(\widetilde{w}_R,\alpha) < 0\}. \end{aligned}$$

Then any alcove $\widetilde{w}'A_\circ$ has either $\widetilde{w}'A_\circ \subseteq K$ or $\widetilde{w}'A_\circ \cap K = \varnothing$. For $\alpha \in \Phi$, we have

$$k(\widetilde{w}_R,\alpha) \leq k(\widetilde{w}',\alpha) \leq k(\widetilde{w},\alpha)$$

whenever $\widetilde{w}'A_\circ \subseteq K$ and $k(\widetilde{w}_R,\alpha) \geq 0$. Similarly

$$k(\widetilde{w},\alpha) \leq k(\widetilde{w}',\alpha) \leq k(\widetilde{w}_R,\alpha)$$

whenever $\widetilde{w}'A_\circ \subseteq K$ and $k(\widetilde{w}_R,\alpha) < 0$. Thus any hyperplane of the affine Coxeter arrangement that separates two alcoves contained in K also separates $\widetilde{w}_R A_\circ$ and $\widetilde{w}A_\circ$. Since no m-Shi hyperplane separates $\widetilde{w}_R A_\circ$ and $\widetilde{w}A_\circ$, no m-Shi hyperplane separates two alcoves contained in K. Since K is convex, there exists a sequence $(\widetilde{w}_1,\widetilde{w}_2,\ldots,\widetilde{w}_l)$ with $\widetilde{w}_1 = \widetilde{w}$, $\widetilde{w}_l = \widetilde{w}_R$, and $\widetilde{w}_i A_\circ \subseteq K$ for all $i \in [l]$, such that $\widetilde{w}_i A_\circ$ shares a facet with $\widetilde{w}_{i+1}A_\circ$ for all $i \in [l-1]$. So the supporting hyperplane of the common facet of $\widetilde{w}_1 A_\circ = \widetilde{w}A_\circ$ and $\widetilde{w}_2 A_\circ$ is a floor of $\widetilde{w}A_\circ$ which is not an m-Shi hyperplane. □

We can now relate the m-Shi alcoves to the Fuß-Catalan ($p = mh+1$) case of the set of p-stable affine Weyl group elements $\widetilde{W}^p$ defined in Section 3.6.1.

Theorem 4.3.6. *An alcove $\widetilde{w}A_\circ$ is an m-Shi alcove if and only if $\widetilde{w} \in \widetilde{W}^{mh+1}$.*

Proof. First note that

$$\widetilde{\Phi}_{mh+1} = \widetilde{\Phi}_1 + m\delta = \widetilde{\Delta} + m\delta.$$

Suppose $\widetilde{w}A_\circ$ is an m-Shi alcove and take

$$\alpha + k\delta \in \widetilde{w}(\widetilde{\Phi}_{mh+1}) = \widetilde{w}(\widetilde{\Delta}+m\delta) = \widetilde{w}(\widetilde{\Delta}) + m\delta.$$

So $\alpha + (k-m)\delta \in \widetilde{w}(\widetilde{\Delta})$ and thus $\widetilde{w}^{-1}(-\alpha+(m-k)\delta) \in -\widetilde{\Delta}$. If $k \geq m$ then $\alpha + k\delta \in \widetilde{\Phi}^+$. Otherwise by Lemma 2.2.2 $H_{-\alpha}^{k-m}$ is a floor of $\widetilde{w}A_\circ$ and thus by Lemma 4.3.5 an m-Shi hyperplane. So $k \geq 0$ and $k > 0$ if $\alpha \in -\Phi^+$. Thus $\alpha + k\delta \in \widetilde{\Phi}^+$. So $\widetilde{w}(\widetilde{\Phi}_{mh+1}) \subseteq \widetilde{\Phi}^+$ and therefore $\widetilde{w} \in \widetilde{W}^{mh+1}$.

Conversely suppose $\widetilde{w} \in \widetilde{W}^{mh+1}$ and H_α^{-k} is a floor of $\widetilde{w}A_\circ$ where $k > 0$. Then by Lemma 2.2.2 we have $\widetilde{w}^{-1}(\alpha + k\delta) \in -\widetilde{\Delta}$. Thus

$$-\alpha + (m-k)\delta \in \widetilde{w}(\widetilde{\Phi}_{mh+1}) \subseteq \widetilde{\Phi}^+.$$

So $k \leq m$ and $k < m$ if $\alpha \in \Phi^+$. Thus H_α^{-k} is an m-Shi hyperplane. So all floors of $\widetilde{w}A_\circ$ are m-Shi hyperplanes and thus by Lemma 4.3.5 $\widetilde{w}A_\circ$ is an m-Shi alcove. □

4.4 ENUMERATIVE CONSEQUENCES

From Theorem 4.3.6 and Corollary 3.6.6 we deduce the following theorem.

Theorem 4.4.1. *The number of m-Shi alcoves equals* $(mh+1)^r$.

We use Theorem 4.3.4 to recover the following result, originally proved by Yoshinaga using the theory of free arrangements [Yos04, Theorem 1.2].

Theorem 4.4.2. *The number of m-Shi regions equals* $(mh+1)^r$.

From Theorem 4.3.6 and Corollary 3.9.4 we deduce the following theorem, due to Athanasiadis [Ath04, Ath05].

Theorem 4.4.3. *The number of dominant m-Shi alcoves (and therefore also the number of dominant m-Shi regions) is the Fuß-Catalan number*

$$\mathsf{Cat}_{\Phi}^{(m)} = \mathsf{Cat}_{mh+1/\Phi} = \frac{1}{|W|}\prod_{i=1}^{r}(mh+1+e_i).$$

THE ZETA MAP 5

In this chapter, which is based on [Thi15], we generalise the zeta map ζ_{HL} of Haglund and Loehr [Hag08, Theorem 5.6] to all irreducible crystallographic root systems Φ, and also to the Fuß-Catalan level of generality. We start by explaining the origin and significance of the zeta map ζ_{HL} in the context of the Hilbert series of the space of diagonal harmonics.

5.1 THE SPACE OF DIAGONAL HARMONICS

Consider the polynomial ring $R := \mathbb{Q}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ in two sets of variables $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. Let the symmetric group S_n act on it by permuting both sets of variables simultaneously. Let I be the ideal generated by the polynomials of positive degree that are invariant under this action. Then R/I is called the *ring of diagonal coinvariants*. As a vector space, we may also realise it as a subspace of R. To do this, consider the inner product on R defined by

$$\langle f, g\rangle := (f(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, \partial_{y_1}, \partial_{y_2}, \ldots, \partial_{y_n})g(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n))(0),$$

where ∂ denotes the partial derivative. Then the orthogonal complement $\mathsf{DH} := I^{\perp}$ of I in R is called the *space of diagonal harmonics* [Hai94]. It is isomorphic to R/I as a bigraded vector space, graded by degree in the x-variables as well as by degree in the y-variables.

5.1.1 *The Hilbert series*

For any $i, j \in \mathbb{N}$, we let DH_{ij} be the homogeneous component of DH that has degree i in the x-variables and degree j in the y-variables. Define the bivariate Hilbert series

$$\mathcal{DH}_n(q,t) := \sum_{i,j\geq 0} \dim(\mathsf{DH}_{ij}) q^i t^j.$$

We will explain two conjectural combinatorial interpretations of $\mathcal{DH}_n(q,t)$. Both are due to Haglund and Loehr [HL05].

> **Example.** For $n = 2$, the space of diagonal harmonics is $\mathsf{DH} = \mathbb{Q}\{1, x_1 - x_2, y_1 - y_2\}$. Its bigraded Hilbert series is $\mathcal{DH}_2(q,t) = 1 + q + t$.

5.2 VERTICALLY LABELLED DYCK PATHS

The first combinatorial interpretation involves an object we have essentially already encountered in Section 3.1.3: vertically labelled Dyck paths. A *Dyck path* of length n is a lattice path in $\mathbb{Z}^2$ consisting of North and East steps that goes from $(0,0)$ to (n,n) and never goes below the diagonal $x = y$. Equivalently, it is a rational $(n+1, n)$-Dyck path with its final step (always an East step) removed. Thus we will not distinguish between Dyck paths and $(n+1, n)$-Dyck paths. We define a *vertically labelled Dyck path* as in Section 3.1.3. In particular, vertically labelled Dyck paths correspond to rational $(n+1, n)$-parking functions. These are also known as *classical parking functions*. So we write $\mathcal{PF}_n := \mathcal{PF}_{n+1/n}$ for the set of vertically labelled Dyck paths of length n. From Corollary 3.1.2 we know that $|\mathcal{PF}_n| = (n+1)^{n-1}$.

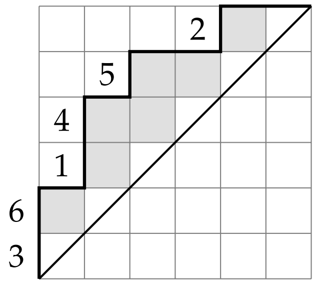

Figure 5.2.1: A vertically labelled Dyck path (P, σ) of length 6 with area vector $(0, 1, 1, 2, 2, 1)$. It has $\operatorname{dinv}'(P, \sigma) = 4$ and $\operatorname{area}(P, \sigma) = 7$.

5.2.1 *The statistics*

If P is a Dyck path corresponding to an increasing classical parking function $(P_1, P_2, \ldots, P_n)$ and $i \in [n]$, let $a_i := i - 1 - P_i$ be the number of boxes between P and the diagonal in the i-th row from the bottom. We call a_i the *area* of row i and the vector $(a_1, a_2, \ldots, a_n)$ the *area vector* of P. For a labelled Dyck path (P, σ) define

$$\operatorname{dinv}'(P, \sigma) := \#\{i < j : a_i = a_j \text{ and } \sigma(i) < \sigma(j)\} + \{i < j : a_i = a_j + 1 \text{ and } \sigma(i) > \sigma(j)\}.$$

We define the *area* of (P, σ) as the number of boxes between P and the diagonal, that is $\operatorname{area}(P, \sigma) := a_1 + a_2 + \ldots + a_n$. Note in particular that $\operatorname{area}(P, \sigma)$ does not depend on σ.

Conjecture 5.2.1 ([Hag08, Conjecture 5.2])**.** *We have*

$$\mathcal{DH}_n(q, t) = \sum_{(P,\sigma) \in \mathcal{PF}_n} q^{\operatorname{dinv}'(P,\sigma)} t^{\operatorname{area}(P,\sigma)}.$$

5.3 DIAGONALLY LABELLED DYCK PATHS

A second equivalent combinatorial interpretation of $\mathcal{DH}_n(q, t)$ is in terms of *diagonally labelled Dyck paths*. A pair (i, j) of positive integers is called a *valley* of a Dyck path D if the i-th East step of D is immediately followed by its j-th North step. The pair (w, D) with $w \in S_n$ is called a diagonally labelled Dyck path if $w(i) < w(j)$ whenever (i, j) is a valley of D. We say that the valley (i, j) is labelled $(w(i), w(j))$. We think of w as labeling the boxes crossed by the diagonal between $(0, 0)$ and (n, n) from bottom to top. So the condition on w is that whenever an East step of D is followed by a North step, the label below the East step is less than the label to the right of the North step. We write $\mathcal{D}_n$ for the set of diagonally labelled Dyck paths on length n.

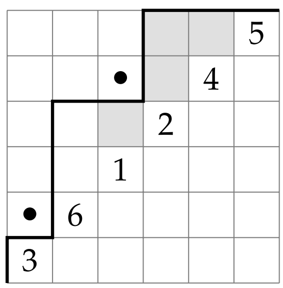

Figure 5.3.1: A diagonally labelled Dyck path (w, D) of length 6. It has $\operatorname{area}'(w, D) = 4$ and $\operatorname{bounce}(w, D) = 7$. The squares contributing to $\operatorname{area}'(w, D)$ are shaded in gray. The valleys are marked by dots.

5.3.1 The statistics

For a diagonally labelled Dyck path (w, D) we define area'(w, D) as the number of boxes below D such that the label below the box is smaller than the label to the right of it.

For the second statistic, we need to define the *bounce path* of a Dyck path D. It is a second Dyck path below D that is determined as follows. Start at $(0, 0)$ and go North until you hit the start of an East step. Then travel East until the diagonal. Then bounce off and go North until you reach the start of another East step. Repeat until you hit (n, n). An integer $i \in [n]$ is called a *bounce* of D if (i, i) lies on the bounce path of D. The *bounce* of a Dyck path is defined as

$$\mathrm{bounce}(D) := \sum_{i \text{ a bounce of } D} n - i$$

The bounce of a diagonally labelled Dyck path (w, D) is $\mathrm{bounce}(w, D) = \mathrm{bounce}(D)$. In particular it does not depend on w.

5.4 THE COMBINATORIAL ZETA MAP

The zeta map ζ_{HL} of Haglund and Loehr is a bijection from $\mathcal{PF}_n$ to $\mathcal{D}_n$ that sends dinv' to area' and area to bounce. Thus ζ_{HL} shows that

$$\sum_{(P,\sigma)\in\mathcal{PF}_n} q^{\mathrm{dinv}'(P,\sigma)} t^{\mathrm{area}(P,\sigma)} = \sum_{(w,D)\in\mathcal{D}_n} q^{\mathrm{area}'(w,D)} t^{\mathrm{bounce}(w,D)}.$$

It is defined as follows. The first ingredient is the zeta map ζ_H of Haglund [Hag08, Theorem 3.15]. It is a bijection from the set of Dyck paths of length n to itself. Given a Dyck path P, iterate the following procedure for $i = 0, 1, \ldots, n$: go through the area vector of P from left to right and draw a North step for every i you see and an East step for every $i - 1$. This gives a Dyck path $\zeta_H(P)$.

The second ingredient is the *diagonal reading word* $\mathsf{drw}(P, \sigma)$ of the vertically labelled Dyck path (P, σ). It is given by first reading the labels of rows of area 0 from bottom to top, then the labels of rows of area 1 from bottom to top, and so on.

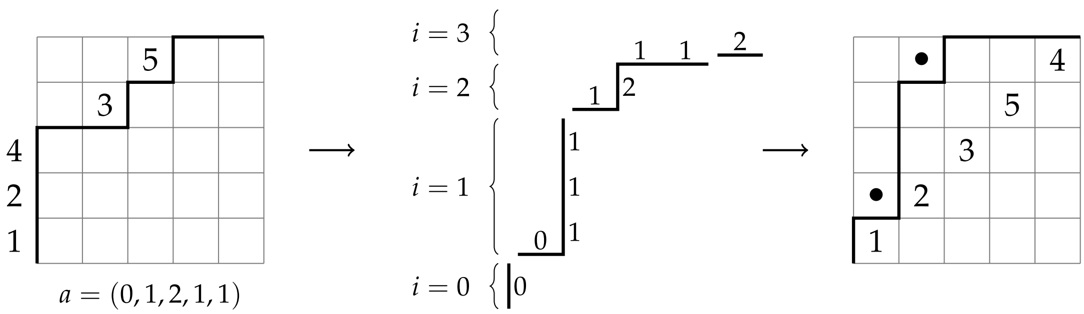

Figure 5.4.1: The zeta map ζ_{HL}: A vertically labelled Dyck path (P, σ) (left), the construction of $\zeta_H(P)$ (middle), and the diagonally labelled Dyck path $\zeta_{HL}(P, \sigma)$ (right).

We define $\zeta_{HL}(P, \sigma) := (\mathsf{drw}(P, \sigma), \zeta_H(P))$. The following is an important property of ζ_{HL}. It inspired the generalisation ζ of ζ_{HL} that will be defined later in this chapter.

Theorem 5.4.1 ([ALW14, Section 5.2]). *For any vertically labelled Dyck path (P, σ) and any pair of positive integers (b, c), the diagonally labelled Dyck path $\zeta_{HL}(P, \sigma) = (\mathsf{drw}(P, \sigma), \zeta_H(P))$ has a valley labelled (b, c) if and only if (P, σ) has a rise labelled (b, c).*

Proof. Let $i \in [n]$ be an index. Let $a = a_i$ be the area of the i-th row of (P, σ) and let $\sigma(i)$ be its label. Suppose that $\sigma(i) = \mathsf{drw}(P, \sigma)(j)$ is the j-th label being read in the diagonal reading word of (P, σ). That means that there are exactly $j - 1$ rows that have either smaller area than row i, or have the same area a and are nearer the bottom. Thus in the construction of $\zeta_H(P)$, the j-th North step is drawn when the area vector entry a_i is read in iteration a, and the j-th East step is drawn when the area vector entry a_i is read in iteration $a + 1$.

Suppose that i is a rise of (P, σ), labelled $(\sigma(i), \sigma(i+1))$. Then $a_{i+1} = a + 1$. So in particular, the label $\sigma(i+1)$ is read later in the diagonal reading word than the label $\sigma(i)$. That is, if $\sigma(i) = \mathsf{drw}(P, \sigma)(j)$ and $\sigma(i+1) = \mathsf{drw}(P, \sigma)(k)$, then $j < k$. In the construction of $\zeta_H(P)$, in the $(a+1)$-st iteration the j-th East step is drawn when a_i is read, and immediately afterwards the k-th North step is drawn when a_{i+1} is read. Thus (j, k) is a valley of $\zeta_H(P)$. In $\zeta_{HL}(P, \sigma) = (\mathsf{drw}(P, \sigma), \zeta_H(P))$ it is labelled $(\mathsf{drw}(P, \sigma)(j), \mathsf{drw}(P, \sigma)(k)) = (\sigma(i), \sigma(i+1))$.

Conversely suppose that (j, k) is a valley of $\zeta_{HL}(P, \sigma)$ labelled $(b, c) = (\mathsf{drw}(P, \sigma)(j), \mathsf{drw}(P, \sigma)(k))$. That is, the j-th East step is immediately followed by the k-th North step. Since every iteration except for the 0-th starts with an East step, both steps must have been drawn in the same iteration, say iteration $a + 1$. Then there is some row i with area a and a row $j > i$ with area $a + 1$ such that no row between i and j has area either a or $a + 1$. This implies $j = i + 1$, so i is a rise of (P, σ). As above, it follows that $\sigma(i) = \mathsf{drw}(P, \sigma)(j)$ and $\sigma(i+1) = \mathsf{drw}(P, \sigma)(k)$, so the rise i is labelled $(\sigma(i), \sigma(i+1)) = (b, c)$. ☐

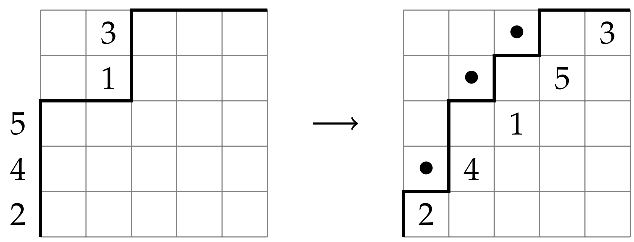

Figure 5.4.2: The zeta map ζ_{HL}: The vertically labelled Dyck path (P, σ) on the left is mapped to the diagonally labelled Dyck path (w, D) on the right.

A Dyck path is uniquely determined by its valleys. Thus Theorem 5.4.1 gives rise to an alternative description of ζ_{HL}. We have $\zeta_{HL}(P, \sigma) = (\mathsf{drw}(P, \sigma), D)$, where D is the unique Dyck path such that $(\mathsf{drw}(P, \sigma), D)$ has a valley labelled (b, c) if and only if (P, σ) has a rise labelled (b, c).

5.5 THE UNIFORM ZETA MAP

We will describe a uniform generalisation ζ of the zeta map ζ_{HL} of Haglund and Loehr to all irreducible crystallographic root systems, and also to the Fuß-Catalan level of generality. Recall from Section 3.7.1 that $\mathcal{PF}_n = \mathcal{PF}_{n+1/n}$ is naturally in bijection with the finite torus $\check{Q}/(n+1)\check{Q}$ of the root system of type A_{n-1}. The Fuß-Catalan generalisation is given by $\check{Q}/(mh+1)\check{Q}$ for m a positive integer. It remains to find an interpretation of $\mathcal{D}_n$ in terms of the root system of type A_{n-1}, and also a Fuß-Catalan generalisation. The next section provides the appropriate uniform generalisation of $\mathcal{D}_n$ to all irreducible crystallographic root systems and also to the Fuß-Catalan level, though a demonstration of this fact will have to wait until later.

5.5.1 *The nonnesting parking functions*

The set of *nonnesting parking functions* Park_Φ of an irreducible crystallographic root system Φ was introduced by Armstrong, Reiner and Rhoades [ARR15]. It was defined in order to combine two desirable properties: being naturally in bijection with the Shi regions of Φ and carrying a natural W-action such that Park_Φ is isomorphic to the finite torus $\check{Q}/(n+1)\check{Q}$ as a W-set. The latter property justifies the name "nonnesting parking functions", since $\mathsf{Park}_\Phi \cong \check{Q}/(n+1)\check{Q} \cong \mathcal{PF}_n$ if Φ is of type A_{n-1}.

The set of *m-nonnesting parking functions* $\mathsf{Park}_\Phi^{(m)}$ is the natural Fuß-Catalan generalisation of Park_Φ. It was introduced by Rhoades [Rho14]. Given a geometric chain $\mathcal{J}$ of m order filters in the root poset of Φ, define $\mathrm{ind}(\mathcal{J})$ as the set of rank m indecomposable elements of $\mathcal{J}$ (see Section 4.3.2) and let

$$W_{\mathcal{J}} = \langle \{s_\alpha : \alpha \in \mathrm{ind}(\mathcal{J})\}\rangle.$$

The set $\mathsf{Park}_\Phi^{(m)}$ of *m-nonnesting parking functions* of Φ is the set of equivalence classes of pairs $(w, \mathcal{J})$ with $w \in W$ and $\mathcal{J}$ a geometric chain of m order filters under the equivalence relation

$$(w_1, \mathcal{J}_1) \sim (w_2, \mathcal{J}_2) \text{ if and only if } \mathcal{J}_1 = \mathcal{J}_2 \text{ and } w_1 W_{\mathcal{J}_1} = w_2 W_{\mathcal{J}_1}.$$

$\mathsf{Park}_\Phi^{(m)}$ is endowed with a left action of W defined by

$$u \cdot [w, \mathcal{J}] := [uw, \mathcal{J}]$$

for $u \in W$.

All the rank m indecomposable elements of a geometric chain of order filters $\mathcal{J} = (J_1, J_2, \ldots, J_m)$ are minimal elements of J_m by Lemma 7.3.2[1]. Thus in particular they are incomparable, that is they form an antichain in the root poset. So there is some $u \in W$ with $I := u(\mathrm{ind}(\mathcal{J})) \subseteq \Delta$ by [Som05, Theorem 6.4]. In particular, $W_{\mathcal{J}}$ is a parabolic subgroup of W and any left coset $wW_{\mathcal{J}}$ of $W_{\mathcal{J}}$ in W has a unique representative w' such that $w'(\mathrm{ind}(\mathcal{J})) \subseteq \Phi^+$.

Lemma 5.5.1. *For any dominant m-Shi region R corresponding to a geometric chain of order filters $\mathcal{J}$ we have*

$$\mathrm{ind}(\mathcal{J}) = \widetilde{w}_R(\widetilde{\Phi}_{mh+1}) \cap \Phi.$$

Proof. For $\alpha \in \Phi^+$, we have the following chain of equivalences.

$$\begin{aligned}
&\alpha \in \mathrm{ind}(\mathcal{J}) \\
&\Leftrightarrow H_\alpha^m \text{ is a floor of } \widetilde{w}_R A_\circ \\
&\Leftrightarrow \widetilde{w}_R^{-1}(-\alpha + m\delta) \in -\widetilde{\Delta} \\
&\Leftrightarrow \widetilde{w}_R^{-1}(-\alpha) \in -\widetilde{\Delta} - m\delta = -\widetilde{\Phi}_{mh+1} \\
&\Leftrightarrow \alpha \in \widetilde{w}_R(\widetilde{\Phi}_{mh+1}).
\end{aligned}$$

Here we used Theorem 4.3.2 and Lemma 2.2.2. □

The following natural bijection relates the m-nonnesting parking functions to the minimal alcoves of the m-Shi arrangement, or equivalently the $(mh+1)$-stable affine Weyl group elements.

Theorem 5.5.2. *The map*

$$\begin{aligned}
\Theta : \mathsf{Park}_\Phi^{(m)} &\to \widetilde{W}^{mh+1} \\
[w, \mathcal{J}] &\mapsto w'\widetilde{w}_R
\end{aligned}$$

is a well-defined bijection. Here w' is the unique representative of $wW_{\mathcal{J}}$ with $w'(\mathrm{ind}(\mathcal{J})) \subseteq \Phi^+$ and $R := \phi^{-1}(\mathcal{J})$ is the dominant m-Shi region corresponding to $\mathcal{J}$.

[1] The proof of this lemma does not depend on any results in this chapter, so there is no circularity.

Proof. The map Θ is well-defined, since if $[w_1, \mathcal{J}_1] = [w_2, \mathcal{J}_2]$ then $\mathcal{J}_1 = \mathcal{J}_2$ and $w_1 W_{\mathcal{J}_1} = w_2 W_{\mathcal{J}_1}$, so $w_1' = w_2'$. Therefore $w_1'\widetilde{w}_{R_1} = w_2'\widetilde{w}_{R_2}$.

To see that $w'\widetilde{w}_R \in \widetilde{W}^{mh+1}$, note that $w'(\widetilde{w}_R(\widetilde{\Phi}_{mh+1}) \cap \Phi) = w'(\mathrm{ind}(\mathcal{J})) \subseteq \Phi^+$ using Lemma 5.5.1. Thus $w'\widetilde{w}_R(\widetilde{\Phi}_{mh+1}) \subseteq \widetilde{\Phi}^+$ and therefore $w'w_R \in \widetilde{W}^{mh+1}$.

To see that Θ is injective, suppose that $\Theta([w_1, \mathcal{J}_1]) = w_1'\widetilde{w}_{R_1} = w_2'\widetilde{w}_{R_2} = \Theta([w_2, \mathcal{J}_2])$. Now $w_1'\widetilde{w}_{R_1}A_\circ \subseteq w_1'C$ and $w_2'\widetilde{w}_{R_2}A_\circ \subseteq w_2'C$, so $w_1' = w_2'$. Thus $\widetilde{w}_{R_1}A_\circ = \widetilde{w}_{R_2}A_\circ$ and therefore $\mathcal{J}_1 = \mathcal{J}_2$. We also get that $w_1 W_{R_1} = w_1' W_{R_1} = w_2' W_{R_1} = w_2 W_{R_1}$, so $[w_1, \mathcal{J}_1] = [w_2, \mathcal{J}_2]$.

To see that Θ is surjective, note that if $\widetilde{w} \in \widetilde{W}^{mh+1}$, say with $\widetilde{w}A_\circ \subseteq wC$, then by Lemma 3.9.2 we have $w^{-1}\widetilde{w}_R \in \widetilde{W}^{mh+1}_{\mathrm{dom}}$. Thus by Lemma 4.3.6 $w^{-1}\widetilde{w}A_\circ$ is the minimal alcove of a dominant m-Shi region R_{dom} corresponding to some geometric chain of order filters $\mathcal{J}$. Furthermore

$$w(\mathrm{ind}(\mathcal{J})) = w(\widetilde{w}_{R_{\mathrm{dom}}}(\widetilde{\Phi}_{mh+1}) \cap \Phi) = \widetilde{w}(\widetilde{\Phi}_{mh+1}) \cap \Phi \subseteq \Phi^+$$

using Lemma 5.5.1 and that $\widetilde{w} \in \widetilde{W}^{mh+1}$. Thus $\Theta([w, R_{\mathrm{dom}}]) = w\widetilde{w}_{R_{\mathrm{dom}}} = \widetilde{w}$. □

A similar bijection using ceilings instead of floors was given for the special case where $m = 1$ in [ARR15, Proposition 10.3]. Note that the proof furnishes a description of Θ^{-1}: we have $\Theta^{-1}(\widetilde{w}) = [w, R_{\mathrm{dom}}]$ where $\widetilde{w}A_\circ \in wC$ and R_{dom} is the m-Shi region containing $w^{-1}\widetilde{w}A_\circ$.

5.5.2 *m-nonnesting parking functions and the finite torus*

In [Rho14, Proposition 9.9] it is shown that there is a W-set isomorphism[2] from $\mathsf{Park}^{(m)}_{\Phi}$ to $\check{Q}/(mh+1)\check{Q}$. The following theorem makes this isomorphism explicit.

Theorem 5.5.3. *The map*

$$\begin{aligned} \Gamma : \mathsf{Park}^{(m)}_{\Phi} &\to \check{Q}/(mh+1)\check{Q} \\ [w, \mathcal{J}] &\mapsto w\widetilde{w}_R\widetilde{w}^{-1}_{mh+1} \cdot 0 + (mh+1)\check{Q}, \end{aligned}$$

where R is the dominant m-Shi region corresponding to $\mathcal{J}$, is a W-set isomorphism. In addition, we have $\Gamma = \mathcal{A} \circ \Theta$.

Proof. We will first show that $\Gamma = \mathcal{A} \circ \Theta$. First note that by Theorem 3.6.8 and Lemma 5.5.1 we have

$$\begin{aligned} &\mathsf{Stab}(\widetilde{w}_R\widetilde{w}^{-1}_{mh+1} \cdot 0 + (mh+1)\check{Q}) \\ &= \mathsf{Stab}(\mathcal{A}(\widetilde{w})) \\ &= \langle \{ s_\beta : \beta \in \widetilde{w}(\widetilde{\Phi}_{mh+1}) \cap \Phi \} \rangle \\ &= \langle \{ s_\beta : \beta \in \mathrm{ind}(\mathcal{J}) \} \rangle \\ &= W_{\mathcal{J}} \end{aligned}$$

Let w' be the unique element of $wW_{\mathcal{J}}$ with $w'(\mathrm{ind}(\mathcal{J})) \subseteq \Phi^+$. We calculate that

$$\begin{aligned} \Gamma([w, \mathcal{J}]) &= w\widetilde{w}_R\widetilde{w}^{-1}_{mh+1} \cdot 0 + (mh+1)\check{Q} \\ &= w'\widetilde{w}_R\widetilde{w}^{-1}_{mh+1} \cdot 0 + (mh+1)\check{Q} \\ &= \mathcal{A}(w'\widetilde{w}_R A_\circ) \\ &= \mathcal{A}(\Theta([w, \mathcal{J}])), \end{aligned}$$

[2] Rhoades mistakenly writes the root lattice Q in place of the coroot lattice $\check{Q}$. However, his result still stands as written: it turns out that $Q/(mh+1)Q$ and $\check{Q}/(mh+1)\check{Q}$ are isomorphic as W-sets.

using that $w^{-1}w' \in W_{\mathcal{J}} = \mathrm{Stab}(\widetilde{w}_R\widetilde{w}_{mh+1}^{-1} \cdot 0 + (mh+1)\check{Q})$. So $\Gamma = \mathcal{A} \circ \Theta$ is a well-defined bijection. Since for $u \in W$ we have

$$\begin{aligned}\Gamma(u \cdot [w, \mathcal{J}]) &= \Gamma([uw, \mathcal{J}]) \\ &= uw\widetilde{w}_R\widetilde{w}_{mh+1}^{-1} \cdot 0 + (mh+1)\check{Q} \\ &= u \cdot \Gamma([w, \mathcal{J}])\end{aligned}$$

we see that Γ is a W-set isomorphism. □

We define the *zeta map* as $\zeta := \Gamma^{-1} = \Theta^{-1} \circ \mathcal{A}^{-1}$.

Theorem 5.5.4. *The map ζ is a W-set isomorphism from $\check{Q}/(mh+1)\check{Q}$ to $\mathsf{Park}_\Phi^{(m)}$.*

5.5.3 *The type of a geometric chain of m order filters*

Recall from Section 3.8.3 the notion of the type of a transitive W-set. The type of a W-orbit $W[e, \mathcal{J}]$ of $\mathsf{Park}_\Phi^{(m)}$ is the type of the stabilizer $W_{\mathcal{J}}$ of $[e, \mathcal{J}]$. So we define the type of a geometric chain of $\mathcal{J}$ order filters to be the type of $W_{\mathcal{J}}$. We get the following theorem.

Theorem 5.5.5. *If $\mathcal{J}$ is a geometric chain of m order filters, then $W_{\mathcal{J}}$ is a parabolic subgroup of W. For any parabolic type $\mathcal{T}$, the number of geometric chains of m order filters of type $\mathcal{T}$ is the rational Kreweras number $\mathsf{Krew}_{mh+1/\Phi}(\mathcal{T})$.*

Proof. The orbits of $\mathsf{Park}_\Phi^{(m)}$ are indexed by the geometric chains of m order filters, and the type of a W-orbit $W[e, \mathcal{J}]$ is the type of $\mathcal{J}$. Thus the number of geometric chain of m order filters of type $\mathcal{T}$ is the number of W-orbits of $\mathsf{Park}_\Phi^{(m)}$ of type $\mathcal{T}$, which equals the number of W-orbits of $\check{Q}/(mh+1)\check{Q}$ of type $\mathcal{T}$ by Theorem 5.5.3. But by Theorem 3.8.3 this equals the rational Kreweras number $\mathsf{Krew}_{mh+1/\Phi}(\mathcal{T})$ if $\mathcal{T}$ is a parabolic type, and 0 otherwise. □

We may also call $\mathsf{Krew}_\Phi^{(m)}(\mathcal{T}) := \mathsf{Krew}_{mh+1/\Phi}(\mathcal{T})$ a *Fuß-Kreweras number*. Note that since $\mathsf{Krew}_{p/\Phi}(\mathcal{T})$ is a polynomial in p, $\mathsf{Krew}_\Phi^{(m)}(\mathcal{T})$ is polynomial in m.

5.5.4 *The rank of a geometric chain of m order filters*

The *rank* $r(W')$ of a parabolic subgroup W' of W is the minimal number of reflections in W' needed to generate W'. Equivalently, $r(W') := |I|$ for any standard parabolic subgroup W_I conjugate to W'. If $X \in \mathcal{L}$ and $W' = \mathsf{Iso}(X)$, then $r(W') = r - \dim(X)$. The rank $r(\mathcal{T})$ of a type $\mathcal{T}$ is the rank of any $W' \in \mathcal{T}$.

If $\mathcal{J}$ is geometric chain of m order filters, recall from Section 5.5.1 that there is some $u \in W$ with $u(\mathrm{ind}(\mathcal{J})) \subseteq \Delta$. So if $I = u(\mathrm{ind}(\mathcal{J}))$, then

$$r(W_{\mathcal{J}}) = r(W_I) = |I| = |\,\mathrm{ind}(\mathcal{J})|.$$

Thus we deduce the following corollary.

Corollary 5.5.6. *For i a nonnegative integer, the number of geometric chain of m order filters with $|\,\mathrm{ind}(\mathcal{J})| = i$ equals*

$$\mathsf{Nar}_\Phi^{(m)}(i) := \sum_{\substack{\mathcal{T} \text{ parabolic type} \\ r(\mathcal{T})=i}} \mathsf{Krew}_\Phi^{(m)}(\mathcal{T}).$$

Proof. This follows from Theorem 5.5.5 by summing over all parabolic types $\mathcal{T}$ of rank i. □

We call the numbers $\mathsf{Nar}_\Phi^{(m)}(i)$ the *Fuß-Narayana numbers* of Φ. They are polynomial in m and were first considered by Athanasiadis [Ath05].

Example. When Φ is of type A_{n-1}, its i-th Fuß-Narayana number is [Ath05, Section 5.1]

$$\mathsf{Nar}_{\Phi}^{(m)}(i) = \frac{1}{n}\binom{n}{i}\binom{mn}{n-i-1}.$$

Remark Armstrong has defined the Fuß-Narayana numbers as $\mathrm{Nar}^{(m)}(W,i)$, the number of *m-noncrossing partitions* of rank i of the Weyl group $W = W(\Phi)$. These will be defined in Chapter 7. For now we content ourselves with showing that his definition agrees with ours.

Theorem 5.5.7. *We have* $\mathsf{Nar}_{\Phi}^{(m)}(i) = \mathrm{Nar}^{(m)}(W,i)$ *for all* $m > 0$, $i \geq 0$ *and* Φ *an irreducible crystallographic root system with Weyl group* W.

Proof. This is a **case-by-case** check. The numbers $\mathrm{Nar}^{(m)}(W,i)$ have been tabulated by Armstrong as polynomials in m for all types in [Arm09, Figure 3.4]. The Fuß-Narayana numbers $\mathsf{Nar}_{\Phi}^{(m)}(i)$ have been calculated by Athanasiadis for the classical types [Ath05, Section 5]. For the exceptional types, they can be calculated using the definition in Corollary 5.5.6 in terms of the Fuß-Kreweras numbers, which in turn can be computed using the definition in Theorem 3.8.3 and the tables of characteristic polynomials in [OS83]. We find that they agree for every type. $\square$

Together with Corollary 5.5.6, we deduce the following theorem.

Theorem 5.5.8 ([Arm09, Conjecture 5.1.22])**.** *For any irreducible crystallographic root system* Φ *with Weyl group* W, *the number* $\mathrm{Nar}^{(m)}(W,i)$ *of m-noncrossing partitions of rank i of W equals the number* $\mathsf{Nar}_{\Phi}^{(m)}(i)$ *of geometric chains of m order filter* $\mathcal{J}$ *in the root poset of* Φ *with* $|\mathrm{ind}(\mathcal{J})| = i$.

It is an important open problem to find a *uniform* proof of this result.

5.6 THE UNIFORM ZETA MAP AND THE COMBINATORIAL ZETA MAP

Our aim for this section is to relate our zeta map ζ from Theorem 5.5.4 to the combinatorial zeta map ζ_{HL} of Haglund and Loehr introduced in Section 5.4. So for this section, we specialise to the case where Φ is the root system of type A_{n-1} and $m = 1$. The content of this section may be captured in the following commutative diagram of bijections:

$$\begin{tikzcd}
\mathsf{Park}_\Phi \arrow[r, "\Theta"] \arrow[d, "\epsilon"'] & \widetilde{W}^{n+1} = \widetilde{S}_n^{n+1} \arrow[r, "\mathcal{A}"] \arrow[dl, "\delta"'] \arrow[dr, "\mathcal{A}_{GMV}"] & \check{Q}/(h+1)\check{Q} \arrow[d, "\chi"] \arrow[ll, bend right, "\zeta"'] \\
\mathcal{D}_n & & \mathcal{PF}_n \arrow[ll, "\zeta_{HL}"']
\end{tikzcd}$$

The first thing we need to do is introduce the bijections ϵ and δ that relate Park_Φ and $\widetilde{S}_n^{n+1}$ to the set $\mathcal{D}_n$ of diagonally labelled Dyck paths of length n.

5.6.1 *Nonnesting parking functions as diagonally labelled Dyck paths*

It is well-known that one can encode the Shi regions of type A_{n-1} as diagonally labelled Dyck paths [Arm13, Theorem 3]. We take a slightly different approach, and instead view diagonally labelled Dyck paths as encoding nonnesting parking functions.

Consider the part of the integer grid $\mathbb{Z}^2$ with $0 \leq x, y \leq n$. We think of the boxes above the diagonal $x = y$ as corresponding to the roots in Φ^+. Say $[i, j]$ is the box whose top right corner is the lattice point (i, j). If $i < j$ we view $[i, j]$ as corresponding to the root $e_i - e_j \in \Phi^+$. So we have $\alpha \leq \beta$ in the root poset if and only if the box corresponding to α is weakly to the right and weakly below the box corresponding to β.

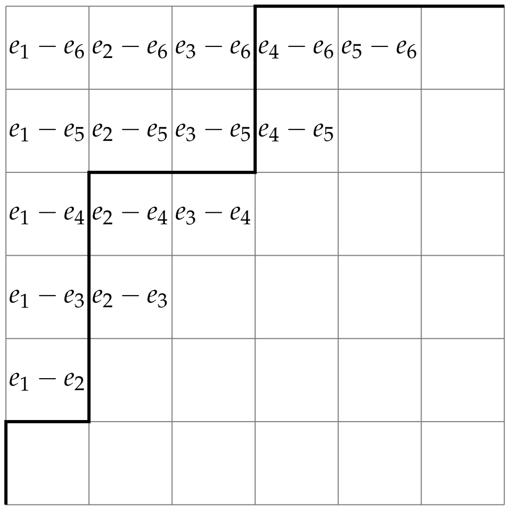

Figure 5.6.1: The Dyck path $D(J)$ corresponding to the order filter J in the root poset of type A_5 whose minimal elements are $e_1 - e_2$ and $e_3 - e_5$.

Note that a geometric chain $\mathcal{J}$ of 1 order filter in the root poset is just a single order filter J. The Dyck path $D(J)$ corresponding to the order filter J is the Dyck path which satisfies

$$\text{The box } [i, j] \text{ is above } D(J) \text{ if and only if } e_i - e_j \in J.$$

Now $\mathrm{ind}(J)$ is exactly the set of minimal elements of J [Ath05]. Thus we have $e_i - e_j \in \mathrm{ind}(J)$ if and only if (i, j) is a valley of $D(J)$.

Take $w \in W$. We have

$$\begin{aligned}
&w(\mathrm{ind}(J)) \subseteq \Phi^+ \\
&\Leftrightarrow w(e_i - e_j) \in \Phi^+ \text{ whenever } (i, j) \text{ is a valley of } D(J) \\
&\Leftrightarrow w(i) < w(j) \text{ whenever } (i, j) \text{ is a valley of } D(J) \\
&\Leftrightarrow (w, D(J)) \text{ is a diagonally labelled Dyck path.}
\end{aligned}$$

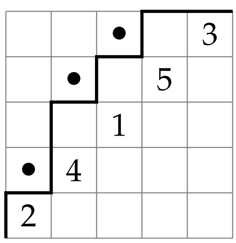

Figure 5.6.2: The diagonally labelled Dyck path $(w, D(J))$ where $w = 24153$ and $\mathrm{ind}(J) = \{e_1 - e_2, e_2 - e_4, e_3 - e_5\}$. The valleys of $D(J)$ are marked by dots.

Lemma 5.6.1. *The map*

$$\begin{aligned}\epsilon : \mathsf{Park}_\Phi &\to \mathcal{D}_n\\ [w,J] &\mapsto (w', D(J)),\end{aligned}$$

where $w' \in wW_J$ is the unique representative with $w'(\mathrm{ind}(J)) \subseteq \Phi^+$, is a bijection.

Proof. The map $J \mapsto D(J)$ is a bijection from order filters in the root poset of type A_{n-1} to Dyck paths of length n. The map ϵ is well-defined since $[w_1, J_1] = [w_2, J_2]$ implies $J_1 = J_2$ and $w_1' = w_2'$, so $\epsilon([w_1, J_1]) = \epsilon([w_2, J_2])$. We see that $(w', D(J)) \in \mathcal{D}_n$ since $w'(\mathrm{ind}(J)) \subseteq \Phi^+$. We see that ϵ is injective since $\epsilon([w_1, J_1]) = \epsilon([w_2, J_2])$ implies $D(J_1) = D(J_2)$, so that $J_1 = J_2$. Furthermore $w_1' = w_2'$, so that $w_1 W_{J_1} = w_2 W_{J_2}$ and thus $[w_1, J_1] = [w_2, J_2]$. We see that ϵ is surjective since for $(w, D) \in \mathcal{D}_n$ we have $(w, D) = \epsilon([w, J])$ where $D = D(J)$. □

Since W_J is generated by the transpositions (ij) such that (i, j) is a valley of $D(J)$ and the condition $w'(\mathrm{ind}(J)) \subseteq \Phi^+$ is equivalent to $w'(i) < w'(j)$ whenever (i, j) is a valley of $D(J)$ we can get w' from w with a simple sorting procedure: for all maximal chains of indices $i_1 < i_2 < \ldots < i_l$ such that (i_j, i_{j+1}) is a valley of $D(J)$ for all $j \in [l-1]$ sort the values of w on positions $i_1, i_2, \ldots, i_l$ increasingly. The result is w'. From this we also get the S_n-action on $\mathcal{D}_n$ that turns ϵ into an S_n-isomorphism: for $u \in S_n$ define

$$u \cdot (w, D) := ((uw)', D)$$

where $(uw)'$ arises from uw through the sorting procedure desribed above. Note the analogy between this action and the S_n-action on $\mathcal{PF}_n$ in terms of vertically labelled Dyck paths that was described in Section 3.1.3.

One may also view diagonally labelled Dyck paths as a combinatorial model for Shi alcoves, or equivalently $(n+1)$-stable affine permutations. The following lemma makes this explicit.

Lemma 5.6.2. *The map*

$$\begin{aligned}\delta : \widetilde{S}_n^{n+1} &\to \mathcal{D}_n\\ \widetilde{w} &\mapsto (w, D),\end{aligned}$$

where $\widetilde{w}A_\circ \in wC$ and $D = D(J) = D(\phi(R_{\mathrm{dom}}))$ is the Dyck path corresponding to the order filter corresponding to the dominant Shi region R_{dom} containing $w^{-1}\widetilde{w}A_\circ$, is a bijection. Furthermore $\delta = \epsilon \circ \Theta^{-1}$.

Proof. An immediate check from the definitions of ϵ and Θ. □

5.6.2 *The zeta maps are equivalent*

The following theorem relates our zeta map ζ from Theorem 5.5.4 to the zeta map ζ_{HL} of Haglund and Loehr. Recall that $\chi = \pi_{\mathcal{PF}}^{-1} \circ \pi_{\check{Q}}$ is the natural S_n-isomorphism from $\check{Q}/(n+1)\check{Q}$ to $\mathcal{PF}_n$.

Theorem 5.6.3. *If Φ is of type A_{n-1} and $m = 1$, then*

$$\begin{aligned}\zeta_{HL} &= \epsilon \circ \zeta \circ \chi^{-1}\\ &= \delta \circ \mathcal{A}_{GMV}^{-1}.\end{aligned}$$

Proof. Define $\zeta' := \epsilon \circ \zeta \circ \chi^{-1}$. We also have

$$\zeta' = \epsilon \circ \zeta \circ \chi^{-1} = \epsilon \circ \Theta^{-1} \circ \mathcal{A}^{-1} \circ \chi^{-1} = \delta \circ \mathcal{A}_{GMV}^{-1}$$

using the definition of ζ, Lemma 5.6.2 and Theorem 3.7.1. We will show that ζ' satisfies the following properties: If $\zeta'(P, \sigma) = (w, D)$ then firstly $w = \mathsf{drw}(P, \sigma)$ and secondly (w, D) has a valley labelled (b,c) if and only if (P,σ) has a rise labelled (b,c) for all b and c. As noted as the end of Section 5.4, these properties define ζ_{HL} uniquely, so we deduce that $\zeta' = \zeta_{HL}$.

First we need to check that if $\zeta'(P,\sigma) = (\delta \circ \mathcal{A}_{GMV}^{-1})(P,\sigma) = (w,D)$, then $w = \mathsf{drw}(P,\sigma)$. Equivalently we need to verify that if $\widetilde{w} \in \widetilde{W}^{n+1}$ with $\widetilde{w}A_\circ \subseteq wC$ and $\mathcal{A}_{GMV}(\widetilde{w}) = (P_{w_R},\sigma)$ then $w = \mathsf{drw}(P_{w_R},\sigma)$.

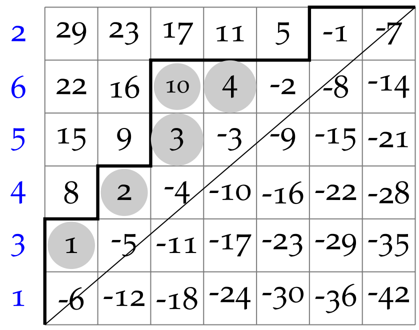

Figure 5.6.3: The vertically labelled 7/6-Dyck path $\mathcal{A}_{GMV}(\widetilde{w})$ for the dominant 7-stable affine permutation $\widetilde{w} = [2,7,3,4,5,0]$. We have that $\widetilde{w}^{-1} = [-4,1,3,4,5,12]$ is affine Grassmanian. The positive beads of the normalized abacus $\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}})$ are shaded in gray.

First suppose that $w = e$ is the identity. That is, $\widetilde{w} \in \widetilde{W}^{n+1}_{\text{dom}}$. So $\widetilde{w}^{-1}$ is *affine Grassmanian*, that is $\widetilde{w}^{-1}(1) < \widetilde{w}^{-1}(2) < \ldots < \widetilde{w}^{-1}(n)$. The set of lowest gaps on the runners of the balanced abacus $\mathsf{A}(\Delta_{\widetilde{w}})$ is

$$\{\widetilde{w}^{-1}(1), \widetilde{w}^{-1}(2), \ldots, \widetilde{w}^{-1}(n)\}.$$

Thus the set of lowest gaps of the normalized abacus $\mathsf{A}(\widetilde{\Delta}_{\widetilde{w}})$ is

$$\{\widetilde{w}^{-1}(1) - M_{\widetilde{w}}, \widetilde{w}^{-1}(2) - M_{\widetilde{w}}, \ldots, \widetilde{w}^{-1}(n) - M_{\widetilde{w}}\},$$

where $M_{\widetilde{w}}$ is the minimal element of $\Delta_{\widetilde{w}}$. This equals the set $\{l_1, l_2, \ldots, l_n\}$ of labels of the boxes to the left of the North steps of the Dyck path $P_{\widetilde{w}}$.

Let $(a_1, a_2, \ldots, a_n)$ be the area vector of $P_{\widetilde{w}}$. Then we have $l_i = na_i + i - 1$. Thus $l_i < l_j$ if and only if either $a_i < a_j$ or $a_i = a_j$ and $i < j$. Furthermore, the label of the i-th North step of $P_{\widetilde{w}}$ is $\sigma(i) = \widetilde{w}(l_i + M_{\widetilde{w}})$. So the j-th label being read in the diagonal reading word is $\mathsf{drw}(P_{\widetilde{w}},\sigma)(j) = \widetilde{w}(l_i + M_{\widetilde{w}})$, where l_i is the j-th smallest element of $\{l_1, l_2, \ldots, l_n\}$. But the j-th smallest element of

$$\{l_1, l_2, \ldots, l_n\} = \{\widetilde{w}^{-1}(1) - M_{\widetilde{w}}, \widetilde{w}^{-1}(2) - M_{\widetilde{w}}, \ldots, \widetilde{w}^{-1}(n) - M_{\widetilde{w}}\}$$

is just $\widetilde{w}^{-1}(j) - M_{\widetilde{w}}$, so $\mathsf{drw}(P_{\widetilde{w}},\sigma)(j) = \widetilde{w}(\widetilde{w}^{-1}(j) - M_{\widetilde{w}} + M_{\widetilde{w}}) = j$. Thus $\mathsf{drw}(P_{\widetilde{w}},\sigma) = e$, as required.

In general if $\widetilde{w}A_\circ \subseteq wC$ then $\widetilde{w} = w\widetilde{w}_D$, where $\widetilde{w}_D \in \widetilde{W}^{n+1}_{\text{dom}}$ using Lemma 3.9.2. We have $\Delta_{\widetilde{w}_D} = \Delta_{\widetilde{w}}$ and thus also $M_{\widetilde{w}_D} = M_{\widetilde{w}}$ and $\widetilde{\Delta}_{\widetilde{w}_D} = \widetilde{\Delta}_{\widetilde{w}}$. Therefore $P_{\widetilde{w}} = P_{\widetilde{w}_D}$ and the tuple $(l_1, l_2, \ldots, l_n)$ is also the same for $\widetilde{w}$ and $\widetilde{w}_D$. Thus the j-th label being read in the diagonal reading word of $\mathcal{A}_{GMV}(\widetilde{w}) = (P_{\widetilde{w}},\sigma)$ is

$$\mathsf{drw}(P_{\widetilde{w}},\sigma)(j) = \widetilde{w}(\widetilde{w}_D^{-1}(j) - M_{\widetilde{w}} + M_{\widetilde{w}}) = w(j).$$

So $\mathsf{drw}(P_{\widetilde{w}},\sigma) = w$, as required.

The second property we need to check is that if $\zeta'(P,\sigma) = (\epsilon \circ \zeta \circ \chi^{-1})(P,\sigma) = (w,D)$ then

(w, D) has a valley labelled (a, b) if and only if (P, σ) has a rise labelled (a, b). But this follows from general considerations: since $\zeta' = \epsilon \circ \zeta \circ \chi^{-1}$ is a composition of S_n-isomorphisms it is itself an S_n-isomorphism. In particular, the S_n-stabilizers of (P, σ) and (w, D) must agree. But (P, σ) has a rise labelled (a, b) if and only if b is the smallest integer with $a < b \leq n$ such that the transposition (ab) fixes (P, σ), and similarly (w, D) has a valley labelled (a, b) if and only if b is the smallest integer with $a < b \leq n$ such that the transposition (ab) fixes (w, D). Thus (w, D) has a valley labelled (a, b) if and only if (P, σ) has a rise labelled (a, b). This concludes the proof. □

5.7 OUTLOOK

Given the uniform zeta map ζ generalising the type A combinatorial zeta map ζ_{HL} to all types, it makes sense to consider finding combinatorial interpretations of ζ also for the other classical types B, C and D. For type C this was accomplished in [ST15].

6 FLOORS AND CEILINGS OF DOMINANT m-SHI REGIONS

In this chapter, which is based on [Thi14a], we prove a conjecture of Armstrong that states that for any two integers k and l the number of dominant m-Shi regions that have exactly l floors of the form H_α^k equals the number of dominant m-Shi regions that have exactly l ceilings of the form H_α^k. To do this, we introduce a uniform bijection that provides even more refined enumerative information.

6.1 THE MAIN RESULT

If M is any set of hyperplanes of the m-Shi arrangement, let $U(M)$ be the set of dominant m-Shi regions R such that all hyperplanes in M are floors of R. Similarly, let $L(M)$ be the set of dominant m-Shi regions R' such that all hyperplanes in M are ceilings of R'. Then we have the following theorem.

Theorem 6.1.1. *For any set $M = \{H_{\alpha_1}^{i_1}, H_{\alpha_2}^{i_2}, \ldots, H_{\alpha_l}^{i_l}\}$ of l hyperplanes with $i_j \in [m]$ and $\alpha_j \in \Phi^+$ for all $j \in [l]$, there is an explicit bijection Θ from $U(M)$ to $L(M)$.*

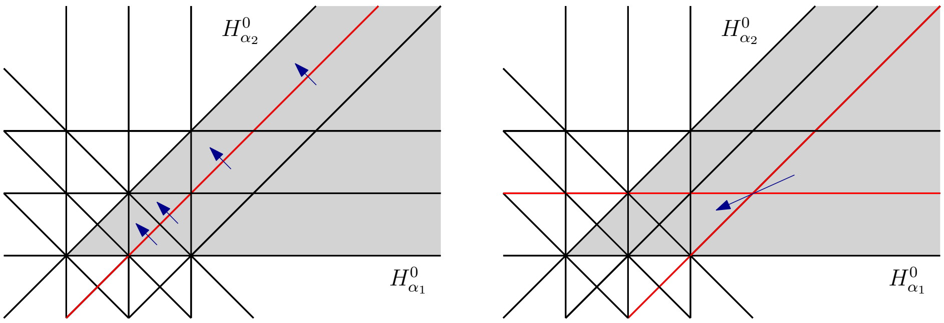

Figure 6.1.1: The bijection Θ for the 2-Shi arrangement of the root system of type B_2, for $M = \{H_{\alpha_2}^1\}$ and for $M = \{H_{\alpha_1}^1, H_{\alpha_2}^2\}$. The dominant chamber is shaded in grey.

See Figure 6.1.1 for an example. From this theorem, we obtain some enumerative corollaries. Say that a hyperplane of the form H_α^k for $\alpha \in \Phi^+$ has *level* k. Let $fl_k(l)$ be the number of dominant m-Shi regions that have exactly l floors of level k, and let $cl_k(l)$ be the number of dominant regions that have exactly l ceilings of level k [Arm09, Definition 5.1.23]. We deduce the following conjecture of Armstrong.

Corollary 6.1.2 ([Arm09, Conjecture 5.1.24])**.** *We have $fl_k(l) = cl_k(l)$ for all $1 \leq k \leq m$ and $l \geq 0$.*

6.2 PRELIMINARIES

Recall from Section 4.3.1 that any dominant m-Shi region R has a unique minimal alcove $A = \widetilde{w}_R A_\circ$ whose address can be obtained from the geometric chain of m order filters $\mathcal{J} = \phi(R)$ corresponding to R. A similar construction associates a unique *maximal* alcove to every *bounded* dominant m-Shi region [AT06]. The ingredients of this are as follows.

For an alcove B and a positive root $\alpha \in \Phi^+$, define

$$r(B, \alpha) = k(B, \alpha) + 1.$$

So we have $r(B, \alpha) - 1 < \langle x, \alpha \rangle < r(B, \alpha)$ for all $x \in B$. Lemma 4.3.1 translates to

Lemma 6.2.1 ([AT06, Lemma 2.3]). *There is an alcove B with $r(B,\alpha) = r_\alpha$ for all $\alpha \in \Phi^+$ if and only if $r_\alpha + r_\beta - 1 \leq r_{\alpha+\beta} \leq r_\alpha + r_\beta$ whenever $\alpha, \beta, \alpha+\beta \in \Phi^+$.*

For a geometric chain of ideals $\mathcal{I} = (I_1, I_2, \ldots, I_m)$, and $\alpha \in \Phi^+$, we define

$$r_\alpha(\mathcal{I}) = \min\{r_1 + r_2 + \ldots + r_l \mid \alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_l \text{ and } \alpha_i \in I_{r_i} \text{ for all } i \in [l]\},$$

where we set $r_\alpha(\mathcal{I}) = \infty$ if α cannot be written as a linear combination of elements in I_m. So $r_\alpha(\mathcal{I}) < \infty$ for all $\alpha \in \Phi^+$ if and only if $\mathcal{I}$ is positive.

The bijection θ^{-1} from Section 4.2 maps a positive geometric chain of ideals $\mathcal{I}$ to the bounded dominant m-Shi region R containing the alcove B with $r(B,\alpha) = r_\alpha(\mathcal{I})$ for all $\alpha \in \Phi^+$ [AT06, Theorem 3.6]. This alcove B is called the *maximal alcove* of R. Its ceilings are exactly the ceilings of R [AT06, Theorem 3.11].

We call $\alpha \in \Phi^+$ a *rank k indecomposable element* [AT06, Definition 3.8] of a geometric chain of ideals $\mathcal{I} = (I_1, I_2, \ldots, I_m)$ if $\alpha \in I_k$ and

1. $r_\alpha(\mathcal{I}) = k$,
2. $\alpha \notin I_i + I_j$ for $i + j = k$ and
3. if $r_{\alpha+\beta}(\mathcal{I}) = t \leq m$ for some $\beta \in \Phi^+$ then $\beta \in I_{t-k}$.

The following theorem relates indecomposable elements to ceilings.

Theorem 6.2.2. *Let R be a dominant m-Shi region, $\mathcal{I} = \theta(R)$ and $\alpha \in \Phi^+$. Then R contains an alcove B such that for all $k \in [m]$ the following are equivalent:*

1. *H_α^k is a ceiling of R,*
2. *α is a rank k indecomposable element of $\mathcal{I}$, and*
3. *H_α^k is a ceiling of B.*

It is already known that Theorem 6.2.2 holds for bounded dominant m-Shi regions [AT06, Theorem 3.11]. In that case, we may take the alcove B to be the maximal alcove of the bounded m-Shi region R.

Our approach to proving Theorem 6.2.2 is to note that when a dominant m-Shi region R is subdivided into $(m+1)$-Shi regions by hyperplanes of the form H_α^{m+1} for $\alpha \in \Phi^+$, at least one of the resulting regions is bounded. We find a dominant $(m+1)$-Shi region $\underline{R}$ which, among the bounded $(m+1)$-Shi regions that are contained in R, is the one furthest away from the origin. We call the maximal alcove B of $\underline{R}$ the *pseudomaximal* alcove of R. It equals the maximal alcove of R if R is bounded. The alcove $B \subseteq R$ will be seen to satisfy the assertion of Theorem 6.2.2. Instead of working directly with the dominant m- and $(m+1)$-Shi regions, we usually phrase our results in terms of the corresponding geometric chains of ideals. We require the following lemmas:

Lemma 6.2.3 ([Ath05, Lemma 2.1 (ii)]). *If $\alpha_1, \alpha_2, \ldots, \alpha_l \in \Phi$ and $\alpha_1 + \alpha_2 + \ldots + \alpha_l = \alpha \in \Phi$, then $\alpha_1 = \alpha$ or there exists i with $2 \leq i \leq l$ such that $\alpha_1 + \alpha_i \in \Phi \cup \{0\}$.*

Lemma 6.2.4 ([AT06, Lemma 3.2]). *For $\alpha \in \Phi^+$ and $r_\alpha(\mathcal{I}) = k \leq m$, we have that $\alpha \in I_k$.*

Lemma 6.2.5 ([AT06, Lemma 3.10]). *Suppose α is an indecomposable element of $\mathcal{I}$. Then*

1. *$r_\alpha(\mathcal{I}) = r_\beta(\mathcal{I}) + r_\gamma(\mathcal{I}) - 1$ if $\alpha = \beta + \gamma$ for $\beta, \gamma \in \Phi^+$ and*
2. *$r_\alpha(\mathcal{I}) + r_\beta(\mathcal{I}) = r_{\alpha+\beta}(\mathcal{I})$ if $\beta, \alpha + \beta \in \Phi^+$.*

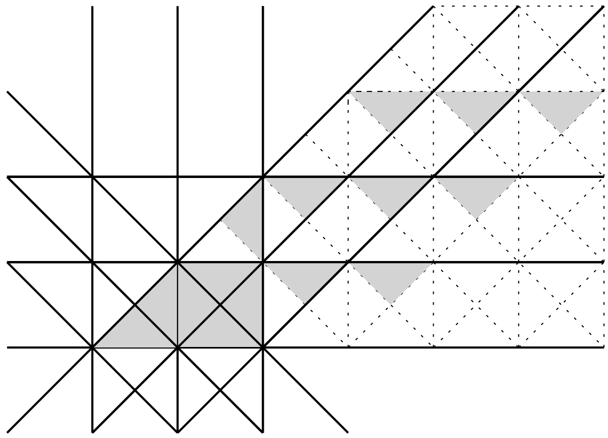

Figure 6.2.1: The dominant 2-Shi regions of the root system of type B_2 together with their pseudomaximal alcove, shaded in grey.

Lemma 6.2.6. *If $\alpha, \beta, \gamma \in \Phi^+$, $\beta+\gamma \in \Phi^+$ and $\alpha \leq \beta+\gamma$, then $\alpha \leq \beta$ or $\alpha \leq \gamma$ or $\alpha = \beta' + \gamma'$ with $\beta', \gamma' \in \Phi^+$, $\beta' \leq \beta$ and $\gamma' \leq \gamma$.*

Proof. Let $\alpha = \beta + \gamma - \sum_{j \in J} \alpha_j$ with $\alpha_j \in \Delta$ for all $j \in J$. We proceed by induction on $|J|$. If $|J| = 0$, we are done. If $|J| = 1$, we have that $\alpha = -\alpha_i + \beta + \gamma$ for some $\alpha_i \in \Delta$. Thus by Lemma 6.2.3, we have either $\alpha = -\alpha_i$ (a contradiction), or $\beta' = \beta - \alpha_i \in \Phi \cup \{0\}$ or $\gamma' = \gamma - \alpha_i \in \Phi \cup \{0\}$. Notice that if $\beta' \neq 0$, then $\beta' \in \Phi^+$, and similarly for γ'. So if $\beta' \in \Phi^+$ we may write $\alpha = \beta' + \gamma$ and otherwise we have $\gamma' \in \Phi^+$ and thus $\alpha = \beta + \gamma'$ as required.

If $|J| > 1$, we have $\alpha + \sum_{j \in J} \alpha_j = \beta + \gamma$, so by Lemma 6.2.3, either $\alpha = \beta + \gamma$, so we are done, or $\alpha + \alpha_j \in \Phi \cup \{0\}$ for some $j \in J$. In the latter case we even have $\alpha + \alpha_j \in \Phi^+$. By induction hypothesis, $\alpha + \alpha_j \leq \beta$ or $\alpha + \alpha_j \leq \gamma$ or $\alpha + \alpha_j = \beta' + \gamma'$ with $\beta', \gamma' \in \Phi^+$, $\beta' \leq \beta$ and $\gamma' \leq \gamma$. In the first two cases, we are done. In the latter case, we have $\alpha = -\alpha_j + \beta' + \gamma'$, so we proceed as in the $|J| = 1$ case. □

We are now ready to define the bounded dominant $(m+1)$-Shi region $\underline{R}$ in terms of the corresponding geometric chain of $m+1$ ideals $\underline{\mathcal{I}}$. For a geometric chain of ideals $\mathcal{I} = (I_1, I_2, \ldots, I_m)$, let $\underline{I}_i = I_i$ for all $i \in [m]$ and let $\underline{I}_{m+1} = \bigcup_{i+j=m+1}((I_i + I_j) \cap \Phi^+) \cup I_m \cup \Delta$. By Lemma 6.2.6, $\underline{I}_{m+1}$ is an ideal. Define $\underline{\mathcal{I}} = (\underline{I}_1, \ldots, \underline{I}_{m+1})$.

Lemma 6.2.7. *If $\mathcal{I} = (I_1, I_2, \ldots, I_m)$ is a geometric chain of m ideals in the root poset of Φ, then $\underline{\mathcal{I}}$ is a positive geometric chain of $m+1$ ideals. The bounded dominant $(m+1)$-Shi region $\underline{R} = \theta^{-1}(\underline{\mathcal{I}})$ is contained in the m-Shi-region $R = \theta^{-1}(\mathcal{I})$.*

Proof. By construction, $\underline{\mathcal{I}}$ is an ascending chain of ideals. If $i + j \leq m$, we have that

$$(\underline{I}_i + \underline{I}_j) \cap \Phi^+ = (I_i + I_j) \cap \Phi^+ \subseteq I_{i+j} = \underline{I}_{i+j}$$

as $\mathcal{I}$ is geometric. If $i + j = m+1$ with $i, j \neq 0$ (otherwise the result is trivial) we have that

$$(\underline{I}_i + \underline{I}_j) \cap \Phi^+ = (I_i + I_j) \cap \Phi^+ \subseteq \bigcup_{i+j=m+1} ((I_i + I_j) \cap \Phi^+) \subseteq \underline{I}_{i+j}.$$

Let $\mathcal{J} = (J_1, J_2, \ldots, J_m)$ be the geometric chain of order filters corresponding to the geometric chain of ideals $\mathcal{I}$. Define $\underline{\mathcal{J}}$ similarly. We need to verify that $(\underline{J}_i + \underline{J}_j) \cap \Phi^+ \subseteq \underline{J}_{i+j}$ for all $i, j \in [m+1]$.

Suppose first that $i + j \leq m$. Then $(\underline{J}_i + \underline{J}_j) \cap \Phi^+ = (J_i + J_j) \cap \Phi^+ \subseteq J_{i+j} = \underline{J}_{i+j}$ since $\mathcal{J}$ is geometric.

Suppose next that $i+j = m+1$. Take any $(m+1)$-Shi region R' that is contained in R. Let $\theta(R') = \mathcal{I}' = (I'_1, I'_2, \ldots, I'_{m+1})$ be the geometric chain of ideals corresponding to R' and let $\mathcal{J}' = (J'_1, J'_2, \ldots, J'_{m+1})$ be the corresponding geometric chain of order filters. Then R and R' are on the same side of each hyperplane of the m-Shi arrangement. Thus $I'_l = I_l$ and $J'_l = J_l$ for $l \in [m]$. Thus we have

$$\underline{I}_{m+1} = \bigcup_{i+j=m+1} ((I_i + I_j) \cap \Phi^+) \cup I_m \cup \Delta = \bigcup_{i+j=m+1} ((I'_i + I'_j) \cap \Phi^+) \cup I'_m \cup \Delta \subseteq I'_{m+1} \cup \Delta$$

since $\mathcal{I}'$ is geometric. Since $\mathcal{J}'$ is geometric, we have

$$(\underline{J}_i + \underline{J}_j) \cap \Phi^+ = (J'_i + J'_j) \cap \Phi^+ \subseteq J'_{i+j} = J'_{m+1}.$$

The sum of two positive roots is never a simple root, so we even have $(\underline{J}_i + \underline{J}_j) \cap \Phi^+ \subseteq J'_{m+1} \backslash \Delta$. But $J'_{m+1} \backslash \Delta \subseteq \underline{J}_{m+1}$, as $\underline{I}_{m+1} \subseteq I'_{m+1} \cup \Delta$. Thus $(\underline{J}_i + \underline{J}_j) \cap \Phi^+ \subseteq \underline{J}_{i+j}$.

Lastly, in the case where $i + j > m+1$, we have $\underline{J}_j \subseteq \underline{J}_{m+1-i}$, so that

$$(\underline{J}_i + \underline{J}_j) \cap \Phi^+ \subseteq (\underline{J}_i + \underline{J}_{m+1-i}) \cap \Phi^+ \subseteq \underline{J}_{m+1} = \underline{J}_{i+j}.$$

Thus the chain of ideals $\underline{\mathcal{I}}$ is geometric. It is also clearly positive, so $\underline{R} = \theta^{-1}(\underline{\mathcal{I}})$ is bounded. Since $\underline{I}_i = I_i$ for $i \in [m]$, $\underline{R}$ and R are on the same side of each hyperplane of the m-Shi arrangement, so $\underline{R}$ is contained in R. $\square$

For a geometric chain of m ideals $\mathcal{I} = (I_1, I_2, \ldots, I_m)$, define $\mathsf{supp}(\mathcal{I}) = I_m \cap \Delta$. In particular, $\mathsf{supp}(\mathcal{I}) = \Delta$ if and only if $\mathcal{I}$ is positive. Let $\mathbb{N}\mathsf{supp}(\mathcal{I})$ be the additive semigroup generated by $\mathsf{supp}(\mathcal{I})$.

Lemma 6.2.8. *If $\alpha \in \mathbb{N}\mathsf{supp}(\mathcal{I})$, then $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I})$. In particular, if $r_\alpha(\underline{\mathcal{I}}) \leq m$, then $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I})$.*

Proof. First note that $\alpha \in \mathbb{N}\mathsf{supp}(\mathcal{I})$ implies that $r_\alpha(\mathcal{I}) < \infty$. So may write $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_l$ with $\alpha_i \in I_{r_i}$ for $i \in [l]$ and $r_1 + r_2 + \ldots + r_l = r_\alpha(\mathcal{I})$. Since $\alpha_i \in I_{r_i} = \underline{I}_{r_i}$ this implies that $r_\alpha(\underline{\mathcal{I}}) \leq r_\alpha(\mathcal{I})$.

We may write $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_l$ with $\alpha_i \in \underline{I}_{r_i}$ for $i \in [l]$ and $r_1 + r_2 + \ldots + r_l = r_\alpha(\underline{\mathcal{I}})$. We wish to show that $r_\alpha(\mathcal{I}) \leq r_\alpha(\underline{\mathcal{I}})$. Thus we seek to write $\alpha = \alpha'_1 + \alpha'_2 + \ldots + \alpha'_{l'}$ with $\alpha'_i \in I_{r'_i}$ for $i \in [l']$ and $r'_1 + r'_2 + \ldots + r'_l = r_\alpha(\underline{\mathcal{I}})$. If $r_p = m+1$ for some $p \in [l]$, then $\alpha_p \in \underline{I}_{m+1} = \bigcup_{i+j=m+1}((I_i + I_j) \cap \Phi^+) \cup I_m \cup \Delta$. If $\alpha_p \in I_m = \underline{I}_m$, we get a contradiction with the minimality of $r_\alpha(\underline{I})$. If $\alpha_p \in \Delta$, then since $\alpha_p \in \mathbb{N}\mathsf{supp}(\mathcal{I})$, we have that $\alpha_p \in \mathsf{supp}(\mathcal{I}) \subseteq I_m$, again a contradiction. So $\alpha_p \in \bigcup_{i+j=m+1}((I_i + I_j) \cap \Phi^+)$. Thus write $\alpha_p = \beta_p + \beta'_p$, where $\beta_p \in I_i$ and $\beta'_p \in I_j$ for some i, j with $i + j = m+1$. So in the sum $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_m$ replace each α_p with $r_p = m+1$ with $\beta_p + \beta'_p$ to obtain (after renaming) $\alpha = \alpha'_1 + \alpha'_2 + \ldots + \alpha'_{l'}$ with $\alpha'_i \in I_{r'_i}$ for $i \in [l']$ and $r'_1 + r'_2 + \ldots + r'_{l'} = r_\alpha(\underline{\mathcal{I}})$, as required.

If $r_\alpha(\underline{\mathcal{I}}) = k \leq m$, then $\alpha \in I_k \subseteq I_m$ by Lemma 6.2.4, so $\alpha \in \mathbb{N}\mathsf{supp}(\mathcal{I})$ and thus $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I})$. $\square$

For R a dominant m-Shi region, define the *pseudomaximal* alcove of R to be the maximal alcove of $\underline{R}$. This term is justified by the following proposition.

Proposition 6.2.9. *If R is a bounded dominant m-Shi region, its pseudomaximal alcove is equal to its maximal alcove.*

Proof. Let A and B be the maximal and pseudomaximal alcoves of R respectively. If $\mathcal{I} = \theta(R)$, then $r(\alpha, A) = r_\alpha(\mathcal{I})$ for all $\alpha \in \Phi^+$. Since B is the maximal alcove of $\underline{R}$, we have $r(\alpha, B) = r_\alpha(\underline{\mathcal{I}})$ for all $\alpha \in \Phi^+$. Now $\mathcal{I}$ is positive since R is bounded, so $\mathsf{supp}(\mathcal{I}) = \Delta$. Thus $r_\alpha(\mathcal{I}) = r_\alpha(\underline{\mathcal{I}})$ for all $\alpha \in \Phi^+$ by Lemma 6.2.8. So $r(\alpha, A) = r(\alpha, B)$ for all $\alpha \in \Phi^+$ and therefore $A = B$. $\square$

Lemma 6.2.10. *Let R be an m-Shi region, let be B be its pseudomaximal alcove and let $t \leq m$ be a positive integer. If $\langle x_0, \alpha \rangle > t$ for some $x_0 \in R$, then $\langle x, \alpha \rangle > t$ for all $x \in B$.*

Proof. Let $\mathcal{I} = \theta(R)$. Since $r(B, \alpha) = r_\alpha(\underline{\mathcal{I}})$ for all $\alpha \in \Phi^+$, it suffices to show that $r_\alpha(\underline{\mathcal{I}}) > t$. If $r_\alpha(\underline{\mathcal{I}}) > m$ this is immediate, so we may assume that $r_\alpha(\underline{\mathcal{I}}) \leq m$. Thus we have $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I})$ by Lemma 6.2.8. Write $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_l$, with $\alpha_i \in I_{r_i}$ for all $i \in [l]$ and $r_1 + r_2 + \ldots + r_l = r_\alpha(\mathcal{I})$. Then $\langle x, \alpha_i \rangle < r_i$ for all $i \in [l]$ and $x \in R$, so $\langle x, \alpha \rangle < r_\alpha(\mathcal{I})$ for all $x \in R$. So if $\langle x_0, \alpha \rangle > t$ for some $x_0 \in R$, then $r_\alpha(\mathcal{I}) > \langle x_0, \alpha \rangle > t$, so $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I}) > t$. □

Lemma 6.2.11. *If α is a rank k indecomposable element of $\mathcal{I}$, then α is a rank k indecomposable element of $\underline{\mathcal{I}}$.*

Proof. Let α be a rank k indecomposable element of $\mathcal{I}$. Then $\alpha \in I_k = \underline{I_k}$, and $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I}) = k$ by Lemma 6.2.8. We have that $\alpha \notin I_i + I_j = \underline{I_i} + \underline{I_j}$ for $i + j = k$. If $r_{\alpha+\beta}(\underline{\mathcal{I}}) = t \leq m + 1$, then $\alpha + \beta \in \underline{I_t}$ by Lemma 6.2.4. So if $t \leq m$, we have $r_{\alpha+\beta}(\mathcal{I}) = r_{\alpha+\beta}(\underline{\mathcal{I}})$ by Lemma 6.2.8. If $t = m + 1$, then $\alpha + \beta \in I_m$ or $\alpha + \beta \in \bigcup_{i+j=m+1}((I_i + I_j) \cap \Phi^+)$, since $\alpha + \beta \notin \Delta$. Either way, $\alpha + \beta \in \mathbb{N}I_m$ so $r_{\alpha+\beta}(\mathcal{I}) = r_{\alpha+\beta}(\underline{\mathcal{I}})$ by Lemma 6.2.8. Thus we have $r_\alpha(\mathcal{I}) + r_\beta(\mathcal{I}) = r_{\alpha+\beta}(\mathcal{I}) = r_{\alpha+\beta}(\underline{\mathcal{I}}) = t$ using Lemma 6.2.5. So $r_\beta(\mathcal{I}) = t - r_\alpha(\mathcal{I}) = t - k$, so $\beta \in I_{t-k} = \underline{I_{t-k}}$ by Lemma 6.2.4. Thus α is a rank k indecomposable element of $\underline{\mathcal{I}}$. □

Lemma 6.2.12. *If $\alpha \in \Phi^+$ and H_α^k is a ceiling of a dominant m-Shi region R, then α is a rank k indecomposable element of $\mathcal{I} = \theta(R)$.*

Proof. Since the origin and R are on the same side of H_α^k, we have that $\langle x, \alpha \rangle < k$ for all $x \in R$, so $\alpha \in I_k$ and thus $r_\alpha(\mathcal{I}) \leq k$. But if $r_\alpha(\mathcal{I}) = i < k$, then $\alpha \in I_i$ by Lemma 6.2.4, so $\langle x, \alpha \rangle < i \leq k - 1$ for all $x \in R$. So H_α^k is not a wall of R, a contradiction. Thus $r_\alpha(\mathcal{I}) = k$.

If $\alpha = \beta + \gamma$ for $\beta \in I_i$ and $\gamma \in I_j$ with $i + j = k$, then the fact that $\langle x, \alpha \rangle < k$ for all $x \in R$ is a consequence of $\langle x, \beta \rangle < i$ and $\langle x, \gamma \rangle < j$ for all $x \in R$, so H_α^k does not support a facet of R. So $\alpha \notin I_i + I_j$ for $i + j = k$.

If $r_{\alpha+\beta}(\mathcal{I}) = t \leq m$, then $\alpha + \beta \in I_t$ by Lemma 6.2.4, so $\langle x, \alpha + \beta \rangle < t$ for all x in R. If also $\langle x, \beta \rangle > t - k$ for all $x \in R$, then $\langle x, \alpha \rangle < k$ for all $x \in R$ is a consequence of these, so H_α^k does not support a facet of R. So $\langle x, \beta \rangle < t - k$ for all $x \in R$, so $\beta \in I_{t-k}$.

Thus α is a rank k indecomposable element of $\mathcal{I}$. □

Proof of Theorem 6.2.2. We take B to be the pseudomaximal alcove of R, that is the maximal alcove of $\underline{R}$. We will show that (1) ⇒ (2) ⇒ (3) ⇒ (1).

The statement that (1) ⇒ (2) is Lemma 6.2.12.

For (2) ⇒ (3), suppose α is a rank k indecomposable element of $\mathcal{I}$. Then by Lemma 6.2.11, α is also a rank k indecomposable element of $\underline{\mathcal{I}}$. So by Lemma 6.2.5, we have $r_\alpha(\underline{\mathcal{I}}) = r_\beta(\underline{\mathcal{I}}) + r_\gamma(\underline{\mathcal{I}}) - 1$ if $\alpha = \beta + \gamma$ for $\beta, \gamma \in \Phi^+$, and also $r_\alpha(\underline{\mathcal{I}}) + r_\beta(\underline{\mathcal{I}}) = r_{\alpha+\beta}(\underline{\mathcal{I}})$ if $\beta, \alpha + \beta \in \Phi^+$. Thus there exists an alcove B' with $r(B', \beta) = r_\beta(\underline{\mathcal{I}})$ for $\beta \neq \alpha$ and $r(B', \alpha) = r_\alpha(\underline{\mathcal{I}}) + 1$ by Lemma 6.2.1. Since $r(B, \beta) = r_\beta(\underline{\mathcal{I}})$ for all $\beta \in \Phi^+$, this means that B' and B are on the same side of each hyperplane of the affine Coxeter arrangement, except for $H_\alpha^{r_\alpha(\mathcal{I})} = H_\alpha^k$. Thus H_α^k is a wall of B. Since H_α^k does not separate B from the origin, it is a ceiling of B.

For (3) ⇒ (1), suppose H_α^k is a ceiling of B. Let B' be the alcove which is the reflection of B in the hyperplane H_α^k. Then $\langle x, \alpha \rangle > k$ for all $x \in B'$, so by Lemma 6.2.10 the alcove B' is not contained in R. Thus H_α^k is a wall of R. It does not separate R from the origin, so it is a ceiling of R. This completes the proof. □

6.3 PROOF OF THE MAIN RESULT

We are now in a position to prove Theorem 6.1.1.

Proof of Theorem 6.1.1. For $l = 0$, the statement is immediate. Suppose that $0 < l \leq r$.

To define the bijection Θ, let $R \in U(M)$ and let A be the minimal alcove of R. The reflections $s_{\alpha_1}^{i_1}, \ldots, s_{\alpha_l}^{i_l}$ in the hyperplanes $H_{\alpha_1}^{i_1}, \ldots, H_{\alpha_l}^{i_l}$ are reflections in facets of the alcove $A = \widetilde{w}_R A_\circ$, so the set $S' := \{s_{\alpha_1}^{i_1}, \ldots, s_{\alpha_l}^{i_l}\}$ equals $\widetilde{w}_R J \widetilde{w}_R^{-1}$ for some $J \subset \widetilde{S}$. Thus the reflection group W' generated by S' is a proper parabolic subgroup of $\widetilde{W}$. In particular, it is finite. With respect to the finite reflection group W', the alcove A is contained in the dominant Weyl chamber, that is the set

$$C' = \{x \in V \mid \langle x, \alpha_j \rangle > i_j \text{ for all } j \in [l]\}.$$

So if w_0' is the longest element of W' with respect to the generating set S', the alcove $A' = w_0'(A)$ is contained in the Weyl chamber

$$w_0'(C) = \{x \in V \mid \langle x, \alpha_j \rangle < i_j \text{ for all } j \in [l]\}$$

of W', so it is on the other side of all the hyperplanes $H_{\alpha_1}^{i_1}, \ldots, H_{\alpha_l}^{i_l}$. A' is an alcove, so it is contained in some m-Shi region R'. Set $\Theta(R) = R'$.

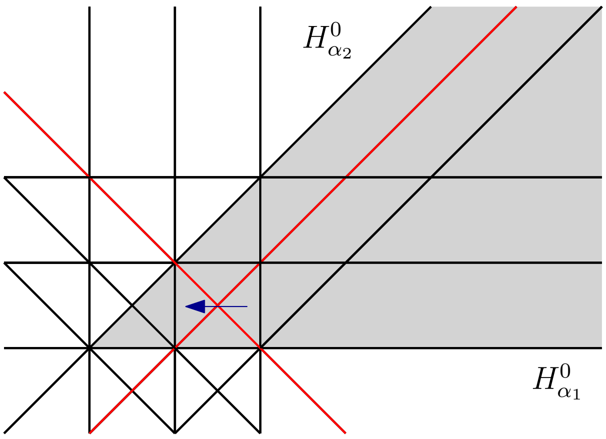

Figure 6.3.1: The bijection Θ for the 2-Shi arrangement of the root system of type B_2 with $M = \{H_{\alpha_2}^1, H_{2\alpha_1+\alpha_2}^2\}$.

Claim 6.3.1. *The m-Shi region R' is dominant and all hyperplanes in M are ceilings of R', that is $R' \in L(M)$, so Θ is well-defined.*

Proof of Claim. The origin is contained in the Weyl chamber $w_0'(C)$ of W'. Thus no reflection in W' fixes the origin. We can write $A' = w_0'(A)$ as $t_n \cdots t_1(A)$ where $t_i \in W'$ is a reflection in a facet of $t_{i-1} \cdots t_1(A)$ for all $i \in [r]$. In fact, if $w_0' = s_1' \cdots s_n'$ with $s_i' \in S'$ for all $i \in [n]$ is a reduced expression for w_0' in W', we can take $t_i = s_1' \cdots s_{i-1}' s_i' s_{i-1}' \cdots s_1'$. So $t_i \cdots t_1(A)$ and $t_{i-1} \cdots t_1(A)$ are on the same side of every hyperplane in the affine Coxeter arrangement of Φ except for the reflecting hyperplane of t_i. Since t_i does not fix the origin, if $t_{i-1} \cdots t_1(A)$ is dominant, then so is $t_i \cdots t_1(A)$. Thus by induction on i, the alcove A' is dominant, so R' is dominant.

Consider the Coxeter arrangement of W', which is the hyperplane arrangement given by the reflecting hyperplanes of all the reflections in W'. The action of W' on V restricts to an action on the set of these hyperplanes. Since $H_{\alpha_1}^{i_1}, \ldots, H_{\alpha_l}^{i_l}$ support facets of A, $w_0'(H_{\alpha_1}^{i_1}), \ldots, w_0'(H_{\alpha_l}^{i_l})$ support facets of $A' = w_0'(A)$. Now the set $\{w_0'(H_{\alpha_1}^{i_1}), \ldots, w_0'(H_{\alpha_l}^{i_l})\}$ is the set of walls of $w_0'(C)$ in the Coxeter arrangement of W', so it equals the set $M = \{H_{\alpha_1}^{i_1}, \ldots, H_{\alpha_l}^{i_l}\}$. Since all hyperplanes in M are floors of A, and A' is on the other side of each of them, they are all ceilings of A'. Thus they are ceilings of R'. □

We show that Θ is a bijection by exhibiting its inverse Ψ, a map from $L(M)$ to $U(M)$. Suppose $R' \in L(M)$. Let B be the alcove in R' given by Theorem 6.2.2. Let R'' be the region that contains $B' = w_0'(B)$. Similarly to the proof of Claim 6.3.1, we have that $R'' \in U(M)$. So let $\Psi(R') = R''$.

Claim 6.3.2. *The maps Θ and Ψ are inverse to each other, so Θ is a bijection.*

Proof of Claim. Suppose $R \in U(M)$, $R' = \Theta(R)$ and $R'' = \Psi(R')$. Use the same notation as above for the alcoves A, A', B and B'. Suppose for contradiction that $R'' \neq R$. Then there is an m-Shi hyperplane $H = H_\alpha^k$ that separates R and R''. So H separates A and B'. Now A and B' are in the dominant Weyl chamber of W', so they are on the same side of each reflecting hyperplane of W'. Thus H is not a reflecting hyperplane of W'. Now we may write A' as $t_n \cdots t_1(A)$, where $t_i \in W'$ is a reflection in a facet of $t_{i-1} \cdots t_1(A)$ for all $i \in [n]$. So $t_i \cdots t_1(A)$ and $t_{i-1} \cdots t_1(A)$ are on the same side of every hyperplane in the affine Coxeter arrangement, except for the reflecting hyperplane of t_i, which cannot be H. Thus by induction on i, the alcove A' is on the same side of H as A. Similarly B is on the same side of H as B'. So A' and B are on different sides of H, a contradiction, as they are contained in the same region, namely R'. Thus $\Psi(\Theta(R)) = R'' = R$, so $\Psi \circ \Theta = id$. Similarly $\Theta \circ \Psi = id$, so Θ and Ψ are inverse to each other, so Θ is a bijection. □

For any dominant alcove, at least one of its $r+1$ facets must either be a floor or contain the origin, and at least one must be a ceiling. So it has at most r ceilings and at most r floors. So any dominant m-Shi region R has at most r ceilings and at most r floors. Thus if $l > r$, both $U(M)$ and $L(M)$ are empty. This completes the proof. □

6.4 COROLLARIES

We deduce some enumerative corollaries of Theorem 6.1.1. For any set M of m-Shi hyperplanes, let $U_=(M)$ be the set of dominant m-Shi regions R such that the floors of R are exactly the hyperplanes in M, and let $L_=(M)$ be the set of dominant m-Shi regions R' such that the ceilings of R' are exactly the hyperplanes in M.

Corollary 6.4.1. *For any set $M = \{H_{\alpha_1}^{i_1}, H_{\alpha_2}^{i_2}, \ldots, H_{\alpha_l}^{i_l}\}$ of l hyperplanes with $i_j \in [m]$ and $\alpha_j \in \Phi^+$ for all $j \in [l]$, we have that $|U_=(M)| = |L_=(M)|$.*

Proof. This follows from Theorem 6.1.1 by an application of the Principle of Inclusion and Exclusion. □

Corollary 6.4.2. *For any tuple $(a_1, a_2, \ldots, a_m)$ of nonnegative integers, the number of dominant regions R that have exactly a_j floors of level j for all $j \in [m]$ is the same as the number of dominant regions R' that have exactly a_j ceilings of level j for all $j \in [m]$.*

Proof. Sum Corollary 6.4.1 over all sets M containing exactly a_j hyperplanes of level j for all $j \in [m]$. □

Proof of Corollary 6.1.2. Set $a_k = l$ and sum Corollary 6.4.2 over all choices of a_j for all $j \neq k$. □

6.5 OUTLOOK

Richard Stanley has asked whether Theorem 6.1.1 also holds for the set of m-Shi regions contained in some non-dominant chamber wC. We suspect the answer to this question to be yes, but we have no proof.

7 CHAPOTON TRIANGLES

In this chapter, which is based on [Thi14b], we introduce the Fuß-Catalan objects $\mathsf{NN}_\Phi^{(m)}$, $\mathsf{NC}_\Phi^{(m)}$ and $\mathsf{Assoc}_\Phi^{(m)}$. With each of these we associate a *Chapoton triangle*, a polynomial in variables x and y that encodes refined enumerative information about it. These are the *H-triangle* $H_\Phi^{(m)}(x,y)$, the *M-triangle* $M_\Phi^{(m)}(x,y)$ and the *F-triangle* $F_\Phi^{(m)}(x,y)$ respectively. They were introduced by Chapoton at the Coxeter-Catalan level of generality [Cha04, Cha06], and later generalised by Armstrong to the corresponding Fuß-Catalan objects [Arm09, Section 5.3].

We will prove the $H = F$ correspondence which relates the H-triangle and the F-triangle by an invertible transformation of variables. It was originally conjectured by Chapoton at the Coxeter-Catalan level [Cha06, Conjecture 6.1] and later generalised by Armstrong to the Fuß-Catalan level [Arm09, Conjecture 5.3.2]. We deduce a number of corollaries from it.

7.1 THREE FUSS-CATALAN OBJECTS AND THEIR CHAPOTON TRIANGLES

For any (not necessarily irreducible) crystallographic root system Φ of rank r and any positive integer m we introduce the three Fuß-Catalan objects $\mathsf{NN}_\Phi^{(m)}$, $\mathsf{NC}_\Phi^{(m)}$ and $\mathsf{Assoc}_\Phi^{(m)}$ together with their Chapoton triangles $H_\Phi^{(m)}(x,y)$, $M_\Phi^{(m)}(x,y)$ and $F_\Phi^{(m)}(x,y)$.

7.1.1 *m-nonnesting partitions*

An *m-nonnesting partition* of Φ is simply a geometric chain $\mathcal{J}$ of m order filters in the root poset of Φ. We let $\mathsf{NN}_\Phi^{(m)}$ denote the set of m-nonnesting partitions of Φ.

We define the H-triangle [Arm09, Definition 5.3.1] as

$$H_\Phi^{(m)}(x,y) = \sum_{\mathcal{J} \in \mathsf{NN}_\Phi^{(m)}} x^{|\operatorname{ind}(\mathcal{J})|} y^{|\operatorname{ind}(\mathcal{J}) \cap \Delta|}.$$

Recall that $\operatorname{ind}(\mathcal{J})$ denotes the set of rank m indecomposable elements of $\mathcal{J}$, as defined in Section 4.3.2.

7.1.2 *m-noncrossing partitions*

Consider the Weyl group W of the root system Φ. A *standard Coxeter element* in this group is a product of all the simple reflections in some order. A *Coxeter element* is any element of W that is conjugate to a standard Coxeter element. If Φ is irreducible, then the order of any Coxeter element of W is the Coxeter number h of Φ.

Let T denote the set of reflections in W. For $w \in W$, define the *absolute length* $l_T(w)$ of w as the minimal l such that $w = t_1 t_2 \cdots t_l$ for some $t_1, t_2, \ldots, t_l \in T$. Define the *absolute order* on W by

$$u \leq_T v \text{ if and only if } l_T(u) + l_T(u^{-1}v) = l_T(v).$$

Fix a Coxeter element $c \in W$. An *m-delta sequence* is a sequence $\delta = (\delta_0, \delta_1, \ldots, \delta_m)$ with $\delta_i \in W$ for all $i \in \{0, 1, \ldots, m\}$ such that $c = \delta_0 \delta_1 \cdots \delta_m$ and $r = l_T(c) = \sum_{i=0}^m l_T(\delta_i)$. Define a partial order on k-delta sequences by

$$\delta \leq \epsilon \text{ if and only if } \delta_i \geq_T \epsilon_i \text{ for all } i \in \{1, 2, \ldots, m\}.$$

The set of m-delta sequences with this partial order is called the poset of *m-divisible non-crossing partitions* $\mathsf{NC}_\Phi^{(m)}$ [Arm09, Definition 3.3.1.]. It is a graded poset with rank function $\mathrm{rk}(\delta) = l_T(\delta_0)$. We drop the choice of the Coxeter element c from the notation, since a different choice of Coxeter element results in a different but isomorphic poset.

We define the M-triangle [Arm09, Definition 5.3.1] as

$$M_\Phi^{(m)}(x,y) = \sum_{\delta,\epsilon \in \mathsf{NC}_\Phi^{(m)}} \mu(\delta,\epsilon) x^{\mathrm{rk}(\delta)} y^{\mathrm{rk}(\epsilon)},$$

where rk is the rank function of the graded poset $\mathsf{NC}_\Phi^{(m)}$ and μ is its Möbius function.

7.1.3 *The m-cluster complex*

Let $\Phi_{\geq -1}^{(m)}$ be the set of *m-coloured almost positive roots* of Φ, containing one uncoloured copy of each negative simple root and m copies of each positive root, each with a different colour from the colour set $\{1, 2, \ldots, m\}$. Then there exists a symmetric binary relation called *compatibility* [FR05, Definition 3.1] on $\Phi_{\geq -1}^{(m)}$. This has the following two important properties: firstly, all uncoloured negative simple roots are pairwise compatible. Secondly, whenever $\alpha \in \Delta$ is an uncoloured simple root and $\beta^{(k)} \in \Phi^+$ is a positive root with colour k, then $-\alpha$ is compatible with $\beta^{(k)}$ if and only if $\alpha \not\leq \beta$. Notice that the colour k of $\beta^{(k)}$ does not matter in this case.

Define a simplicial complex $\mathsf{Assoc}_\Phi^{(m)}$ as the set of all subsets $A \subseteq \Phi_{\geq -1}^{(m)}$ such that all m-coloured almost positive roots in A are pairwise compatible. This is the *m-cluster complex* of Φ. All of its facets have cardinality r.

We define the F-triangle [Arm09, Definition 5.3.1] as

$$F_\Phi^{(m)}(x,y) = \sum_{A \in \mathsf{Assoc}_\Phi^{(m)}} x^{|A \cap \Phi^+|} y^{|A \cap -\Delta|}$$

Define $f_{i,j}^{(m)}(\Phi) := [x^i y^j] F_\Phi^{(m)}(x,y)$ as the coefficient of $x^i y^j$ in $F_\Phi^{(m)}(x,y)$. Then $f_{i,j}^{(m)}(\Phi)$ is the number of faces of $\mathsf{Assoc}_\Phi^{(m)}$ containing exactly i coloured positive roots and exactly j uncoloured negative simple roots.

7.2 THE H=F CORRESPONDENCE

Our main aim for this chapter is to prove the following theorem, known as the $H = F$ correspondence.

Theorem 7.2.1 ([Arm09, Conjecture 5.3.2]). *If Φ is a crystallographic root system of rank r, then*

$$H_\Phi^{(m)}(x,y) = (x-1)^r F_\Phi^{(m)}\left(\frac{1}{x-1}, \frac{1+(y-1)x}{x-1}\right).$$

Our strategy is as follows. First we will show that the identity holds when specialised to $y = 1$. Then we introduce a bijection that leads to a differential equation for the H-triangle. The F-triangle is known to satisfy a similar differential equation, and putting all of these facts together allows us to prove the $H = F$ correspondence by induction on r.

7.2.1 *The h-vector of* $\mathsf{Assoc}_\Phi^{(m)}$

The *h-vector* $\left(h_0^{(m)}(\Phi), h_1^{(m)}(\Phi), \ldots, h_r^{(m)}(\Phi)\right)$ of $\mathsf{Assoc}_\Phi^{(m)}$ is defined by the relation

$$\sum_{i=0}^{r} h_i^{(m)}(\Phi) x^{r-i} = \sum_{i,j} f_{i,j}^{(m)}(\Phi)(x-1)^{r-(i+j)}.$$

There is a uniform proof that $h_i^{(m)}(\Phi)$ equals the number $\mathrm{Nar}^{(m)}(W, r-i)$ of m-noncrossing partitions of rank $r-i$ for all $i \in \{0,1,2,\ldots,r\}$ [STW15, Corollary 5.43]. Thus Theorem 5.5.7 implies the following result relating the h-vector of $\mathsf{Assoc}_\Phi^{(m)}$ to the Fuß-Narayana numbers.

Theorem 7.2.2. *For every irreducible crystallographic root system Φ of rank r and $i \in \{0,1,\ldots,r\}$ we have*

$$h_i^{(m)}(\Phi) = \mathsf{Nar}_\Phi^{(m)}(r-i).$$

The following lemma is the specialisation of the $H = F$ correspondence at $y = 1$.

Lemma 7.2.3. *If Φ is a crystallographic root system of rank n, then*

$$H_\Phi^{(m)}(x,1) = (x-1)^r F_\Phi^{(m)}\left(\frac{1}{x-1}, \frac{1}{x-1}\right).$$

Proof. Suppose first Φ is irreducible. Then we have

$$(x-1)^r F_\Phi^{(m)}\left(\frac{1}{x-1}, \frac{1}{x-1}\right) = \sum_{i,j} f_{i,j}(x-1)^{r-(i+j)} = \sum_{i=0}^{r} h_i^{(m)}(\Phi) x^{r-i},$$

where $\left(h_0^{(m)}(\Phi), h_1^{(m)}(\Phi), \ldots, h_r^{(m)}(\Phi)\right)$ is the h-vector of $\mathsf{Assoc}_\Phi^{(m)}$. So

$$[x^i](x-1)^r F_\Phi^{(m)}\left(\frac{1}{x-1}, \frac{1}{x-1}\right) = h_{r-i}^{(m)}(\Phi) = \mathsf{Nar}_\Phi^{(m)}(i),$$

by Theorem 7.2.2. But

$$[x^i] H_\Phi^{(m)}(x,1) = \mathsf{Nar}_\Phi^{(m)}(i)$$

by Corollary 5.5.6, as required.

If Φ is reducible, say $\Phi = \Phi_1 \cup \Phi_2$ with $\Phi_1 \perp \Phi_2$, then [Kra06a, Proposition F]

$$H_\Phi^{(m)}(x,y) = H_{\Phi_1}^{(m)}(x,y) H_{\Phi_2}^{(m)}(x,y) \text{ and } F_\Phi^{(m)}(x,y) = F_{\Phi_1}^{(m)}(x,y) F_{\Phi_2}^{(m)}(x,y).$$

Thus the result follows from the irreducible case. □

7.3 THE BIJECTION

Apart from its specialisation at $y = 1$ (Lemma 7.2.3), the other main ingredient in the proof of the $H = F$ correspondence is the following bijection. For $I \subset \Delta$ define $\Phi(I) := \Phi \cap \mathbb{R}I$. Then $\Phi(I)$ is itself a crystallographic root system, and I is a set of simple roots for it.

Theorem 7.3.1. *For every simple root $\alpha \in \Delta$, there exists a bijection*

$$D_\alpha : \{\mathcal{J} \in \mathsf{NN}_{\Phi(\Delta)}^{(m)} : \alpha \in J_m\} \to \mathsf{NN}_{\Phi(\Delta \setminus \{\alpha\})}^{(m)}.$$

The rank k indecomposable elements of $D_\alpha(\mathcal{J})$ are exactly the rank k indecomposable elements of $\mathcal{J}$ if $k < m$. The rank m indecomposable elements of $D_\alpha(\mathcal{J})$ are exactly the rank m indecomposable elements of $\mathcal{J}$ except for α.

In order to prove this, we first need a basic lemma, implicit in [Ath05].

Lemma 7.3.2. *The rank k indecomposable elements of an m-generalised nonnesting partition $\mathcal{J} \in \mathsf{NN}_{\Phi}^{(m)}$ are minimal elements of J_k.*

Proof. Let $\alpha \in J_k$ be an indecomposable element. Suppose for contradiction that α is not minimal in J_k, say $\alpha > \beta \in J_k$. Then $\alpha = \beta + \sum_{i=1}^{l} \alpha_i$, where $\alpha_i \in \Delta$ for all $i \in [l]$. So $\sum_{i=1}^{l} \alpha_i \in \Phi$ or $\beta + \sum_{i \neq j} \alpha_i \in \Phi$ for some $j \in [l]$, by [Ath05, Lemma 2.1 (i)]. In the first case, $\alpha = \beta + \sum_{i=1}^{l} \alpha_i$, with $\beta \in J_k$ and $\sum_{i=1}^{l} \alpha_i \in J_0$, so α is not indecomposable. In the second case, $\alpha = \beta + \sum_{i \neq j} \alpha_i + \alpha_j$, with $\beta + \sum_{i \neq j} \alpha_i \in J_k$ and $\alpha_j \in J_0$, so α is not indecomposable. $\square$

Proof of Theorem 7.3.1. Let $J(\alpha)$ be the order filter in the root poset generated by α, that is $J(\alpha) = \{\beta \in \Phi^+ : \beta \geq \alpha\}$. Define $\partial_\alpha(J_i) = J_i \backslash J(\alpha) = J_i \cap \Phi(\Delta \backslash \{\alpha\})$, where $\Phi(\Delta \backslash \{\alpha\})$ is the root system with simple system $\Delta \backslash \{\alpha\}$. Then let $D_\alpha(\mathcal{J}) = (\partial_\alpha(J_1), \partial_\alpha(J_2), \ldots, \partial_\alpha(J_m))$.

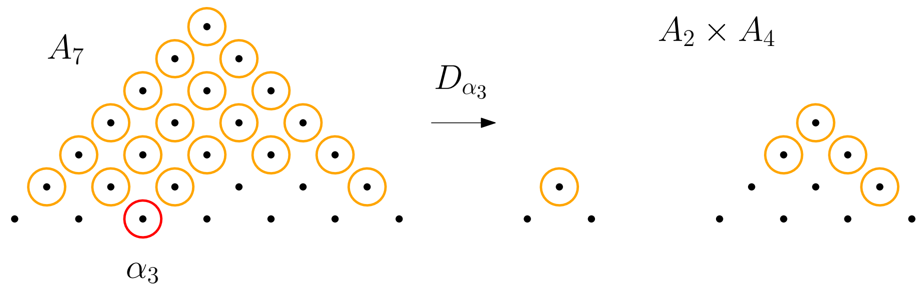

Figure 7.3.1: The bijection D_{α_3} for the root system of type A_7, applied to an order filter J.

We claim that $D_\alpha(\mathcal{J})$ is an m-nonnesting partition of $\Phi(\Delta \backslash \{\alpha\})$ and thus D_α is well-defined.

In order to see this, first observe that every $\partial_\alpha(J_i)$ is an order filter in the root poset of $\Phi(\Delta \backslash \{\alpha\})$, and the $\partial_\alpha(J_i)$ form a (multi)chain under inclusion. For all $i, j \in \{0, 1, \ldots, m\}$, we have

$$(\partial_\alpha(J_i) + \partial_\alpha(J_j)) \cap \Phi^+(\Delta \backslash \{\alpha\}) \subseteq (J_i + J_j) \cap \Phi^+(\Delta \backslash \{\alpha\}) \subseteq J_{i+j} \cap \Phi^+(\Delta \backslash \{\alpha\}) = \partial_\alpha(J_{i+j}).$$

For convenience, let us denote $\Phi(\Delta \backslash \{\alpha\}) \backslash \partial_\alpha(J_i)$ by $\partial_\alpha(I_i)$. Then $\partial_\alpha(I_i) = I_i$ for all $i \in \{0, 1, \ldots, m\}$, so $(\partial_\alpha(I_i) + \partial_\alpha(I_j)) \cap \Phi^+(\Delta \backslash \{\alpha\}) \subseteq \partial_\alpha(I_{i+j})$ for all i, j with $i + j \leq m$. So $D_\alpha(\mathcal{J})$ is a geometric chain of order filters in the root poset of $\Phi(\Delta \backslash \{\alpha\})$, and the claim follows.

Now define a map U_α from $\mathsf{NN}_{\Phi(\Delta \backslash \{\alpha\})}^{(m)}$ to $\{\mathcal{J} \in \mathsf{NN}_{\Phi(\Delta)}^{(m)} : \alpha \in J_m\}$ by $\sigma_\alpha(J_i) = J_i \cup J(\alpha)$, and $U_\alpha(\mathcal{J}) = (\sigma_\alpha(J_1), \sigma_\alpha(J_2), \ldots, \sigma_\alpha(J_m))$.

We claim that $U_\alpha(\mathcal{J})$ is an m-nonnesting partition of $\Phi(\Delta)$ and thus U_α is well-defined.

In order to see this, first observe that every $\sigma_\alpha(J_i)$ is an order filter in the root poset of $\Phi(\Delta)$ and the $\sigma_\alpha(J_i)$ form a (mulit)chain under inclusion. For all $i, j \in \{0, 1, \ldots, m\}$,

$$\begin{aligned}(\sigma_\alpha(J_i) + \sigma_\alpha(J_j)) \cap \Phi^+(\Delta) &= ((J_i \cup J(\alpha)) + (J_j \cup J(\alpha)) \cap \Phi^+(\Delta) \\ &\subseteq ((J_i + J_j) \cap \Phi^+(\Delta)) \cup J(\alpha) \subseteq J_{i+j} \cup J(\alpha) = \sigma_\alpha(J_{i+j}).\end{aligned}$$

For convenience, let us denote $\Phi(\Delta) \backslash \sigma_\alpha(J_i)$ by $\sigma_\alpha(I_i)$. Then $\sigma_\alpha(I_i) = I_i$ for all $i \in \{0, 1, \ldots, m\}$, so

$$(\sigma_\alpha(I_i) + \sigma_\alpha(I_j)) \cap \Phi^+(\Delta) = (\sigma_\alpha(I_i) + \sigma_\alpha(I_j)) \cap \Phi^+(\Delta \backslash \{\alpha\}) \subseteq \sigma_\alpha(I_{i+j})$$

for all i, j with $i + j \leq m$. So $U_\alpha(\mathcal{J})$ is a geometric chain of order filters in the root poset of $\Phi(\Delta)$, and the claim follows.

Now D_α and U_α are inverse to each other, so D_α is a bijection, as required.

We claim that for $\beta \in \Phi^+$, β is a rank k indecomposable element of $D_\alpha(\mathcal{J})$ if and only if β is a rank k indecomposable element of $\mathcal{J}$ and $\beta \neq \alpha$.

In order to see this, first notice that for $\beta \in \Phi(\Delta\backslash\{\alpha\})$, $k_\beta(D_\alpha(\mathcal{J})) = k_\beta(\mathcal{J})$. Now for the "if" direction, suppose $\beta \neq \alpha$ is a rank k indecomposable element of $\mathcal{J}$. The only element in $J_k \backslash \partial_\alpha(J_k) = J(\alpha)$ that can be indecomposable of rank k in $\mathcal{J}$ is α, since all other elements are not minimal in J_k, so not indecomposable by Lemma 7.3.2. So $\beta \in \partial_\alpha(J_k)$. If β were not indecomposable in $D_\alpha(\mathcal{J})$, then either $\beta = \gamma + \delta$ for $\gamma \in \partial_\alpha(J_i)$, $\delta \in \partial_\alpha(J_j)$, with $i + j = k$, in contradiction to β being indecomposable in $\mathcal{J}$, or there is a $\gamma \in \Phi(\Delta\backslash\{\alpha\})\backslash\partial_\alpha(J_{t-k})$ with $\beta + \gamma \in \partial_\alpha(J_t)$ and $k_{\beta+\gamma}(D_\alpha(\mathcal{J})) = t$, for some $k \leq t \leq m$, also in contradiction to β being indecomposable in $\mathcal{J}$. So β is rank k indecomposable in $D_\alpha(\mathcal{J})$.

For the "only if" direction, suppose β is a rank k indecomposable element of $D_\alpha(\mathcal{J})$, and suppose for contradiction that β were not indecomposable in $\mathcal{J}$. If $\beta = \gamma + \delta$ for $\gamma \in J_i$, $\delta \in J_j$, with $i + j = k$, then $\alpha \not\leq \gamma$ and $\alpha \not\leq \delta$, so $\gamma \in \partial_\alpha(J_i)$ and $\delta \in \partial_\alpha(J_j)$, a contradiction to β being indecomposable in $D_\alpha(\mathcal{J})$. If $\beta + \gamma \in J_t$ and $k_{\beta+\gamma}(\mathcal{J}) = t$ for some $k \leq t \leq m$, and $\gamma \in \Phi(\Delta\backslash\{\alpha\})$, then $\gamma \in \partial_\alpha(J_{t-k}) \subseteq J_{t-k}$, as β is indecomposable in $D_\alpha(\mathcal{J})$. If $\beta + \gamma \in J_t$ for some $k \leq t \leq m$ and $\gamma \notin \Phi(\Delta\backslash\{\alpha\})$, then $\gamma \in J(\alpha)$, so $\gamma \in J_m \subseteq J_{t-k}$. So β is indecomposable in $\mathcal{J}$. This establishes the claim.

Thus D_α is a bijection having the desired properties. □

7.4 PROOF OF THE H=F CORRESPONDENCE

To prove the $H = F$ correspondence, we set up differential equations for the H-triangle and the F-triangle. We use these together with Lemma 7.2.3 and induction on r to deduce the result.

Lemma 7.4.1 ([Kra06a, Proposition F (2)]). *If Φ is a crystallographic root system, then*

$$\frac{\partial}{\partial y} F^{(m)}_{\Phi(\Delta)} = \sum_{\alpha\in\Delta} F^{(m)}_{\Phi(\Delta\backslash\{\alpha\})}(x, y).$$

Proof. As mentioned in [Kra06a], this can be proven in the same way as the $m = 1$ case, which is due to Chapoton [Cha04, Proposition 3]. For completeness, as well as to highlight the analogy to the proof of Lemma 7.4.2, we give the proof here.

We wish to show that

$$j f^{(m)}_{i,j}(\Phi) = \sum_{\alpha\in\Delta} f^{(m)}_{i,j-1}(\Phi(\Delta\backslash\{\alpha\}))$$

for all i, j, that is we seek a bijection

$$\varphi : \{(A, -\alpha) : A \in \mathsf{Assoc}^{(m)}_{\Phi(\Delta)} \text{ and } -\alpha \in A \cap (-\Delta)\} \to \amalg_{\alpha\in\Delta}\mathsf{Assoc}^{(m)}_{\Phi(\Delta\backslash\{\alpha\}}$$

such that $\varphi(A, -\alpha)$ contains the same number of coloured positive roots as A, but exactly one less uncoloured negative simple root. For this it is sufficient to find for each $\alpha \in \Delta$ a bijection

$$\varphi_\alpha : \{A \in \mathsf{Assoc}^{(m)}_{\Phi(\Delta)} : -\alpha \in A\} \to \mathsf{Assoc}^{(m)}_{\Phi(\Delta\backslash\{\alpha\}}$$

with the same property. By [FR05, Proposition 3.5], we may take

$$\varphi_\alpha : A \mapsto A\backslash\{-\alpha\}.$$

□

Lemma 7.4.2. *If Φ is a crystallographic root system, then*

$$\frac{\partial}{\partial y} H^{(m)}_{\Phi(\Delta)}(x, y) = x \sum_{\alpha\in\Delta} H^{(m)}_{\Phi(\Delta\backslash\{\alpha\})}(x, y).$$

Proof. Analogously to Lemma 7.4.1, we seek a bijection

$$D : \{(\mathcal{J},\alpha) : \mathcal{J} \in \mathsf{NN}^{(m)}_{\Phi(\Delta)} \text{ and } \alpha \in \mathrm{ind}(\mathcal{J}) \cap \Delta\} \to \amalg_{\alpha\in\Delta}\mathsf{NN}^{(m)}_{\Phi(\Delta\setminus\{\alpha\})}$$

such that $D(\mathcal{J},\alpha)$ has exactly one less simple rank m indecomposable element and exactly one less rank m indecomposable element than $\mathcal{J}$. We note that any simple root $\alpha \in J_m \cap \Delta$ is automatically rank m indecomposable, so it is sufficient to find for each $\alpha \in \Delta$ a bijection

$$D_\alpha : \{\mathcal{J} \in \mathsf{NN}^{(m)}_{\Phi(\Delta)} : -\alpha \in J_m\} \to \mathsf{NN}^{(m)}_{\Phi(\Delta\setminus\{\alpha\})}$$

with the same property. Such a bijection is given in Theorem 7.3.1. □

We are now in a position to prove the $H = F$ correspondence.

Proof of Theorem 7.2.1. We proceed by induction on r. If $r = 0$, both sides are equal to 1, so the result holds. If $r > 0$,

$$\frac{\partial}{\partial y}H^{(m)}_{\Phi(\Delta)}(x,y) = x\sum_{\alpha\in S} H^{(m)}_{\Phi(\Delta\setminus\{\alpha\})}(x,y),$$

by Lemma 7.4.2. By induction hypothesis, this is further equal to

$$x\sum_{\alpha\in\Delta}(x-1)^{r-1}F^{(m)}_{\Phi(\Delta\setminus\{\alpha\})}\left(\frac{1}{x-1},\frac{1+(y-1)x}{x-1}\right),$$

which equals

$$\frac{\partial}{\partial y}(x-1)^r F^{(m)}_{\Phi(S)}\left(\frac{1}{x-1},\frac{1+(y-1)x}{x-1}\right)$$

by Lemma 7.4.1. But

$$H^{(m)}_{\Phi}(x,1) = (x-1)^r F^{(m)}_{\Phi}\left(\frac{1}{x-1},\frac{1}{x-1}\right)$$

by Lemma 7.2.3, so

$$H^{(m)}_{\Phi}(x,y) = (x-1)^r F^{(m)}_{\Phi}\left(\frac{1}{x-1},\frac{1+(y-1)x}{x-1}\right),$$

since the derivatives with respect to y as well as the specialisations at $y = 1$ of both sides agree. □

7.5 COROLLARIES OF THE H=F CORRESPONDENCE

Specialising Theorem 7.2.1 to $m = 1$, we can now prove Chapoton's original conjecture.

Corollary 7.5.1 ([Chao6, Conjecture 6.1])**.** *If Φ is a crystallographic root system of rank r, then*

$$H^{(1)}_{\Phi}(x,y) = (1-x)^r F^{(1)}_{\Phi}\left(\frac{x}{1-x},\frac{xy}{1-x}\right).$$

Proof. We have

$$H^{(1)}_{\Phi}(x,y) = (x-1)^r F^{(1)}_{\Phi}\left(\frac{1}{x-1},\frac{1+(y-1)x}{x-1}\right). \tag{7.5.1}$$

But we also have [Chao4, Proposition 5]

$$F^{(1)}_{\Phi}(x,y) = (-1)^r F^{(1)}_{\Phi}(-1-x,-1-y). \tag{7.5.2}$$

Substituting (7.5.2) into (7.5.1), we obtain

$$H^{(1)}_{\Phi}(x,y) = (1-x)^r F^{(1)}_{\Phi}\left(\frac{x}{1-x},\frac{xy}{1-x}\right).$$

□

Using the $M = F$ (ex-)conjecture, we can also relate the H-triangle to the M-triangle.

Corollary 7.5.2 ([Arm09, Conjecture 5.3.2])**.** *If Φ is a crystallographic root system of rank r, then*

$$H_{\Phi}^{(m)}(x,y) = (1+(y-1)x)^r M_{\Phi}^{(m)}\left(\frac{y}{y-1}, \frac{(y-1)x}{1+(y-1)x}\right).$$

Proof. We have

$$H_{\Phi}^{(m)}(x,y) = (x-1)^r F_{\Phi}^{(m)}\left(\frac{1}{x-1}, \frac{1+(y-1)x}{x-1}\right). \tag{7.5.3}$$

But we also have [Kra06a, Conjecture FM] [Tza08, Theorem 1.2]

$$F_{\Phi}^{(m)}(x,y) = y^r M_{\Phi}^{(m)}\left(\frac{1+y}{y-x}, \frac{y-x}{y}\right). \tag{7.5.4}$$

Substituting (7.5.4) into (7.5.3), we obtain

$$H_{\Phi}^{(m)}(x,y) = (1+(y-1)x)^r M_{\Phi}^{(m)}\left(\frac{y}{y-1}, \frac{(y-1)x}{1+(y-1)x}\right).$$

□

The coefficients of $F_{\Phi}^{(m)}(x,y)$ are known to be polynomials in m [Kra06a], so the coefficients of $H_{\Phi}^{(m)}(x,y)$ are also polynomials in m. Thus it makes sense to consider $H_{\Phi}^{(m)}(x,y)$ even if m is not a positive integer. We can use Corollary 7.5.2 to transfer a remarkable instance of combinatorial reciprocity observed by Krattenthaler [Kra06b, Theorem 8] for the M-triangle to the H-triangle.

Corollary 7.5.3. *If Φ is a crystallographic root system of rank r, then*

$$H_{\Phi}^{(m)}(x,y) = (-1)^r H_{\Phi}^{(-m)}\left(1-x, \frac{-xy}{1-x}\right).$$

Proof. We have

$$H_{\Phi}^{(m)}(x,y) = (1+(y-1)x)^r M_{\Phi}^{(m)}\left(\frac{y}{y-1}, \frac{(y-1)x}{1+(y-1)x}\right). \tag{7.5.5}$$

But we also have [Kra06b, Theorem 8] [Tza08, Theorem 1.2]

$$M_{\Phi}^{(m)}(x,y) = y^r M_{\Phi}^{(-m)}\left(xy, \frac{1}{y}\right). \tag{7.5.6}$$

Substituting (7.5.6) into (7.5.5), we obtain

$$H_{\Phi}^{(m)}(x,y) = ((y-1)x)^r M_{\Phi}^{(-m)}\left(\frac{xy}{1+(y-1)x}, \frac{1+(y-1)x}{(y-1)x}\right). \tag{7.5.7}$$

Inverting (7.5.5), we get

$$M_{\Phi}^{(m)}(x,y) = (1-y)^r H_{\Phi}^{(m)}\left(\frac{y(x-1)}{1-y}, \frac{x}{x-1}\right). \tag{7.5.8}$$

Substituting (7.5.8) into (7.5.7), we obtain

$$H_{\Phi}^{(m)}(x,y) = (-1)^r H_{\Phi}^{(-m)}\left(1-x, \frac{-xy}{1-x}\right).$$

□

For $m = 1$, we can transfer a duality for the F-triangle to the H-triangle.

Corollary 7.5.4.

$$H_{\Phi}^{(1)}(x,y) = x^r H_{\Phi}^{(1)}\left(\frac{1}{x}, 1+(y-1)x\right).$$

Proof. Inverting Theorem 7.2.1, we get

$$F_{\Phi}^{(1)}(x,y) = x^r H_{\Phi}^{(1)}\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right). \tag{7.5.9}$$

Thus

$$\begin{aligned} H_{\Phi}^{(1)}(x,y) &= (1-x)^r F_{\Phi}^{(1)}\left(\frac{x}{1-x}, \frac{xy}{1-x}\right) \\ &= x^r H_{\Phi}^{(1)}\left(\frac{1}{x}, 1+(y-1)x\right), \end{aligned}$$

using Corollary 7.5.1 and (7.5.9). □

7.6 OUTLOOK

Chapoton has recently introduced an F-triangle and an H-triangle for *quadrangulations* and *serpent nests* respectively [Cha15]. He conjectures that the same $H = F$ correspondence holds between these also [Cha15, Conjecture 4.5]. This very intriguing conjecture remains open.

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