Suites combinatoires modulo puissances de 3

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Congruences for Catalan and Motzkin numbers and related sequences

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Abstract

We prove various congruences for Catalan and Motzkin numbers as well as related sequences. The common thread is that all these sequences can be expressed in terms of binomial coefficients. Our techniques are combinatorial and algebraic: group actions, induction, and Lucas' congruence for binomial coefficients come into play. A number of our results settle conjectures of Cloitre and Zumkeller. The Thue–Morse sequence appears in several contexts.

Central trinomial coefficients modulo 3

Theorem (Deutsch and Sagan)

Let T_n denote the n-th central trinomial coefficient, that is, the coefficient of z^n in $(1+z+z^2)^n$. Then

$$T_n \equiv \begin{cases} 1 \pmod 3, & \textit{if } n \in T(01), \\ 0 \pmod 3, & \textit{otherwise}. \end{cases}$$

Here, $T(01)$ denotes the set of all positive integers n, which have only digits 0 and 1 in their ternary expansion.

Motzkin numbers modulo 3

Theorem (Deutsch and Sagan)

The Motzkin numbers M_n satisfy

$$M_n \equiv \begin{cases} 1 \pmod 3, & \text{if } n \in 3T(01) \text{ or } n \in 3T(01)-2, \\ -1 \pmod 3, & \text{if } n \in 3T(01)-1, \\ 0 \pmod 3, & \text{otherwise.} \end{cases}$$

Here, $T(01)$ denotes the set of all positive integers n, which have only digits 0 and 1 in their ternary expansion.

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Theorem (Deutsch and Sagan)

The central binomial coefficients satisfy

$$\binom{2n}{n} \equiv \begin{cases} (-1)^{\delta_3(n)} \pmod 3, & \textit{if } n \in T(01), \\ 0 \pmod 3, & \textit{otherwise.} \end{cases}$$

Here, $T(01)$ denotes the set of all positive integers n, which have only digits 0 and 1 in their ternary expansion, and $\delta_3(n)$ denotes the number of 1s in the ternary expansion of n.

Catalan numbers modulo 3

Theorem (Deutsch and Sagan)

The Catalan numbers C_n satisfy

$$C_n \equiv \begin{cases} (-1)^{\delta_3^*(n+1)} \pmod 3, & \text{if } n \in T^*(01) - 1, \\ 0 \pmod 3, & \text{otherwise.} \end{cases}$$

Here, $T^*(01)$ denotes the set of all positive integers n, where all digits in their ternary expansion are 0 or 1 except for the right-most digit, and $\delta_3^*(n)$ denotes the number of 1s in the ternary expansion of n ignoring the right-most digit.

Central Eulerian numbers modulo 3

Let $A(n,k)$ denote the number of permutations of $\{1,2,\dots,n\}$ with exactly $k-1$ descents.

Theorem (Deutsch and Sagan)

The central Eulerian numbers $A(2n-1,n)$ and $A(2n,n)$ satisfy

$$A(2n-1,n) \equiv \begin{cases} 1 \pmod 3, & \text{if } n \in T(01)+1, \\ 0 \pmod 3, & \text{otherwise.} \end{cases}$$

and

$$A(2n,n) \equiv \begin{cases} 1 \pmod 3, & \text{if } n \in T(01)+1, \\ -1 \pmod 3, & \text{if } n \in T(01) \text{ or } n \in T(01)+2, \\ 0 \pmod 3, & \text{otherwise.} \end{cases}$$

Here, $T(01)$ denotes the set of all positive integers n, which have only digits 0 and 1 in their ternary expansion.

The paper by Deutsch and Sagan contains results of similar nature for *Motzkin prefix numbers*, *Riordan numbers*, *sums of central binomial coefficients*, *central Delannoy numbers*, *Schröder numbers*, and *hex tree numbers*.

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The goal of today:

Show that:

- All these results are trivial$^{\copyright}$!
- There is a meta-theorem which covers them all.
- One can generalize everything to congruences modulo any power of 3.

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Generating Functions!!

Let us have another look at the central trinomial numbers theorem:

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Let T_n denote the n-th central trinomial coefficient, that is, the coefficient of z^n in $(1+z+z^2)^n$. Then

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where $T(01)$ denotes the set of all positive integers n, which have only digits 0 and 1 in their ternary expansion.

In other words: Let

$$\Psi(z) = \sum_{k\ge0} \sum_{n_1>\cdots>n_k\ge0} z^{3^{n_1}+3^{n_2}+\cdots+3^{n_k}} = \prod_{j=0}^{\infty}(1+z^{3^j})$$
$$= 1+z+z^3+z^4+z^9+z^{10}+z^{12}+z^{13}+\cdots.$$

Then: $\sum_{n\ge0} T_n z^n = \Psi(z)$ modulo 3.

A functional equation modulo 3 satisfied by $\Psi(z)$

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Lemma

The series $\Psi(z) = \prod_{j=0}^{\infty}(1 + z^{3^j})$ *satisfies*

$$\Psi^2(z) = \frac{1}{1+z} \quad \text{modulo } 3.$$

Proof.

We have

$$\begin{aligned}
\Psi^2(z) &= \prod_{j=0}^{\infty}(1 + z^{3^j})^2 = \frac{1}{1+z}(1+z)\prod_{j=0}^{\infty}(1 + z^{3^j})^2 \\
&= \frac{1}{1+z}(1+z)^3\prod_{j=1}^{\infty}(1 + z^{3^j})^2 = \frac{1}{1+z}(1+z^3)\Psi^2(z^3) \quad \text{modulo } 3 \\
&= \frac{1}{1+z}(1+z^9)\Psi^2(z^9) \quad \text{modulo } 3 \\
&= \cdots \\
&= \frac{1}{1+z} \quad \text{modulo } 3.
\end{aligned}$$

A functional equation modulo 3 satisfied by $\Psi(z)$

Lemma

The series $\Psi(z)=\prod_{j=0}^{\infty}(1+z^{3^j})$ *satisfies*

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It is well-known that the generating function $T(z) \sum_{n\ge0} T_n z^n$ is given by $T(z) = 1/\sqrt{1-2z-3z^2}$, or, phrased differently,

$$(1-2z-3z^2)T^2(z) - 1 = 0.$$

Morover, this functional equation determines $T(z)$ uniquely.

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$$(1-2z-3z^2)T^2(z) - 1 = 0.$$

Morover, this functional equation determines $T(z)$ uniquely. Taken modulo 3, the above functional equation becomes:

$$(1+z)T^2(z) - 1 = 0 \quad \text{modulo } 3.$$

Consequently:

$$\sum_{n\ge0} T_n z^n = \Psi(z) \quad \text{modulo } 3.$$

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Equivalently:

$$\begin{aligned} \sum_{n\ge0} M_n z^n &= z^{-1} - z^{-2} + (1 - z^{-1} + z^{-2})\Psi(z^3) \quad \text{modulo } 3 \\ &= z^{-1} - z^{-2} + (z^{-1} + z^{-2})(1+z)\Psi(z^3) \quad \text{modulo } 3 \\ &= z^{-1} - z^{-2} + (z^{-1} + z^{-2})\Psi(z) \quad \text{modulo } 3. \end{aligned}$$

Want to prove:

$$\sum_{n\geq 0} M_n z^n = z^{-1} - z^{-2} + (z^{-1} + z^{-2})\Psi(z) \quad \text{modulo } 3.$$

It is well-known (and easy to see) that the generating function $M(z) = \sum_{n\geq 0} M_n z^n$ satisfies

$$z^2 M^2(z) + (z-1)M(z) + 1 = 0.$$

Hence, to verify the claim above, we substitute in the left-hand side:

$$z^2 M^2(z) + (z-1)M(z) + 1 = z^2 \Big(z^{-1} - z^{-2} + (z^{-1} + z^{-2})\Psi(z)\Big)^2 \\ + (z-1)\Big(z^{-1} - z^{-2} + (z^{-1} + z^{-2})\Psi(z)\Big) + 1.$$

This vanishes indeed modulo 3, once we invoke the relation

$$\Psi^2(z) = \frac{1}{1+z} \quad \text{modulo } 3.$$

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Let us have another look at the central binomial coefficients theorem:

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where $\delta_3(n)$ denotes the number of 1s in the ternary expansion of n.

Equivalently:

$$\sum_{n\ge0} \binom{2n}{n} z^n = \Psi(-z) \quad \text{modulo } 3.$$

Want to prove:

$$\sum_{n\ge0}\binom{2n}{n}z^n=\Psi(-z)\quad\text{modulo }3.$$

It is well-known that

$$CB(z)=\sum_{n\ge0}\binom{2n}{n}z^n=\frac{1}{\sqrt{1-4z}},$$

and, hence,

$$(1-4z)CB^2(z)-1=0.$$

In view of

$$\Psi^2(-z)=\frac{1}{1-z}\quad\text{modulo }3,$$

this is obvious.

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The Catalan numbers C_n satisfy

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Here, $T^*(01)$ denotes the set of all positive integers n, where all digits in their ternary expansion are 0 or 1 except for the right-most digit, and $\delta_3^*(n)$ denotes the number of 1s in the ternary expansion of n ignoring the right-most digit.

Equivalently:

$$\begin{aligned}\sum_{n\ge 0} C_n z^n &= -z^{-1} + (z^{-1}+1+z)\Psi(-z^3) \quad \text{modulo } 3\\ &= -z^{-1} + z^{-1}(1-z)^2\Psi(-z^3) \quad \text{modulo } 3\\ &= -z^{-1} + z^{-1}(1-z)\Psi(-z) \quad \text{modulo } 3.\end{aligned}$$

Want to prove:

$$\sum_{n\ge0} C_n z^n = -z^{-1} + z^{-1}(1-z)\Psi(-z) \quad \text{modulo } 3.$$

It is well-known that the generating function $C(z) = \sum_{n\ge0} C_n z^n$ satisfies

$$zC^2(z) - C(z) + 1 = 0.$$

Hence, to verify the claim above, we substitute in the left-hand side:

$$zC^2(z) - C(z) + 1 = z\Big(- z^{-1} + z^{-1}(1-z)\Psi(-z)\Big)^2 \\ - \Big(- z^{-1} + z^{-1}(1-z)\Psi(-z)\Big) + 1.$$

This vanishes indeed modulo 3, once we invoke the relation

$$\Psi^2(-z) = \frac{1}{1-z} \quad \text{modulo } 3.$$

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What are the common features?

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- In each case, one could express the generating function, after reduction of its coefficients modulo 3, as a linear expression in $\Psi(\pm z)$.

Can this be so many accidents?

A meta-theorem

Theorem

Let $F(z)$ be a formal power series with integer coefficients which satisfies a quadratic equation

$$c_2(z)F^2(z)+c_1(z)F(z)+c_0(z)=0 \quad \text{modulo } 3,$$

where

1. $c_2(z)=z^{e_1}(1+\varepsilon z^\gamma)^{e_2}$ modulo 3, *with non-negative integers* e_1,e_2 *and* $\varepsilon\in\{1,-1\}$*;*
2. $c_1^2(z)-c_0(z)c_2(z)=z^{2f_1}(1+\varepsilon z^\gamma)^{2f_2+1}$ modulo 3, *with non-negative integers* f_1,f_2*.*

Then

$$F(z)=\frac{c_1(z)}{c_2(z)}\pm\frac{z^{f_1}(1+\varepsilon z^\gamma)^{f_2+1}}{c_2(z)}\Psi(\varepsilon z^\gamma) \quad \text{modulo } 3.$$

Proof.

Proof.

. . . . $\square$

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. . . . $\square$

The corresponding choices of $c_2(z), c_1(z), c_0(z)$ are:

	$c_2(z)$	$c_1(z)$	$c_0(z)$	$c_1^2(z) - c_0(z)c_2(z) \mod 3$
trinomial	$1-2z-3z^2$	0	-1	$1+z$
Motzkin	z^2	$z-1$	1	$1+z$
cent.bin.	$1-4z$	0	-1	$1-z$
Catalan	z	-1	1	$1-z$
Motz.pref.	$z-3z^2$	$1-3z$	-1	$1+z$
Riordan	$z+z^2$	$1+z$	1	$1+z$
Delannoy	$1-6z+z^2$	0	-1	$1+z^2$
Schröder	z	$z-1$	1	$1+z^2$
hex tree	z^2	$3z-1$	1	$1-z^2$
...	...	...	...	...

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Certainly:

$$\left(\Psi^2(z) - \frac{1}{1+z}\right)^{\alpha} = 0 \quad \text{modulo } 3^{\alpha}.$$

So, in particular,

$$\left(\Psi^2(z) - \frac{1}{1+z}\right)^{3^{\alpha}} = 0 \quad \text{modulo } 3^{3^{\alpha}}.$$

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So, in particular,

$$\left(\Psi^2(z) - \frac{1}{1+z}\right)^{3^{\alpha}} = 0 \quad \text{modulo } 3^{3^{\alpha}}.$$

Expressing the generating function as a polynomial in $\Psi(z)$ makes of course only sense, if we also know how to extract coefficients from powers $\Psi^e(z)$ modulo a given power of 3; more on this later.

The “method” for proving congruences modulo 3^k

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We suppose that the generating function $F(z) = \sum_{n\ge0} f_n\, z^n$ satisfies a differential equation of the form

$$\mathcal{P}(z; F(z), F'(z), F''(z), \dots, F^{(s)}(z)) = 0,$$

where $\mathcal{P}$ is a polynomial with integer coefficients. We want to solve the problem of determining the coefficients f_n modulo powers of 3.

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Idea:

Make the Ansatz

$$F(z) = \sum_{i=0}^{2\cdot 3^\alpha - 1} a_i(z)\Psi^i(z) \quad \text{modulo } 3^{3^\alpha},$$

where the $a_i(z)$'s are (at this point) undetermined elements of $\mathbb{Z}[z, z^{-1}, (1+z)^{-1}]$.

Then, gradually determine approximations $a_{i,\beta}(z)$ to $a_i(z)$ such that the differential equation above holds modulo 3^β, for $\beta = 1, 2, \dots, 3^\alpha$.

The "method" for proving congruences modulo 3^k

The base step:

Substitute

$$F(z)=\sum_{i=0}^{2\cdot 3^\alpha-1} a_{i,1}(z)\Psi^i(z) \quad \text{modulo } 3$$

into the differential equation, considered modulo 3,

$$\mathcal{P}(z;F(z),F'(z),F''(z),\dots,F^{(s)}(z))=0 \quad \text{modulo } 3,$$

use $\Psi'(z)=\Psi(z)/(1+z)$ modulo 3, reduce high powers of $\Psi(z)$ modulo the polynomial relation of degree $2\cdot 3^\alpha$ satisfied by $\Psi(z)$, and compare coefficients of powers $\Psi^k(z)$, $k=0,1,\dots,2\cdot 3^\alpha-1$. This yields a system of $2\cdot 3^\alpha$ (algebraic differential) equations (modulo 3) for the unknown Laurent polynomials $a_{i,1}(z)$, $i=0,1,\dots,2\cdot 3^\alpha-1$, which may or may not have a solution.

The “method” for proving congruences modulo 3^k

The iteration:

Provided we have already found $a_{i,\beta}(z)$, $i = 0, 1, \dots, 2 \cdot 3^\alpha - 1$, such that

$$F(z) = \sum_{i=0}^{2\cdot3^\alpha-1} a_{i,\beta}(z)\Psi^i(z)$$

solves our differential equation modulo 3^β, we put

$$a_{i,\beta+1}(z) := a_{i,\beta}(z) + 3^\beta b_{i,\beta+1}(z), \quad i = 0, 1, \dots, 2 \cdot 3^\alpha - 1,$$

where the $b_{i,\beta+1}(z)$'s are (at this point) undetermined Laurent polynomials in z. Next we substitute

$$F(z) = \sum_{i=0}^{2\cdot3^\alpha-1} a_{i,\beta+1}(z)\Psi^i(z)$$

in the differential equation.

The "method" for proving congruences modulo 3^k

The iteration:

One uses
$$\Psi'(z) = \Psi(z) \sum_{j=0}^{\beta} \frac{3^j z^{3^j - 1}}{1 + z^{3^j}} \quad \text{modulo } 3^{\beta+1},$$
one reduces high powers of $\Psi(z)$ using the polynomial relation satisfied by $\Psi(z)$, and one compares coefficients of powers $\Psi^j(z)$, $j = 0, 1, \ldots, 2 \cdot 3^\alpha - 1$. After simplification, this yields a system of $2 \cdot 3^\alpha$ (linear differential) equations (modulo 3) for the unknown Laurent polynomials $b_{i,\beta+1}(z)$, $i = 0, 1, \ldots, 2 \cdot 3^\alpha - 1$, which may or may not have a solution.

Theorem

Let α *be some positive integer. Furthermore, suppose that the formal power series* $F(z)$ *with integer coefficients satisfies the functional-differential equation*

$$c_2(z)F^2(z) + c_1(z)F(z) + c_0(z) + 3\mathcal{Q}(z; F(z), F'(z), F''(z), \dots, F^{(s)}(z)) = 0,$$

where

1. $c_2(z) = z^{e_1}(1 + \varepsilon z^{\gamma})^{e_2}$ modulo 3, *with non-negative integers* e_1, e_2 *and* $\varepsilon \in \{1, -1\}$;
2. $c_1^2(z) - c_0(z)c_2(z) = z^{2f_1}(1 + \varepsilon z^{\gamma})^{2f_2+1}$ modulo 3, *with non-negative integers* f_1, f_2;
3. $\mathcal{Q}$ *is a polynomial with integer coefficients.*

Then $F(z)$*, when coefficients are reduced modulo* 3^{3^α}*, can be expressed as a polynomial in* $\Psi(\varepsilon z^\gamma)$ *of the form*

$$F(z) = a_0(z) + \sum_{i=0}^{2\cdot 3^\alpha - 1} a_i(z)\Psi^i(\varepsilon z^\gamma) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$*,* $i = 0, 1, \dots, 2 \cdot 3^\alpha - 1$*, are Laurent polynomials in* z *and* $1 + \varepsilon z^\gamma$*.*

Sketch of proof.

Base step:

$$F(z)=\frac{c_1(z)}{c_2(z)}\pm\frac{z^{f_1}(1+\varepsilon z^{\gamma})^{f_2+(3^{\alpha}+1)/2}}{c_2(z)}\Psi^{3^{\alpha}}(\varepsilon z^{\gamma})$$

solves the equation modulo 3.

Iteration step: Works smoothly; in fact, the system of equations which one has to solve is already in diagonal form for each iteration.

$\square$

Motzkin numbers modulo 27

Theorem

We have

$$\sum_{n\ge0} M_n\,z^n = 13z^{-1}+14z^{-2}+\left(9z+12+24z^{-1}+21z^{-2}\right)\Psi(z^3)$$
$$+\left(9z^5+12z^4+10z^3+23z^2+25z+19+14z^{-1}+4z^{-2}\right)\Psi^3(z^3)$$
$$-\big(9z^7+3z^6+24z^5+30z^4+6z^3$$
$$+21z^2+6z+3+24z^{-1}+12z^{-2}\big)\,\Psi^5(z^3)$$
$$\text{modulo } 27.$$

Central trinomial numbers modulo 27

Theorem

We have

$$\sum_{n\ge0} T_n\, z^n = -\left(9z^2+24z+15\right)\Psi(z^3) + \left(15z^5+25z^4+4z^3+12z^2+10z+19\right)\Psi^3(z^3) + \left(9z^8+6z^7+6z^6+9z^5+21z^4+3z^3+15z+24\right)\Psi^5(z^3) \quad \text{modulo } 27.$$

Central binomial coefficients modulo 27

Theorem

We have

$$\sum_{n\ge0}\binom{2n}{n}z^n=\left(9\frac{1+z}{1-z}+3\right)\Psi(-z)-(4z+8)\Psi^3(-z)$$
$$-\left(12z^2+12z+3\right)\Psi^5(-z)\quad\text{modulo }27.$$

Catalan numbers modulo 27

Theorem

We have

$$\sum_{n\ge0} C_n\, z^n = -13z^{-1} - 3\left(4+2z^{-1}\right)\Psi(-z)$$
$$+\left(-8z-14+4z^{-1}\right)\Psi^3(-z)$$
$$+3\left(z^2-6z+9-4z^{-1}\right)\Psi^5(-z) \quad \text{modulo } 27.$$

Coefficient extraction

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Can we extract coefficients from powers of $\Psi(z)$ (modulo 3^k)?

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For accomplishing this, we need an extension of the relation

$$\Psi^2(z) = \frac{1}{1+z} \quad \text{modulo } 3$$

to higher powers of 3. This extension comes from the identity

$$\Psi^2(z) = \frac{1}{1+z} \sum_{s\ge 0} \sum_{k_1>\cdots>k_s\ge 0} 3^s \prod_{j=1}^{s} \frac{z^{3^{k_j}}\left(1+z^{3^{k_j}}\right)}{1+z^{3^{k_j+1}}}.$$

The identity again:

$$\Psi^2(z) = \frac{1}{1+z} \sum_{s\ge 0} 3^s \sum_{k_1>\cdots>k_s\ge 0} \prod_{j=1}^{s} \frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}.$$

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Let us write

$$\widetilde{H}_{a_1,a_2,\dots,a_s}(z) := \sum_{k_1>\cdots>k_s\ge 0} \prod_{j=1}^{s} \left(\frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}} \right)^{a_j}.$$

Using this notation, the above identity can be rephrased as

$$\Psi^2(z) = \frac{1}{1+z} \sum_{s\ge 0} 3^s \widetilde{H}_{\underbrace{1,1,\dots,1}_{s \text{ times}}}(z).$$

It is not difficult to see that powers of $\Psi(z)$ can be expressed using the series $\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$.

The identity again:

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It is not difficult to see that powers of $\Psi(z)$ can be expressed in the form

$$\Psi^{2K}(z) = \frac{1}{(1+z)^K} \sum_{r=1}^{K} \sum_{a_1,\dots,a_r\ge1} c_{2K}(a_1,a_2,\dots,a_r) \widetilde{H}_{a_1,a_2,\dots,a_r}(z),$$

respectively

$$\Psi^{2K+1}(z) = \frac{1}{(1+z)^K} \Psi(z) \sum_{r=1}^{K} \sum_{a_1,\dots,a_r\ge1} c_{2K}(a_1,a_2,\dots,a_r) \widetilde{H}_{a_1,a_2,\dots,a_r}(z),$$

where the coefficients $c_{2K}(a_1,a_2,\dots,a_r)$ are suitable combinatorial coefficients, which can be written down explicitly.

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where the coefficients $c_{2K}(a_1,a_2,\dots,a_r)$ are suitable combinatorial coefficients, which can be written down explicitly.
Consequently, the coefficient extraction problem will be solved if we are able to say how to extract coefficients from the series

$$(1+z)^K\widetilde{H}_{a_1,a_2,\dots,a_r}(z) \quad \text{and} \quad (1+z)^K\Psi(z)\widetilde{H}_{a_1,a_2,\dots,a_r}(z).$$

Theorem

The binomial coefficient $\binom{2n}{n}$*, when reduced modulo* 9*, equals*

1. 0*, if, and only if,* n *contains at least two digits* 2 *or the string* 12 *in its* 3*-adic expansion;*
2. 3*, if, and only if,* n *contains the string* 02*, no other digit* 2*, and an odd number of digits* 1 *in its* 3*-adic expansion;*
3. 6*, if, and only if,* n *contains the string* 02*, no other digit* 2*, and an even number of digits* 1 *in its* 3*-adic expansion;*
4. 1*, if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\{0\} \cup \bigcup_{k \ge 0} \left(11^*00^*\right)^{3k+2} 11^*0^*,$$

where the number of digits 1 *is even;*

5. 4, *if, and only if, the* 3*-adic expansion of n is an element of*

$$\bigcup_{k\ge0}\left(11^*00^*\right)^{3k+1}11^*0^*,$$

where the number of digits 1 *is even;*

6. 7, *if, and only if, the* 3*-adic expansion of n is an element of*

$$\bigcup_{k\ge0}\left(11^*00^*\right)^{3k}11^*0^*,$$

where the number of digits 1 *is even;*

7. 2, *if, and only if, the* 3*-adic expansion of n is an element of*

$$\bigcup_{k\ge0}\left(11^*00^*\right)^{3k}11^*0^*,$$

where the number of digits 1 *is odd;*

8. 5, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\bigcup_{k\ge0}\left(11^*00^*\right)^{3k+1}11^*0^*,$$

where the number of digits 1 *is odd;*

9. 8, *if, and only if, the* 3*-adic expansion of* n *is an element of*

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where the number of digits 1 *is odd.*

Central Eulerian numbers

Central Eulerian numbers

The *Eulerian number* $A(n, k)$ is defined as the number of permutations of $\{1, 2, \ldots, n\}$ with exactly $k - 1$ descents. It is well-known that

$$A(n, k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{n+1}{k-j} j^n.$$

We are interested in analysing *central Eulerian numbers*, that is, the numbers $A(2n, n) = A(2n, n + 1)$ and $A(2n - 1, n)$, modulo powers of 3.

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We are interested in analysing *central Eulerian numbers*, that is, the numbers $A(2n,n)=A(2n,n+1)$ and $A(2n-1,n)$, modulo powers of 3.

Problem: There is (provably?) no functional or differential equation for the corresponding generating functions $\sum_{n\ge0}A(2n,n)z^n$ or $\sum_{n\ge0}A(2n-1,n)z^n$.

However: If one considers (the coefficients in the) generating functions $\sum_{n\ge0} A(2n,n)z^n$ and $\sum_{n\ge0} A(2n-1,n)z^n$ modulo a fixed power of 3, 3^k say, then they do satisfy functional equations, modulo 3^k!

First key observation

Let us consider $A(2n, n) = A(2n, n+1)$, given explicitly by

$$A(2n, n+1) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{2n+1}{n+1-j} j^{2n}.$$

Since $\varphi(3^\beta) = 2 \cdot 3^{\beta-1}$ (with $\varphi(\,.\,)$ denoting the Euler totient function), we have

$$A(2n, n+1) \equiv \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{2n+1}{n+1-j} j^{2s} \pmod{3^\beta}$$

$$\text{for} \quad n \equiv s \pmod{3^{\beta-1}} \text{ and } n, s \geq \tfrac{1}{2}(\beta - 1).$$

Second key observation

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Proposition

For any positive integer s, we have

$$\sum_{n\geq 0} z^n \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{2n+1}{n+1-j} j^{2s} = \frac{1}{2}\left(1+\sqrt{1+4z}\right)\left(1+3p_s(z)\right),$$

where $p_s(z)$ is a polynomial in z with integer coefficients, and which satisfies $p_s(0)=0$.

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where $p_s(z)$ is a polynomial in z with integer coefficients, and which satisfies $p_s(0)=0$.

That is, if we denote the generating function on the left-hand side by $E_s(z)$, then it satisfies the equation

$$E_s^2(z) - E_s(z) - z - 3p_s(z)E_s(z) - z3p_s(z)\left(2+3p_s(z)\right) = 0.$$

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This is an equation of the form

$$c_2(z)F^2(z) + c_1(z)F(z)(z) + c_0(z) + 3\mathcal{Q}(\dots) = 0$$

as in our theorem!!

Recipe for treating central Eulerian numbers modulo 3^β

- First consider $A(2n, n+1)$ only for $n \equiv s \pmod{3^{\beta-1}}$, with s fixed.
- Solve the functional equation for $E_s(z)$ modulo 3^β.
- Only the coefficients of z^n in $E_s(z)$ with $n \equiv s \pmod{3^{\beta-1}}$ are of interest to us; compute the corresponding section of the series.
- Add the various sections for $s = 0, 1, 2, \dots, 3^{\beta-1} - 1$. This yields the desired polynomial in $\Psi(z)$.

Something similar works for $A(2n-1, n)$.

The corresponding results modulo 27

Theorem

We have

$$\sum_{n\ge0} A(2n,n+1)\,z^n = 14 + 3\left(3z^2-4z+2\right)\Psi(z) + \left(21z^3+20z^2+13z+23\right)\Psi^3(z) + 3\left(6z^4+4z^3+3z^2+4\right)\Psi^5(z) \quad \text{modulo } 27.$$

The corresponding results modulo 27

Theorem

We have

$$\sum_{n\geq 0} A(2n, n+1)\, z^n = 14 + 3\left(3z^2 - 4z + 2\right)\Psi(z)$$

$$+ \left(21z^3 + 20z^2 + 13z + 23\right)\Psi^3(z)$$

$$+ 3\left(6z^4 + 4z^3 + 3z^2 + 4\right)\Psi^5(z) \quad \text{modulo } 27.$$

Theorem

We have

$$\sum_{n\geq 0} A(2n-1, n)\, z^n = -3z\left(3z^2 + 5\right)\Psi(z)$$

$$+ z\left(24z^3 + 15z^2 + 10z + 19\right)\Psi^3(z)$$

$$+ 3z\left(3z^4 + 6z^3 + 2z^2 + 7z + 8\right)\Psi^5(z) \quad \text{modulo } 27.$$